SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(1), p. 27–30.

4001. *Proposed by Cristinel Mortici and Leonard Giugiuc.*

Let $a, b, c, d \in \mathbb{R}$ with $d > 2$ such that

$$(2d+1)\cdot\frac{a}{6}+\frac{b}{2}+\frac{c}{d+1}=0.$$

Prove that there exists $t \in (0, d)$ such that $at^2 + bt + c = 0$.

We received four correct solutions and one solution that was almost complete. The first solution is due to Digby Smith and the second consists of ingredients of others.

Solution 1, by Digby Smith.

The result actually holds when $d > 1$. Let

$$2v = d+1 \quad \text{and} \quad 3w = 2d+1.$$

Then $0 < v < w < d$ with $6(w - v) = d - 1$. The given condition can be rewritten as

$$0 = avw + bv + c = v(aw + b) + c.$$

Let $f(t) = at^2 + bt + c$. Then

$$f(w) = aw^2 + bw + c = w(aw + b) + c = -\frac{wc}{v} + c = -\frac{c(w-v)}{v} = -\frac{c(d-1)}{6v}.$$

If $c \neq 0$, then it follows that $f(0) = c$ and $f(w)$ have opposite signs, so that $f(t)$ has a real root in the interval $(0, w) \subseteq (0, d)$.

If $c = 0$, then $f(w) = w(aw + b) = 0$ since $v(aw + b) = 0$.

Solution 2.

Use the notation of Solution 1. Again $0 < v < w < d$. Furthermore, when $a = 0$, $f(v) = 0$. Otherwise, we may assume that $a > 0$, in which case $f(v) < avw + bv + c = 0$.

When $c = 0$, then $f(w) = 0$. When $c > 0$, then $f(0) > 0$ and $f(t)$ has a root in $(0, v)$. Finally, when $c < 0$, then $avw + bv = -c > 0$ and

$$f(w) = aw^2 + bw - avw - bv = (w - v)(aw + b) > 0$$

and $f(t)$ has a root in (v, w).

4002. *Proposed by Henry Aniobi.*

Let f be a convex function on an interval I. Let $x_1 \leq x_2 \leq \ldots \leq x_n$ and $y_1 \leq y_2 \leq \ldots \leq y_n$ be numbers such that $x_i + y_j$ is always in I for all $1 \leq i, j \leq n$. Let $z_1, z_2, \ldots, z_n$ be an arbitrary permutation of $y_1, y_2, \ldots, y_n$. Show that

$$\begin{aligned} f(x_1+y_1)+\ldots+f(x_n+y_n) &\geq f(x_1+z_1)+\ldots+f(x_n+z_n) \\ &\geq f(x_1+y_n)+f(x_2+y_{n-1})+\ldots+f(x_n+y_1); \end{aligned}$$

We received five submissions of which four were correct and complete. We present the solution by Joseph DiMuro.

We can prove the above statement by proving the following simpler statement:

Claim. Let $x_1 < x_2$ and $y_1 < y_2$ be numbers such that $x_i + y_j$ is always in I. Then

$$f(x_1+y_1)+f(x_2+y_2) \geq f(x_1+y_2)+f(x_2+y_1).$$

The reason why this suffices: if we choose a permutation $z_1, z_2, \ldots, z_n$ such that $z_i > z_j$ for some $i < j$, then we will have

$$f(x_i+z_i)+f(x_j+z_j) \leq f(x_i+z_j)+f(x_j+z_i).$$

We would then be able to interchange z_i and z_j without decreasing the overall sum. Thus, a permutation $z_1, z_2, \ldots, z_n$ that gives us the largest overall sum is one where $z_i \leq z_j$ whenever $i < j$; that is, $z_i = y_i$ for all i. Similarly, a permutation $z_1, z_2, \ldots, z_n$ that gives us the smallest overall sum is one where $z_i \geq z_j$ whenever $i < j$; that is, $z_i = y_{n-i+1}$ for all i.

Proof of claim. By the definition of convexity, for all $a, b \in I$ and for all $t \in [0,1]$, we have

$$f(ta+(1-t)b) \leq tf(a)+(1-t)f(b).$$

Let $a = x_1 + y_1$ and $b = x_2 + y_2$. Then $a < x_1 + y_2 < b$, so for some $t \in [0,1]$, we have $x_1 + y_2 = ta + (1-t)b$. From that, we have:

$$\begin{aligned} x_2+y_1 &= (x_1+y_1+x_2+y_2)-(x_1+y_2) \\ &= (a+b)-(ta+(1-t)b) \\ &= (1-t)a+tb. \end{aligned}$$

Therefore,

$$\begin{aligned} f(x_1+y_2)+f(x_2+y_1) &= f(ta+(1-t)b)+f((1-t)a+tb) \\ &\leq (tf(a)+(1-t)f(b))+((1-t)f(a)+tf(b)) \\ &= f(a)+f(b) \\ &= f(x_1+y_1)+f(x_2+y_2), \end{aligned}$$

completing the proof of the claim.

4003. *Proposed by Martin Lukarevski.*

Show that for any triangle ABC, the following inequality holds

$$\sin A\sin B\sin C\left(\frac{1}{\sin A+\sin B}+\frac{1}{\sin B+\sin C}+\frac{1}{\sin C+\sin A}\right)$$
$$\leq\frac{3}{4}(\cos A+\cos B+\cos C).$$

We received 13 correct solutions. We present the solution by John G. Heuver, modified slightly by the editor.

Let r, R and s denote the inradius, the circumradius and the semiperimeter of $\triangle ABC$, respectively. The following identities and inequalities are well known:

$$\sum\sin^2 A=\frac{s^2-4Rr-r^2}{2R^2}, \tag{1}$$
$$\sum\sin A\sin B=\frac{s^2+4Rr+r^2}{4R^2}, \tag{2}$$
$$\sum\cos A=\frac{R+r}{R}, \tag{3}$$
$$R\geq 2r \quad \text{Euler's inequality} \tag{4}$$
$$s^2\leq 4R^2+4Rr+3r^2 \quad \text{Gerretsen's inequality,} \tag{5}$$

(where all the summations are taken over all angles of $\triangle ABC$).

Let L denote the left-hand side of the given inequality. By the AM-GM Inequality, we have $\sin A+\sin B\geq 2\sqrt{\sin A\sin B}$. Hence, by Cauchy-Schwarz Inequality we have

$$L\leq\frac{1}{2}\sum(\sin A)\sqrt{\sin B\sin C}\leq\frac{1}{2}\sqrt{\sum\sin^2 A}\sqrt{\sum\sin A\sin B}. \tag{6}$$

By (1) and (5), we have

$$\sum\sin^2 A\leq\frac{4R^2+2r^2}{2R^2}. \tag{7}$$

By (2) and (5), we have

$$\sum\sin A\sin B\leq\frac{4R^2+8Rr+4r^2}{4R^2}. \tag{8}$$

Using (6), (7) and (8) followed by (3) and (4), we then have

$$\begin{aligned}L\leq\frac{1}{2}\sqrt{\frac{4R^2+2r^2}{2R^2}}\cdot\sqrt{\frac{4R^2+8Rr+4r^2}{4R^2}}&=\frac{1}{2}\sqrt{2+\left(\frac{r}{R}\right)^2}\cdot\frac{R+r}{R}\\&\leq\frac{1}{2}\sqrt{2+\left(\frac{1}{2}\right)^2}\cdot\sum\cos A\\&=\frac{3}{4}\sum\cos A,\end{aligned}$$

which completes the proof.

Editor's comment. Digby Smith remembered that the following problem proposed by Jack Garfunkel and George Tsintsifas appeared in the August–September 1982 issue (Vol. 8, no. 7, p. 210) of ***Crux*** and a solution given by Vedula N. Murty appeared in the November 1983 issue (Vol. 9, no. 9, p. 282):

$$\frac{4}{9}\sum \sin B \sin C \leq \prod \cos\frac{B-C}{2} \leq \frac{2}{3}\sum \cos A.$$

Smith gave a proof by first showing that $2L \leq \sum \sin B \sin C$, which together with the above inequality yields the result.

4004. *Proposed by George Apostolopoulos.*

Let x, y, z be positive real numbers such that $x + y + z = 2$. Prove that

$$\frac{x^5}{yz(x^2+y^2)} + \frac{y^5}{zx(y^2+z^2)} + \frac{z^5}{xy(z^2+x^2)} \geq 1.$$

We received 16 correct submissions. We present 3 solutions.

Solution 1, by Arkady Alt.

Since by Cauchy's Inequality

$$\sum_{cyc} \frac{x^5}{yz\left(x^2+y^2\right)} = \sum_{cyc} \frac{x^6}{xyz\left(x^2+y^2\right)} \geq \frac{\left(x^3+y^3+z^3\right)^2}{\sum_{cyc} xyz\left(x^2+y^2\right)},$$

it suffices to prove the inequality

$$\frac{\left(x^3+y^3+z^3\right)^2}{\sum_{cyc} xyz\left(x^2+y^2\right)} \geq 1.$$

We have the following equivalences:

$$\begin{aligned}\frac{\left(x^3+y^3+z^3\right)^2}{\sum_{cyc} xyz\left(x^2+y^2\right)} \geq 1 \quad &\iff \quad \left(x^3+y^3+z^3\right)^2 \geq 2xyz\left(x^2+y^2+z^2\right)\\ &\iff \quad \left(x^3+y^3+z^3\right)^2 \geq xyz\left(x+y+z\right)\left(x^2+y^2+z^2\right),\end{aligned}$$

where the latter inequality holds because by AM-GM Inequality

$$x^3+y^3+z^3 \geq 3xyz$$

and by Chebyshev's Inequality

$$x^3+y^3+z^3 \geq \frac{(x+y+z)\left(x^2+y^2+z^2\right)}{3}.$$

Solution 2, by Michel Bataille.

Let $a = \frac{x}{2},\ b = \frac{y}{2}$ and $c = \frac{z}{2}$. With these notations, we are required to prove

$$\frac{a^6}{a^2+b^2} + \frac{b^6}{b^2+c^2} + \frac{c^6}{c^2+a^2} \geq \frac{abc}{2} \tag{1}$$

under the conditions $a, b, c > 0$ and $a + b + c = 1$.

The Cauchy-Schwarz inequality gives

$$\left(\frac{a^6}{a^2+b^2} + \frac{b^6}{b^2+c^2} + \frac{c^6}{c^2+a^2}\right)((a^2+b^2)+(b^2+c^2)+(c^2+a^2)) \geq (a^3+b^3+c^3)^2.$$

Hence, (1) will follow if we prove

$$\frac{(a^3+b^3+c^3)^2}{a^2+b^2+c^2} \geq abc.$$

Since $abc \leq \frac{a^3+b^3+c^3}{3}$, it is sufficient to show that

$$3(a^3+b^3+c^3) \geq a^2+b^2+c^2.$$

Now, the latter follows from

$$\begin{aligned}3(a^3+b^3+c^3) &\geq 2(a^3+b^3+c^3) + 3abc\\ &= a^3+b^3+c^3+(a^3+b^3+c^3+3abc)\\ &\geq a^3+b^3+c^3+ab^2+a^2b+bc^2+b^2c+ca^2+c^2a \quad \text{(Schur's ineq.)}\\ &= (a^2+b^2+c^2)(a+b+c) = a^2+b^2+c^2 \quad (\text{since } a+b+c=1)\end{aligned}$$

so we are done.

Solution 3, by Oliver Geupel.

By hypothesis $x + y + z = 2$ and by the Cauchy-Schwarz inequality we have

$$\begin{aligned}&\left(\frac{x^5}{yz(x^2+y^2)} + \frac{y^5}{zx(y^2+z^2)} + \frac{z^5}{xy(z^2+x^2)}\right)(x+y+z)xyz(x^2+y^2+z^2)\\ &= \left(\sum_{\text{cyc}} \frac{x^5}{yz(x^2+y^2)}\right)\left(\sum_{\text{cyc}} xyz(x^2+y^2)\right) \geq (x^3+y^3+z^3)^2.\end{aligned}$$

By the power mean inequality, it holds

$$\left(\frac{x^3+y^3+z^3}{3}\right)^{1/3} \geq \left(\frac{x^2+y^2+z^2}{3}\right)^{1/2} \geq \frac{x+y+z}{3} \geq (xyz)^{1/3}.$$

Putting together we obtain

$$\begin{aligned}
&\frac{x^5}{yz(x^2+y^2)}+\frac{y^5}{zx(y^2+z^2)}+\frac{z^5}{xy(z^2+x^2)}\\
&\qquad\ge\frac{(x^3+y^3+z^3)^2}{(x+y+z)xyz(x^2+y^2+z^2)}\\
&\qquad=\frac{(x^3+y^3+z^3)^{1/3}}{x+y+z}\cdot\frac{x^3+y^3+z^3}{xyz}\cdot\frac{(x^3+y^3+z^3)^{2/3}}{x^2+y^2+z^2}\\
&\qquad\ge 3^{-2/3}\cdot 3\cdot 3^{-1/3}=1.
\end{aligned}$$

Hence the result. By the equality condition of the power mean inequality, the equality holds if and only if $x=y=z=2/3$.

4005. *Proposed by Michel Bataille.*

Let a,b,c be the sides of a triangle with area F. Suppose that some positive real numbers x,y,z satisfy the equations

$$x+y+z=4 \quad \text{and}$$
$$2xb^2c^2+2yc^2a^2+2za^2b^2-\left(\frac{4-yz}{x}\,a^4+\frac{4-zx}{y}\,b^4+\frac{4-xy}{z}\,c^4\right)=16F^2.$$

Show that the triangle is acute and find x,y,z.

We present the proposer's solution — no others were submitted.

The second equation gives

$$\begin{aligned}
&(xyz)(16F^2)\\
&=xyz(2xb^2c^2+2yc^2a^2+2za^2b^2)-yz(4-yz)a^4-zx(4-zx)b^4-xy(4-xy)c^4\\
&=(a^2yz+b^2zx+c^2xy)^2-(4a^4yz+4b^4zx+4c^4xy)\\
&=\left(\frac{x}{2}(b^2z+c^2y)+\frac{y}{2}(c^2x+a^2z)+\frac{z}{2}(b^2x+a^2y)\right)^2-(4a^4yz+4b^4zx+4c^4xy).
\end{aligned}$$

Since $t\mapsto t^2$ is a convex function and $x+y+z=4$, Jensen's inequality yields

$$\begin{aligned}
&\frac{x}{4}(b^2z+c^2y)^2+\frac{y}{4}(c^2x+a^2z)^2+\frac{z}{4}(b^2x+a^2y)^2\\
&\qquad\ge\left(\frac{x}{4}(b^2z+c^2y)+\frac{y}{4}(c^2x+a^2z)+\frac{z}{4}(b^2x+a^2y)\right)^2 \qquad (1)
\end{aligned}$$

and it follows that

$$\begin{aligned}
&(xyz)(16F^2)\\
&\le x(b^2z+c^2y)^2+y(c^2x+a^2z)^2+z(b^2x+a^2y)^2-(4a^4yz+4b^4zx+4c^4xy)\\
&=a^4yz(y+z-4)+b^4zx(z+x-4)+c^4xy(x+y-4)+xyz(2b^2c^2+2c^2a^2+2a^2b^2)\\
&=xyz(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)\\
&=(xyz)(16F^2).
\end{aligned}$$

Thus, equality must hold in (1) and because $t \mapsto t^2$ is a strictly convex function, this calls for

$$b^2z + c^2y = c^2x + a^2z = b^2x + a^2y.$$

Setting these three expressions equal to λ and solving for x, y, z yields

$$\begin{aligned} x &= \lambda\frac{b^2+c^2-a^2}{2b^2c^2} = \frac{\lambda\cos A}{bc},\\ y &= \lambda\frac{c^2+a^2-b^2}{2c^2a^2} = \frac{\lambda\cos B}{ca},\\ z &= \lambda\frac{a^2+b^2-c^2}{2a^2b^2} = \frac{\lambda\cos C}{ab}. \end{aligned}$$

(As usual, A, B, C denote the angles of the triangle opposite sides a, b, c, respectively.) Since at most one of A, B, C is not acute and x, y, z are positive, we conclude that $\cos A, \cos B, \cos C$, and λ are positive. Thus, the triangle is acute.

In addition, we have

$$4 = \frac{\lambda\cos A}{bc} + \frac{\lambda\cos B}{ca} + \frac{\lambda\cos C}{ab}.$$

Since $a\cos A + b\cos B + c\cos C = \dfrac{2F}{R}$ and $4RF = abc$ (where R is the circumradius of the triangle), we readily find $\lambda = \dfrac{a^2b^2c^2}{2F^2}$ and obtain

$$x = \frac{a^2(b^2+c^2-a^2)}{4F^2}, \quad y = \frac{b^2(c^2+a^2-b^2)}{4F^2}, \quad z = \frac{c^2(a^2+b^2-c^2)}{4F^2}.$$

Note that conversely, if given an acute triangle, then these numbers x, y, z are positive and satisfy the two equations: $x + y + z = 4$ is readily checked; also we have $b^2z + c^2y = c^2x + a^2z = b^2x + a^2y = \lambda$, hence the calculations made at the beginning (with equality in (1)) show that the second equation holds as well.

4006. *Proposed by Dragolijub Milošević.*

Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{2}{xy+yz+zx} - \frac{1}{x+y+z} \le \frac{1}{3}.$$

We received 15 correct solutions from 14 submitters. Ten of these solutions were along the lines of the solution presented below, with variations in how they justified the ancillary inequalities and how straightforwardly they handled the algebra. In addition, there was a MAPLE-based solution, which seemed heavy-handed for this problem. There were four other solutions that were defective in some way. We present the solution by Henry Ricardo.

Let $p = x+y+z$, $q = xy+yz+zx$ and $r = xyz = 1$. Observe that $q^2 \geq 3rp = 3p$, since, by the arithmetic-geometric means inequality,

$$\begin{aligned} q^2 &= \frac{1}{2}(x^2y^2+y^2z^2) + \frac{1}{2}(y^2z^2+z^2x^2) + \frac{1}{2}(z^2x^2+x^2y^2) + 2xyz(x+y+z) \\ &\geq xy^2z + yz^2x + zx^2y + 2xyz(x+y+z) \\ &= 3xyz(x+y+z) = 3rp = 3p. \end{aligned}$$

The difference between the two sides of the inequality is one-third of

$$1 - \frac{6}{q} + \frac{3}{p} \geq 1 - \frac{6}{q} + \frac{9}{q^2} = \left(1 - \frac{3}{q}\right)^2 \geq 0,$$

and the result follows with equality if and only if $x = y = z = 1$.

Editor's comment. Oliver Geupel notes that this problem is equivalent to a problem proposed by Vasile Cîrtoaje and Mircea Lascu for the Junior TST 2003 Romania. It is also Problem 72 in Chapter 20 of *Inequalities, Theorems, Techniques and Selected Problems* by Zdravko Cvetkovski (Springer, 2012).

4007. *Proposed by Mihaela Berindeanu.*

Show that for any numbers $a, b, c > 0$ such that $a^2+b^2+c^2 = 12$, we have

$$(a^3+4a+8)(b^3+4b+8)(c^3+4c+8) \leq 24^3.$$

We received nine submission of which eight were correct and complete. We present two solutions.

Solution 1, by Ángel Plaza.

By taking logarithms, the proposed inequality may be written as

$$\frac{\ln\left(a^3+4a+8\right) + \ln\left(b^3+4b+8\right) + \ln\left(c^3+4c+8\right)}{3} \leq \ln 24.$$

Changing variables $a^2 = x$, $b^2 = y$, $c^2 = z$ the problem becomes:

For any $x, y, z > 0$ such that $x+y+z = 12$, prove that

$$\frac{\ln\left(x^{3/2}+4x^{1/2}+8\right) + \ln\left(y^{3/2}+4y^{1/2}+8\right) + \ln\left(z^{3/2}+4z^{1/2}+8\right)}{3} \leq \ln 24.$$

Let us consider function $f(x) = \ln\left(x^{3/2}+4x^{1/2}+8\right)$ for $x > 0$. Then

$$f''(x) = \frac{-8x^{3/2} - 3x^{5/2} + 12x - 16\sqrt{x} - 16}{2x^{3/2}\left(x^{3/2}+4\sqrt{x}+8\right)^2}$$

and since $f''(x) < 0$ for $x > 0$, the function f is concave. By Jensen's inequality

$$\frac{f(x)+f(y)+f(z)}{3} \leq f\left(\frac{x+y+z}{3}\right) = f(12/3) = f(4) = \ln 24.$$

Solution 2, by the proposer.

Observe that $(a-2)^4 \geq 0$ implies that $a^4 - 8a^3 + 24a^2 - 32a + 16 \geq 0$, that is

$$a^4 + 24a^2 + 80 \geq 8a^3 + 32a + 64,$$

which gives

$$(a^2+4)(a^2+20) \geq 8(a^3+4a+8)$$

and hence

$$a^3 + 4a + 8 \leq \frac{(a^2+4)(a^2+20)}{8}.$$

So,

$$(a^3+4a+8)(b^3+4b+8)(c^3+4c+8) \leq \frac{(a^2+4)(a^2+20)}{8} \cdot \frac{(b^2+4)(b^2+20)}{8} \cdot \frac{(c^2+4)(c^2+20)}{8},$$

but we know that $\sqrt[3]{xyz} \leq \dfrac{x+y+z}{3}$, therefore

$$(a^2+4)(b^2+4)(c^2+4) \leq \left(\frac{a^2+b^2+c^2+12}{3}\right)^3 = 8^3$$

and

$$(a^2+20)(b^2+20)(c^2+20) \leq \left(\frac{a^2+b^2+c^2+60}{3}\right)^3 = 24^3.$$

Finally,

$$\left(a^3+4a+8\right)\left(b^3+4b+8\right)\left(c^3+4c+8\right) \leq \frac{8^3 \cdot 24^3}{8^3} = 24^3.$$

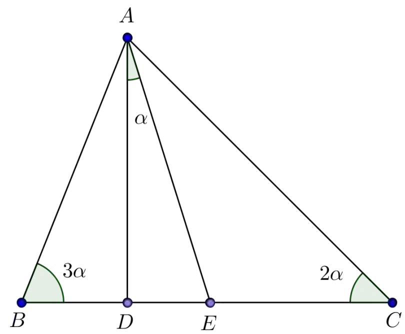
Editor's Comments. Ángel Plaza sent two solutions: the second solution consists in taking the logarithms of the given inequality, setting $a^2 = x, b^2 = y, c^2 = z$, considering the concave function (on $x \in (0,12)$) $f(x) = \ln(x^{3/2} + 4x^{1/2} + 8)$ and using Jensen's Inequality. A very similar approach was also used by Šefket Arslanagić.

4008. *Proposed by Mehmet Şahin.*

Let ABC be a triangle with $\angle ACB = 2\alpha$, $\angle ABC = 3\alpha$, AD is an altitude and AE is a median such that $\angle DAE = \alpha$. If $|BC| = a, |CA| = b, |AB| = c$, prove that

$$\frac{a}{b} = 1 + \sqrt{2\left(\frac{c}{b}\right)^2 - 1}.$$

We received 15 correct solutions and one incorrect submission. We present the solution given by Titu Zvonaru, modified slightly by the editor.



We have $AD = c\sin 3\alpha, BD = c\cos 3\alpha$, so $DE = \dfrac{a}{2} - \cos 3\alpha$. By the law of sines, we have $\dfrac{a}{\sin(180^\circ - 5\alpha)} = \dfrac{c}{\sin 2\alpha}$, so $a = \dfrac{c\sin 5\alpha}{\sin 2\alpha}$. Then

$$\begin{aligned}\frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \frac{DE}{AD} = \frac{\frac{a}{2} - \cos 3\alpha}{c\sin 3\alpha} &= \frac{\sin 5\alpha - 2\sin 2\alpha\cos 3\alpha}{2\sin 2\alpha\sin 3\alpha}\\ &= \frac{\sin 5\alpha - (\sin 5\alpha + \sin(-\alpha))}{2\sin 2\alpha\sin 3\alpha}\\ &= \frac{\sin\alpha}{2\sin 2\alpha\sin 3\alpha},\end{aligned}$$

so that

$$\cos\alpha = 2\sin 2\alpha\sin 3\alpha = \cos\alpha - \cos 5\alpha,$$

which implies that

$$\cos 5\alpha = 0, \quad \text{so} \quad 5\alpha = 90^\circ, \quad \text{or} \quad \alpha = 18^\circ.$$

Hence, $\angle BAC = 180^\circ - 5\alpha = 90^\circ$, $\angle ABC = 3\alpha = 54^\circ$ and $\angle ACB = 2\alpha = 36^\circ$. Since $\cos 36^\circ = \dfrac{1+\sqrt{5}}{4}$, we have

$$b = a\cos 2\alpha = \left(\frac{1+\sqrt{5}}{4}\right)a,$$

so

$$c = \sqrt{a^2 - b^2} = \sqrt{a^2 - \frac{3+\sqrt{5}}{8}a^2} = a\sqrt{\frac{5-\sqrt{5}}{8}}.$$

Now, $\dfrac{a}{b} = \dfrac{4}{1+\sqrt{5}} = \sqrt{5} - 1$ and

$$2\left(\frac{c}{b}\right)^2 - 1 = 2\left(\frac{5-\sqrt{5}}{8}\right)\left(\frac{4}{1+\sqrt{5}}\right)^2 - 1 = \frac{14-6\sqrt{5}}{6+2\sqrt{5}} = \frac{7-3\sqrt{5}}{3+\sqrt{5}}.$$

Therefore, we have the following equivalences:

$$\begin{aligned} &\frac{a}{b} = 1 + \sqrt{2\left(\frac{c}{b}\right)^2 - 1} \\ \iff\quad & \sqrt{5} - 1 = 1 + \sqrt{\frac{7 - 3\sqrt{5}}{3 + \sqrt{5}}} \\ \iff\quad & (\sqrt{5} - 2)^2 = \frac{7 - 3\sqrt{5}}{3 + \sqrt{5}} \\ \iff\quad & (9 - 4\sqrt{5})(3 + \sqrt{5}) = 7 - 3\sqrt{5} \\ \iff\quad & 7 - 3\sqrt{5} = 7 - 3\sqrt{5}, \end{aligned}$$

which is true and our proof is complete.

4009. *Proposed by George Apostolopoulos.*

Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC. Prove that

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \le \frac{R}{2r^2},$$

where r and R are inradius and circumradius of ABC, respectively.

We received eleven solutions, of which ten were correct. We present two solutions.

Solution 1, by Arkady Alt.

