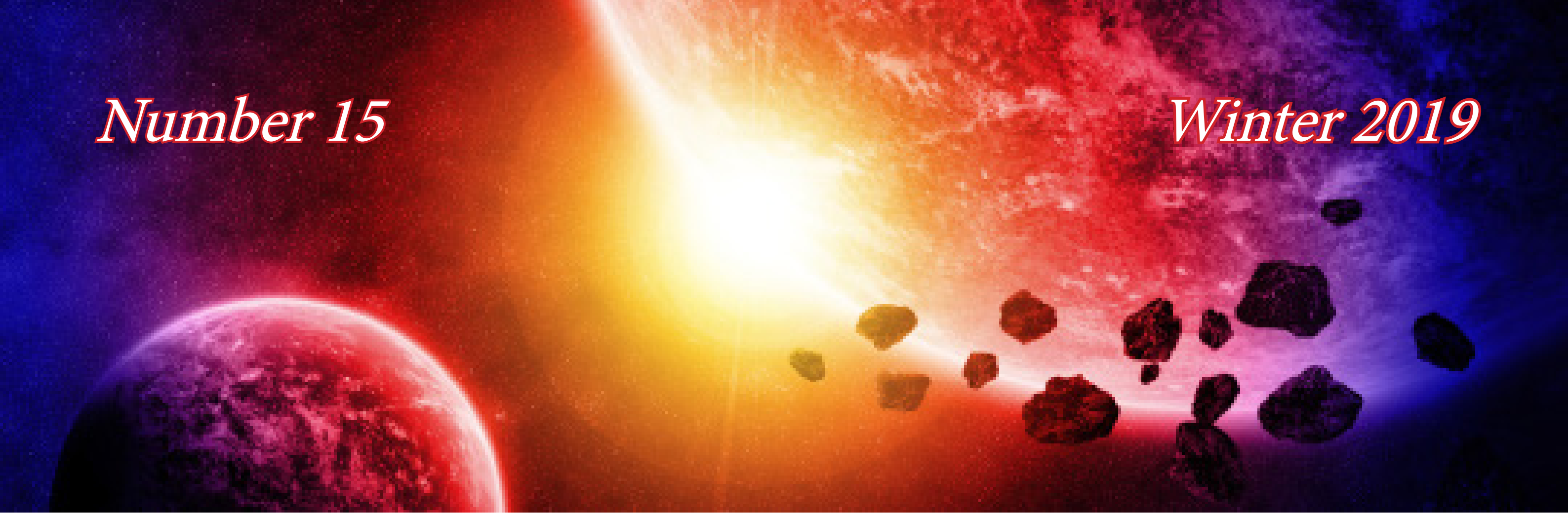

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PROBLEMS FOR JUNIORS

JP.211. Prove that there are infinitely many triples (a, b, c) of positive integers satisfying:

$$\frac{a^3+b^3+c^3}{3}-abc=a+b+c$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.212. Find all real roots of the following equation:

$$(x^3-2)^3+(x^2-2)^2=0$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.213. Prove that in any ABC triangle the following inequality holds:

$$\frac{r}{4R}(7R^2-4r^2)\le\sum m_a^2\tan^2\frac{B}{2}\tan^2\frac{C}{2}\le 4R^2-13r^2$$

Proposed by Marin Chirciu - Romania

JP.214. Prove that in any ABC triangle the following inequality holds:

$$\frac{27r^3}{2R}\le\sum m_a^2\sin^2\frac{A}{2}\le\frac{27R^2}{16}$$

Proposed by Marin Chirciu - Romania

JP.215. Prove that in any ABC triangle the following inequality holds:

$$(4R+r)^2\cdot\frac{r}{2R}\le\sum m_a^2\cos^2\frac{A}{2}\le(4R+r)^2\cdot\frac{1}{16}\left(5-\frac{2r}{R}\right)$$

Proposed by Marin Chirciu - Romania

JP.216. Prove that in any ABC triangle the following inequality holds:

$$\frac{(4R+r)^2}{r(R+r)}(-2R^2+17r^2)\le\sum m_a^2\cot^2\frac{B}{2}\cot^2\frac{C}{2}\le\frac{3(4R+r)^2}{r^2(2R-r)}(R^3-5r^3)$$

Proposed by Marin Chirciu - Romania

JP.217. Prove that in any ABC triangle the following inequality holds:

$$n\sum\sin^2 A - k\sum\cos^3 A \leq \frac{3}{8}(6n-k), \text{ where } n,k\geq 0$$

Proposed by Marin Chirciu - Romania

JP.218. Let a, b and c be positive real numbers. Prove that:
(a) $\frac{a^4+b^4}{(a^2-ab+b^2)^2}+\frac{b^4+c^4}{(b^2-bc+c^2)^2}+\frac{c^4+a^4}{(c^2-ca+a^2)^2}\leq 6$
(b) $\sqrt{\frac{a^5+b^5}{a^2+b^2}}+\sqrt{\frac{b^5+c^5}{b^2+c^2}}+\sqrt{\frac{c^5+a^5}{c^2+a^2}}\geq 3\sqrt{abc}$

Proposed by George Apostolopoulos - Messolonghi - Greece

JP.219. Let be $a,b,c>0$ such that: $a^2b^2+b^2c^2+c^2a^2=3a^2b^2c^2$. Find the maximum value of:

$$P=\frac{ab}{2a^6-a^5+b^4+a^2+1}+\frac{bc}{2b^6-b^5+c^4+b^2+1}+$$
$$+\frac{ca}{2c^6-c^5+a^4+c^2+1}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.220. Let a,b,c be positive real numbers. Prove that:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq\frac{4(a^2+b^2+c^2)}{ab+bc+ca}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.221. Let ABC be an acute-angled triangle. The perpendiculars from O on ΔABC sides, intersect BC, AC and AB sides in A_1, A_2, A_3 and the circumcircle of ΔABC in the points A_2, B_2, C_2. Prove that:

$$A_1A_2^n + B_1B_2^n + C_1C_2^n \geq 3r^n, \forall n\in\mathbb{N}^*$$

Proposed by Marian Ursărescu - Romania

JP.222. In ABC triangle the following relationship holds:

$$a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}}+b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}}+b\left(\frac{c}{b}\right)^{\frac{h_a}{w_a}}+c\left(\frac{b}{c}\right)^{\frac{h_a}{w_a}}+c\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}}+a\left(\frac{c}{a}\right)^{\frac{h_b}{w_b}}\leq 4s$$

Proposed by Daniel Sitaru - Romania

JP.223. Let a,b,c be the lengths of the sides of a triangle with circumradius R. Prove that:

$$a(a^3+(b+c)^3)+b(b^3+(c+a)^3)+c(c^3+(a+b)^3)\leq 243R^4$$

Proposed by George Apostolopoulos - Messolonghi - Greece

JP.224. Let a,b,c be the lengths of the sides of a triangle with circumradius R. Prove that:

$$\frac{(\frac{a+b}{c})^3+(\frac{b+c}{a})^3+(\frac{c+a}{b})^3+3}{\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}}\le(3R)^4$$

Proposed by George Apostolopoulos - Messolonghi - Greece

JP.225. Solve the following system of equations:

$$\begin{cases}x^3+2x+3=8y^3-6xy+4y\\ \sqrt{x^2-2y+2}+\sqrt{x^2-4y+4}=x^2-3y+4\end{cases}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

PROBLEMS FOR SENIORS

SP.211. Find all real roots of the following equation:

$$\sqrt[2n]{2-x^2}+\sqrt[2n]{2|x|-1}=(x^2-1)^{2m}+2$$

where m,n are positive integers.

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.212. Evaluate:

$$\lim_{n\to\infty}\frac{\lfloor e^{\frac{1}{n}}\rfloor+\lfloor e^{\frac{2}{n}}\rfloor+\cdots+\lfloor e^{\frac{n}{n}}\rfloor}{n}$$

where $\lfloor x\rfloor$ denotes the integer part of x.