Let F, s and h_a, h_b, h_c be the area, semiperimeter, and altitudes of the triangle. Since $m_x \ge h_x, x \in \{a, b, c\}$ and

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{a}{2F} + \frac{b}{2F} + \frac{c}{2F} = \frac{s}{2F} = \frac{1}{r}$$

then

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \le \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r} \le \frac{R}{2r^2}$$

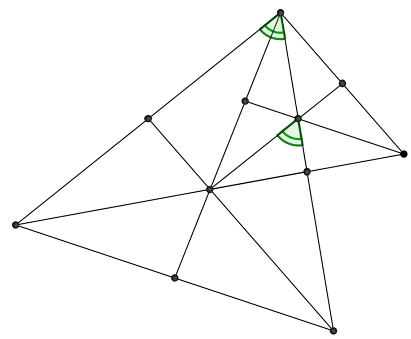
because

$$\frac{1}{r} \le \frac{R}{2r^2} \iff 2r \le R,$$

by Euler's Inequality.

Solution 2, by Edmund Swylan.

We take it as known that the triangle with side lengths $2m_a$, $2m_b$, $2m_c$ has medians of lengths $\frac{3}{2}a$, $\frac{3}{2}b$, $\frac{3}{2}c$. (See the drawing below.)



Let the area of $\triangle ABC$ be F. The area of the big triangle is then $3F$. Let the altitudes of the big triangle be H_a, H_b, H_c.

We have that $\dfrac{6F}{2m_x} = H_x$ and $H_x \leq \dfrac{3}{2}x$, for each $x \in \{a, b, c\}$. Therefore,

$$3F(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}) \leq \frac{3}{2}(a+b+c);$$

equality occurs if and only if the big triangle, and consequently $\triangle ABC$ too, is equilateral. Finally,

$$\frac{3}{2}(a+b+c) = 3F\frac{1}{r} \leq 3F\frac{1}{r}\frac{R}{2r} = 3F\frac{R}{2r^2};$$

equality occurs if and only if $\triangle ABC$ is equilateral.

4010. *Proposed by Ovidiu Furdui.*

Let $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ be a continuous function. Calculate

$$\lim_{n\to\infty} n \int_0^{\frac{\pi}{2}} \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)^{2n} f(x)\mathrm{d}x.$$

There were eight submitted solutions for this problem, all of which were correct. We present two solutions.

Solution 1, by the group of M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal, expanded slightly by the editor.

The value of the required limit is $\dfrac{1}{4}\left(f(0) + f\left(\dfrac{\pi}{2}\right)\right)$. Indeed, if we denote by L the limit, then from the identity

$$\frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)} = \tan\left(\frac{\pi}{4} - x\right),$$

we have

$$\begin{aligned}
L &= \lim_{n\to\infty} n \int_0^{\pi/2} \left(\tan\left(\frac{\pi}{4}-x\right)\right)^{2n} f(x)\ dx \\
&= \lim_{n\to\infty} n \int_{-\pi/4}^{\pi/4} (\tan(s))^{2n} f\left(\frac{\pi}{4}-s\right)\ ds \\
&= \lim_{n\to\infty} n \int_0^{\pi/4} (\tan(s))^{2n} \left(f\left(\frac{\pi}{4}-s\right)+f\left(\frac{\pi}{4}+s\right)\right)\ ds \\
&= \lim_{n\to\infty} \int_0^1 \frac{r^{1+1/n}}{1+r^{2/n}} \left(f\left(\frac{\pi}{4}-\arctan(r^{1/n})\right)+f\left(\frac{\pi}{4}+\arctan(r^{1/n})\right)\right)\ dr,
\end{aligned}$$

where we have used symmetry, and in the last step we have used the change of variable $r=(\tan(s))^n$.

Since f is a continuous function, $\exists M$ such that $|f(x)| \le M$, for $x \in [0, \frac{\pi}{2}]$, and

$$\left|\frac{r^{1+1/n}}{1+r^{2/n}} \left(f\left(\frac{\pi}{4}-\arctan(r^{1/n})\right)+f\left(\frac{\pi}{4}+\arctan(r^{1/n})\right)\right)\right| \le M$$

for all $r \in [0,1]$, using the bound for f and that the fraction in r is bounded above by $r/(1+r^2)$ (which is bounded by $1/2$, by looking at $(r-1)^2 \ge 0$). In this way, we can apply the dominated convergence theorem to obtain

$$\begin{aligned}
L &= \int_0^1 \lim_{n\to\infty} \frac{r^{1+1/n}}{1+r^{2/n}} \left(f\left(\frac{\pi}{4}-\arctan(r^{1/n})\right)+f\left(\frac{\pi}{4}+\arctan(r^{1/n})\right)\right)\ dr \\
&= \frac{1}{2}\left(f(0)+f\left(\frac{\pi}{2}\right)\right)\int_0^1 r\ dr = \frac{1}{4}\left(f(0)+f\left(\frac{\pi}{2}\right)\right).
\end{aligned}$$

Solution 2, by Michel Bataille.

We show that the required limit is $\frac{f(0)+f(\pi/2)}{4}$. Let

$$I_n = \int_0^{\frac{\pi}{2}} \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)^{2n} f(x)\,dx = \int_0^{\frac{\pi}{2}} \left(\tan\left(\frac{\pi}{4}-x\right)\right)^{2n} f(x)\,dx.$$

The change of variables $x = \frac{\pi}{4} - \tan^{-1}(y)$ yields

$$I_n = \int_{-1}^1 \frac{y^{2n}}{1+y^2} f\left(\frac{\pi}{4}-\tan^{-1}(y)\right)\ dy.$$

But we have

$$\int_{-1}^0 \frac{y^{2n}}{1+y^2} f\left(\frac{\pi}{4}-\tan^{-1}(y)\right)\ dy = \int_0^1 \frac{u^{2n}}{1+u^2} f\left(\frac{\pi}{4}+\tan^{-1}(u)\right)\ du$$

so that

$$I_n = \int_0^1 y^{2n} g(y)\,dy,$$

where $g(y) = \left(f\left(\frac{\pi}{4} - \tan^{-1}(y)\right) + f\left(\frac{\pi}{4} + \tan^{-1}(y)\right)\right) \cdot \frac{1}{y^2+1}$. It is known that if g is continuous on $[0,1]$, then $\lim\limits_{n\to\infty} n\int_0^1 x^n g(x)\,dx = g(1)$ [for completeness, a quick proof is given at the end]. From this result, it follows that

$$\lim_{n\to\infty}(2n)\cdot I_n = g(1) = \frac{f(0)+f(\pi/2)}{2}$$

and so

$$\lim_{n\to\infty} n\cdot I_n = \frac{f(0)+f(\pi/2)}{4},$$

as claimed.

For the proof of the property used above, let $\epsilon > 0$. Using the continuity of g, we choose $\delta \in (0,1)$ such that $|g(x) - g(1)| \le \epsilon$ whenever $x \in [\delta, 1]$. Then we have

$$\begin{aligned}\left|(n+1)\int_0^1 x^n \cdot g(x)\,dx - g(1)\right| &= \left|(n+1)\int_0^1 x^n\cdot(g(x)-g(1))\,dx\right| \\ &\le (n+1)\int_0^\delta x^n|g(x)-g(1)|\,dx + (n+1)\int_\delta^1 x^n|g(x)-g(1)|\,dx \\ &\le M\cdot\delta^{n+1} + \epsilon\end{aligned}$$

where M denotes the maximum of the continuous function $x \mapsto |g(x) - g(1)|$ on $[0,1]$. Since $0 < \delta < 1$, we deduce $\limsup\limits_{n\to\infty} |(n+1)\int_0^1 x^n \cdot g(x)\,dx - g(1)| \le \epsilon$. Since the latter holds for any positive ϵ, we must have

$$\lim_{n\to\infty}\left((n+1)\int_0^1 x^n\cdot g(x)\,dx - g(1)\right) = 0,$$

and the result follows.

Editor's Comments. This type of problem has its roots in Fourier analysis, where we are interested in limits such as the one in the problem statement. This particular limit picks out half the arithmetic mean of the function's value at the endpoints of the interval $[0, \frac{\pi}{2}]$; more classical Fourier analysis will focus on limits which pick out the function's value at a specific point, like the 'Dirac delta' distribution. All solutions aside from Solution 1 essentially followed Bataille's Solution 2, including the proposer's; À. Plaza's solution used a limit result from the proposer's own book (O. Furdui, *Limits, Series and Fractional Part Integrals*, Springer, Second ed., 2013) to skip a substitution. The techniques in both solutions (i.e. utilizing dominated convergence in a clever way, and separating an integral up into two parts which are handled using the two different functions involved in the integrand) are common techniques in classical analysis. A. Stadler's approach (namely, using 'Big O' notation instead of more precise estimates) is equally successful and is common in analytic number theory.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

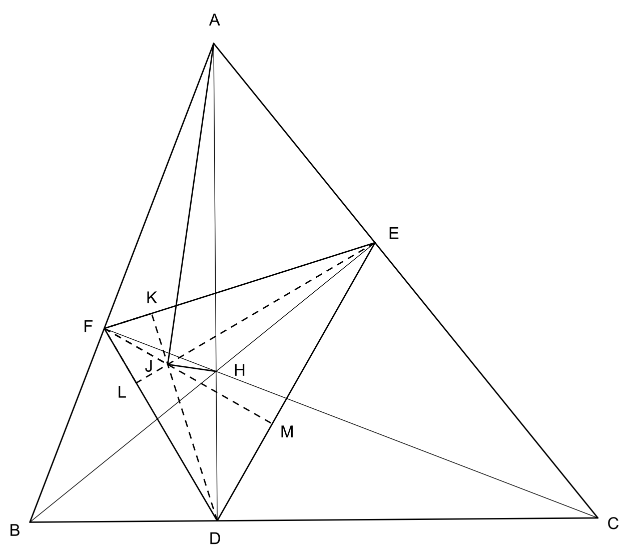
Statements of the problems in this section originally appear in 2015: 41(2), p. 71–74.

4011. *Proposed by Abdilkadir Altinas.*

In non-equilateral triangle ABC, let H be the orthocentre of ABC and J be the orthocentre of the orthic triangle DEF of ABC (that is the triangle formed by the feet of the altitudes of ABC). If $\angle BAC = 60^\circ$, show that $AJ \perp HJ$.

We received nine solutions. Eight of the solutions used angle-chasing in cyclic quadrilaterals, and one solution used barycentric coordinates. The former type of solutions were simpler, but they all missed the fact that there are subtleties if the orthocenter is not interior to the triangle.

We present the solution by Ricardo Barroso Campos slightly modified by the editor.



Since $\angle HFA = \angle HEA = 90^\circ$, quadrilateral $AFHE$ is cyclic. Hence

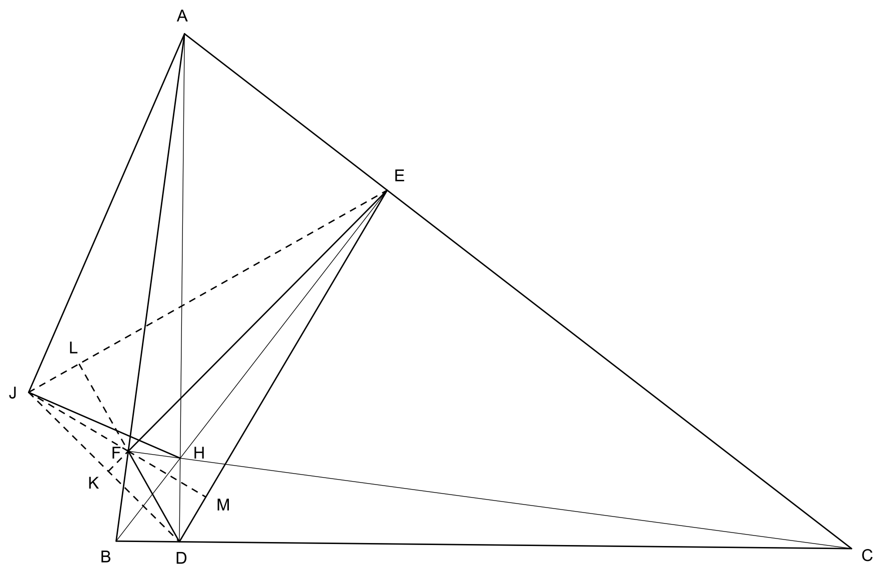
$$\angle EHF = 180^\circ - \angle CAB = 120^\circ.$$

Denote by Γ the circumcircle of $AFHE$, and note that AH is the diameter of this circle (since $\angle AFH = 90^\circ$).

In the cyclic quadrilateral $BDHF$, $\angle BDF = \angle BHF = 180^\circ - \angle EHF = 60^\circ$. Similarly, from the cyclic quadrilateral $CDHE$, we get $\angle CDE = 60^\circ$. Hence $\angle FDE = 180^\circ - (\angle BDF + \angle CDE) = 60^\circ$.

Denote the feet of the altitudes from D, E, and F by K, L and M respectively, as in the diagram. $DLJM$ is cyclic, thus $\angle LJM = 180^\circ - \angle FDE = 120^\circ$. Hence $\angle FJE = \angle LJM = 120^\circ$, which implies that the quadrilateral $AFJE$ is cyclic (since $\angle FJE + \angle FAE = 180^\circ$). It follows that J is on the circle Γ. Hence, since AH is the diameter of Γ, we get $\angle AJH = 90^\circ$, so $AJ \perp HJ$.

Editor's Comments. The provided solution fails if one of the orthocentres is not interior to its triangle. The following diagram shows the case where the point J is not interior to DEF (for one, $\angle FJE = 60^\circ$, not 120°). Note however that it is not difficult to adjust the solution for these cases.



4012. *Proposed by Leonard Giugiuc.*

Let n be an integer with $n \geq 3$. Consider real numbers a_k, $1 \leq k \leq n$ such that

$$a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq 1 \geq a_n \geq 0 \quad \text{and} \quad \sum_{k=1}^{n} a_k = n.$$

Prove that

$$\frac{(n-2)(n+1)}{2} \leq \sum_{1 \leq i < j \leq n} a_i a_j \leq \frac{n(n-1)}{2}.$$

We received eight submissions of which seven were correct and complete. We present the solution by Ivan Chan Kai Chin.

Since we have

$$2 \cdot \sum_{1 \leq i \leq j \leq n} a_i a_j = (\sum_{k=1}^{n} a_k)^2 - \sum_{k=1}^{n} a_k^2 = n^2 - \sum_{k=1}^{n} a_k^2,$$

it suffices to prove that

$$n \le \sum_{k=1}^{n} a_k^2 \le n + 2.$$

The left inequality holds by Cauchy-Schwarz, since

$$\sum_{k=1}^{n} a_k^2 \ge \frac{1}{n} \cdot (\sum_{k=1}^{n} a_k)^2 = n,$$

with equality when $a_1 = a_2 = \cdots = a_n = 1$.

For the other inequality, set $b_k = a_k - 1$ for all $1 \le k \le n$. Then $b_k \ge 0$ for all $1 \le k \le n-1$, $-1 \le b_n \le 0$, and $\sum_{k=1}^{n} b_i = 0$. We have

$$\sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} (1 + b_k)^2 = \sum_{k=1}^{n-1} (1 + b_k)^2 + (1 - (b_1 + b_2 + \cdots + b_{n-1}))^2. \qquad (1)$$

Define the quantity $S = b_1 + b_2 + \cdots + b_{n-1} \le 1$, and

$$f(b_1, b_2, \cdots, b_{n-1}) = \sum_{k=1}^{n-1} (1 + b_k)^2.$$

For any $1 \le i \le j \le n-1$,

$$f(\cdots b_i, \cdots, b_j, \cdots) \le f(\cdots b_i + b_j, \cdots, 0, \cdots),$$

since

$$(1 + b_i)^2 + (1 + b_j)^2 \le (1 + b_i + b_j)^2 + 1 \iff 2b_i b_j \ge 0$$

holds for all b_i, b_j, $1 \le i \le j \le n-1$. Thus we have

$$f(b_1, b_2, \cdots, b_{n-1}) \le f(b_1 + b_2 + \cdots b_{n-1}, 0, \cdots, 0) = f(S, 0, \cdots, 0)$$

and (1) becomes

$$\begin{aligned} f(S, 0, \cdots, 0) + (1 - S)^2 &= (1 + S)^2 + (n - 2) + (1 - S)^2 \\ &= n + 2S^2 \\ &\le n + 2 \end{aligned}$$

Equality holds when $S = 1$, $b_2 = b_3 = \cdots = b_{n-1} = 0$, $b_1 = 1$ and $b_n = -1$, which corresponds to $a_1 = 2, a_2 = a_3 = \cdots = a_{n-1} = 1, a_n = 0$.

4013. *Proposed by Mehmet Şahin.*

Let a, b, c be the sides of triangle ABC, D be the foot of the altitude from A and E be the midpoint of BC. Define $\theta = \angle DAE$ and suppose that $\angle ACB = 2\theta$. Prove that the sides of the triangle satisfy

$$(a - b)^2 = 2c^2 - b^2.$$

We received 16 submissions. Among them one simply stated that the claim was incorrect and provided a counterexample, while 15 proved the claim under the additional assumption that $b > c$; moreover, 5 proved that for the claim to be correct, the assumption that $b > c$ is both necessary and sufficient, and 3 of those submissions went on to provide a complete description of triangles that satisfy the given hypotheses.

We present the solution by Joel Schlosberg, supplemented by ideas from C. R. Pranesachar.

We shall prove that if a triangle satisfies $\angle ACB = 2\angle DAE$ and, moreover, $b > c$, then $(a-b)^2 = 2c^2 - b^2$; if $b < c$ then $a = 2b$ (and the claimed equation fails to hold). Note that if $b = c$ then $A = D = E$ is the midpoint of the segment BC, and the triangle is degenerate.

Scale $\triangle ABC$ so that $b = 1$. By right-angle trigonometry, $AD = \sin 2\theta = 2\sin\theta\cos\theta$, so that

$$AD^2 = 4\sin^2\theta(1-\sin^2\theta).$$

Use signed lengths for segments on BC, with BC positive. Then

$$DE = pAD\tan\theta = 2p\sin^2\theta,$$

where $p = 1$ if B and D are on one side of E, and C is on the other (which happens if and only if $b > c$); otherwise, when E is between B and D (and, equivalently, $b < c$) then we set $p = -1$. Furthermore, we have

$$DC = \cos 2\theta = 1 - 2\sin^2\theta$$

$$EC = DC - DE = 1 - (2+2p)\sin^2\theta$$
$$a = BC = 2EC = 2 - (4+4p)\sin^2\theta$$
$$BD = BC - DC = 1 - (2+4p)\sin^2\theta.$$

By the Pythagorean theorem (using $p^2 = 1$),

$$c^2 = AD^2 + BD^2 = 1 - 8p\sin^2\theta + (16+16p)\sin^4\theta, \tag{1}$$

while (using $b = 1$)

$$\frac{(a-b)^2+b^2}{2} = \frac{1}{2}a^2 - ab + b^2 = 1 - (4+4p)\sin^2\theta + (16+16p)\sin^4\theta. \tag{2}$$

Comparing equations (1) and (2), we see that $(a-b)^2 = 2c^2 - b^2$ iff $p = 1$. On the other hand, setting $p = -1$ and $b = 1$ in equation (2) we get $a = 2$ and deduce that $a = 2b$.

Editor's Comments. This problem should be compared with problem 4008 whose solution appeared in the previous issue. It dealt with triangles for which $\angle ACB = 2\angle DAE$ and, in addition, $\angle ABC = 3\angle DAE$. One finds that this can happen if and only if $\angle A = 90°, \angle B = 54°$, and $\angle C = 36°$; of course this implies that $b > c$ and, consequently, that $(a-b)^2 = 2c^2 - b^2$.

4014. *Proposed by Mihaela Berinedanu.*

Let n be a natural number and let x, y and z be positive real numbers such that $x + y + z + nxyz = n + 3$. Prove that

$$(1 + \frac{y}{x} + nyz)(1 + \frac{z}{y} + nzx)(1 + \frac{x}{z} + nxy) \geq (n+2)^3$$

and determine when equality holds.

We received six correct solutions. We present the solution by Dionne Bailey, Elsie Campbell and Charles Diminnie (joint).

The arithmetic-geometric means inequality yields that

$$n+3 = x+y+z+nxyz \geq (n+3)\left[x \cdot y \cdot z \cdot (xyz)^n\right]^{1/n+3} = (n+3)\left[xyz\right]^{(n+1)/(n+3)},$$

so that $xyz \leq 1$.

The inequality is equivalent to

$$(x + y + nxyz)(y + z + nxyz)(z + x + nxyz) \geq (n+2)^3(xyz)$$

or

$$(n + 3 - z)(n + 3 - x)(n + 3 - y) \geq (n+2)^3 xyz.$$

Using the arithmetic-geometric means inequality and the fact that $xyz \leq (xyz)^{2/3}$, we obtain that

$$\begin{aligned}
&(n+3-x)(n+3-y)(n+3-z) \\
&\qquad = (n+3)^3 - (n+3)^2(x+y+z) + (n+3)(xy+yz+zx) - xyz \\
&\qquad \geq (n+3)^3 - (n+3)^2[(n+3) - nxyz] + 3(n+3)(xyz)^{2/3} - xyz \\
&\qquad \geq n(n+3)^2 xyz + 3(n+3)xyz - xyz \\
&\qquad = (n^3 + 6n^2 + 9n + 3n + 9 - 1)xyz = (n+2)^3 xyz,
\end{aligned}$$

as desired, with equality if and only if $x = y = z = 1$.

4015. *Proposed by Michel Bataille.*

Find all real numbers a such that

$$a \cos x + (1-a) \cos \frac{x}{3} > \frac{\sin x}{x}$$

for every nonzero x of the interval $(-\frac{3\pi}{2}, \frac{3\pi}{2})$.

There were four submitted solutions for this problem, all of which were correct. We present the solution by Joel Schlosberg.

We will prove that the inequality

$$a\cos(x) + (1-a)\cos\left(\frac{x}{3}\right) > \frac{\sin(x)}{x} \tag{1}$$

holds for $x \in (-\frac{3\pi}{2}, \frac{3\pi}{2}) \setminus \{0\}$ if and only if $a \in (-\infty, 1/4]$.

Substituting $y = x/3$ and dividing by y^2, the above inequality is equivalent to

$$a\left(\frac{\cos(y) - \cos(3y)}{y^2}\right) < \frac{3y\cos(y) - \sin(3y)}{3y^3}, \tag{2}$$

for $|y| \in (0, \pi/2)$. By repeated applications of l'Hospital's Rule,

$$\lim_{y\to 0} \frac{\cos(y) - \cos(3y)}{y^2} = \lim_{y\to 0} \frac{-\sin(y) + 3\sin(3y)}{2y} = \lim_{y\to 0} \frac{-\cos(y) + 9\cos(3y)}{2} = 4,$$

and similarly,

$$\begin{aligned}\lim_{y\to 0} \frac{3y\cos(y) - \sin(3y)}{3y^3} &= \lim_{y\to 0} \frac{3\cos(y) - 3y\sin(y) - 3\cos(3y)}{9y^2} \\ &= \lim_{y\to 0} \frac{-6\sin(y) - 3y\cos(y) + 9\sin(3y)}{18y} \\ &= \lim_{y\to 0} \frac{-9\cos(y) + 3y\sin(y) + 27\cos(3y)}{18} = 1.\end{aligned}$$

Suppose that (1) holds for $x \in (-\frac{3\pi}{2}, \frac{3\pi}{2}) \setminus \{0\}$. Then the rewritten inequality (2) holds for non-zero y in a neighbourhood of 0. Taking $y \to 0$ on both sides yields $4a \le 1$, and so $a \in (-\infty, 1/4]$.

Since both sides of (2) are even functions of y, it is sufficient to prove (2) for $y \in (0, \pi/2)$. We let

$$f(x) = 9y\cos(y) - 4\sin(3y) + 3y\cos(3y).$$

For $y \in (0, \pi/2)$, it is well-known that $\tan(y) > y$, so

$$\begin{aligned}f'(y) &= 9(\cos(y) - y\sin(y) - \cos(3y) - y\sin(3y)) \\ &= 9(4(1-\cos^2(y))\cos(y) - 4y(1-\sin^2(y))\sin(y)) \\ &= 36(\sin^2(y)\cos(y) - y\cos^2(y)\sin(y)) \\ &= 36\cos^2(y)\sin(y)(\tan(y) - y) > 0\end{aligned}$$

(via triple-angle formulas). Therefore, for $y \in (0, \pi/2)$, $f(y) > f(0) = 0$, which is equivalent to (2) when $a = 1/4$. Hence (1) holds for $a = 1/4$.

Suppose now that $a < 1/4$. For $y \in (0, \pi/2)$, we have

$$\cos(y) - \cos(3y) = 4(1-\cos^2(y))\cos(y) = 4\sin^2(y)\cos(y) > 0,$$

so that

$$a\left(\frac{\cos(y)-\cos(3y)}{y^2}\right) < \frac{1}{4}\left(\frac{\cos(y)-\cos(3y)}{y^2}\right) < \frac{3y\cos(y)-\sin(3y)}{3y^3},$$

and we are done.

Editor's Comments. All four solution methods involved similar elements: trigonometric identities used to rewrite the inequality, a limit (either by power series or by L'Hospital's Rule), and some calculus. Deiermann noted that if we set the right-hand side of the original inequality equal to 1 for $x = 0$, then we may allow equality at $x = 0$. Deiermann also suggested a generalization, propped up by Mathematica: if $n \geq 3$, then we have

$$a\cos(x) + (1-a)\cos\left(\frac{x}{n}\right) > \frac{\sin(x)}{x}.$$

for all non-zero $x \in (-\frac{3\pi}{2}, \frac{3\pi}{2}) \setminus \{0\}$ if and only if $a \leq \frac{n^2-3}{3(n^2-1)}$. A quick sketch of the argument by the editor seems to indicate that it is true, but the conclusion of the proof is still out of reach.

4016. *Proposed by George Apostolopoulos.*

Let x, y, z be positive real numbers. Find the maximal value of the expression

$$\frac{x+2y}{2x+3y+z} + \frac{y+2z}{2y+3z+x} + \frac{z+2x}{2z+3x+y}.$$

We received 21 submissions, all of which were correct. We present two solutions.

Solution 1, by Arkady Alt.

Let $S(x,y,z)$ denote the given expression. Then by using Cauchy-Schwarz Inequality we have

$$\begin{aligned} 3 - S(x,y,z) &= \sum_{cyc}\left(1 - \frac{x+2y}{2x+3y+z}\right) \\ &= \sum_{cyc}\frac{x+y+z}{2x+3y+z} \\ &= \frac{6(x+y+z)}{6}\sum_{cyc}\frac{1}{2x+3y+z} \\ &= \frac{1}{6}\sum_{cyc}(2x+3y+z)\cdot\sum_{cyc}\frac{1}{2x+3y+z} \\ &\geq \frac{1}{6}\cdot 9 = \frac{3}{2}. \end{aligned}$$

Hence, $S(x,y,z) \leq \dfrac{3}{2}$ and $S(x,x,x) = \dfrac{3}{2}$.

Solution 2, by Šefket Arslanagić.

Since the given inequality is homogeneous, we may assume that $x+y+z=1$. By the AM-HM Inequality, we have

$$\begin{aligned} S(x,y,z) &= 3 - \left(\frac{1}{1+x+2y} + \frac{1}{1+y+2z} + \frac{1}{1+z+2x}\right) \\ &\leq 3 - \frac{9}{(1+x+2y)+(1+y+2z)+(1+z+2x)} \\ &= 3 - \frac{9}{6} = \frac{3}{2}. \end{aligned}$$

Hence, the maximum value of $S(x,y,z)$ is $\frac{3}{2}$ attained when $x=y=z$.

Editor's Comments. Kee-Wai Lau made an interesting and not-so-easy-to-see observation that

$$S(x,y,z) - \frac{3}{2} = -\frac{\sum(3x+y+2z)(x+y-2z)^2}{6\prod(2x+3y+z)} \leq 0.$$

4017. *Proposed by Michel Bataille.*

Let P be a point of the incircle γ of a triangle ABC. The perpendiculars to BC, CA and AB through P meet γ again at U, V and W, respectively. Prove that one of the numbers $PU \cdot BC, PV \cdot CA, PW \cdot AB$ is the sum of the other two.

From the 6 correct submissions we received, we present a composite of the similar solutions by Šefket Arslanagić, Ricard Peiró i Estruch, and Joel Schlosberg.

Since $PV \perp CA$ and $PW \perp AB$, $\angle VPW$ is either equal to or supplementary to $\angle BAC$, so

$$\sin\angle VPW = \sin\angle BAC = \sin A;$$

similarly,

$$\sin\angle WPU = \sin B \quad \text{and} \quad \sin\angle UPV = \sin C.$$

Moreover, because $PUVW$ is cyclic we have

$$\sin\angle VPW = \sin\angle VUW, \ \sin\angle WPU = \sin\angle WVU, \ \sin\angle UPV = \sin\angle UWV.$$

Finally, the Law of Sines applied to ΔUVW implies

$$\frac{VW}{WU} = \frac{\sin\angle VUW}{\sin\angle WVU} \quad \text{and} \quad \frac{UV}{WU} = \frac{\sin\angle UWV}{\sin\angle WVU},$$

while applied to ΔABC implies

$$\frac{\sin A}{\sin B} = \frac{BC}{CA} \quad \text{and} \quad \frac{\sin C}{\sin B} = \frac{AB}{CA}.$$

Let us suppose that the diagram has been labeled so that the quadrilateral $PUVW$ is cyclic in that order, whence PV is the diagonal, and Ptolemy's theorem says that $PV \cdot WU = PU \cdot VW + PW \cdot UV$. Putting it all together, we get

$$\begin{aligned} PV &= PU \cdot \frac{VW}{WU} + PW \cdot \frac{UV}{WU} \\ &= PU \cdot \frac{\sin \angle VUW}{\sin \angle WVU} + PW \cdot \frac{\sin \angle UWV}{\sin \angle WVU} \\ &= PU \cdot \frac{\sin \angle VPW}{\sin \angle WPU} + PW \cdot \frac{\sin \angle UPV}{\sin \angle WPU} \\ &= PU \cdot \frac{\sin A}{\sin B} + PW \cdot \frac{\sin C}{\sin B} \\ &= PU \cdot \frac{BC}{CA} + PW \cdot \frac{AB}{CA}. \end{aligned}$$

Thus, we conclude that $PV \cdot CA = PU \cdot BC + PW \cdot AB$; in other words, the product involving the diagonal of the quadrilateral equals the sum of the products involving the sides.

Additionally, the proposer observed (and proved) that the area of ΔUVW is independent of the choice of P on γ.

4018. *Proposed by Ovidiu Furdui.*

Let

$$I_n = \int_0^1 \cdots \int_0^1 \ln(x_1 x_2 \cdots x_n) \ln(1 - x_1 x_2 \cdots x_n) dx_1 dx_2 \cdots dx_n,$$

where $n \geq 1$ is an integer. Prove that this integral converges and find its value.

We received three solutions, all of which were correct and complete. We present the solution by the proposer.

The integral equals

$$n\left(n + 1 - \zeta(2) - \zeta(3) - \cdots - \zeta(n+1)\right),$$

where ζ denotes the Riemann zeta function.

We have, based on symmetry reasons, that for all $i, j = 1, 2, \ldots, n$

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 \ln x_i \ln(1 - x_1 x_2 \cdots x_n) dx_1 dx_2 \cdots dx_n \\ &\quad = \int_0^1 \cdots \int_0^1 \ln x_j \ln(1 - x_1 x_2 \cdots x_n) dx_1 dx_2 \cdots dx_n, \end{aligned}$$

and this implies that

$$\begin{aligned}
I_n &= n\int_0^1 \cdots \int_0^1 \ln x_1 \ln(1 - x_1 x_2 \cdots x_n) dx_1 dx_2 \cdots dx_n \\
&= n\int_0^1 \cdots \int_0^1 -\ln x_1 \sum_{k=1}^{\infty} \frac{(x_1 \cdots x_n)^k}{k} dx_1 dx_2 \cdots dx_n \\
&\overset{(*)}{=} n\sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 \cdots \int_0^1 -\ln x_1 (x_1 \cdots x_n)^k dx_1 dx_2 \cdots dx_n \\
&= n\sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 (-x_1^k \ln x_1) \mathrm{d}x_1 \int_0^1 x_2^k \mathrm{d}x_2 \cdots \int_0^1 x_n^k \mathrm{d}x_n \\
&= n\sum_{k=1}^{\infty} \frac{1}{k(k+1)^{n+1}}.
\end{aligned}$$

We used at step (*) Tonelli's Theorem for nonnegative functions, which allows us to interchange the integration sign and the summation sign.

Let $S_{n+1} = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)^{n+1}}$. Since

$$\frac{1}{k(k+1)^{n+1}} = \frac{1}{k(k+1)^n} - \frac{1}{(k+1)^{n+1}},$$

we have, by summation, that $S_{n+1} = S_n - (\zeta(n+1) - 1)$. This implies, since $S_1 = 1$, that

$$S_{n+1} = S_1 - (\zeta(2) + \zeta(3) + \cdots + \zeta(n+1) - n) = n + 1 - \zeta(2) - \zeta(3) - \cdots - \zeta(n+1).$$

Hence

$$I_n = n\left(n + 1 - \zeta(2) - \zeta(3) - \cdots - \zeta(n+1)\right),$$

and the problem is solved.

4019. *Proposed by George Apostolopoulos.*

A triangle with side lengths a, b, c has perimeter 3. Prove that

$$a^3 + b^3 + c^3 + a^4 + b^4 + c^4 \geq 2(a^2b^2 + b^2c^2 + c^2a^2).$$

We received 21 correct solutions. We present the solution by AN-anduud Problem Solving Group.

The claimed inequality is equivalent to

$$(a^3 + b^3 + c^3)(a + b + c) + 3(a^4 + b^4 + c^4) \geq 6(a^2b^2 + b^2c^2 + c^2a^2)$$

or

$$[(a^3b + b^3a) + (a^3c + c^3a) + (b^3c + c^3b)] + [2(a^4 + b^4) + 2(b^4 + c^4) + 2(c^4 + a^4)] \geq 6(a^2b^2 + b^2c^2 + c^2a^2).$$

By the AM-GM Inequality we have

$$a^3b + b^3a \geq 2a^2b^2,\ a^3c + c^3a \geq 2a^2c^2,\ b^3c + c^3b \geq 2b^2c^2$$

and

$$a^4 + b^4 \geq 2a^2b^2,\ b^4 + c^4 \geq 2b^2c^2,\ c^4 + a^4 \geq 2a^2c^2.$$

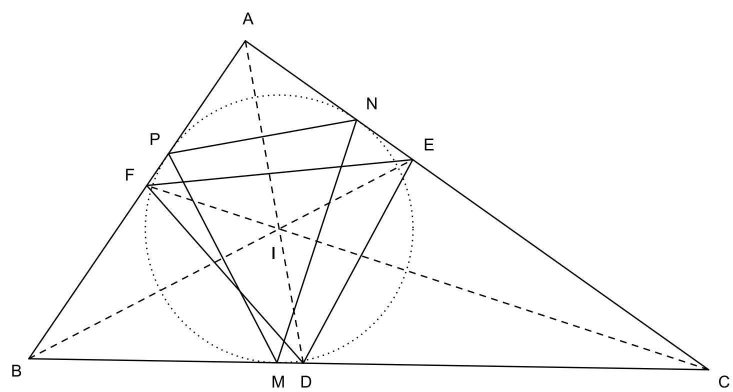
Adding the above inequalities, we obtain the desired inequality. Equality holds if and only if $a = b = c = 1$.

4020. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let ABC be a triangle and let the internal bisectors from A, B and C intersect the sides BC, CA and AB in D, E and F, respectively. The incircle of ΔABC touches the sides BC, CA and AB in M, N, and P, respectively. Prove that $[MNP] \leq [DEF]$, where $[\cdot]$ denotes the area of the specified triangle.

We received eleven submissions, of which nine were correct and complete. We present the solution by Šefket Arslanagić, slightly modified by the editor.

Denote by α, β and γ the angles BAC, ABC and respectively ACB of the triangle, and let r be the radius of the incircle.



From the quadrilateral $PIMB$ note that $\angle PIM = 180^\circ - \angle ABC = 180^\circ - \beta$, whence

$$[PIM] = \frac{PI \cdot MI}{2} \sin(\angle PIM) = \frac{r^2}{2} \sin(180^\circ - \beta) = \frac{r^2}{2} \sin \beta.$$

Similarly, we calculate $[MIN]$ and $[NIP]$, and we get

$$\begin{aligned} [MNP] &= [PIM] + [MIN] + [NIP] \\ &= \frac{r^2}{2} \cdot (\sin \beta + \sin \gamma + \sin \alpha). \end{aligned} \tag{1}$$

On the other hand, we have $\angle FID = \angle AIC = 180^\circ - \frac{\alpha}{2} - \frac{\gamma}{2}$, and so

$$[FID] = \frac{ID \cdot IF}{2} \sin(\angle FID) = \frac{ID \cdot IF}{2} \sin \frac{\alpha+\gamma}{2}.$$

Similarly, calculate the area of $\triangle EIF$ and $\triangle DIE$. We have

$$\begin{aligned}[DEF] &= [DIE] + [EIF] + [FID] \\ &= \frac{ID \cdot IE}{2} \sin \frac{\alpha+\beta}{2} + \frac{IE \cdot EF}{2} \sin \frac{\beta+\gamma}{2} + \frac{ID \cdot IF}{2} \sin \frac{\alpha+\gamma}{2}.\end{aligned} \tag{2}$$

The triangles $\triangle PIF$, $\triangle MID$ and $\triangle NIE$ are all right-angled triangles, from which it follows that $IF \geq r$, $ID \geq r$ and $IE \geq r$. Hence, from the formula for $[DEF]$ above we get

$$[DEF] \geq \frac{r^2}{2} \left(\sin \frac{\alpha+\beta}{2} + \sin \frac{\beta+\gamma}{2} + \sin \frac{\alpha+\gamma}{2} \right).$$

Comparing this with the formula for $[MNP]$ in (1), in order to show that $[MNP] \leq [DEF]$ it is sufficient to show that

$$\sin\alpha + \sin\beta + \sin\gamma \leq \sin \frac{\alpha+\beta}{2} + \sin \frac{\beta+\gamma}{2} + \sin \frac{\alpha+\gamma}{2}. \tag{3}$$

However, using the sum to product trigonometric formula, we have

$$\sin\alpha + \sin\beta = 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} \leq 2 \sin \frac{\alpha+\beta}{2},$$

where the inequality follows from the fact that $\sin \frac{\alpha+\beta}{2} \geq 0$ and $\cos \frac{\alpha-\beta}{2} \leq 1$. Similarly we have $\sin\beta + \sin\gamma \leq 2 \sin \frac{\beta+\gamma}{2}$ and $\sin\gamma + \sin\alpha \leq 2 \sin \frac{\alpha+\gamma}{2}$, which we can add to get the inequality in (3), the final step we need in order to conclude that $[MNP] \leq [DEF]$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(3), p. 119–123.

4021. *Proposed by Arkady Alt.*

Let $(\overline{a}_n)_{n\geq 0}$ be a sequence of Fibonacci vectors defined recursively by $\overline{a}_0 = \overline{a}, \overline{a}_1 = \overline{b}$ and $\overline{a}_{n+1} = \overline{a}_n + \overline{a}_{n-1}$ for all integers $n \geq 1$. Prove that, for all integers $n \geq 1$, the sum of vectors $\overline{a}_0 + \overline{a}_1 + \cdots + \overline{a}_{4n+1}$ equals $k\overline{a}_i$ for some i and constant k.

We received nine correct solutions. We present the solution by David Stone and John Hawkins (joint).

We shall prove that $\bar{a}_0 + \bar{a}_1 + \cdots + \bar{a}_{4n+1} = L_{2n+1}\bar{a}_{2n+2}$, where L_k is the kth Lucas number. We use some easily proven results. Here, F_k is the kth Fibonacci number.

1. $F_0 + F_1 + \cdots + F_m = F_{m+2} - 1$.
2. $F_{4n+2} = L_{2n+1}F_{2n+1}$
3. $F_{4n+3} = L_{2n+1}F_{2n+2} + 1$
4. $\bar{a}_k = F_{k-1}\bar{a}_0 + F_k\bar{a}_1$ for $k \geq 1$.

Therefore,

$$\begin{aligned}
\sum_{k=0}^{m} \bar{a}_k &= \bar{a}_0 + \sum_{k=1}^{m} (F_{k-1}\bar{a}_0 + F_k\bar{a}_1) \\
&= \bar{a}_0 + \left(\sum_{k=1}^{m} F_{k-1}\right)\bar{a}_0 + \left(\sum_{k=1}^{m}\right)\bar{a}_1 \\
&= \bar{a}_0 + (F_{m+1} - 1)\bar{a}_0 + (F_{m+2} - 1)\bar{a}_1 \\
&= F_{m+1}\bar{a}_0 + F_{m+2}\bar{a}_1 - \bar{a}_1 \\
&= \bar{a}_{m+2} - \bar{a}_1
\end{aligned}$$

Hence, with $m = 4n + 1$,

$$\begin{aligned}
\sum_{k=0}^{4n+1} \bar{a}_k = \bar{a}_{4n+3} - \bar{a}_1 &= F_{4n+2}\bar{a}_0 + F_{4n+3}\bar{a}_1 - \bar{a}_1 \\
&= (L_{2n+1}F_{2n+1})\bar{a}_0 + (L_{2n+1}F_{2n+2})\bar{a}_1 \\
&= L_{2n+1}(F_{2n+1}\bar{a}_0 + F_{2n+2}\bar{a}_1) \\
&= L_{2n+1}\bar{a}_{2n+2}.
\end{aligned}$$

Editor's Comments. Various solvers expressed the coefficient of $\bar{a}_{2n+2}$ as L_{2n+1}, $F_{2n} + F_{2n+2}$, and $\frac{F_{4n+2}}{F_{2n+1}}$ and variations of these resulting from different indexing of the Fibonacci sequence. Swylan pointed out that if the word 'constant' is interpreted to mean 'independent of n', then the claim of the problem is false. Perhaps 'scalar' would have been a better word.

4022. *Proposed by Leonard Giugiuc.*

In a triangle ABC, let internal angle bisectors from angles A, B and C intersect the sides BC, CA and AB in points D, E and F and let the incircle of ΔABC touch the sides in M, N, and P, respectively. Show that

$$\frac{PA}{PB} + \frac{MB}{MC} + \frac{NC}{NA} \geq \frac{FA}{FB} + \frac{DB}{DC} + \frac{EC}{EA}.$$

We received eleven submissions, of which seven were correct, two were incorrect, and two were incomplete. We present the solution by Titu Zvonaru.

Define $x = NA = PA, \quad y = PB = MB,$ and $z = MC = NC$; then

$$BC = y + z, \quad CA = z + x, \quad \text{and} \quad AB = x + y.$$

By the angle bisector theorem we have

$$\frac{FA}{FB} = \frac{CA}{BC} = \frac{z+x}{y+z}, \quad \frac{DB}{DC} = \frac{AB}{CA} = \frac{x+y}{z+x}, \quad \text{and} \quad \frac{EC}{EA} = \frac{BC}{AB} = \frac{y+z}{x+y}.$$

We therefore have to prove that for positive real numbers x, y, z,

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{z+x}{y+z} + \frac{x+y}{z+x} + \frac{y+z}{x+y}\ . \tag{1}$$

After clearing denominators what we must prove reduces to

$$x^2y^4 + y^2z^4 + z^2x^4 + x^3y^3 + y^3z^3 + z^3x^3 \ \geq\ x^3yz^2 + x^2y^3z + xy^2z^3 + 3x^2y^2z^2. \tag{2}$$

By the AM-GM inequality we have

$$\begin{aligned} x^2y^4 + y^2z^4 + z^2x^4 &\geq 3x^2y^2z^2, \\ x^3y^3 + z^3x^3 + z^3x^3 &\geq 3x^3yz^2, \\ y^3z^3 + x^3y^3 + x^3y^3 &\geq 3x^2y^3z, \text{ and} \\ z^3x^3 + y^3z^3 + y^3z^3 &\geq 3xy^2z^3\ , \end{aligned}$$

which together imply that (2) holds. Equality holds if and only if $x = y = z$, which immediately implies that the triangle is equilateral.

Editor's Comments. Most submissions reduced our problem to equation (1), but then algebra caused difficulties with two of the faulty arguments. The solution from Salem Malikić neatly avoided calculations by remarking that (1) is known;

see, for example, the Belarussian IMO Team preparation tests of 1997, where calculations are much simplified by exploiting the cyclic symmetry of the inequalities. Beware, however, that one must not assume noncyclic symmetry (as in one of the incomplete submissions).

4023. *Proposed by Ali Behrouz.*

Find all functions $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}$ with $x > y$, we have

$$f\left(\frac{x}{x-y}\right) + f(xf(y)) = f(xf(x)).$$

We received two correct submissions. We present the solution by Joseph Ling.

It is easy to see that $f(x) = \dfrac{1}{x}$ $\forall x > 0$ satisfies

$$f\left(\frac{x}{x-y}\right) + f(xf(y)) = f(xf(x)) \tag{1}$$

whenever $0 < y < x$. We show that there are no other solutions $f : (0, \infty) \to (0, \infty)$ to (1).

First, we note that f is one-to-one. For if $0 < y < x$ are such that $f(y) = f(x)$, then (1) implies that $f\left(\frac{x}{x-y}\right) = 0$, which is impossible.

Second, we note that $f(x) \le \frac{1}{x}$ for all $x > 0$. For if there exists $x > 0$ such that $f(x) > \frac{1}{x}$, then $y = x - \frac{1}{f(x)}$ satisfies $0 < y < x$. But then $\frac{x}{x-y} = xf(x)$ and (1) will imply that $f(xf(y)) = 0$, which is impossible.

Now, suppose that $0 < y_1 < y_2$. Consider $x = y_2 + \frac{1}{f(y_1)}$. Then $0 < y_1 < y_2 < x$ and (1) implies that

$$f\left(\frac{x}{x-y_2}\right) + f(xf(y_2)) = f(xf(x)) = f\left(\frac{x}{x-y_1}\right) + f(xf(y_1)). \tag{2}$$

By the definition of x, $\frac{x}{x-y_2} = xf(y_1)$. So, (2) is reduced to $f(xf(y_2)) = f\left(\frac{x}{x-y_1}\right)$. Since f is one-to-one, we have $xf(y_2) = \frac{x}{x-y_1}$, and so, $x = y_1 + \frac{1}{f(y_2)}$. Using this and the definition of x, we see that $\frac{1}{f(y_1)} - y_1 = \frac{1}{f(y_2)} - y_2$. Since y_1 and y_2 are arbitrary, $\frac{1}{f(y)} - y$ is independent of y, and so, it must be some constant, say, c. That is,

$$f(y) = \frac{1}{y+c}$$

for all $y > 0$.

It remains to show that $c = 0$. Since $f(x) \le \frac{1}{x}$ for all x, $c \ge 0$. Furthermore, if

$c > 0$, then for any $0 < y < x$, we have

$$\begin{aligned} f\left(\frac{x}{x-y}\right) + f(xf(y)) &= \frac{x-y}{x+c(x-y)} + \frac{y+c}{x+c(y+c)} \\ &> \frac{x-y}{x+c(x+c)} + \frac{y+c}{x+c(x+c)} = \frac{x+c}{x+c(x+c)} \\ &= f(xf(x)), \end{aligned}$$

a contradiction to (1). So, $c = 0$ and our proof is complete.

4024. *Proposed by Leonard Giugiuc.*

Let a, b, c and d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$abc + abd + acd + bcd + 4 \geq a + b + c + d$$

and determine when equality holds.

We received three correct solutions. We present the solution by Titu Zvononu, modified by the editor.

We consider several cases separately.

Case 1. If $a, b, c, d \geq 0$, then by the Cauchy-Schwarz Inequality, we have

$$(a + b + c + d)^2 \leq (1^2 + 1^2 + 1^2 + 1^2)(a^2 + b^2 + c^2 + d^2) = 16.$$

Thus, $a + b + c + d \leq 4$ from which the result follows immediately.

Case 2. If $a, b, c, d \leq 0$, we set $x = -a$, $y = -b$, $z = -c$, and $t = -d$. Then, $x, y, z, t \geq 0$ and we would now like to show that

$$-(xyz + xyt + xzt + yzt) + 4 \geq -(x + y + z + t) \quad \text{or}$$

$$2(x + y + z + t) - (xyz + xyt + xzt + yzt) + 4 \geq x + y + z + t. \tag{1}$$

Since we know $x^2 + y^2 + z^2 + t^2 = 4$, we have by the first case (with a, b, c, d replaced with x, y, z, t, respectively and symbolically), that

$$x + y + z + t \leq 4. \tag{2}$$

Then,

$$\begin{aligned} &4(x + y + z + t) - 2(xyz + xyt + xzt + yzt) \\ &= (x^2 + y^2 + z^2 + t^2)(x + y + z + t) - 2(xyz + xyt + xzt + yzt) \\ &= x(y - z)^2 + y(x - t)^2 + z(x - t)^2 + t(y - z)^2 + x(x^2 + t^2) \\ &\qquad + y(y^2 + z^2) + z(z^2 + y^2) + t(t^2 + x^2) \\ &\geq 0, \end{aligned}$$

which together with (2) implies (1).

Case 3. If one of a, b, c, d is nonnegative and the other three are nonpositive. Due to the symmetry in the given equation and the one we wish to prove, we may assume that $a \geq 0$ and $b, c, d \leq 0$. Here we let $y = -b$, $z = -c$, and $t = -d$. Then, $a, y, z, t \geq 0$ with $a^2 + y^2 + z^2 + t^2 = 4$ and we wish to prove $ayz + ayt + azt - yzt + 4 \geq a - y - z - t$ or

$$2(y + z + t) + ayz + ayt + azt - yzt + 4 \geq a + y + z + t. \tag{3}$$

Now,

$$\begin{aligned}
&4(y + z + t) + 2(ayz + ayt + azt - yzt) \\
&= (a^2 + y^2 + z^2 + t^2)(y + z + t) + 2(ayz + ayt + azt - yzt) \\
&\geq y(z^2 + t^2) - 2yzt \\
&= y(z - t)^2 \\
&\geq 0,
\end{aligned}$$

so, $2(y + z + t) + ayz + ayt + azt - yzt + 4 \geq 4$, thus, establishing (3), as desired, since $a + y + z + t \leq 4$ (see Case 1).

Case 4. If $a, b \geq 0$ and $c, d \leq 0$, we set $z = -c$ and $t = -d$. Then $a, b, z, t \geq 0$ such that $a^2 + b^2 + z^2 + t^2 = 4$ and we would like to show that

$$-abz - abt + azt + bzt + 4 \geq a + b - z - t \quad \text{or}$$

$$2(z + t) - abz - abt + azt + bzt + 4 \geq a + b + z + t. \tag{4}$$

Now,

$$\begin{aligned}
&4(z + t) - 2(abz + abt - azt - bzt) \\
&= (a^2 + b^2 + z^2 + t^2)(z + t) - 2(abz + abt - azt - bzt) \\
&= z(a - b)^2 + t(a - b)^2 + (z + t)(z^2 + t^2) + 2azt + 2bzt \\
&\geq 0.
\end{aligned}$$

So, $2(z+t) - abz - abt + azt + bzt \geq 4$, thus establishing (4) since $a + b + z + t \leq 4$.

Case 5. If $a, b, c \geq 0$ and $d \leq 0$, we set $t = -d$. Then $a, b, c, t \geq 0$ with $a^2 + b^2 + c^2 + t^2 = 4$ and we would like to show that

$$abc - abt - act - bct + 4 \geq a + b + c - t \quad \text{or}$$

$$2t + abc - abt - act - bct + 4 \geq a + b + c + t. \tag{5}$$

Since $a + b + c + t \leq 4$, to establish (5), it suffices to show that

$$(a^2 + b^2 + c^2 + t^2)t + 2(abc - abt - act - bct) \geq 0. \tag{6}$$

We let L denote the left-hand side of (6) and assume, without loss of generality, that $a \geq b \geq c$. Note that

$$\begin{aligned} L &= t(b-c)^2 + t(a-t)^2 + 2a(t-b)(t-c) \qquad (7)\\ \text{and } L &= t(a-b)^2 + t(c-t)^2 + 2c(t-a)(t-b). \qquad (8) \end{aligned}$$

If $t \leq b$, then from (7) we can see that $L \geq 0$ and if $t \geq b$, then $L \geq 0$ from (8). Hence, we can conclude that (6) is true, as desired.

Examining the five cases, it is readily seen that equality can only hold in Case 5 when $a = b = c = t$; that is, if and only if $(a, b, c, d) = (1, 1, 1, -1)$ and all its permutations.

4025. *Proposed by Dragoljub Miloşević.*

Prove that for positive numbers a, b and c, we have

$$\sqrt[3]{\left(\frac{a}{2b+c}\right)^2} + \sqrt[3]{\left(\frac{b}{2c+a}\right)^2} + \sqrt[3]{\left(\frac{c}{2a+b}\right)^2} \geq \sqrt[3]{3}.$$

We received eleven correct solutions. We present the solution by Salem Madikić.