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.213. Prove that in any ABC triangle the following inequality holds:

$$\frac{9r^2}{4R^2}(2R^2-5r^2)\le\sum m_a^2\sin^2\frac{B}{2}\sin^2\frac{C}{2}\le\frac{1}{4R^2}(4R^4-37r^4)$$

Proposed by Marin Chirciu - Romania

SP.214. Prove that in any ABC triangle the following inequality holds:

$$\frac{3r^2}{4R^2}(4R+r)^2\le\sum m_a^2\cos^2\frac{B}{2}\cos^2\frac{C}{2}<\frac{3}{16}(4R+r)^2$$

Proposed by Marin Chirciu - Romania

SP.215. Let a, b, c be positive real numbers such that $a + b + c + 1 = 4abc$. Prove that:

$$\frac{a^2b}{b+5c} + \frac{b^2c}{c+5a} + \frac{c^2a}{a+5b} \geq \frac{1}{2}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.216. Let I be the incentre of a triangle ABC with inradius r, and let K, L, M be the intersection points of the segments AI, BI, CI with the incribed circle of the triangle ABC, respectively. Prove that:

$$AK^n + BL^n + CM^n \geq 3 \cdot r^n$$

for each positive integer n.

Proposed by George Apostolopoulos - Messolonghi - Greece

SP.217. Let a, b, c be positive real numbers such that: $(a^3 + b^3)(b^3 + c^3)(c^3 + a^3) = 8$. Find the minimum value of:

$$T = \frac{a}{(b^2+bc+c^2)(a+2b)^2} + \frac{b}{(c^2+ca+a^2)(b+2c)^2} +$$

$$+ \frac{c}{(a^2+ab+b^2)(c+2a)^2}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.218. Let x, y, z be positive real numbers such that: $x^2 + y^2 + z^2 = 3$. Find the minimum of the expression:

$$P = \frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}} + 3yz} + \frac{y}{\sqrt[4]{\frac{z^8+x^8}{2}} + 3zx} + \frac{z}{\sqrt[4]{\frac{x^8+y^8}{2}} + 3xy}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.219. Prove the following inequality:

$$\sum_{k=1}^{n} \frac{a_k^2}{a_k + (n+1)(s - a_k)} \geq \frac{1}{n^2} \sum_{k=1}^{n} a_k$$

where $a_1, a_2, \ldots, a_n$ are any strictly positive real numbers and we make the notation $s = a_1 + a_2 + \ldots + a_n$

Proposed by Vasile Mircea Popa - Romania

SP.220. Let $a, b, c > 0$ such that: $a+b+c=3$. Find the minimum of the expression:

$$P = \frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}+$$
$$+\frac{(a+b)(b+c)(c+a)}{24}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.221. Prove that in any ΔABC the following inequality holds:

$$\sqrt[3]{(\pi-A)^{m_a^2}\cdot(\pi-B)^{m_b^2}\cdot(\pi-C)^{m_c^2}} \geq \sqrt[4]{(\pi-A)^{a^2}(\pi-B)^{b^2}(\pi-C)^{c^2}},$$

A, B, C the measures in radians of the angles.

Proposed by Marian Ursărescu - Romania

SP.222. Let ABC be a triangle and A', B', C' the intersection points of the simedians with circumcircle. Prove that:

$$\frac{6r}{R^2} \leq \frac{1}{KA'}+\frac{1}{KB'}+\frac{1}{KC'} \leq \frac{3R}{4r^2}$$

Proposed by Marian Ursărescu - Romania

SP.223. In ΔABC the following relationship holds:

$$\left(a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}}+b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}}\right)\left(b\left(\frac{c}{b}\right)^{\frac{h_a}{w_a}}+c\left(\frac{b}{c}\right)^{\frac{h_a}{w_a}}\right)\left(c\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}}+a\left(\frac{c}{a}\right)^{\frac{h_b}{w_b}}\right) \geq 8abc$$

Proposed by Daniel Sitaru - Romania

SP.224. In ΔABC the following relationship holds:

$$\frac{(s^2+r_ar_b)(s^2+r_br_c)(s^2+r_cr_a)}{(s^2-r_ar_b)(s^2-r_br_c)(s^2-r_cr_a)} \geq 8$$

Proposed by Daniel Sitaru - Romania

SP.225. Let a, b, c, d be positive real numbers with $abcd = 1$. Prove that:

$$\sum_{cyc}\frac{1}{a(b+c+d)} \leq \frac{1}{9}\left(\sum_{cyc}\frac{1}{a^2}+2\sum_{cyc}a^2\right)$$