Let $f(a, b, c)$ denote the left hand side of the given inequality. By the AM-GM Inequality, we have

$$\begin{aligned} \sqrt[3]{\left(\frac{a}{2b+c}\right)^2} = \frac{a}{\sqrt[3]{a(2b+c)^2}} &= \frac{a}{\sqrt[3]{9}\sqrt[3]{a \cdot \frac{2b+c}{3} \cdot \frac{2b+c}{3}}}\\ &\geq \frac{3a}{\sqrt[3]{9}\left(a + \frac{2b+c}{3} + \frac{2b+c}{3}\right)}\\ &= \frac{3\sqrt[3]{3}a}{3a+4b+2c}. \end{aligned}$$

Using similar inequalities involving the other two summands, we then have

$$f(a, b, c) \geq 3\sqrt[3]{3} \sum_{\text{cyc}} \frac{a}{3a+4b+2c}. \qquad (1)$$

Now, by the Cauchy-Schwarz Inequality, we have

$$\left(\sum \left(\sqrt{\frac{a}{3a+4b+2c}}\right)^2\right)\left(\sum \left(\sqrt{a(3a+4b+2c)}\right)^2\right) \geq \sum a^2.$$

So,

$$\sum \frac{a}{3a+4b+2c} \geq \frac{\sum a^2}{\sum a(3a+4b+2c)} = \frac{\sum a^2}{3\sum a^2 + 6\sum ab} = \frac{1}{3}. \qquad (2)$$

Substituting (2) into (1), $f(a, b, c) \geq \sqrt[3]{3}$ follows immediately.

To achieve equality, we must have $3a = 2b+c$, $3b = 2c+a$, and $3c = 2a+b$. Without loss of generality, we may assume that $\max\{a, b, c\} = a$. Then, $3a = 2b+c$ implies $2(a-b) + (a-c) = 0$, so $a = b = c$. Conversely, it is readily checked that if $a = b = c$, then equality holds.

Editor's Comments. Using convexity and Jensen's Inequality, Stadler proved that in general, $\sum \left(\frac{a}{2b+c}\right)^k \geq 3^{1-k}$ for all $k \geq 0$.

4026. *Proposed by Roy Barbara.*

Prove or disprove the following property: if r is any non-zero rational number, then the real number $x = (1+r)^{1/3} + (1-r)^{1/3}$ is irrational.

We received two correct solutions. We present the solution by Joseph DiMuro.

Assume both r and x are rational numbers with $r \neq 0$. Setting $y_1 = (1+r)^{1/3}$ and $y_2 = (1-r)^{1/3}$ we can show that

$$\frac{x^3-2}{3x} = y_1 y_2.$$

That means that y_1 and y_2 are the two solutions for y of $y^2 - xy + \frac{x^3-2}{3x} = 0$. But using the quadratic formula, we also obtain

$$y = \frac{x \pm \sqrt{x^2 - \frac{4x^3-8}{3x}}}{2} = \frac{x}{2} \pm \sqrt{\frac{8-x^3}{12x}}.$$

This shows that if x is rational then y_1 and y_2 are contained in quadratic extensions of $\mathbb{Q}$. On the other hand, if r is rational then $y_1 = (1+r)^{1/3}$ and $y_2 = (1-r)^{1/3}$ are contained in cubic extensions of $\mathbb{Q}$ as well. Both of these can only be true if y_1 and y_2 are rational numbers themselves.

Let $r = \frac{a}{b}$, where a, b are relatively prime non-zero integers. Then $y_1 = (\frac{b+a}{b})^{1/3}$ and $y_2 = (\frac{b-a}{b})^{1/3}$. The fractions $\frac{b+a}{b}$ and $\frac{b-a}{b}$ are in lowest terms, so for them to be perfect cubes, their numerators and denominators must be perfect cubes. Then we have an arithmetic progression $b-a, b, b+a$ of cubes, which is known to be impossible (e.g. see P. Dénes, Über die Diophantische Gleichung $x^l + y^l = cz^l$, *Acta. Math.* **88** (1952) 241-251).

Editor's Comments. The statement that there is no arithmetic progression of three cubes can be proven with elementary number theory and is an interesting exercise.

4027. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a+b+c = 3$. Prove that

$$\frac{ab}{a+ab+b} + \frac{bc}{b+bc+c} + \frac{ac}{a+ac+c} \leq 1.$$

We received 24 submissions of which 22 were correct and complete. We present 5 solutions, each of them insightful in a different way.

Solution 1, by Ali Adnan.

Observe that

$$\sum_{cyc} \frac{ab}{a+ab+b} \leq 1 \quad \Longleftrightarrow \quad \sum_{cyc} \frac{9}{\frac{a+b}{ab}+1} \leq 9. \tag{1}$$

Now, from Cauchy-Schwarz Inequality,

$$\frac{9}{\frac{a+b}{ab}+1} = \frac{9}{\frac{1}{a}+\frac{1}{b}+1} \leq a+b+1,$$

and adding up analogous such inequalities cyclically, (1) follows.

Solution 2, by Ali Adnan.

We note that the inequality is equivalent to

$$\textstyle\sum_{cyc} \frac{a+b}{a+b+ab} \geq 2 \quad \Longleftrightarrow \quad \textstyle\sum_{cyc} \frac{1}{2+\frac{2ab}{a+b}} \geq 1,$$

which follows easily from the AM-HM and Cauchy-Schwarz Inequalities:

$$\textstyle\sum_{cyc} \frac{1}{2+\frac{2ab}{a+b}} \geq \textstyle\sum_{cyc} \frac{1}{2+\frac{a+b}{2}} \geq \frac{(1+1+1)^2}{6+a+b+c} = 1,$$

thus completing the proof.

Solution 3, by Henry Ricardo.

We have

$$\begin{aligned}\sum_{cyclic} \frac{ab}{a+ab+b} &= \sum_{cyclic} \frac{1}{\frac{1}{b}+1+\frac{1}{a}} = \frac{1}{3}\sum_{cyclic} \frac{3}{\frac{1}{b}+1+\frac{1}{a}} \\ &\leq \frac{1}{3}\sum_{cyclic} \frac{a+b+1}{3} = \frac{1}{3}\left(\frac{2(a+b+c)+3}{3}\right) = 1,\end{aligned}$$

where we have used the harmonic mean-arithmetic mean inequality.

Equality holds if and only if $a=b=c=1$.

Solution 4, by Salem Malikić.

Using the inequality between arithmetic and geometric mean for positive reals x and y we have

$$x+xy+y \geq 3\sqrt[3]{x^2y^2}$$

with equality if and only if $x=xy=y$, that gives $x=y=1$ and implying that

$$\frac{xy}{x+xy+y} \leq \frac{\sqrt[3]{xy}}{3}.$$

Using this inequality we have

$$\frac{ab}{a+ab+b}+\frac{bc}{b+bc+c}+\frac{ca}{c+ca+a}\le\frac{\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}}{3}.$$

Now, using Power-mean inequality, we have

$$\frac{\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}}{3}\le\sqrt[3]{\frac{ab+bc+ca}{3}}\le\sqrt[3]{\frac{\frac{(a+b+c)^2}{3}}{3}}=1.$$

where in the last step we used the well known inequality

$$3(ab+bc+ca)\le(a+b+c)^2.$$

This completes our proof.

In order to achieve equality we must have $a=b=c=1$. It is easy to verify that this is indeed an equality case.

Solution 5, by Leonard Giugiuc.

The inequality is equivalent to

$$\frac{1}{\frac{1}{a}+\frac{1}{b}+1}+\frac{1}{\frac{1}{b}+\frac{1}{c}+1}+\frac{1}{\frac{1}{c}+\frac{1}{a}+1}\le 1.$$

By AM-HM Inequality,

$$\frac{1}{a}+\frac{1}{b}\ge\frac{4}{a+b},\qquad \frac{1}{b}+\frac{1}{c}\ge\frac{4}{b+c},\qquad \frac{1}{c}+\frac{1}{a}\ge\frac{4}{c+a}.$$

From here,

$$\frac{1}{\frac{1}{a}+\frac{1}{b}+1}+\frac{1}{\frac{1}{b}+\frac{1}{c}+1}+\frac{1}{\frac{1}{c}+\frac{1}{a}+1}\le\frac{1}{\frac{4}{a+b}+1}+\frac{1}{\frac{4}{b+c}+1}+\frac{1}{\frac{4}{c+a}+1}.$$

But

$$\frac{1}{\frac{4}{a+b}+1}+\frac{1}{\frac{4}{b+c}+1}+\frac{1}{\frac{4}{c+a}+1}=\frac{a+b}{a+b+4}+\frac{b+c}{b+c+4}+\frac{c+a}{c+a+4}.$$

Since the function $f(x)=\dfrac{x}{x+4}$ is concave if $x>0$, then by Jensen's Inequality we get

$$f(a+b)+f(b+c)+f(c+a)\le 3f\left(\frac{2(a+b+c)}{3}\right)=3f(2)=1.$$

So,

$$\frac{a+b}{a+b+4}+\frac{b+c}{b+c+4}+\frac{c+a}{c+a+4}\le 1.$$

4028. *Proposed by Michel Bataille.*

In 3-dimensional Euclidean space, a line ℓ meets orthogonally two distinct parallel planes $\mathcal{P}$ and $\mathcal{P}'$ at H and H'. Let r and r' be positive real numbers with $r \leq r'$; let $\mathcal{C}$ be the circle in $\mathcal{P}$ with center H, radius r, and let $\mathcal{C}'$ in $\mathcal{P}'$ be similarly defined. For a fixed point M' on $\mathcal{C}'$, find the maximum distance between the lines ℓ and MM' as M moves about the circle $\mathcal{C}$ (where the distance between two lines is the minimum distance from a point of one line to a point of the other).

We received four correct solutions and will feature two of them that are quite similar except that the first makes use of coordinates.

Solution 1, by Oliver Geupel.

We prove that the required maximum distance is r. We use Cartesian coordinates such that $H' = (0,0,0)$, $M' = (r',0,0)$, and $H = (0,0,h)$ where $h \in \mathbb{R}$. For every point M on $\mathcal{C}$, the distance between ℓ and MM' is not greater than $|MH| = r$ (because that distance is, by definition, the length of the shortest among all segments joining a point of ℓ to a point of MM', which is therefore at most $|MH|$). Moreover, the distance between two non-intersecting lines is measured along a line that is perpendicular to both. Put

$$M = \left(\frac{r}{r'} \cdot r,\ \sqrt{1 - \frac{r^2}{r'^2}} \cdot r,\ h \right),$$

which is on $\mathcal{C}$. We have $\overrightarrow{HM} \cdot \overrightarrow{HH'} = 0$ and

$$\overrightarrow{M'M} \cdot \overrightarrow{HM} = \left(\frac{r^2}{r'} - r',\ \sqrt{1 - \frac{r^2}{r'^2}} \cdot r,\ h \right) \cdot \left(\frac{r^2}{r'},\ \sqrt{1 - \frac{r^2}{r'^2}} \cdot r,\ 0 \right) = 0,$$

so that HM is perpendicular to both ℓ and MM'. Therefore the distance between the lines ℓ and MM' is $|HM| = r$, which completes the proof.

Solution 2, by Edmund Swylan.

For every point M on $\mathcal{C}$, let $\mathcal{Q}$ be the plane orthogonal to ℓ that contains a point P of MM' nearest to ℓ. Let $O, \mathcal{D}, \mathcal{D}', N, N'$ be the orthogonal projections of $\ell, \mathcal{C}, \mathcal{C}', M, M'$, respectively, onto $\mathcal{Q}$. The distance between the lines ℓ and MM' projects to $|PO|$. Our problem is thereby reduced to a 2-dimensional problem:

> Given circles $\mathcal{D}$ and $\mathcal{D}'$ in the same plane with common centre O and radii r and r', a fixed point N' on $\mathcal{D}'$, and a point N moving about $\mathcal{D}$, what is the maximum distance from O to NN'?

The answer is r.

For $r < r'$ the maximum is achieved if and only if NN' is tangent to $\mathcal{D}$. For $r = r'$ it is achieved if and only if $N = N'$ and the line NN' degenerates into a point which occurs when MM' is parallel to ℓ.

4029. *Proposed by Paul Bracken.*

Suppose $a > 0$. Find the solutions of the following equation in the interval $(0, \infty)$:

$$\frac{1}{x+1} + \sum_{n=1}^{\infty} \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} = x - a.$$

We received four correct solutions and will feature two different ones.

Solution 1. We present a composite of the very similar solutions by Arkady Alt and the proposer, Paul Bracken. Another similar solution was received from Oliver Geupel.

It is clear that

$$\frac{1}{x} - \frac{1}{x+1} = \frac{1}{x(x+1)}, \quad \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)(x+2)} = \frac{2}{x(x+1)(x+2)},$$

and

$$\frac{n!}{x(x+1)(x+2)\cdots(x+n)} - \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} = \frac{(n+1)!}{x(x+1)(x+2)\cdots(x+n+1)}.$$

It therefore follows by induction that

$$\frac{1}{x} - \frac{1}{x+1} - \sum_{k=1}^{n-1} \frac{k!}{(x+1)(x+2)\dots(x+k+1)} = \frac{n!}{x(x+1)(x+2)\dots(x+n)}.$$

However, for $x > 0$,

$$\lim_{n\to\infty} \frac{n!}{x(x+1)\dots(x+n)} = 0,$$

since

$$\frac{n!}{x(x+1)(x+2)\cdots(x+n)} = \frac{1}{x\,(x+1)\left(\frac{x}{2}+1\right)\cdots\left(\frac{x}{n}+1\right)}$$

and

$$(x+1)\left(\frac{x}{2}+1\right)\cdots\left(\frac{x}{n}+1\right) > 1 + x\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

Hence the left-hand side of the original equation is given by

$$\frac{1}{x+1} + \sum_{n=1}^{\infty} \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} = \frac{1}{x}.$$

Therefore the original equation is equivalent to $x^2 - ax - 1 = 0$. This quadratic equation has the following unique solution in $(0, \infty)$:

$$x_r = \frac{1}{2}(a + \sqrt{a^2+4}).$$

Solution 2, by Albert Stadler.

We note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} \\
&= \sum_{n=1}^{\infty} \frac{\Gamma(x+1)\Gamma(n+1)}{\Gamma(x+1)(x+2)\cdots(x+n+1)} \\
&= \sum_{n=1}^{\infty} \frac{\Gamma(x+1)\Gamma(n+1)}{\Gamma(x+n+1)} \\
&= \sum_{n=1}^{\infty} \beta(x+1, n+1) \\
&= \sum_{n=1}^{\infty} \int_0^1 t^x(1-t)^n \, dt \\
&= \int_0^1 t^{x-1}(1-t) \, dt \\
&= \frac{1}{x} - \frac{1}{x+1}, \quad x > 0.
\end{aligned}$$

The original equation is therefore equivalent to $\frac{1}{x} = x - a$ or $x^2 - ax - 1 = 0$. This quadratic equation has exactly one positive root, which is $x_r = \frac{1}{2}(a + \sqrt{a^2+4})$.

4030. *Proposed by Paolo Perfetti.*

a) Prove that $4^{\cos t} + 4^{\sin t} \geq 5$ for $t \in [0, \frac{\pi}{4}]$.

b) Prove that $6^{\cos t} + 6^{\sin t} \geq 7$ for $t \in [0, \frac{\pi}{4}]$.

There were six submitted solutions for this problem, four of which were correct. We present the solution by Michel Bataille.

Lemma. Let $u(t) = \sin t \cos t(\cos t + \sin t)$. Then, u is an increasing function on $[0, \frac{\pi}{4}]$ with $u(0) = 0$ and $u(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$.

Proof. $u(0) = 0, u(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ are immediate. A simple calculation gives the derivative of u:

$$u'(t) = (\cos t - \sin t)((\cos t + \sin t)^2 + \sin t \cos t).$$

For $t \in (0, \frac{\pi}{4}), \cos t > \sin t$, hence $u'(t) > 0$ and so u is increasing on $[0, \frac{\pi}{4}]$. □

a) Let $f(t) = 4^{\cos t} + 4^{\sin t}$. We show that f is increasing on $[0, \frac{\pi}{4}]$ (the required result then follows since $f(0) = 5$). To this end, we prove that $f'(t) > 0$ for all $t \in (0, \frac{\pi}{4})$. We calculate

$$f'(t) = (\ln 4)4^{\cos t} \cos t \left(4^{\sin t - \cos t} - \frac{\sin t}{\cos t}\right)$$

so that it is sufficient to prove that $\phi(t) > 0$ for $t \in (0, \frac{\pi}{4})$ where

$$\phi(t) = (\sin t - \cos t)(\ln 4) - \ln(\sin t) + \ln(\cos t).$$

Now, we easily obtain $\phi'(t) = \frac{(\ln 4)u(t)-1}{\sin t \cos t}$ with, from the lemma,

$$(\ln 4)u(t) - 1 < \frac{\sqrt{2}\ln 4}{2} - 1 < 0.$$

Therefore, $\phi'(t) < 0$ for $t \in (0, \frac{\pi}{4})$ and $\phi(t) > \phi\left(\frac{\pi}{4}\right) = 0$, as desired.

b) Similarly, we introduce $g(t) = 6^{\cos t} + 6^{\sin t}$ whose derivative has the same sign as $\psi(t) = (\sin t - \cos t)(\ln 6) - \ln(\sin t) + \ln(\cos t)$. Here,

$$\psi'(t) = \frac{\ln 6}{\sin t \cos t} \cdot \left(u(t) - \frac{1}{\ln 6}\right),$$

and since $0 < \frac{1}{\ln 6} < \frac{\sqrt{2}}{2}$, $u(t) - \frac{1}{\ln 6}$ (and so $\psi'(t)$) vanishes at a unique t_0 in $(0, \frac{\pi}{4})$. From the lemma, we deduce that $\psi'(t) < 0$ if $0 < t < t_0$ and $\psi'(t) > 0$ if $t_0 < t < \frac{\pi}{4}$. Thus, ψ is decreasing on $(0, t_0]$ and increasing on $[t_0, \frac{\pi}{4})$. Since $\psi(\frac{\pi}{4}) = 0$, we must have $\psi(t_0) < 0$, and since $\lim_{t \to 0^+} \psi(t) = \infty$, we deduce that for some $\alpha \in (0, t_0)$, we have $\psi(t) > 0$ if $t \in (0, \alpha)$, $\psi(\alpha) = 0$ and $\psi(t) < 0$ if $t \in (\alpha, \frac{\pi}{4})$. Thus, $g'(t) > 0$ if $t \in (0, \alpha)$ and $g'(t) < 0$ if $t \in (\alpha, \frac{\pi}{4})$ and so $g(t) \geq (\min(g(0), g(\pi/4)) = 7$ for all $t \in [0, \frac{\pi}{4}]$.

Editor's Comments. It turns out that AM-GM is too weak to prove this inequality when used right at the beginning; the resulting right-hand-side is too small. However, one may use AM-GM in a step of the proof, as A. Stadler did, and have things work out well; the Stadler solution is an impressive use of Taylor series and clever bounds. As well, the 'general' inequality, $a^{\cos(t)} + a^{\sin(t)} \geq a + 1$, is not true over the required interval for every $a > 1$; plotting it for $a = 10$, for example, shows this.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(4), p. 169–172.

4031. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Prove that

$$\frac{2F_1^4+F_2^4+F_3^4}{F_1^2+F_3^2}+\frac{2F_2^4+F_3^4+F_4^4}{F_2^2+F_4^2}+\cdots+\frac{2F_n^4+F_1^4+F_2^4}{F_n^2+F_2^2}>2F_nF_{n+1},$$

where F_n represents the nth Fibonacci number ($F_0=0, F_1=1$ and $F_{n+2}=F_n+F_{n+1}$ for all $n\geq 1$).

We received five correct solutions. We present two solutions.

Editor's comments. When $n=1$, the interpretation of the left side is not clear, while when $n=2$, we obtain equality. Therefore, we suppose that $n\geq 3$.

Solution 1, by Adnan Ali and the proposers (independently).

Observe that, for positive x, y, z,

$$\frac{2x^2+y^2+z^2}{x+z}\geq x+y$$

with equality if and only if $x=y=z$, since this inequality is equivalent to $(x-y)^2+(y-z)^2+(z-x)^2\geq 0$. It follows that the left side of the inequality is greater than

$$(F_1^2+F_2^2)+(F_2^2+F_3^2)+\cdots+(F_{n-1}^2+F_n^2)+(F_n^2+F_1^2)=2\sum_{k=1}^{n}F_k^2.$$

Since $\sum_{k=1}^{n}F_k^2=F_nF_{n+1}$ (easily obtained by induction for $n\geq 1$), the result follows.

Solution 2, by Arkady Alt.

For positive x, y, z,

$$\frac{x^2}{y+z}\geq\frac{4x-y-z}{4}$$

with equality iff $2x=y+z$. This implies that

$$\begin{aligned}\frac{2F_i^4+F_j^4+F_k^4}{F_i^2+F_k^2}&>\frac{2(4F_i^2-F_i^2-F_k^2)+(4F_j^2-F_i^2-F_k^2)+(4F_k^2-F_i^2-F_k^2)}{4}\\&=F_i^2+F_j^2,\end{aligned}$$

for distinct i, j, k (since not all of F_i, F_j, F_k are equal to $F_i + F_k$). Thus the left side is greater than $2\sum_{k=1}^{n} F_k^2 = 2F_nF_{n+1}$.

4032. *Proposed by Dan Stefan Marinescu and Leonard Giugiuc.*

Prove that in any triangle ABC with sides a, b and c, inradius r and exradii r_a, r_b, r_c, we have:

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq 2\sqrt{3r(r_a + r_b + r_c)}.$$

We received 13 correct solutions. We present two solutions.

Solution 1, by Titu Zvonaru.

Using Ravi's substitutions ($a = y + z$, $b = z + x$, $c = x + y$, with $x, y, z > 0$), we have

$$[ABC] = \sqrt{xyz(x+y+z)},\ r = \sqrt{\frac{xyz}{x+y+z}},\ r_a = \frac{\sqrt{xyz(x+y+z)}}{x},$$

so that

$$r(r_a + r_b + r_c) = xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = xy + yz + zx.$$

We have to prove that

$$\sqrt{(x+y)(y+z)} + \sqrt{(y+z)(z+x)} + \sqrt{(z+x)(x+y)} \geq 2\sqrt{3(xy+yz+zx)}.$$

Using Minkowski's inequality and the inequality $(x+y+z)^2 \geq 3(xy+yz+zx)$, we obtain

$$\begin{aligned}
&\sqrt{(x+y)(y+z)} + \sqrt{(y+z)(z+x)} + \sqrt{(z+x)(x+y)}\\
&= \sqrt{x^2 + (xy+yz+zx)} + \sqrt{y^2 + (xy+yz+zx)} + \sqrt{z^2 + (xy+yz+zx)}\\
&\geq \sqrt{(x+y+z)^2 + (3\sqrt{xy+yz+zx})^2}\\
&\geq \sqrt{3(xy+yz+zx) + 9(xy+yz+zx)}\\
&= 2\sqrt{3(xy+yz+zx)}.
\end{aligned}$$

Equality holds if and only if $x = y = z$; that is, if and only if triangle ABC is equilateral.

Solution 2, composite of similar solutions by Sĕfket Arslanagic and Kee-Wai Lau.

By the known equality

$$r_a + r_b + r_c = 4R + r,$$

the given inequality is equivalent to

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq 2\sqrt{3r(4R + r)}.$$

By the AM-GM inequality,

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq 3 \cdot \sqrt[3]{abc}.$$

It therefore suffices to show that

$$3 \cdot \sqrt[3]{abc} \geq 2 \cdot \sqrt{3r(4R + r)},$$

which is successively equivalent to

$$\begin{aligned} 3 \cdot \sqrt[3]{4Rrs} &\geq 2 \cdot \sqrt{3r(4R + r)} \\ 3^6 \cdot 16R^2r^2s^2 &\geq 2^6 \cdot 27r^3(4R + r)^3 \\ 27R^2s^2 &\geq 4r(4R + r)^3. \end{aligned}$$

By the inequalities $s^2 \geq 16Rr - 5r^2$ due to J.C. Gerretsen and $R \geq 2r$ due to L. Euler, we have

$$\begin{aligned} 27R^2s^2 - 4r(4R + r)^3 &\geq 27(16Rr - 5r^2)R^2 - 4r(4R + r)^3 \\ &= r(R - 2r)(176R^2 + 25Rr + 2r^2) \\ &\geq 0. \end{aligned}$$

This proves the inequality of the problem. Equality holds for the equilateral triangle.

4033. *Proposed by Salem Malikic.*

Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be positive real numbers and $x_1, \ldots, x_n$ be real numbers such that $x_1 + \cdots + x_n = 1$ and $\alpha_i x_i + \beta_i \geq 0$ for all $i = 1, \ldots, n$. Find the maximum value of

$$\sqrt{\alpha_1 x_1 + \beta_1} + \sqrt{\alpha_2 x_2 + \beta_2} + \cdots + \sqrt{\alpha_n x_n + \beta_n}.$$

We received eight correct solutions and one incorrect solution. We present a composite of three nearly identical solutions given independently by Adnan Ali, Joe Schlosberg, and Titu Zvonaru.

Let M denote the required maximum value. Applying the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} &\sqrt{\alpha_1 x_1 + \beta_1} + \sqrt{\alpha_2 x_2 + \beta_2} + \cdots + \sqrt{\alpha_n x_n + \beta_n} \\ &\quad = \sqrt{\alpha_1}\sqrt{x_1 + \frac{\beta_1}{\alpha_1}} + \sqrt{\alpha_2}\sqrt{x_2 + \frac{\beta_2}{\alpha_2}} + \cdots + \sqrt{\alpha_n}\sqrt{x_n + \frac{\beta_n}{\alpha_n}} \\ &\quad \geq \sqrt{(\alpha_1 + \alpha_2 + \cdots + \alpha_n)(x_1 + x_2 + \cdots + x_n + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \cdots + \frac{\beta_n}{\alpha_n})} \\ &\quad = \sqrt{(\alpha_1 + \alpha_2 + \cdots + \alpha_n)(1 + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \cdots + \frac{\beta_n}{\alpha_n})}. \end{aligned}$$

The equality holds if and only if for some k we have

$$\frac{x_1+\frac{\beta_1}{\alpha_1}}{\alpha_1}=\frac{x_2+\frac{\beta_2}{\alpha_2}}{\alpha_2}=\cdots=\frac{x_n+\frac{\beta_n}{\alpha_n}}{\alpha_n}=k.$$

Since

$$k=\frac{\sum x_i+\sum\frac{\beta_i}{\alpha_i}}{\sum\alpha_i}=\frac{1+\sum\frac{\beta_i}{\alpha_i}}{\sum\alpha_i},$$

we have

$$x_i=\alpha_i\cdot\frac{1+\sum\frac{\beta_i}{\alpha_i}}{\sum\alpha_i}-\frac{\beta_i}{\alpha_i},\quad \text{for}\quad i=1,2,\ldots,n. \tag{1}$$

Therefore,

$$M=\sqrt{(\alpha_1+\alpha_2+\cdots+\alpha_n)(1+\frac{\beta_1}{\alpha_1}+\frac{\beta_2}{\alpha_2}+\cdots+\frac{\beta_n}{\alpha_n})}$$

attained when x_i are as in (1).

4034. *Proposed by Michel Bataille.*

Evaluate

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{16^n}\sum_{k=0}^{2n}\frac{(-1)^k}{2n+1-k}\binom{2k}{k}\binom{4n-2k}{2n-k}.$$

We received three correct solutions. We present the solution by Ángel Plaza.

Let us denote

$$a_n=\sum_{k=0}^{n}\frac{(-1)^k}{n+1-k}\binom{2k}{k}\binom{2n-2k}{n-k},$$

so the proposed expression reads as

$$\sum_{n=0}^{\infty}a_{2n}\left(\frac{-1}{16}\right)^n.$$

We will use the snake oil method to find the generating function of the sequence with general term a_n. If $F(x)$ is its generating function then

$$\begin{aligned}F(x) &= \sum_{n\geq 0} a_n x^n \\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{(-1)^k}{n+1-k}\binom{2k}{k}\binom{2n-2k}{n-k}\right)x^n \\ &= \sum_{k\geq 0}(-1)^k\binom{2k}{k}\sum_{n\geq k}\frac{1}{n+1-k}\binom{2n-2k}{n-k}x^n \\ &= \sum_{k\geq 0}(-1)^k\binom{2k}{k}x^k\sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n} \\ &= \sum_{k\geq 0}\binom{2k}{k}(-x)^k\frac{2}{1+\sqrt{1-4x}} \\ &= \frac{1}{\sqrt{1+4x}}\cdot\frac{2}{1+\sqrt{1-4x}},\end{aligned}$$

where we used the generating functions of the Catalan number and of the central binomial coefficients (both with radius of convergence $|x| < \frac{1}{4}$) in the last two steps. Now

$$\sum_{n\geq 0} a_{2n}x^n = \frac{F(\sqrt{x})+F(-\sqrt{x})}{2},$$

so the proposed sum is equal to

$$\frac{1}{\sqrt{1+i}}\cdot\frac{1}{1+\sqrt{1-i}}+\frac{1}{\sqrt{1-i}}\cdot\frac{1}{1+\sqrt{1+i}},$$

which can be calculated to $\sqrt{2}-\sqrt{\sqrt{2}-1}$.

4035. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let a and b be two real numbers such that $ab = 225$. Find all real solutions (in real 2×2 matrices) to the matrix equation

$$X^3 - 5X^2 + 6X = \begin{pmatrix} 15 & a \\ b & 15 \end{pmatrix}.$$

We received four submissions for this question, of which three were correct and complete. We present the solution by Michel Bataille.

We will show that the solutions are the three matrices

$$X_1 = \begin{pmatrix} 5/2 & a/6 \\ b/6 & 5/2 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 4 & a/15 \\ b/15 & 4 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 7/2 & a/10 \\ b/10 & 7/2 \end{pmatrix}.$$

Simple calculations show that these three matrices satisfy the given equation.

Let $A = \begin{pmatrix} 15 & a \\ b & 15 \end{pmatrix}$ and let I_2 be the 2×2 unit matrix. Since $\det(xI_2 - A) = x^2 - 30x$, the eigenvalues of A are 0 and 30.

Let X be a solution of the given equation and let λ be an eigenvalue of X. Then $\lambda^3 - 5\lambda^2 + 6\lambda$ is an eigenvalue of A, hence $\lambda^3 - 5\lambda^2 + 6\lambda \in \{0, 30\}$. Thus,

$$\lambda(\lambda - 2)(\lambda - 3) = 0 \quad \text{or} \quad (\lambda - 5)(\lambda^2 + 6) = 0$$

and the possible eigenvalues of X are $0, 2, 3, 5, i\sqrt{6}, -i\sqrt{6}$.

Noting that the characteristic polynomial $\chi(x)$ of the real matrix X is in $\mathbb{R}[x]$, if $i\sqrt{6}$ (resp. $-i\sqrt{6}$) is an eigenvalue of X, so is its complex conjugate and then X is similar to $\begin{pmatrix} i\sqrt{6} & 0 \\ 0 & -i\sqrt{6} \end{pmatrix}$. But then $A = X^3 - 5X^2 + 6X$ is similar to

$$\begin{pmatrix} -6i\sqrt{6} & 0 \\ 0 & 6i\sqrt{6} \end{pmatrix} - 5\begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix} + 6\begin{pmatrix} i\sqrt{6} & 0 \\ 0 & -i\sqrt{6} \end{pmatrix} = \begin{pmatrix} 30 & 0 \\ 0 & 30 \end{pmatrix},$$

a contradiction since 0 is an eigenvalue of A.

As a result, if λ_1, λ_2 are the eigenvalues of X (possibly $\lambda_1 = \lambda_2$), we have $\lambda_1, \lambda_2 \in \{0, 2, 3, 5\}$. Since the eigenvalues of X are real, X is similar to an upper triangular real matrix, say,

$$X = P\begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix}P^{-1}$$

for some invertible real matrix P and some real number α. Then

$$A = X^3 - 5X^2 + 6X = P\begin{pmatrix} \lambda_1^3 - 5\lambda_1^2 + 6\lambda_1 & \beta \\ 0 & \lambda_2^3 - 5\lambda_2^2 + 6\lambda_2 \end{pmatrix}P^{-1}$$

where $\beta \in \mathbb{R}$. Taking traces, it follows that

$$30 = (\lambda_1^3 - 5\lambda_1^2 + 6\lambda_1) + (\lambda_2^3 - 5\lambda_2^2 + 6\lambda_2) \qquad (1).$$

After trying all possibilities for λ_1 and λ_2, we realize that $\lambda_1 \neq \lambda_2$ and $\{\lambda_1, \lambda_2\}$ is either $\{0, 5\}$ or $\{2, 5\}$ or $\{3, 5\}$. We consider the three cases in turn:

- if $\{\lambda_1, \lambda_2\} = \{0, 5\}$, then $\chi(x) = x^2 - 5x$. Note that

  $$x^3 - 5x^2 + 6x = x(x^2 - 5x) + 6x = x\chi(x) + 6x.$$

  By the Cayley-Hamilton theorem, $\chi(X) = 0$; replacing x by X in the above we get $A = X\chi(X) + 6X = 6X$, hence $X = \frac{1}{6}A = \begin{pmatrix} 5/2 & a/6 \\ b/6 & 5/2 \end{pmatrix}$.

- if $\{\lambda_1, \lambda_2\} = \{2, 5\}$, then $\chi(x) = (x - 2)(x - 5)$ and

  $$x^3 - 5x^2 + 6x = (x + 2)\chi(x) + 10x - 20.$$

  Therefore $A = 10X - 20I_2$ and

  $$X = \frac{1}{10}(A + 20I_2) = \begin{pmatrix} 7/2 & a/10 \\ b/10 & 7/2 \end{pmatrix}.$$

- if $\{\lambda_1, \lambda_2\} = \{3, 5\}$, then $\chi(x) = (x-3)(x-5)$ and
$$x^3 - 5x^2 + 6x = (x+3)\chi(x) + 15x - 45.$$
Hence $A = 15X - 45$, which gives us
$$X = \frac{1}{15}(A + 45I_2) = \begin{pmatrix} 4 & a/15 \\ b/15 & 4 \end{pmatrix}.$$

The proof is complete.

4036. *Proposed by Arkady Alt.*

Let a, b and c be non-negative real numbers. Prove that for any real $k \geq \frac{11}{24}$ we have:

$$k(ab + bc + ca)(a + b + c) - (a^2c + b^2a + c^2b) \leq \frac{(3k-1)(a+b+c)^3}{9}.$$

We received five submissions all of which are correct. We present the solution by the proposer, slightly modified by the editor.

Due to cyclic symmetry of the functions involved, we may assume that $c = \min\{a, b, c\}$.

Let $x = a - c$ and $y = b - c$. Then $x, y, c \geq 0, a = x + c,\ b = y + c$, and $a + b + c = x + y + 3c$.

The given inequality is equivalent to

$$(3k-1)(a+b+c)^3 - 9k(ab+bc+ca)(a+b+c) + 9\left(a^2c + b^2a + c^2b\right) \geq 0$$

or

$$\begin{aligned}(3k-1)(x+y+3c)^3 - 9k((x+c)(y+c) + c(x+y+2c))(x+y+3c) \\ + 9((x+c)^2c + (y+c)^2(x+c) + c^2(y+c)) \geq 0. \quad (1)\end{aligned}$$

Let $F(x, y, c)$ denote the expression on the left hand side of (1), and set $p = x + y$ and $q = xy$.
Since $9k((x+c)(y+c) + c(x+y+2c))(x+y+3c) = 9k(3c^2 + 2pc + q)(p+3c)$ and

$$\begin{aligned}&9((x+c)^2c + (y+c)^2(x+c) + c^2(y+c)) \\ &= 9(cx^2 + cy^2 + 2cxy + 3c^2(x+y) + xy^2 + 3c^3) \\ &= 9(cp^2 + 3c^2p + 3c^3 + xy^2) \\ &= 9cp^2 + 27c^2p + 27c^3 + 9xy^2 \\ &= (p+3c)^3 - p^3 + 9xy^2,\end{aligned}$$

we have

$$\begin{aligned}F(x,y,c) &= (3k-1)(p+3c)^3 - 9k(3c^2+2pc+q)(p+3c) + (p+3c)^3 - p^3 + 9xy^2 \\ &= 3k(p+3c)^3 - 9k(3c^2+2pc+q)(p+3c) - p^3 + 9xy^2 \\ &= 3k(p^3+9cp^2+27c^2p+27c^3) - 9k(2cp^2+9c^2p+9c^3+pq+3cq) \\ &\qquad - p^3 + qxy^2 \\ &= (3k-1)p^3 + 9ckp^2 - 27ckq - 9kpq + 9xy^2 \\ &= (3k-1)(x+y)^3 + 9ck(x+y)^2 - 27ckxy - 9kxy(x+y) + 9xy^2 \\ &= (3k-1)(x^3+y^3+3xy(x+y)) + 9ck(x^2+2xy+y^2) - 27ckxy \\ &\qquad - 9kxy(x+y) + 9xy^2 \\ &= (3k-1)x^3 + 6xy^2 - 3x^2y + 9ck(x^2-xy+y^2) + (3k-1)y^3. \qquad (2)\end{aligned}$$

Clearly, $9ck(x^2-xy+y^2)+(3k-1)y^3 \geq 0$. Furthermore,

$$(3k-1)x^3 + 6xy^2 - 3x^2y = x((3k-1)x^2 - 3xy + 6y^2) \geq 0$$

since the discriminant of $(3k-1)x^2 - 3xy + 6y^2$ is

$$9y^2 - 24(3k-1)y^2 = 3(11-24k)y^2 \leq 0$$

and $3k-1>0$. Hence, from (2) we conclude that $F(x,y,c) \geq 0$ which by (1) completes the proof.

4037. *Proposed by Michel Bataille.*

Let P be a point of the incircle γ of a triangle ABC. The perpendiculars to BC, CA and AB through P meet γ again at U, V and W, respectively. Prove that the area of UVW is independent of the chosen point P on γ.

We received six correct and complete solutions. We present the solution by Oliver Geupel. Ricard Peiró and Prithwijt De submitted similar solutions.

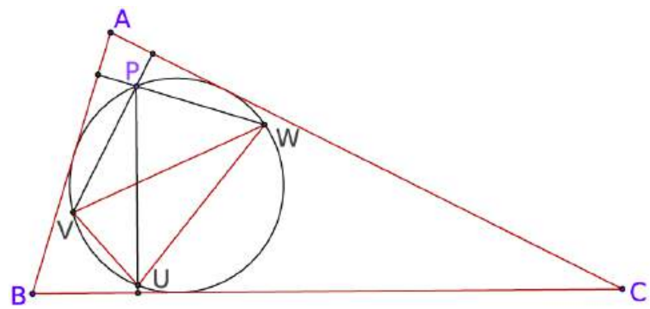
We prove that triangle UVW is similar to triangle ABC. As a consequence, since γ is the circumcircle of UVW, the area of triangle UVW is

$$[UVW] = \frac{r^2}{R^2}[ABC],$$

where r and R denote the inradius and the circumradius, respectively, of $\triangle ABC$.

Let $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{U}$, $\hat{V}$, and $\hat{W}$ denote measures of the the interior angles of the triangles ABC and UVW. Since $PV \perp AC$ and $PW \perp AB$, the size of $\angle VPW$ is $180^\circ - \hat{A}$. Also, since the points P, U, V, and W are concyclic, $\angle VUW$ is equal to either $\angle VPW$ or $180^\circ - \angle VPW$. Hence, $\hat{U} \in \{\hat{A},\ 180^\circ - \hat{A}\}$. Analogously, $\hat{V} \in \{\hat{B},\ 180^\circ - \hat{B}\}$ and $\hat{W} \in \{\hat{C},\ 180^\circ - \hat{C}\}$.

We show that $(\hat{U}, \hat{V}, \hat{W}) = (\hat{A}, \hat{B}, \hat{C})$.



Assume the contrary. Then, there is no loss of generality in assuming that $\hat{U} \neq \hat{A}$. Thus, $\hat{U} = 180^\circ - \hat{A}$.

If $(\hat{V}, \hat{W}) = (\hat{B}, \hat{C})$, then we obtain $180^\circ = \hat{U} + \hat{V} + \hat{W} = 180^\circ - \hat{A} + \hat{B} + \hat{C}$, so that $\hat{A} = \hat{B} + \hat{C} = 90^\circ = 180^\circ - \hat{A} = \hat{U}$, contradicting our assumption that $\hat{U} \neq \hat{A}$. Hence, $(\hat{V}, \hat{W}) \neq (\hat{B}, \hat{C})$. There is no loss of generality in assuming that $\hat{V} = 180^\circ - \hat{B}$. But then, $\hat{U} + \hat{V} = (180^\circ - A) + (180^\circ - B) > 180^\circ$, a contradiction.

Editor's Comments. We were pleased to find that among the six solutions submitted, four different formulas for the area of a triangle were used.

4038. *Proposed by George Apostolopoulos.*

Let x, y, z be positive real numbers such that $x + y + z = xyz$. Find the minimum value of the expression

$$\sqrt{\frac{1}{3}x^4 + 1} + \sqrt{\frac{1}{3}y^4 + 1} + \sqrt{\frac{1}{3}z^4 + 1}.$$

There were 21 correct solutions, with four from one submitter and three from another. An additional solution was incorrect.

Solution 1, by Arkady Alt, Šefket Arslanagić, and Daniel Dan (independently).

Since $xyz = x+y+z \geq 3\sqrt[3]{xyz}$, if follows that $x+y+z = xyz \geq 3\sqrt{3}$. Applying the inequality of the root mean square and arithmetic mean, we have, for $t = x, y, z$,

$$\sqrt{\frac{1}{3}t^4 + 1} = \sqrt{\left(\frac{t^2}{3}\right)^2 + \left(\frac{t^2}{3}\right)^2 + \left(\frac{t^2}{3}\right)^2 + 1}$$
$$\geq \frac{(t^2/3) + (t^2/3) + (t^2/3) + 1}{2} = \frac{t^2 + 1}{2}$$

with equality iff $t = \sqrt{3}$. (Alternatively, the inequality $\sqrt{\frac{1}{3}t^4 + 1} \geq \frac{t^2+1}{2}$ is equivalent to $(t^2 - 3)^2 \geq 0$.) Therefore, the left side of the inequality is not less than

$$\frac{1}{2}(3 + x^2 + y^2 + z^2) \geq \frac{1}{2}\left(3 + \frac{(x+y+z)^2}{3}\right) \geq \frac{1}{2}(3 + (27/3)) = 6.$$

Since equality occurs when $x = y = z = \sqrt{3}$, the desired minimum is 6.

Solution 2, by Šefket Arslanagić and Salem Malikić (independently).

As before, $x + y + z \geq 3\sqrt{3}$. We begin by establishing that

$$\sqrt{\frac{1}{3}u^4 + 1} \geq u\sqrt{3} - 1,$$

with equality iff $u = \sqrt{3}$. The result is clear when $u < 1/\sqrt{3}$. When $u \geq 1/\sqrt{3}$, the inequality is equivalent to

$$\frac{1}{3}u^4 + 2\sqrt{3}u \geq 3u^2.$$

By the arithmetic-geometric means inequality, we obtain that

$$\frac{1}{3}u^4 + 2\sqrt{3}u = \frac{1}{3}u^4 + u\sqrt{3} + u\sqrt{3} \geq 3\sqrt[3]{\frac{1}{3}u^4 \cdot u\sqrt{3} \cdot u\sqrt{3}} = 3u^2$$

as desired.

The left side of the inequality of the problem is not less than

$$\sqrt{3}(x + y + z) - 3 = 9 - 3 = 6,$$

with equality iff $x = y = z = \sqrt{3}$. The desired minimum is 6.

Solution 3, by Titu Zvonaru.

From the triangle inequality in Euclidean space $\mathbb{R}^2$,

$$\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|_2 \leq \|\mathbf{a}\|_2 + \|\mathbf{b}\|_2 + \|\mathbf{c}\|_2,$$

applied to $\mathbf{a} = (x^2/\sqrt{3}, 1)$, $\mathbf{b} = (y^2/\sqrt{3}, 1)$, $\mathbf{c} = (z^2/\sqrt{3}, 1)$, we have that

$$\begin{aligned}\sqrt{\frac{x^4}{3} + 1} + \sqrt{\frac{y^4}{3} + 1} + \sqrt{\frac{z^4}{3} + 1} &\geq \sqrt{\left(\frac{x^2}{\sqrt{3}} + \frac{y^2}{\sqrt{3}} + \frac{z^2}{\sqrt{3}}\right)^2 + (1 + 1 + 1)^2} \\ &= \sqrt{\frac{(x^2 + y^2 + z^2)^2}{3} + 9},\end{aligned}$$

with equality iff $x = y = z = \sqrt{3}$. Using $x^2 + y^2 + z^2 \geq xy + yz + zx$, the arithmetic-harmonic means inequality and $x + y + z = xyz$ in turn, we obtain that

$$x^2 + y^2 + z^2 \geq xy + yz + zx \geq \frac{9xyz}{x + y + z} = 9.$$

Thus the left side of the inequality is not less than

$$\sqrt{\frac{81}{3} + 9} = 6,$$

with equality iff $x = y = z = \sqrt{3}$. The desired minimum is 6.

Solution 4, by Ali Adnan.

By the Cauchy-Schwarz inequality,

$$\sqrt{\frac{1}{3}t^4+1}\cdot\sqrt{3+1}\geq t^2+1,$$

and by the arithmetic-geometric means inequality,

$$(x^2+y^2+z^2)(x+y+z)\geq 3(xyz)^{2/3}\cdot 3(xyz)^{1/3}=9xyz.$$

Therefore

$$\begin{aligned}\sqrt{\frac{1}{3}x^4+1}+\sqrt{\frac{1}{3}y^4+1}+\sqrt{\frac{1}{3}z^4+1}&\geq\frac{1}{2}(x^2+y^2+z^2)+\frac{3}{2}\\&\geq\frac{1}{2}\left(\frac{9xyz}{x+y+z}\right)+\frac{3}{2}=6,\end{aligned}$$

with equality if and only if $x=y=z=\sqrt{3}$.

Editor's comments. Seven solvers applied Jensen's Inequality to obtain the result, the function $\sqrt{(x^4/3)+1}$ being convex. Some solvers noted that, under the stated constraint, we can write $(x,y,z)=(\tan\alpha,\tan\beta,\tan\gamma)$ with $\alpha+\beta+\gamma=\pi$ and each angle less than $\pi/2$, or $(x,y,z)=(\cot\lambda+\cot\mu+\cot\nu)$ with $\lambda+\mu+\nu=\pi/2$, and then obtain a trigonometric inequality.

4039. *Proposed by Abdilkadir Altinaş.*

In a triangle ABC, let $\angle CAB=48^\circ$ and $\angle CBA=12^\circ$. Suppose D is a point on AB such that $CD=1$ and $AB=\sqrt{3}$. Find $\angle DCB$.

We received 17 correct solutions and will feature the solution submitted by the Skidmore College Problem Group.

Let $\angle DCB=\gamma$; we shall show that $\gamma=6^\circ$.

By the Law of Sines applied to triangle ABC, $\frac{\sin 12^\circ}{AC}=\frac{\sin 120^\circ}{\sqrt{3}}=\frac{1}{2}$, so we have

$$AC=2\sin 12^\circ.$$

For triangle ACD, $\frac{\sin(12^\circ+\gamma)}{AC}=\frac{\sin 48^\circ}{1}$, whence

$$\sin(12^\circ+\gamma)=2\sin 48^\circ\sin 12^\circ=\cos 36^\circ-\cos 60^\circ=\cos 36^\circ-\frac{1}{2}.$$

Let $\phi=\frac{1+\sqrt{5}}{2}$ $(=\phi^2-1)$ be the golden section (that is, the ratio of a diagonal of the regular pentagon to a side). We know that

$$\sin 18^\circ=\frac{1}{2\phi}=\frac{\phi}{2}-\frac{1}{2}\quad\text{and}\quad\cos 36^\circ=\frac{\phi}{2}.$$

[For example, if the regular pentagon $PQRST$ has unit sides, then the isosceles triangle PRS has apex angle 36° and sides ϕ and 1 (which provides the value for

$\sin 18^\circ$), while the isosceles triangle PQT with base angle 36° also has sides 1 and ϕ (which gives $\cos 36^\circ$).] Thus $\cos 36^\circ = \sin 18^\circ + \frac{1}{2}$, and

$$\sin(12^\circ + \gamma) = \sin 18^\circ = \sin 162^\circ.$$

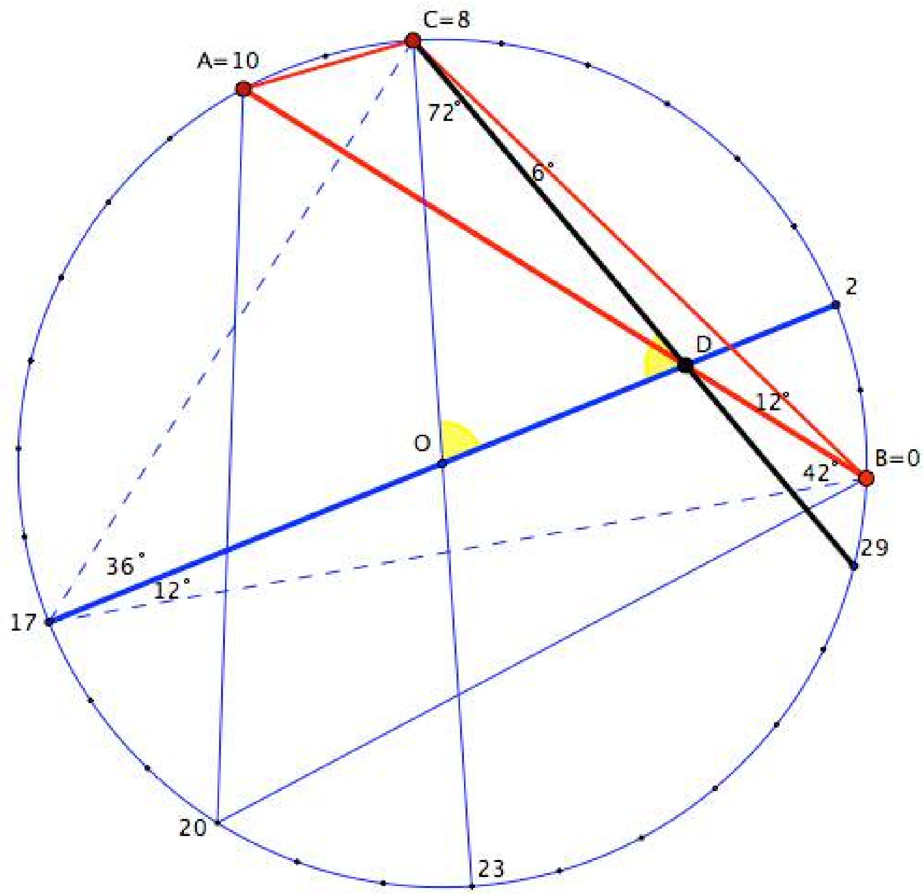
But $\gamma < 120^\circ$, which implies that $\gamma = 6^\circ$, as claimed.

Editor's comments. In a remark added to his solution, Dag Jonsson observed that there would be a second solution should we allow D to be a point of the line AB (instead of restricting it to the segment AB). Vertex A would then lie between D and B; the argument of the featured solution remains valid so that $\angle ADC = 18^\circ$, whence $\angle DCA = 30^\circ$ and, finally, $\angle DCB = 150^\circ$.

Returning to the case where D lies between A and B, we note that the degenerate quadrangle $ADBC$ of our problem is an example of an *adventitious quadrangle*. Adventitious quadrangles were first defined by Colin Tripp [5], but his narrow definition was later extended to include any quadrangle for which the angles formed by sides and diagonals are all rational multiples of 180°. Rigby [3] observed that the problem of classifying the adventitious quadrangles is equivalent to classifying all triple intersections of diagonals in regular polygons, a problem solved many years earlier by Gerrit Bol [1]. A complete account, including an elementary summary and a 15-item bibliography, was provided by Poonen and Rubinstein in [2]. Triangle ABC of our problem can be inscribed in a regular 30-gon that has a unit circumradius. Label its vertices from 0 to 29 and place A at 10, B at 0, and C at 8. Note that the angle subtended at a vertex of the 30-gon by any nonadjacent edge is 6°, which immediately implies that the angles of ΔABC are indeed $48^\circ, 12^\circ$, and 120°. Because AB is the edge of an inscribed equilateral triangle (with vertices numbered 0, 10, 20), we have $AB = \sqrt{3}$ as desired. Let D' be the point of intersection of the diameter 2, 17 and the diagonal 8, 29. If O is the circumcentre, then triangle $OD'C$ is isosceles (with angles $36^\circ, 72^\circ$, and 72°), whence $CD' = CO = 1$. It remains to prove that $D' \in AB$, which will immediately imply that $D' = D$ (and, consequently, that $\angle DCB = \angle D'CB = 6^\circ$). On p. 223 of [4] Rigby declares (with a slightly different labeling) that the diagonals 29, 8; 0, 10; and 2, 17 are indeed concurrent. While the author provides a "geometric" proof that these three lines are concurrent, it is perhaps more efficient to use trigonometry. Applying the sine form of Ceva's theorem to the triangle whose vertices are those numbered 0, 8, 17 with cevians joining 0 to 10, 8 to 29, and 17 to 2 (as in the accompanying figure), we must show that the product

$$\frac{\sin 42}{\sin 12} \cdot \frac{\sin 6}{\sin 72} \cdot \frac{\sin 36}{\sin 12}$$

equals 1. This is easily accomplished using an argument similar to that of our featured solution.