Proposed by George Apostolopoulos - Messolonghi - Greece

UNDERGRADUATE PROBLEMS

UP.211. Calculate the integral:

$$\int_0^1 \frac{\sqrt{x}\ln x}{x^2+1}dx$$

Proposed by Vasile Mircea Popa - Romania

UP.212. Calculate the limit of the sequence $(a_n)_{n\geq 1}$ defined by the following relationship:

$$a_n = \frac{1}{n}\int_1^2 \ln(1+e^{n\cdot \arctan x})dx$$

Proposed by Vasile Mircea Popa - Romania

UP.213. Let $A \in M_3(\mathbb{R})$ invertible such that: $\text{Tr}\, A = \text{Tr}\, A^{-1} = 1$. Prove that:

$$\det(A^2+A+I_3) \geq 3\det A$$

Proposed by Marian Ursărescu - Romania

UP.214. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^n-1}}{e^n}\right)$$

Proposed by Daniel Sitaru - Romania

UP.215. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\int_a^b\int_a^b\Big(\frac{\cot x+\cot y+\tan(x+y)}{\cot x\cot y\tan(x+y)}\Big)dxdy \leq \frac{\pi(b-a)}{2}$$

Proposed by Daniel Sitaru - Romania

UP.216. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\int_a^b\int_a^b \frac{(1+\tan x)(1+\tan y)(1+\tan(\frac{\pi}{4}-x-y))}{1+\tan x\tan y\tan(\frac{\pi}{4}-x-y)}dxdy \leq \pi(b-a)$$

Proposed by Daniel Sitaru - Romania

UP.217. Find:

$$\Omega = \int\left(\tan\Big(\frac{\pi-9x}{3}\Big)\tan\Big(\frac{\pi-3x}{3}\Big)\tan x\tan\Big(\frac{\pi+3x}{3}\Big)\tan\Big(\frac{\pi+9x}{3}\Big)\right)dx$$

Proposed by Daniel Sitaru - Romania

UP.218. Let be $G = \{a + b\sqrt[3]{5} + c\sqrt[3]{25} | a, b, c \in \mathbb{Q}\}$. Prove that if $x \in G$ then $x^{2019} \in G$

Proposed by Daniel Sitaru - Romania

UP.219. Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$$\frac{a^3}{b^4c(a^2+ac+c^2)} + \frac{b^3}{c^4a(b^2+ba+a^2)} + \frac{c^3}{a^4b(c^2+cb+b^2)} \geq 1$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

UP.220. If $e_n = (1+\frac{1}{n})^n; n \in \mathbb{N}^*$ then find:

$$\Omega = \lim_{n\to\infty}\left((e - e_n)\cdot e^{H_n}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.221. If $(x_n)_{n\geq 1} \subset (0,\infty); \lim_{n\to\infty}\left(\frac{x_n}{\sqrt{n}}\cdot e^{2\sqrt{n}}\right) = b \in (0,\infty)$; $a_n = \sum_{k=1}^{n}\frac{1}{\sqrt{k}}$ then find:

$$\Omega = \lim_{n\to\infty}\left((e^{a_{n+1}} - e^{a_n})\cdot x_n\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.222. If $a > 0; (x_n)_{n\geq 1} \subset (0,\infty)$ such that:

$$\log(n + ax_n) = H_n - \gamma \text{ then find } \Omega = \lim_{n\to\infty} x_n$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.223. If $(a_n)_{n\geq 1}; (b_n)_{n\geq 1} \subset (0,\infty)$ such that:

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\cdot\frac{1}{n\sqrt{n}}\right) = a > 0; \lim_{n\to\infty}\left(\frac{b_{n+1}}{b_n}\cdot\sqrt{n}\right) = b > 0$$

then find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n]{a_nb_n}\cdot\left(\left(1+\frac{1}{n}\right)^{n+1} - e\right)\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.224. If $(a_n)_{n\geq 1}; (b_n)_{n\geq 1} \subset (0,\infty)$ such that:

$$\lim_{n\to\infty}\left(\frac{a_n}{n}\right) = a > 0; \lim_{n\to\infty}\left(\frac{b_{n+1}}{a_nb_n}\right) = b > 0 \text{ then find:}$$

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

UP.225. If $m \in \mathbb{N}$ then in ΔABC the following relationship holds:

$$3^m\left(\left(\frac{a}{h_a}\cot A\right)^{m+1}+\left(\frac{b}{h_b}\cot B\right)^{m+1}+\left(\frac{c}{h_c}\cot C\right)^{m+1}\right)\geq m+2$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

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