References

[1] G. Bol, Beantwoording van prijsvraag no. 17, *Nieuw archief voor Wiskunde*, **18** (1936) 14-66.

[2] Bjorn Poonen and Michael Rubinstein, The Number of Intersection Points Made by the Diagonals of a Regular Polygon. *Siam Journal of Discrete Mathematics* **11**:1 (February 1998) 135-156.

[3] J.F. Rigby, Adventitious Quadrangles: A Geometrical Approach. *Mathematical Gazette* **62**:421 (October 1978) 183-191.

[4] J.F. Rigby, Multiple Intersections of Diagonals of Regular Polygons and Related Topics. *Geometriae Dedicata* **9**:2 (1980) 207-238.

[5] C.E. Tripp, Adventitious Angles. *Mathematical Gazette* **59**:408 (June 1975) 98-106.

4040. *Proposed by Ali Behrouz.*

Find all functions $f : \mathbb{N} \mapsto \mathbb{N}$ such that

$$(f(a)+b)f(a+f(b)) = (a+f(b))^2 \quad \forall a,b \in \mathbb{N}.$$

There were nine submitted solutions for this problem, all of which were correct. We present three solutions; the first solution is for the case that 0 *is the first natural number, and the latter two are for the case that* 1 *is the first natural number.*

Solution 1, by Leonard Giugiuc.

Suppose that 0 is the first natural number. Let $a = b = 0$; we have:

$$f(0)f\big(f(0)\big) = f(0)^2,$$

which implies that either $f(0) = 0$ or $f\big(f(0)\big) = f(0)$. If $f\big(f(0)\big) = f(0)$, then we replace a with 0 and b with $f(0)$ in the given relation and obtain:

$$2f(0)f(0) = f(0)^2,$$

which implies that $f(0)^2 = 0$. Hence no matter what, $f(0) = 0$. Now replace b with 0 and obtain:

$$f(a)^2 = a^2,$$

for all $a \geq 0$, and so $f(a) = a$ for all $a \geq 0$, because $f(a) \geq 0$. Thus the required function is the identity (which clearly satisfies the given relation).

Solution 2, by Adnan Ali.

Suppose that 1 is the first natural number. Denote by $P(a,b)$ the above functional equation for all $a,b \in \mathbb{N}$. Now let $f(1) = k$, where k is a natural number. Then $P(1,1)$ gives $f(k+1) = k+1$. Using this, $P(1,k+1)$ implies

$$(2k+1)f(k+2) = (k+2)^2.$$

This means that $(2k+1)$ divides $(k+2)^2$. This means that $2k+1$ divides

$$4(k+2)^2 - 8(2k+1) - (2k+1)(2k-1) = 9.$$

This forces $2k+1$ to be either 3 or 9, which gives $f(1) = 1$ or $f(1) = 4$.

If $f(1) = 4$, then from $P(1,1)$ we get $f(5) = 5$. Next $P(5,1)$ gives

$$(f(5)+1)f(5+f(1)) = (5+f(1))^2 = 81,$$

implying that $6f(9) = 81$, which is clearly impossible as $6 \nmid 81$ and $f(9) \in \mathbb{N}$.

Thus we must have $f(1) = 1$. Now we prove by induction that $f(1) = 1$ implies that $f(n) = n$ for all $n \in \mathbb{N}$. The case $n = 1$ is already true. Now assume that $f(a) = a$ for some $a \geq 1$. Then from $P(a,1)$:

$$(f(a)+1)f(a+f(1)) = (a+f(1))^2,$$

which implies that $f(a+1)=a+1$, and consequently by induction $f(n)=n$ for all $n \in \mathbb{N}$. So, in summary the only function satisfying the equation is the function $f(n)=n$ for all $n \in \mathbb{N}$.

Solution 3, by Joseph Ling.

It is easy to see that $f(n)=n$ for all n satisfies the required relation for all $a, b \in \mathbb{N}$ We show that there are no other solutions.

First, we note that f is one-to-one. For if b_1 and b_2 are such that $f(b_1)=f(b_2)$, then for all a, we have

$$f(a)+b_1=\frac{(a+f(b_1))^2}{f(a+f(b_1))}=\frac{(a+f(b_2))^2}{f(a+f(b_2))}=f(a)+b_2,$$

implying that $b_1=b_2$.

Next, we note that f is (strictly) increasing. For if there exist $b_1<b_2$ with $f(b_1)>f(b_2)$, then for every $a \in \mathbb{N}$, we consider $a'=a+f(b_1)-f(b_2)$. We have $a' \in \mathbb{N}$, with $a'+f(b_2)=a+f(b_1)$, and consequently,

$$f(a')+b_2=\frac{(a'+f(b_2))^2}{f(a'+f(b_2))}=\frac{(a+f(b_1))^2}{f(a+f(b_1))}=f(a)+b_1<f(a)+b_2$$

implying that $f(a')<f(a)$. But then this means that the range of f has no smallest element, contradicting the well-ordering principle.

Now, since f is strictly increasing, a simple induction shows that $f(n) \geq n$ for all $n \in \mathbb{N}$. It follows that $f(a+f(b)) \geq a+f(b)$ for all $a, b \in \mathbb{N}$. Consequently, the given relation implies that $f(a)+b \leq a+f(b)$ for all $a, b \in \mathbb{N}$. But then by symmetry, we conclude that in fact, $f(a)+b=a+f(b)$ for all $a, b \in \mathbb{N}$. Therefore,

$$f(n)=n+k$$

for all $n \in \mathbb{N}$, where $k=f(1)-1$. Applying this to the relation, we get

$$(a+b+k)(a+b+2k)=(a+b+k)^2$$

for all $a, b \in \mathbb{N}$. But then this implies that $k=0$, and so, $f(n)=n$ for all $n \in \mathbb{N}$.

Editor's Comments. Most of the solvers, and the proposer, assumed simply that 0 was not a natural number; only Giugiuc solved the problem with both interpretations. Assuming that 0 is natural means there is more known information, and thus it is reasonable that the solution is easier than in the other case. Regarding the other case, the solution by Ali uses divisibility, a number-theoretic property, whereas the solution by Ling uses the function-theoretic properties of solutions to this relation. Other solutions use the relation differently (some use it to find additivity of the function, others show that there are infinitely many primes satisfying $f(n)=n$, etc.).

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(5), p. 212–215.

4041. *Proposed by Arkady Alt.*

Let a, b and c be the side lengths of a triangle ABC. Let AA', BB' and CC' be the heights of the triangle and let $a_p = B'C', b_p = C'A'$ and $c_p = A'B'$ be the sides of the orthic triangle. Prove that:

a) $a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p) = 3abc$;

b) $a_p + b_p + c_p \leq s$, where s is the semiperimeter of ABC.

We received 15 correct solutions and present the solution by Michel Bataille.

We show (a) and (b) in the case when ΔABC has no obtuse angle and provide a counter-example in the opposite case.

First, suppose that angles A, B, C are acute. Since $\Delta AB'B$ is right-angled with $\angle AB'B = 90°$, we have $AB' = c \cdot \cos A$. Similarly, $AC' = b \cdot \cos A$, and it follows that

$$\begin{aligned} B'C'^2 &= c^2 \cos^2 A + b^2 \cos^2 A - 2bc \cos^3 A \\ &= (c^2 + b^2 - 2bc \cos A) \cos^2 A = a^2 \cos^2 A \end{aligned}$$

and so $a_p = B'C' = a \cos A$. In a similar way, we obtain $b_p = A'C' = b \cos B$ and $c_p = A'B' = c \cos C$.

Now we calculate $X = a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p)$ as follows:

$$\begin{aligned} X &= a^2 b \cos B + a^2 c \cos C + b^2 c \cos C + b^2 a \cos A + bc^2 \cos B + c^2 a \cos A \\ &= ab(a \cos B + b \cos A) + bc(b \cos C + c \cos B) + ca(c \cos A + a \cos C) \\ &= abc + bca + cab = 3abc, \end{aligned}$$

as desired. Denoting by r and R the inradius and the circumradius of ΔABC and using the Law of Sines, we get

$$\begin{aligned} a_p + b_p + c_p &= a \cos A + b \cos B + c \cos C \\ &= R(\sin 2A + \sin 2B + \sin 2C) \\ &= 4R \sin A \sin B \sin C \\ &= 4R \cdot \frac{abc}{8R^3} = \frac{4rRs}{2R^2} = s \cdot \frac{2r}{R} \end{aligned}$$

and the result $a_p + b_p + c_p \leq s$ follows from Euler's inequality $2r \leq R$.

If ΔABC is right-angled, say $\angle BAC = 90^\circ$, results (a) and (b) continue to hold if we take, as is natural, $a_p = 0$, $b_p = c_p = h$, where $h = AA'$. Indeed, we have $3abc = 3a \cdot ah = 3a^2h$ and

$$\begin{aligned} a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p) &= a^2 \cdot 2h + b^2 \cdot h + c^2 \cdot h \\ &= h(b^2 + c^2 + 2a^2) = 3a^2h. \end{aligned}$$

Also, the inequality $a_p + b_p + c_p \le s$ rewrites as $4h \le a+b+c$ or $4bc \le a^2 + a(b+c)$. Since $b + c \ge 2\sqrt{bc}$ and $a^2 = b^2 + c^2 \ge 2bc$, we have

$$a^2 + a(b+c) \ge 2bc + 2\sqrt{2bc} \cdot 2\sqrt{bc} = (2 + 2\sqrt{2})bc \ge 4bc.$$

None of these results is correct, however, if an angle of ΔABC is obtuse, as the following example shows. Consider a triangle ABC with $\angle BAC = 120^\circ$ and $AB = AC$. Then $b = c$, $a = c\sqrt{3}$, and $a_p = b_p = c_p = \frac{a}{2} = \frac{c\sqrt{3}}{2}$. One easily finds that $3abc = 3c^3\sqrt{3}$, while

$$a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p) = 5c^3\sqrt{3}.$$

Also,

$$a_p + b_p + c_p = \frac{3c\sqrt{3}}{2} > \frac{(2+\sqrt{3})c}{2} = s.$$

4042. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Let a, b and c be real numbers in $[0, \pi/2]$ such that $a + b + c = \pi$. Prove the inequality

$$2\sqrt{2}\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2} \ge \sqrt{\cos a \cos b \cos c}.$$

We received 14 correct solutions. We present the solution by Scott Brown. Similar solutions came from Arslanagić Šefket, Michel Bataille, Andrea Fanchini, and John Heuvel.

In [1] and [2] respectively, we find the identities

$$\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2} = \frac{r}{4R} \tag{1}$$

and

$$\cos a \cos b \cos c = \frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2}, \tag{2}$$

where R, r, and s are the circumradius, inradius, and semiperimeter of the triangle. We square both sides of the original inequality to obtain the equivalent statement

$$8\sin^2\frac{a}{2}\sin^2\frac{b}{2}\sin^2\frac{c}{2} \le \cos a \cos b \cos c,$$

into which we substitute the identities (1) and (2). The resulting inequality is equivalent to one due to Gerretsen [3]:

$$s^2 \leq 4R^2 + 4Rr + 3r^2.$$

References

[1] Anders Bager. "A family of goniometric inequalities." Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., No. 338 352 (1971), p. 10.

[2] Anders Bager. "Another family of goniometric inequalities." Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., No. 412 460 (1973), p. 209.

[3] D. S. Mitrinovic et al. *Recent Advances in Geometric Inequalities.* Kluwer, Dordrecht, 1989.

Editor's Comment. Digby Smith pointed out that the inequality is equivalent to ***Crux*** Problem **974**, proposed by Jack Garfunkel in Volume 10, (8), October 1984, and solved by Murray Klamkin in Volume 11 (10), December 1985. The solution to **974** is based on ***Crux*** Problem **836**, proposed by Vedula N. Murty in Volume 9 (4), April 1983, and solved, again by Klamkin, in Volume 10 (7), August 1984.

4043. *Proposed by Michel Bataille.*

Suppose that the lines m and n intersect at A and are not perpendicular. Let B be a point on n, with $B \neq A$. If F is a point of m, distinct from A, show that there exists a unique conic $\mathcal{C}_F$ with focus F and focal axis BF, intersecting n orthogonally at A. Given $\epsilon > 0$, how many of the conics $\mathcal{C}_F$ have eccentricity ϵ?

We recieved two correct solutions and present the solution submitted by the proposer.

Since $m \neq n$, the perpendicular to m through F and the perpendicular t to n at A intersect, say at K. Note that K is distinct from both F and A (since $F \neq A$). Define p to be the perpendicular to BF through K. Then $A \notin p$ (otherwise we would have $t \perp BF$, implying $n \parallel BF$, a contradiction). We also have $F \notin p$ (otherwise $KF \perp BF$, implying $BF = m$ and $B \in m$, contradicting $B \neq A$).

We first show uniqueness: Suppose that $\mathcal{C}_F$ exists. Note that t is the tangent to $\mathcal{C}_F$ at A. Since $\angle KFA = 90^\circ$ and $K \in t$, K must be on the directrix of $\mathcal{C}_F$ associated with F (see [2], Theorem 1 p. 14). Thus, $\mathcal{C}_F$ must be the unique conic with focus F, directrix p and eccentricity $\frac{AF}{d(A,p)}$. Conversely, because the line p misses the distinct points A and F, we can consider the unique conic with focus F, directrix p and eccentricity $\frac{AF}{d(A,p)}$. This conic passes through A (by the definition of eccentricity) and is tangent to AK at A (since $K \in p$ and $\angle KFA = 90^\circ$); it therefore intersects n orthogonally at A. Also, its focal axis is BF (since $BF \perp p$). Thus, this conic satisfies the required conditions for $\mathcal{C}_F$.

Note that the eccentricity of $\mathcal{C}_F$ is also equal to $\frac{FB}{FA}$ (see [2], Theorem 4, p. 18) and that if F, F' are two distinct points on m ($F, F' \neq A$), then the conics $\mathcal{C}_F$ and

$\mathcal{C}_{F'}$ are distinct (their focal axes are distinct). From these remarks, we see that there are as many conics $\mathcal{C}_F$ with eccentricity ϵ as points of $M \in m$ that belong to the locus $\mathcal{E}$ of points for which $\frac{MB}{MA} = \epsilon$. If $\epsilon = 1$, $\mathcal{E}$ is the perpendicular bisector of AB; it intersects m (since m and n are not perpendicular), so that exactly one conic $\mathcal{C}_F$ is a parabola. If $\epsilon \neq 1$, then $\mathcal{E}$ is a circle—the circle of Apollonius—which can intersect m in at most two points. The collection of all these circles (as ϵ varies over the positive real numbers except 1) forms a nonintersecting pencil of circles with limiting points A and B, one through each point of the plane not on the perpendicular bisector of AB (see [1], Section 6.6). It follows that there are at most two conics $\mathcal{C}_F$ corresponding to a given value of ϵ.

To be more specific, $\mathcal{E}$ has diameter JJ' where J, J' are the points of n defined by $(1+\epsilon)\overrightarrow{AJ} = \overrightarrow{AB}$ and $(1-\epsilon)\overrightarrow{AJ'} = \overrightarrow{AB}$. The centre U of $\mathcal{E}$ is such that $(1-\epsilon^2)\overrightarrow{AU} = \overrightarrow{AB}$; its radius is $\rho = JU = \epsilon AU = \frac{\epsilon AB}{|1-\epsilon^2|}$. If H, H' are the orthogonal projections of B, U onto m, respectively, then $\frac{UH'}{BH} = \frac{AU}{AB} = \frac{1}{|1-\epsilon^2|}$; hence $UH' = \frac{BH}{|1-\epsilon^2|}$ (where BH is the distance from B to m which, of course, is always less than AB). We conclude that no, one, or two conics $\mathcal{C}_F$ have eccentricity ϵ according as UH' is greater than, equal to, or less than ρ, which is equivalent to ϵ less than, equal to, or greater than $\frac{BH}{AB}$. So, for example, $\mathcal{C}_F$ could never be a circle (for which $\epsilon = 0$).

References

[1] H.S.M. Coxeter, *Introduction to Geometry*, Wiley, 1961.

[2] C. V. Durell, *A Concise Geometrical Conics*, MacMillan, 1952.

4044. *Proposed by Dragoljub Milošević.*

Let x, y, z be positive real numbers such that $x + y + z = 1$. Prove that

$$\frac{x+1}{x^3+1} + \frac{y+1}{y^3+1} + \frac{z+1}{z^3+1} \leq \frac{27}{7}.$$

We received 24 submissions, of which 22 were correct and complete. There were two main approaches: Jensen's inequality, or comparing each term to a linear function. We present two solutions, one for each approach.

Solution 1, by Fernando Ballesta Yagüe.

As $x + y + z = 1$ and x, y, z are positive, we have $x, y, z \in (0, 1)$. For x in the interval $(0, 1)$, consider the rational function

$$f(x) = \frac{x+1}{x^3+1} = \frac{1}{x^2 - x + 1}.$$

Let's take the second derivative to check its convexity:

$$f'(x) = \frac{-2x+1}{(x^2-x+1)^2},$$
$$f''(x) = \frac{-2(x^2-x+1)^2 - (-2x+1)\cdot 2\cdot (x^2-x+1)\cdot(2x-1)}{(x^2-x+1)^4} = \frac{6x(x-1)}{(x^2-x+1)^3}.$$

Since $x \in (0,1)$, we have $x-1<0$, but $6x>0$ and $x^2-x+1>0$ (note that $-x+1>0$ for $x\in(0,1)$). So in the interval $x\in(0,1)$ we have $f''(x)<0$ and hence f is concave. By Jensen's Inequality for concave functions,

$$\frac{1}{3}(f(x)+f(y)+f(z)) \le f\left(\frac{1}{3}(x+y+z)\right) = f\left(\frac{1}{3}\right) = \frac{9}{7};$$

in other words,

$$\frac{1}{x^2-x+1}+\frac{1}{y^2-y+1}+\frac{1}{z^2-z+1} \le \frac{27}{7},$$

which is equivalent to the inequality we wanted to prove. Note that equality holds when $x=y=z=\frac{1}{3}$.

Solution 2, by Paul Bracken.

As in the previous solution, define $f(x)=\frac{1}{x^2-x+1}$ and show that f is concave for $x\in(0,1)$. Therefore, if $t(x)$ is a tangent line to $f(x)$ at some point $x_0\in(0,1)$ then the inequality $f(x)\le t(x)$ holds for $x\in(0,1)$. Let us calculate the tangent line to $f(x)$ at $x_0=\frac{1}{3}$:

$$t(x) = f'\left(\frac{1}{3}\right)\cdot\left(x-\frac{1}{3}\right)+f\left(\frac{1}{3}\right) = \frac{27}{49}x+\frac{54}{49}.$$

The inequality $f(x)\le t(x)$ then gives us $\frac{x+1}{x^3+1}\le\frac{27}{49}x+\frac{54}{49}$ for $x\in(0,1)$. We obtain similar inequalities by replacing x by y and z respectively, then add the three inequalities to get

$$\frac{x+1}{x^3+1}+\frac{y+1}{y^3+1}+\frac{z+1}{z^3+1} \le \frac{27}{49}(x+y+z)+3\cdot\frac{54}{49} = \frac{27}{7},$$

where for the last equality we used $x+y+z=1$.

4045. *Proposed by Galav Kapoor.*

Suppose that we have a natural number n such that $n\ge 10$. Show that by changing at most one digit of n, we can compose a number of the form $x^2+y^2+10z^2$, where x, y, z are integers.

We received two correct solutions. We present the solution by Roy Barbara.

Recall that Legendre's three-square theorem states that a natural number is a sum of three squares if and only if it is not of the form $4^m(8k+7)$. In particular, any natural number of the form $4k+2$ is a sum of three squares.

Now let $2k+1$ be any odd natural number. Then we can write $4k+2 = a^2+b^2+c^2$. Using $a^2 + b^2 + c^2 \equiv 2 \pmod 4$, it is clear that exactly one of a, b, c is even, say c. Setting $x = \frac{1}{2}(a+b)$, $y = \frac{1}{2}(a-b)$, $c = 2z$ yields

$$4k + 2 = 2x^2 + 2y^2 + 4z^2,$$

whence

$$2k + 1 = x^2 + y^2 + 2z^2.$$

Finally let $n \geq 10$. By changing the last digit of n to a 5 (if necessary), we obtain a number of the form $10k + 5$ for which we have

$$10k + 5 = 5x^2 + 5y^2 + 10z^2 = (2x + y)^2 + (2y - x)^2 + 10z^2.$$

4046. *Proposed by Michel Bataille.*

Let a, b, c be nonnegative real numbers such that $\sqrt{a} + \sqrt{b} + \sqrt{c} \geq 1$. Prove that

$$a^2 + b^2 + c^2 + 7(ab + bc + ca) \geq \sqrt{8(a + b)(b + c)(c + a)}.$$

Two correct solutions were received. A purported counterexample that was submitted had an error. We present both solutions.

Solution 1, by Madhav R. Modak.

$$\begin{aligned}
&\sqrt{8(a+b)(b+c)(c+a)} \\
&\leq (\sqrt{a}+\sqrt{b}+\sqrt{c})\sqrt{8(a+b)(b+c)(c+a)} \\
&= \sqrt{(4ab+4ca)[2(a+b)(c+a)]} + \sqrt{(4bc+4ab)[2(a+b)(b+c)]} \\
&\qquad\qquad + \sqrt{(4ca+4bc)[2(b+c)(c+a)]} \\
&\leq \frac{1}{2}[(4ab+4ca)+2(a+b)(c+a)] + \frac{1}{2}[(4bc+4ab)+2(a+b)(b+c)] \\
&\qquad\qquad + \frac{1}{2}[(4ca+4bc)+2(b+c)(c+a)] \\
&= 4(ab+bc+ca) + (a^2+ab+ca+bc) + (b^2+ab+bc+ca) + (c^2+ca+bc+ab) \\
&= a^2+b^2+c^2+7(ab+bc+ca),
\end{aligned}$$

which yields the desired result.

Solution 2, by the proposer.

Since

$$(a^2 + 3ab + 3ca + bc)^2 = 8a(a + b)(b + c)(c + a) + (a - b)^2(a - c)^2,$$

it follows that

$$a^2 + 3ab + 3ca + bc \geq \sqrt{a}(\sqrt{8(a+b)(b+c)(c+a)}).$$

Similarly

$$b^2 + 3ab + 3bc + ca \geq \sqrt{b}(\sqrt{8(a+b)(b+c)(c+a)})$$

and

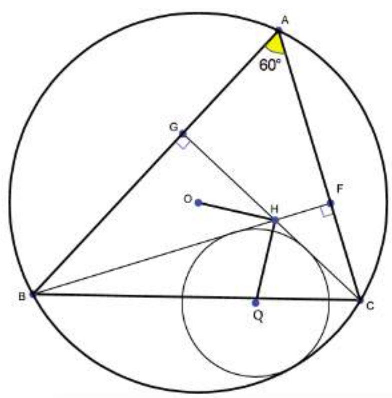
$$c^2 + 3ca + 3bc + ab \geq \sqrt{c}(\sqrt{8(a+b)(b+c)(c+a)}).$$

Adding these three inequalities yields the result.

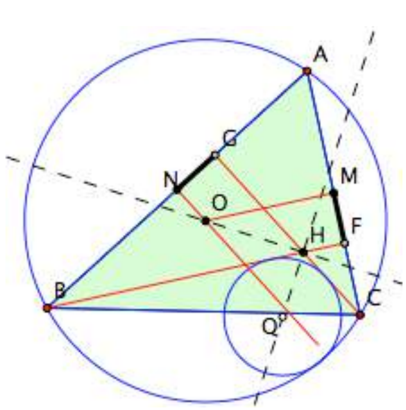
Editor's comment. Equality holds if and only if $a = b = c = 1/9$.

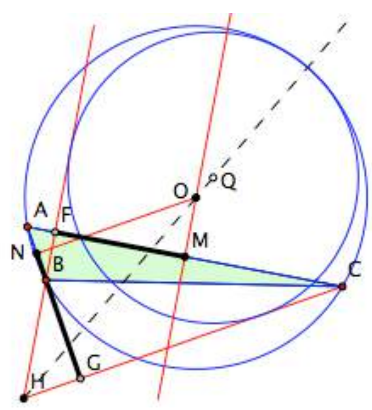
4047★. *Proposed by Abdilkadir Altintaş.*

Let ABC be a triangle with circumcircle O, orthocenter H and $\angle BAC = 60°$. Suppose the circle with centre Q is tangent to BH, CH and the circumcircle of ABC. Show that $OH \perp HQ$.



All 14 submissions we received were correct. We feature two solutions.





Solution 1 is a composite of solutions by Václav Konečný and Edmund Swylan.

The statement of the problem is faulty: Because the plane is partitioned into as many as eight regions by the circumcircle of triangle ABC and the lines HB and HC, there could be eight tritangent circles and, consequently, eight choices for Q, of which some lie on the Euler line OH (in which case the lines OH and HQ would be coincident, not perpendicular).

[*Editor's comment*: Since the centres of all circles tangent to HB and HC would lie on a bisector of $\angle BHC$, the requirement that the circle be tangent also to the circumcircle was perhaps included to limit the choice of tritangent circle to the incircle of the curvilinear triangle HBC (formed by the line segments HB and HC and the circular arc BC). Then the problem has been correctly stated for an acute ΔABC, but it is still not correct when there is an obtuse angle at B or C.]

The exact location of Q is not relevant to the correct theorem:

> For any circle tangent to the lines HB and HC, its centre Q must belong to one of the two bisectors of $\angle BHC$, and so must O.

The claim for Q is a familiar theorem, while the claim for O depends on $\angle BAC = 60^\circ$ and must be proved.

As in the figure, denote the midpoints of AC and AB by M and N, respectively, and the feet of the altitudes to these lines by F and G. Then the segments FM and GN are congruent:

$$FM = |FA - MA| = \left|AB \cdot \cos 60^\circ - \frac{AC}{2}\right| = \frac{1}{2}|AB - AC|$$

and

$$GN = |GA - NA| = \left|AC \cdot \cos 60^\circ - \frac{AB}{2}\right| = \frac{1}{2}|AC - AB|.$$

Then the lines BF, CG, MO, NO form the sides of a rhombus for which the line OH is a diagonal. Thus OH bisects one of the angles formed by the lines HF and HG, as claimed.

Solution 2 is a composite of similar solutions by Šefket Arslanagić, Ricardo Barroso Campos, Prithwijit De (done independently), and Adnan Ibrić with Salem Malikić.

As in the figure that accompanies the statement of the problem, we assume that the given triangle is acute, and that F and G are the feet of the altitudes from B and C, respectively. Observe that $\angle BHC = \angle FHG = 120^\circ$ (since $\angle A$ is 60° and is opposite $\angle FHG$ in the circle whose diameter is AH). Furthermore, $\angle BOC = 120^\circ$ also (because O is the centre of the circumcircle so that the angle there is twice the angle $BAC = 60^\circ$ which is inscribed in that circle). Because O and H are on the same side of BC, it follows that B, O, H, C are concyclic. Finally, note that because ΔBOC is isosceles, $\angle OCB = 30^\circ$. Since HQ is the bisector of $\angle BHC$,

$$\angle BHQ = \frac{1}{2}\angle BHC = 60^\circ.$$

Combine this with

$$\angle OHB = \angle OCB = 30^\circ,$$

and conclude that

$$\angle OHQ = \angle OHB + \angle BHQ = 90^\circ.$$

Editor's Comments. Essentially the same problem has appeared before in ***Crux*** [1988: 165; 1990: 103] as Problem **M1046**, which was taken from the 1987 U.S.S.R journal *Kvant*:

> If $\angle A = 60^\circ$ then one of the bisectors of the angle between the altitudes from B and C passes through O.

This and related properties were discussed under the heading "Property 3" in the article "Recurring ***Crux*** Configurations 3: Triangles Whose Angles Satisfy $2B = C + A$" [2011: 350].

4048. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let $n \geq 2$ be an integer and let $a_k \geq 1$ be real numbers, $1 \leq k \leq n$. Prove the inequality

$$a_1a_2\cdots a_n - \frac{1}{a_1a_2\cdots a_n} \geq \left(a_1 - \frac{1}{a_1}\right) + \left(a_2 - \frac{1}{a_2}\right) + \cdots + \left(a_n - \frac{1}{a_n}\right)$$

and study equality cases.

Thirteen solutions were received, all of which established the inequality. Two of them did not get all the possible conditions for equality, while three others neglected to consider when equality occurred. The solutions were all similar to the one presented below.

Let

$$f(x) = x - \frac{1}{x}$$

and observe that, for $x, y \geq 1$,

$$f(xy) - f(x) - f(y) = (xy)^{-1}(xy-1)(x-1)(y-1) \geq 0$$

with equality if and only if at least one of x and y is equal to 1.

We establish the result by induction.

The foregoing shows that it is true for $n = 2$. Suppose that the inequality holds for $n = m \geq 2$ with equality iff all but at most one of $a_1, a_2, \ldots, a_m$ is equal to 1. Then, by the foregoing property of f and the result for $n = m$,

$$f(a_1a_2\cdots a_ma_{m+1}) \geq f(a_1a_2\cdots a_m) + f(a_{m+1}) \geq \sum_{k=1}^{m} f(a_k) + f(a_{m+1}) = \sum_{k=1}^{m+1} f(a_k).$$

Equality holds if and only if either

- $a_1a_2\cdots a_m = 1$, in which case $a_1 = a_2 = \cdots = a_m = 1$, or
- $a_{m+1} = 1$ and $f(a_1a_2\cdots a_m) = \sum_{k=1}^{m} f(a_k)$.

In either case, all but at most one of $a_1, a_2, \ldots, a_{m+1}$ is equal to 1.

Editor's comments. One can also peel off the last two terms in the product so that the induction step becomes

$$f(a_1a_2\ldots a_ma_{m+1}) \geq \sum_{k=1}^{m-1} f(a_k) + f(a_ma_{m+1}) \geq \sum_{k=1}^{m+1} f(a_k).$$

Edmund Swyland observed that if, for any i and j, you replaced the pair (a_i, a_j) by $(a_ia_j, 1)$, the left side $f(a_1a_2\cdots a_n)$ of the inequality remained unchanged, but the right side increased. Thus we can reduce the problem to establishing that it holds when all but two of the a_i are equal to 1, and this now involves dealing with the case $n = 2$.

Kee-Wai Lau pointed out that an easy induction argument yields

$$f(a_1a_2\cdots a_n) - \sum_{k=1}^{n} f(a_k) = \sum_{k=2}^{n} \frac{(a_1a_2\cdots a_{k-1}-1)(a_k-1)(a_1a_2\cdots a_k-1)}{a_1a_2\cdots a_k}.$$

4049. *Proposed by Mihaela Berindeanu.*

Evaluate

$$\int \frac{\sin x - x\cos x}{(x+\sin x)(x+2\sin x)} dx$$

for all $x \in (0, \pi/2)$.

We received 16 submissions all of which were correct. We present a composite of the nearly identical solutions given by Adnan Ali, Michel Bataille, Prithwijit De, Joseph Ling and Albert Stadler, all done independently

Let I denote the given integral. Since it is readily checked that

$$(1+\cos x)(x+2\sin x) - (1+2\cos x)(x+\sin x) = \sin x - x\cos x,$$

we have

$$\begin{aligned} I &= \int \left(\frac{1+\cos x}{x+\sin x} - \frac{1+2\cos x}{x+2\sin x} \right) dx \\ &= \ln(x+\sin x) - \ln(x+2\sin x) + C \\ &= \ln\left(\frac{x+\sin x}{x+2\sin x} \right) + C, \end{aligned}$$

where C is an arbitrary constant.

4050. *Proposed by Mehtaab Sawhney.*

Prove that

$$\sum_{k=0}^{2n} \binom{4n}{k, k, 2n-k, 2n-k} = \binom{4n}{2n}^2$$

for all nonnegative integers n.

We received twelve correct solutions which were split between an arithmetic proof and a proof by double counting, so we present a solution of each type.

Solution 1, by C.R. Pranesachar.

We have

$$\begin{aligned}\sum_{k=0}^{2n} \binom{4n}{k, k, 2n-k, 2n-k} &= \sum_{k=0}^{2n} \binom{4n}{2n}\binom{2n}{k}^2 \\ &= \binom{4n}{2n} \sum_{k=0}^{2n} \binom{2n}{k}\binom{2n}{2n-k} \\ &= \binom{4n}{2n}^2,\end{aligned}$$

where the last equality is due to Vandermonde's identity.

Solution 2, by Joseph DiMuro.

Let's say we have a classroom with $4n$ students. The teacher wants to choose $2n$ of them to work on one project and $2n$ of them to work on a second project (independently of each other – students may be assigned to both or neither of the projects). In how many ways can the teacher assign students to the projects?

On one hand there are $\binom{4n}{2n}$ ways to choose the students for each of the two projects, thus $\binom{4n}{2n}^2$ possibilities altogether.

On the other hand note that if k students are assigned to both projects then $2n-k$ will be assigned to just the first project, $2n-k$ to just the second project, and k to neither project. So the teacher can proceed as follows: first decide on the number k of students that will be assigned to both projects, then partition the class into four groups – of size k (both projects), $2n-k$ (first project), $2n-k$ (second project), and k (neither project). There are

$$\sum_{k=0}^{2n} \binom{4n}{k, k, 2n-k, 2n-k}$$

ways to do this. Thus the two sides in the problem are equal.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015 : 41(6), p. 260–263.

4051. *Proposed by Arkady Alt.*

Let a, b and c be the side lengths of a triangle. Prove that

$$(a+b+c)\left(a^2b^2+b^2c^2+c^2a^2\right) \geq 3abc\left(a^2+b^2+c^2\right).$$

We received eleven correct solutions. We present two solutions.

Solution 1, by Michel Bataille.

A key to a solution is contained in solution 2 to ***Crux*** problem **3991** published in 41 (1) (December 2015).

Setting $a = \frac{y+z}{2}$, $b = \frac{z+x}{2}$, $c = \frac{x+y}{2}$ transforms the proposed inequality into

$$x^5+y^5+z^5+x^2y^2z+x^2yz^2+xy^2z^2 \geq x^3y^2+x^2y^3+y^3z^2+y^2x^3+z^3x^2+z^2x^3, \quad (1)$$

where x, y, z are positive real numbers. The general Schur inequality is

$$u^r(u-v)(u-w)+v^r(v-w)(v-u)+w^r(w-u)(w-v) \geq 0$$

for $u, v, w \geq 0$ and r real. We take $r = \frac{1}{2}$ and $u = x^2$, $v = y^2$, $w = z^2$ and obtain

$$x(x^2-y^2)(x^2-z^2)+y(y^2-z^2)(y^2-x^2)+z(z^2-x^2)(z^2-y^2) \geq 0.$$

Expanding and arranging directly leads to (1).

Solution 2, by Titu Zvonaru.

By Consequence 16.3, p. 156 from [1], all symmetric three-variable polynomials of degree less than or equal to five achieve their maximum and minimum values on $\mathbb{R}^*$ at (a, b, c) if and only if $(a-b)(b-c)(c-a) = 0$ or $abc = 0$. It thus suffices to prove the given inequality for $b = c$ and $c = 0$.

If $b = c$, then we have to prove that

$$\begin{aligned} (a+2b)(2a^2b^2+b^4) &\geq 3ab^2(a^2+2b^2) \\ 2a^3b^2+ab^4+4a^2b^3+2b^5 &\geq 3a^3b^2+6ab^4 \\ 4a^2b^3+2b^5 &\geq a^3b^2+5ab^4 \\ b^2(a-b)^2(2b-a) &\geq 0, \end{aligned}$$

which is true by the triangle inequality.

If $c = 0$, then we have to prove that

$$(a+b)a^2b^2 \geq 0,$$

which is true.

Equality holds if and only if $a = b = c$.

[1] Z. Cvetkovski, *Inequalities - Theorems, Techniques and Selected Problems*, Springer-Verlag, 2012.

4052. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let $k < 0$ be a fixed real number. Let a, b, c and d be real numbers such that $a + b + c + d = 0$ and $ab + bc + cd + da + ac + bd = k$. Prove that $abcd \geq -k^2/12$ and determine when equality holds.

We received six submissions, of which five were correct and complete. We present the solution by Oliver Geupel, slightly modified by the editor.

It is sufficient to consider the case when three of the numbers a, b, c, d have the same sign. Otherwise, $abcd \geq 0 > -k^2/12$, and we're done.

By symmetry, there is no loss of generality in assuming that it is a, b, and c which have the same sign. Then $a + b + c + d = 0$ gives us $d = -(a + b + c)$, so

$$\begin{aligned} k &= ab + bc + cd + da + ac + bd \\ &= (a+b+c)d + (ab+bc+ca) \\ &= -(a+b+c)^2 + (ab+bc+ca) \\ &= -(a^2+b^2+c^2+ab+bc+ca), \end{aligned}$$

and the inequality we want to prove can be rewritten as

$$-abc(a+b+c) \geq -(a^2+b^2+c^2+ab+bc+ca)^2/12,$$

or, equivalently,

$$12abc(a+b+c) \leq (a^2+b^2+c^2+ab+bc+ca)^2. \tag{1}$$

Let $x = |a|$, $y = |b|$, $z = |c|$, so that x, y, z are nonnegative real numbers. Since a, b and c have the same sign by assumption,

$$\begin{aligned} &abc(a+b+c) = xyz(x+y+z), \text{ and} \\ &a^2+b^2+c^2+ab+bc+ca = x^2+y^2+z^2+xy+yz+zx, \end{aligned}$$

so it is sufficient to prove that (1) holds with a,b and c replaced by x, y and z, respectively.

By the AM-GM Inequality (where the right hand side is treated as a sum of 8 terms, with terms repeated as indicated by the coefficients), we have

$$8x^2yz \le x^4 + 2x^3y + 2x^3z + 3y^2z^2. \tag{2}$$

The equality holds only when $x^4 = x^3y = x^3z = y^2z^2$, that is when either $x = y = z$ or $x = yz = 0$. Summing up inequality (2) and its two cyclic variants, and adding terms to both sides so we can complete the square on the right hand side, we obtain

$$12xyz(x+y+z) \le (x^2+y^2+z^2+xy+yz+zx)^2, \tag{3}$$

where the equality holds if and only if $x = y = z$. This shows that (1) holds, and thus concludes the proof that $abcd \ge -k^2/12$.

It follows from the preceding steps that the equality holds if and only if three of the four numbers a, b, c, and d are equal. A straightforward computation (from $a+b+c+d = 0$ and $abcd = -k^2/12$) shows that the common value is $\pm\sqrt{-k/6}$, and that the fourth number has the value $\mp 3\sqrt{-k/6}$.

4053. *Proposed by Šefket Arslanagić.*

Prove that

$$\frac{\cos\alpha\cos\beta}{\cos\gamma} + \frac{\cos\beta\cos\gamma}{\cos\alpha} + \frac{\cos\alpha\cos\gamma}{\cos\beta} \ge \frac{3}{2},$$

where α, β and γ are angles of an acute triangle.

We received 13 correct solutions. We present a composite of essentially the same solution by José Luis Díaz-Barrero, Dionne Bailey, Elsie Campbell, and Charles R. Diminnie (joint), Henry Ricardo, and Lorian Saceanu.

Since $\alpha, \beta, \gamma \in (0, \frac{\pi}{2})$ we have

$$\begin{aligned}
\frac{\cos\alpha\cos\beta}{\cos\gamma} &= \frac{\cos\alpha\cos\beta}{\cos\gamma} \cdot \frac{\tan\alpha + \tan\beta}{\tan\alpha + \tan\beta} \\
&= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\gamma} \cdot \frac{1}{\tan\alpha + \tan\beta} \\
&= \frac{\sin(\pi-\gamma)}{\cos\gamma} \cdot \frac{1}{\tan\alpha + \tan\beta} \\
&= \frac{\tan\gamma}{\tan\alpha + \tan\beta}.
\end{aligned}$$

By the well-known Nesbitt's Inequality which states that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}$$

with equality if and only if $a = b = c$, we then have

$$\sum_{\text{cyc}} \frac{\cos\alpha\cos\beta}{\cos\gamma} = \sum_{\text{cyc}} \frac{\tan\gamma}{\tan\alpha + \tan\beta} \ge \frac{3}{2}$$

with equality if and only if $\alpha = \beta = \gamma$; in other words, if the given triangle is equilateral.

Editor's comment. Both Ricardo and Diminnie et al pointed out that the current problem is the special case when $m = 0$ of problem #5381 in the January 2016 issue of *School, Science and Mathematics* :

> If A, B, C are angles of an acute triangle, then
> $$\sum \left(\frac{\cos A \cos B}{\cos C} \right)^{m+1} \geq \frac{3}{2^{m+1}}$$
> for all nonnegative integers m.

Interestingly, this general problem was proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu, two regular contributors to the ***Crux*** problem section. Ricardo actually gave a proof to the general inequality as well as a proof for the above case. The general proof uses the same argument in the featured solution above together with the power mean inequality and it appeared as solution 3 in the April 2016 issue of that journal (see `www.ssma.org/publications`).

4054. *Proposed by Mihaela Berindeanu.*

Find a prime p such that the number

$$(p^2 - 4)^2 - 117(p^2 - 4) + 990$$

has a minimum digit sum.

We received eight correct and complete solutions, all of which were very similar. We present the solution by Joseph DiMuro.

Let

$$f(p) = (p^2 - 4)^2 - 117(p^2 - 4) + 990.$$

If $p \neq 3$ is a prime, then $p = 3n \pm 1$ for some integer n. Then

$$p^2 - 4 = 9n^2 \pm 6n - 3,$$

a multiple of 3 ; thus, $(p^2-4)^2$ is a multiple of 9. But 117 and 990 are also multiples of 9. So $f(p)$ is a multiple of 9, as is its digit sum.

If we had $f(p) = 0$, then by the quadratic formula we would have

$$p^2 - 4 = \frac{117 \pm \sqrt{9729}}{2},$$

which has no integer solutions, so $f(p) \neq 0$. Therefore, the digit sum of $f(p)$ must be at least 9 when p is a prime other than 3.

However, the digit sum of $f(3) = 430$ is 7, so the minimum digit sum is 7, attained for $p = 3$.

4055. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Prove that if $x, y > 0, x \neq y$ and $0 < a < b < \frac{1}{2} < c < d < 1$ then :

$$x\Big[\Big(\frac{y}{x}\Big)^a+\Big(\frac{y}{x}\Big)^d-\Big(\frac{y}{x}\Big)^b-\Big(\frac{y}{x}\Big)^c\Big] > y\Big[\Big(\frac{x}{y}\Big)^b+\Big(\frac{x}{y}\Big)^c-\Big(\frac{x}{y}\Big)^a-\Big(\frac{x}{y}\Big)^d\Big].$$

We received three correct solutions and feature two of them that are similar.

Solution 1, by Michel Bataille.

With $t = \frac{x}{y}$, the inequality can be re-written as

$$(t^d + t^{1-d}) - (t^c + t^{1-c}) > (t^b + t^{1-b}) - (t^a + t^{1-a}). \tag{1}$$

Let us fix $t > 0, t \neq 1$ and set $f(u) = t^u$ and $g(u) = f(u) + f(1-u)$ so that (1) is just

$$g(d) - g(c) > g(b) - g(a). \tag{2}$$

From the Mean Value Theorem, we have

$$g(b) - g(a) = (b-a)g'(\alpha), \qquad g(d) - g(c) = (d-c)g'(\beta)$$

for some $\alpha \in (a, b)$ and $\beta \in (c, d)$.

Since $g'(u) = (\ln t)(t^u - t^{1-u}) = (\ln t)(f(u) - f(1-u))$, (2) becomes

$$(d-c)(\ln t)(f(\beta) - f(1-\beta)) > (b-a)(\ln t)(f(\alpha) - f(1-\alpha)). \tag{3}$$

Applying the Mean Value Theorem again, we have $f(\beta)-f(1-\beta) = (2\beta-1)(\ln t)t^\sigma$ and $f(\alpha) - f(1-\alpha) = (2\alpha-1)(\ln t)t^\tau$ with σ between β and $1-\beta$ and τ between α and $1-\alpha$.

Substituting into (3) and because $(\ln t)^2 > 0$, we are reduced to proving

$$(d-c)(2\beta-1)t^\sigma > (b-a)(2\alpha-1)t^\tau. \tag{4}$$

Now, on the one hand $d-c > 0,\ t^\sigma > 0,\ 2\beta - 1 > 0$ (note that $\beta \in (c, d)$, hence $\beta > \frac{1}{2}$) and on the other hand, $b-a > 0,\ t^\tau > 0,\ 2\alpha - 1 < 0$ (since $\alpha \in (a, b)$). Thus,

$$(d-c)(2\beta-1)t^\sigma > 0 > (b-a)(2\alpha-1)t^\tau$$

and (4) follows.

Solution 2, by Daniel Sitaru and Leonard Giugiuc.

Let $f : [0, 1] \to \mathbb{R}, f(\alpha) = \frac{x^{1-\alpha}y^\alpha + x^\alpha y^{1-\alpha}}{2}$, with $x, y \in (0, \infty), x \neq y$. We have :

$$\lim_{\substack{\alpha\to0\\ \alpha>0}} f(\alpha) = \frac{x+y}{2}, \quad \lim_{\substack{\alpha\to1\\ \alpha<1}} f(\alpha) = \frac{x+y}{2}.$$

Since

$$f'(\alpha) = \frac{1}{2}(\ln y - \ln x)(x^{1-\alpha}y^{\alpha} - x^{\alpha}y^{1-\alpha}),$$

then

$$f'(\alpha) = 0 \quad \Rightarrow \quad x^{1-\alpha}y^{\alpha} = x^{\alpha}y^{1-\alpha} \quad \Rightarrow \quad \left(\frac{x}{y}\right)^{1-2\alpha} = 0 \Rightarrow a = \frac{1}{2}.$$

Therefore, $\min f(\alpha) = f\left(\frac{1}{2}\right) = \sqrt{xy}$ and $f(a) > f(b) > f\left(\frac{1}{2}\right)$, $\quad f\left(\frac{1}{2}\right) < f(c) < f(d)$. By adding, we get $f(a) + f(d) > f(b) + f(c)$ or

$$\begin{aligned} \frac{x^{1-a}y^a + x^a y^{1-a}}{2} + \frac{x^{1-d}y^d + x^d y^{1-d}}{2} &> \frac{x^{1-b}y^b + x^b y^{1-b}}{2} \\ &> \frac{x^{1-b}y^b + x^b y^{1-b}}{2} + \frac{x^{1-c}y^c + x^c y^{1-c}}{2}. \end{aligned}$$

This results in

$$x\left[\left(\frac{y}{x}\right)^a + \left(\frac{y}{x}\right)^d - \left(\frac{y}{x}\right)^b - \left(\frac{y}{x}\right)^c\right] > y\left[\left(\frac{x}{y}\right)^b + \left(\frac{x}{y}\right)^c - \left(\frac{x}{y}\right)^a - \left(\frac{x}{y}\right)^d\right].$$

4056. *Proposed by Idrissi Abdelkrim-Amine.*

Let n be an integer, $n \geq 2$. Consider real numbers a_k, $1 \leq k \leq n$ such that $a_1 \geq 1 \geq a_2 \geq \ldots \geq a_n > 0$ and $a_1 a_2 \ldots a_n = 1$. Prove that

$$\sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} \frac{1}{a_k}.$$

We received eight solutions of which six were correct. We present the solution by Roy Barbara.

Note first that if $0 < a, b \leq 1$, then

$$a - \frac{1}{a} + b - \frac{1}{b} \geq ab - \frac{1}{ab}. \tag{1}$$

Indeed, multiplying by ab, (1) is equivalent, in succession, to

$$\begin{array}{rccl} & a^2b - b + ab^2 - a & \geq & (ab)^2 - 1 \\ \text{or} & (a+b)(ab-1) & \geq & (ab+1)(ab-1) \\ \text{or} & (1-ab)(1-a)(1-b) & \geq & 0, \end{array}$$

which is true.

We now prove the given inequality by using induction on $n \geq 2$.

The case when $n = 2$ is trivial. Suppose the inequality holds for some $n \geq 2$, and let a_i, $i = 1, 2, \cdots, n+1$ satisfy $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1}$ and $a_1 a_2 \cdots a_n a_{n+1} = 1$. We need to prove that

$$\left(\sum_{k=1}^{n-1} a_k - \sum_{k=1}^{n-1} \frac{1}{a_k}\right) + \left(a_n - \frac{1}{a_n}\right) + \left(a_{n+1} - \frac{1}{a_{n+1}}\right) \geq 0. \tag{2}$$

Set $b_n = a_n a_{n+1}$.

Then clearly $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_{n-1} \geq b_n$ and $a_1 a_2 \cdots a_{n-1} b_n = 1$.

By the induction hypothesis, we have

$$\left(\sum_{k=1}^{n-1} a_k - \sum_{k=1}^{n-1} \frac{1}{a_k}\right) + \left(b_n - \frac{1}{b_n}\right) \geq 0. \tag{3}$$

Hence, to get (2), it suffices to prove that

$$\left(a_n - \frac{1}{a_n}\right) + \left(a_{n+1} - \frac{1}{a_{n+1}}\right) \geq \left(b_n - \frac{1}{b_n}\right),$$

that is,

$$\left(a_n - \frac{1}{a_n}\right) + \left(a_{n+1} - \frac{1}{a_{n+1}}\right) \geq \left(a_n a_{n+1} - \frac{1}{a_n a_{n+1}}\right),$$

which is true by (1) as $0 < a_n, a_{n+1} \leq 1$.

Editor's comment. The proposer of the current problem remarked that the given inequality was inspired by the following inequality due to Leonard Giugiuc : $(\sum a_k)^2 \geq n \sum \frac{1}{a_k}$, where the a_k's satisfy the same conditions given in the current problem.

4057. *Proposed by Eeshan Banerjee.*

Let ABC be a non-obtuse triangle with circumradius R, inradius r and area Δ. Prove that

$$\Delta < \left(\frac{\frac{1}{r} + 3R + 3}{7}\right)^7.$$

We received four correct solutions. We present a composite of very similar solutions by Michel Bataille and Andrea Fanchini.

With AM-GM, we have that

$$\frac{\frac{1}{r} + R + R + R + 1 + 1 + 1}{7} \geq \sqrt[7]{\frac{R^3}{r}},$$

so it suffices to prove that

$$\Delta < \frac{R^3}{r}.$$

From Euler's inequality $R \geq 2r$ and the inequality $R \geq \frac{2s}{3\sqrt{3}}$, we have

$$\frac{R^3}{r} \geq \frac{4r^2 \cdot 2s}{r \cdot 3\sqrt{3}} = \frac{8\sqrt{3}}{9} \cdot rs = \frac{8\sqrt{3}}{9} \cdot \Delta > \Delta,$$

completing the proof.

4058. *Proposed by Francisco Javier García Capitán.*

Let ABC be a triangle. For any X on line BC, let X_b and X_c be the circumcenters of the triangles ABX and AXC, and let P be the intersection point of BX_c and CX_b. Prove that the locus of P as X varies along the line BC is the conic through the centroid, orthocenter, and vertices B and C, and whose tangents at these vertices are the corresponding symmedians. (Recall that a symmedian is the reflection of a median in the bisector of the corresponding angle.)

We received two submissions, both of which were correct. We feature the solution by Michel Bataille with a few details added from the proposer's solution.

As usual, set $BC = a, CA = b, AB = c$. Should $a = b$, then CX_b will be the perpendicular bisector of AB, which immediately implies that this line is the locus of P; similarly, should $a = c$, then the locus of P would be the perpendicular bisector of AC. Assume therefore that $a \neq b, c$. We shall see that under this further assumption the locus is a hyperbola. For our argument we shall use barycentric coordinates relative to (A, B, C), and the following notation :

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2}.$$

For later use, here are a few readily checked relations satisfied by these numbers :

$$S_B + S_C = a^2, \quad S_C + S_A = b^2, \quad S_A + S_B = c^2,$$

and

$$\begin{aligned} a^2 S_A + S_B S_C &= b^2 S_B + S_C S_A = c^2 S_C + S_A S_B \\ &= S_A S_B + S_B S_C + S_C S_A = \frac{1}{2}(a^2 S_A + b^2 S_B + c^2 S_C). \end{aligned}$$

If $(f : g : h)$ is the point at infinity of a line ℓ, then $(gS_B - hS_C : hS_C - fS_A : fS_A - gS_B)$ is the point at infinity of the perpendiculars to ℓ. With the help of this property, we easily obtain the equations of the perpendicular bisector ℓ_1 of AB and ℓ_2 of AC,

$$\ell_1 : c^2 x - c^2 y + (a^2 - b^2) z = 0, \qquad \ell_2 : b^2 x + (a^2 - c^2) y - b^2 z = 0.$$

If $X = (0 : \beta : \gamma)$ with $\beta + \gamma = 1$, the point at infinity on AX is $(-1 : \beta : \gamma)$, hence the one on the perpendiculars to AX is $(\beta S_B - \gamma S_C : \gamma S_C + S_A : -S_A - \beta S_B)$. It follows that the perpendicular bisector m of AX is

$$x(\beta^2 S_B + \gamma^2 S_C + S_A) + y(\gamma^2 a^2 - c^2) + z(\beta^2 a^2 - b^2) = 0.$$

Note that

$$\begin{aligned}&\beta^2 S_B + \gamma^2 S_C + S_A\\&= \beta^2 S_B + (1-\beta)^2 S_C + S_A\\&= a^2\beta^2 - 2\beta S_C + b^2 = a^2\beta^2 - \beta(a^2+b^2-c^2) + b^2 = \beta c^2 + \gamma b^2 - a^2\beta\gamma.\end{aligned}$$

Let $x_1 = a^2(S_A + \beta S_B),\ y_1 = a^2\beta S_A + b^2 S_B,\ z_1 = c^2(S_C - a^2\beta)$. It is rather long but easy to check that (x_1, y_1, z_1) satisfies both the equations of ℓ_1 and m. Thus $X_b = (x_1 : y_1 : z_1)$. Similarly, we obtain $X_c = (x_2 : y_2 : z_2)$ with $x_2 = a^2(S_A + \gamma S_C),\ y_2 = b^2(S_B - a^2\gamma),\ z_2 = a^2\gamma S_A + c^2 S_C$.

The equations of $CX_b : xy_1 - yx_1 = 0$ and $BX_c : xz_2 - zx_2 = 0$ then provide $P = (u : v : w) = (x_1x_2 : x_2y_1 : x_1z_2)$ so that

$$v = \frac{a^2\beta S_A + b^2 S_B}{a^2(S_A + \beta S_B)}\cdot u, \quad w = \frac{a^2\gamma S_A + c^2 S_C}{a^2(S_A + \gamma S_C)}\cdot u,$$

from which we obtain

$$a^2\beta = \frac{b^2 S_B u - a^2 S_A v}{vS_B - uS_A}, \qquad a^2\gamma = \frac{c^2 S_C u - a^2 S_A w}{wS_C - uS_A}.$$

Eliminating β, γ (through $\beta + \gamma = 1$) yields a necessary and sufficient condition on u, v, w for P to belong to the desired locus, namely

$$\begin{aligned}u^2 S_A(a^2 S_A + b^2 S_B + c^2 S_C) - uvc^2(a^2 S_A + S_B S_C) - wub^2(a^2 S_A + S_B S_C)\\+vwa^2(S_A S_B + S_B S_C + S_C S_A) = 0;\end{aligned}$$

that is, $2u^2 S_A - c^2 uv - b^2 uw + a^2 vw = 0$.

Thus, the locus of P is the conic Γ with equation

$$(b^2 + c^2 - a^2)x^2 - c^2 xy + a^2 yz - b^2 zx = 0.$$

Note that because we have assumed that $a \neq b, c$, the discriminant of the conic (namely, $\frac{a^2}{4}(b^2 - a^2)(c^2 - a^2)$) is nonzero ; because the coefficient of y^2 is zero, this nondegenerate conic must be a hyperbola, as claimed. Let $\mathcal{C}(x, y, z)$ be the left-hand side of the equation. We readily find that

$$\mathcal{C}(0,1,0) = \mathcal{C}(0,0,1) = \mathcal{C}(1,1,1) = \mathcal{C}(S_B S_C, S_C S_A, S_A S_B) = 0,$$

hence Γ passes through B, C, G, H (respectively), where $G = (1 : 1 : 1)$ and $H = (S_B S_C : S_C S_A : S_A S_B)$ denote the centroid and the orthocenter of ABC. In addition, the equation of the tangent to Γ at $(x_0 : y_0 : z_0)$ is

$$2xx_0 S_A - \frac{1}{2}c^2(x_0 y + xy_0) - \frac{1}{2}b^2(x_0 z + xz_0) + \frac{1}{2}a^2(y_0 z + yz_0) = 0.$$

In particular, the tangent to Γ at B is $c^2 x - a^2 z = 0$, a line passing through the Lemoine point $K = (a^2 : b^2 : c^2)$ and through B. Therefore this tangent is the symmedian through vertex B. Similarly, the tangent to Γ at C is the symmedian through vertex C.

Editor's comments. The proposer noted the following theorem to be a consequence of his problem :

> If a conic passes through the vertices B and C of a triangle ABC, while the tangents at those points are the corresponding symmedians, then the centroid of the triangle lies on the conic if and only if the orthocenter does also.

4059. *Proposed by Marcel Chiriţa.*

Let $a, b \in (0, \infty)$, $a \neq b$. Determine the functions $f : \mathbb{R} \mapsto \mathbb{R} \setminus \{0\}$ such that

$$f(ax) = e^x f(bx), \quad \forall x \in \mathbb{R}.$$

We received four submissions of which three were correct and complete. We present the solution by Michel Bataille.

We show the following :

> Let $\mathcal{P}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R} \setminus \{0\}$ that are periodic with period $\ln(b/a)$ and let $u : t \mapsto u(t) = \ln(|t|)$ for $t \neq 0$. Then the solutions are the functions $t \mapsto g(t) \cdot e^{\frac{t}{a-b}}$ where the function g is defined by
>
> $$g(t) = p(u(t)) \ (t > 0), \quad g(0) = \alpha, \quad g(t) = q(u(t)) \ (t < 0)$$
>
> for some $p, q \in \mathcal{P}$ and some $\alpha \in \mathbb{R} \setminus \{0\}$.

First, a remark : if we set $g(x) = f(x) \cdot e^{-\frac{x}{a-b}}$, a simple calculation shows that solving the given equation boils down to solving the functional equation $g(ax) = g(bx)$ for functions $g : \mathbb{R} \to \mathbb{R} \setminus \{0\}$, f being then defined by $f(x) = g(x) \cdot e^{\frac{x}{a-b}}$ for $x \in \mathbb{R}$. Since $\frac{x}{a}$ takes all real values when x does, substituting $\frac{x}{a}$ for x even reduces the problem to seeking functions $g : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ such that

$$g(x) = g\left(x \cdot \frac{b}{a}\right) \tag{1}$$

for all $x \in \mathbb{R}$. Let g be a solution. Then, for $x > 0$, we have

$$g(e^{\ln x}) = g(e^{\ln x + \ln(b/a)}),$$

that is, $p(\ln x) = p(\ln x + \ln(b/a))$ if we set $p = g \circ \exp$. Since $\ln x$ takes all real values as x describes $(0, \infty)$, it follows that p is periodic with period $\ln(b/a)$ and does not take the value 0, i.e. $p \in \mathcal{P}$, and that $g(x) = p(\ln x) = p(\ln(|x|)$ for positive x.

Similarly, for $x < 0$, we have

$$g(-e^{\ln(-x)}) = g(-e^{\ln(-x) + \ln(b/a)})$$

that is, $q(\ln(-x)) = q(\ln(-x) + \ln(b/a))$ where $q = g \circ (-\exp)$ is an element of $\mathcal{P}$, so that $g(x) = q(\ln(-x)) = q(\ln(|x|)$.

Conversely, define g by

$$g(t) = p(u(t)) \ (t > 0), \quad g(0) = \alpha, \quad g(t) = q(u(t)) \ (t < 0)$$

where $p, q \in \mathcal{P}$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Then, g takes only nonzero values and, if $x > 0$, $g\left(x \cdot \left(\frac{b}{a}\right)\right) = p\left(\ln\left(x \cdot \left(\frac{b}{a}\right)\right)\right) = p(\ln x + \ln(b/a)) = p(\ln x) = g(x)$ and if $x < 0$, then $g\left(x \cdot \left(\frac{b}{a}\right)\right) = q\left(\ln\left(-x \cdot \left(\frac{b}{a}\right)\right)\right) = q(\ln(-x) + \ln(b/a)) = q(\ln(-x)) = g(x)$. Thus, the equality $g\left(x \cdot \left(\frac{b}{a}\right)\right) = g(x)$ holds for all real x (it is obvious for $x = 0$) and so g satisfies (1) with $g(x) \neq 0$ for all real x. The proof is complete.

Remark. The solutions which are continuous at 0 are the functions $t \mapsto \alpha e^{\frac{t}{a-b}}$ where α is a nonzero real constant : with the above notations, f is continuous at 0 if and only if g is. So, we consider a solution g of (1), with g continuous at 0. Suppose first that $b < a$. Then,

$$g\left(x \cdot \left(\frac{b}{a}\right)^k\right) = g\left(x \cdot \left(\frac{b}{a}\right)^{k+1}\right)$$

if k is a positive integer; hence, by an immediate induction, we see that $g(x) = g\left(x \cdot \left(\frac{b}{a}\right)^n\right)$ for all positive integers n. Since $0 < \frac{b}{a} < 1$, we have $\lim\limits_{n\to\infty} x \cdot \left(\frac{b}{a}\right)^n = 0$ and so $\lim\limits_{n\to\infty} g\left(x \cdot \left(\frac{b}{a}\right)^n\right) = g(0)$. As a result, $g(x) = g(0)$ for any $x \in \mathbb{R}$. If $b > a$, the treatment is similar using the equation $g(x) = g\left(x \cdot \frac{a}{b}\right)$ which holds for all x as well. Thus g is a constant function. Conversely, any constant function $\mathbb{R} \to \mathbb{R} \setminus \{0\}$ is obviously a solution of (1).

Editor's comments. Roy Barbara proposed and solved the following generalization of the problem. Set $k = a/b$ and $t = bx$ ($t \in \mathbb{R}$). If $\varphi : \mathbb{R} \to \mathbb{R}^*$ is a function with $\varphi(0) = 1$ and $k \in (0, \infty)$, $k \neq 1$, determine all the functions $f : \mathbb{R} \to \mathbb{R}^*$ satisfying

$$f(kt) = \varphi(t) \cdot f(t) \qquad \forall t \in \mathbb{R}.$$

Unfortunately, Marcel Chiriţa passed away on 29 February 2016 and he cannot enjoy the beautiful solution given above. We will miss him and we will miss his precious contribution to the journal.

4060. *Proposed by Michel Bataille.*

Let

$$f(x, y) = \frac{xy(x + y)}{(1 - x - y)^3}.$$

Find the range of $f(x, y)$ when its domain is restricted to the circle S that satisfies the equation $x^2 + y^2 = 1 - 2x - 2y$.

Five correct solutions were submitted from four people. Two others made incorrect submissions. We present two solutions.

Solution 1, following the approach of Kee-Wai Lau.

Suppose that $(x,y)\in S$ and let $t=x+y$. Then

$$(t+2)^2=2(x^2+y^2+2x+2y-1)+6-(x-y)^2=6-(x-y)^2\le 6,$$

so that $-2-\sqrt{6}\le t\le -2+\sqrt{6}<1$, with equality possible only if $x=y$. Since

$$2xy=(x+y)^2+2(x+y)-1=t^2+2t-1,$$

we have

$$f(x,y)=\frac{t(t^2+2t-1)}{2(1-t)^3}.$$

The condition for S can be written as

$$(x+1)^2+(y+1)^2=3,$$

so that S is a circle with centre $(-1,-1)$ that intersects the line $y=x$ at

$$\left(\frac{1}{2}(-2-\sqrt{6}),\frac{1}{2}(-2-\sqrt{6})\right)\quad\text{and}\quad\left(\frac{1}{2}(-2+\sqrt{6}),\frac{1}{2}(-2+\sqrt{6})\right).$$

These points correspond to the limiting values of t. From the geometry, it is easily seen that as (x,y) ranges over S, the variable t assumes all values in the closed interval $[-2-\sqrt{6},-2+\sqrt{6}]$. [Alternatively, using the theory of the quadratic, one can determine that the system

$$\begin{cases}(x+1)+(y+1)=t+2,\\(x+1)^2+(y+1)^2=3,\end{cases}$$

is solvable for real values of x and y if and only if $(t+2)^2\le 6$.]

Since $f(x,y)=-\sqrt{6}/18$ when $t=-2-\sqrt{6}$ and $f(x,y)=\sqrt{6}/18$ when $t=-2+\sqrt{6}$, and since f is continuous in t, the range of f includes the closed interval $[-\sqrt{6}/18,\sqrt{6}/18]$.

Observe that

$$\begin{aligned}f(x,y)+\frac{\sqrt{6}}{18}&=\frac{(9-\sqrt{6})t^3+(18+3\sqrt{6})t^2-(9+3\sqrt{6})t+\sqrt{6}}{18(1-t)^3}\\&=\frac{(t+2+\sqrt{6})[(9-\sqrt{6})t^2+(6-4\sqrt{6})t+(3-\sqrt{6}]}{18(1-t)^3}.\end{aligned}$$

Since the quadratic factor, having zero discriminant, is the square of a linear polynomial, and since $1-t>0$, $f(x,y)\ge-\sqrt{6}/18$ when $t\ge -2-\sqrt{6}$. Likewise

$$\begin{aligned}f(x,y)-\frac{\sqrt{6}}{18}&=\frac{(9+\sqrt{6})t^3+(18-3\sqrt{6})t^3+(-9+3\sqrt{6})t-\sqrt{6}}{18(1-t)^3}\\&=\frac{(t+2-\sqrt{6})[(9+\sqrt{6})t^2+(6+4\sqrt{6})t+(3+\sqrt{6})]}{18(1-t)^3}\le 0,\end{aligned}$$

so that $f(x,y) \leq \sqrt{6}/18$ when $t \leq -2+\sqrt{6}$. Thus the range of f is exactly $[-\sqrt{6}/18, \sqrt{6}/18]$.

Solution 2, by the proposer.

Let $u = x^2 + y^2$ and

$$a = \frac{2x}{u+1} \qquad b = \frac{2y}{u+1} \qquad c = \frac{u-1}{u+1}.$$

When $(x,y) \in S$, we have that

$$u + 1 = 2(1-x-y) \quad \text{and} \quad u - 1 = x^2 + y^2 - 1 = -2(x+y).$$

It can be checked that

$$a+b+c = 0, \quad a^2+b^2+c^2 = 1 \quad \text{and so} \quad ab+bc+ca = -\frac{1}{2}.$$

Also

$$abc = -\frac{xy(x+y)}{(1-x-y)^3} = -f(x,y).$$

Thus, the real numbers a, b, c are the roots of the polynomial

$$t^3 - \frac{1}{2}x + f(x,y).$$

Since the roots are all real, we must have $4 \cdot (1/8) \geq 27(f(x,y))^2$, so that

$$-\frac{1}{\sqrt{54}} \leq f(x,y) \leq \frac{1}{\sqrt{54}}$$

and $f(x,y)$ belongs to the closed interval $[-\sqrt{6}/18, \sqrt{6}/18]$.

Conversely, let $p \in [-\sqrt{6}/18, \sqrt{6}/18]$. We show that $p = f(x,y)$ for some $(x,y) \in S$. This is true for $p = 0$, since $f(0, -1+\sqrt{2}) = 0$. Suppose $p \neq 0$. Consider the polynomial $t^3 - \frac{1}{2}t + p$ and let a, b, c be its roots. Since the discriminant condition $4 \cdot (1/8) \geq 27p^2$ holds, the roots are all real. Since $a+b+c = 0$ and $ab+bc+ca = -1/2$, we have that $a^2+b^2+c^2 = 1$. Since $abc \neq 0$, $-1 < c < 1$, so that $c = (v-1)/(v+1)$ for some $v > 0$. Now, let

$$x = \frac{a(1+v)}{2} \qquad \text{and} \qquad y = \frac{b(1+v)}{2}.$$

Then $x^2 + y^2 = v = 1 - 2x - 2y$, so that $(x,y) \in S$. Moreover

$$\begin{aligned} f(x,y) &= \frac{xy(x+y)}{(1-x-y)^3} = \frac{ab}{4}(v+1)^2 \cdot \frac{(v+1)(a+b)}{2} \cdot \frac{8}{(v+1)^3} \\ &= ab(a+b) = -abc = p. \end{aligned}$$

Editor's Comments. The proposer submitted a second solution that followed the same strategy as Lau, except that he analyzed the behaviour of the function $t(t^2 + 2t-1)(1-t)^{-3}$ by calculus. Paul Bracken used Lagrange Multipliers and located the maximum and minimum values of $f(x,y)$ as well as eight other critical points on S given by the equations $(x+1)^2+(y+1)^2 = 3$ and $(x^2+y^2)(x+y) = xy+x+y$.

Paul Deiermann looked at the more general restriction $(x+1)^2 + (y+1)^2 = a$ where $0 < a < 9/2$, and parameterized the points of S by

$$(x,y) = (-1+\sqrt{a}\cos\theta, -1+\sqrt{a}\sin\theta).$$

He found $f(x,y)$ to be equal to

$$\frac{1}{2}\cdot\frac{[q^2-2q+2-a][-2+q]}{(3-q)^3},$$

where $q = \sqrt{2a}\cos(\theta-\pi/4)$ satisfies $-\sqrt{2a} \le q \le \sqrt{2a}$. He then analyzed this function, identifying its values at the endpoints and the two critical points of the interval $[-\sqrt{2a}, \sqrt{2a}]$. In the case $a = 3$ of the problem, the global maximum is achieved at both an endpoint and a critical point, while the global minimum is achieved at the other endpoint and critical point. He adds that *Mathematica* graphs suggest that $a = 3$ is the only value of a where f achieves both global extrema at an endpoint and a critical point at the same time.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015 : 41(7), p. 302–305.

4061. *Proposed by Leonard Giugiuc.*

Let ABC be a non-obtuse triangle none of whose angles are less than $\frac{\pi}{4}$. Find the minimum value of $\sin A \sin B \sin C$.

We received twelve submissions, of which 10 were correct and one was faulty. We present two solutions.

Solution 1, by Adnan Ali.

Let us first fix angle A and determine the values of B and C for which the product $\sin A \sin B \sin C$ is smallest. Since A is fixed, this product is minimized if and only if

$$\sin B \sin C = \frac{\cos(B-C) - \cos(B+C)}{2} = \frac{\cos(B-C) - \cos(\pi - A)}{2}$$

is minimized. But $\cos(\pi - A)$ is fixed and so it is enough to minimize $\cos(B - C)$. Because the triangle is not obtuse and all angles are not less than $\pi/4$, we have $|B - C| \leq \pi/4$. Since the cosine function is decreasing over $[0, \pi/2]$, $\cos(B - C)$ is minimized if $|B - C|$ is maximum, and that happens when $B = \pi/4$ and $C = 3\pi/4 - A$, or vice-versa. So now we have reduced the problem of finding the minimum value of the given product to finding the minimum value of

$$\sin A \sin(\pi/4) \sin(3\pi/4 - A), \quad \pi/4 \leq A \leq \pi/2.$$

This is quickly done by minimizing $\sin A \sin(3\pi/4 - A) = \frac{\cos(3\pi/4-2A)-\cos(3\pi/4)}{2}$, which is same as minimizing $\cos(3\pi/4-2A)$, where $\pi/4 \leq A \leq \pi/2$. The bounds on A imply that $-\pi/4 \leq 3\pi/4 - 2A \leq \pi/4$, and so the minimum value of $\cos(3\pi/4 - 2A)$ is achieved for $3\pi/4 - 2A = -\pi/4$ or $\pi/4$; each of these values leads to an isosceles right triangle. Thus, the minimum value of $\sin A \sin B \sin C$ is $1/2$, achieved for an isosceles right triangle ABC.

Solution 2, by Daniel Dan.

We use the identity $\sin A \sin B \sin C = \frac{1}{4}(\sin 2A + \sin 2B + \sin 2C)$. Define

$$f(x) : \left[\frac{\pi}{2}, \pi\right] \to [0, 1], \quad f(x) = \sin x,$$

and note that the function is concave; in particular, every point of the graph of $f(x)$ except for its end points, namely $\left(\frac{\pi}{2}, 1\right)$ and $(\pi, 0)$, lies above the line

$$g(x) = -\frac{2}{\pi}x + 2$$

that joins those end points. Consequently, we have

$$\begin{aligned}\frac{1}{4}\left((f(2A)+f(2B)+f(2C)\right) &\geq \frac{1}{4}\left((g(2A)+g(2B)+g(2C)\right)\\ &= \frac{1}{4}\left(-\frac{2(2A+2B+2C)}{\pi}+6\right)=\frac{1}{2}.\end{aligned}$$

We conclude that the product $\sin A \sin B \sin C$ cannot be less than $\frac{1}{2}$ when all three angles are restricted to the domain $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. The minimum is achieved if and only if $f(x) = g(x)$ for x equal to $2A, 2B$, and $2C$; because $A + B + C = \pi$, this is possible only if one of the angles is $\frac{\pi}{2}$ while the other two are $\frac{\pi}{4}$.

4062. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let L_n denote the nth Lucas number defined by $L_0 = 2, L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. Prove that

$$\frac{L_n^4+L_{n+1}^4}{L_nL_{n+1}}+\frac{L_{n+1}^4+L_{n+3}^4}{L_{n+1}L_{n+3}}+\frac{L_{n+3}^4+L_n^4}{L_{n+3}L_n}\geq \frac{2}{3}L_{n+4}^2.$$

We received ten correct and complete solutions. We present the solutions of Arkady Alt, who like most submitters used standard inequalities for a simple proof, and a slightly modified version of the solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, who made heavier use of the given recursion to find a stronger bound.

Solution 1, by Arkady Alt.

Since $a^4 + b^4 \geq ab(a^2 + b^2)$ (as this can be rewritten as $(a^2 + ab + b^2)(a - b)^2 \geq 0$) and $a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3}$ for all $a, b, c \in \mathbb{R}$, we have

$$\begin{aligned}&\frac{L_n^4+L_{n+1}^4}{L_nL_{n+1}}+\frac{L_{n+1}^4+L_{n+3}^4}{L_{n+1}L_{n+3}}+\frac{L_{n+3}^4+L_n^4}{L_{n+3}L_n}\\ &\geq \frac{L_nL_{n+1}(L_n^2+L_{n+1}^2)}{L_nL_{n+1}}+\frac{L_{n+1}L_{n+3}(L_{n+1}^2+L_{n+3}^2)}{L_{n+1}L_{n+3}}+\frac{L_{n+3}L_n(L_{n+3}^2+L_n^2)}{L_{n+3}L_n}\\ &= 2(L_n^2+L_{n+1}^2+L_{n+3}^2)\\ &\geq 2\frac{(L_n+L_{n+1}+L_{n+3})^2}{3}=\frac{2(L_{n+2}+L_{n+3})^2}{3}\\ &= \frac{2L_{n+4}^2}{3}\end{aligned}$$

Solution 2, by Dionne Bailey, Elsie Campbell, and Charles Diminnie.

More generally, we will show that for all $n \geq 0$,

$$\frac{L_n^4+L_{n+1}^4}{L_nL_{n+1}}+\frac{L_{n+1}^4+L_{n+3}^4}{L_{n+1}L_{n+3}}+\frac{L_{n+3}^4+L_n^4}{L_{n+3}L_n}> 2L_{n+4}^2.$$

We can check by hand that this holds for $n \leq 2$.

For $n \geq 3$ we first use the Arithmetic Mean - Geometric Mean inequality to obtain

$$\begin{aligned} x^4 + y^4 &= 2x^2y^2 + (x^2 - y^2)^2 \\ &= 2x^2y^2 + (x+y)^2(x-y)^2 \\ &\geq 2x^2y^2 + 4xy(x-y)^2 \\ &= xy(2xy + 4(x-y)^2) \end{aligned}$$

and hence

$$\frac{x^4 + y^4}{xy} \geq 2xy + 4(x-y)^2.$$

Using this property and the recursion for the Lucas numbers (multiple times, when necessary), we get

$$\frac{L_n^4 + L_{n+1}^4}{L_n L_{n+1}} \geq 2L_n L_{n+1} + 4(L_{n+1} - L_n)^2 = 4L_{n+1}^2 - 6L_{n+1}L_n + 4L_n^2,$$

$$\frac{L_{n+1}^4 + L_{n+3}^4}{L_{n+1} L_{n+3}} \geq 2L_{n+1} L_{n+3} + 4(L_{n+3} - L_{n+1})^2 = 8L_{n+1}^2 + 10L_{n+1}L_n + 4L_n^2,$$

$$\frac{L_{n+3}^4 + L_n^4}{L_{n+3} L_n} \geq 2L_{n+3} L_n + 4(L_{n+3} - L_n)^2 = 16L_{n+1}^2 + 4L_{n+1}L_n + 2L_n^2,$$

and

$$2L_{n+4}^2 = 18L_{n+1}^2 + 24L_{n+1}L_n + 8L_n^2.$$

Combining these, we obtain

$$\begin{aligned} \frac{L_n^4 + L_{n+1}^4}{L_n L_{n+1}} + \frac{L_{n+1}^4 + L_{n+3}^4}{L_{n+1} L_{n+3}} + \frac{L_{n+3}^4 + L_n^4}{L_{n+3} L_n} &\geq 28L_{n+1}^2 + 8L_{n+1}L_n + 10L_n^2 \\ &= 2L_{n+4}^2 + 10L_{n+1}^2 - 16L_{n+1}L_n + 2L_n^2 \\ &= 2L_{n+4}^2 - 6L_{n+1}L_n + 10L_{n+1}L_{n-1} + 2L_n^2 \\ &= 2L_{n+4}^2 + 4L_{n+1}L_{n-1} - 6L_{n+1}L_{n-2} + 2L_n^2 \\ &= 2L_{n+4}^2 + 4L_{n+1}L_{n-1} - 4L_{n+1}L_{n-2} + 2L_nL_{n-1} - 2L_{n-1}L_{n-2} \\ &= 2L_{n+4}^2 + 4L_{n+1}L_{n-3} + L_{n-1}^2 \\ &> 2L_{n+4}^2. \end{aligned}$$

4063. *Proposed by Marcel Chiriţă.*

Let a, b, c be real numbers greater than or equal to 3. Show that

$$\min\left(\frac{a^2b^2 + 3b^2}{b^2 + 27}, \frac{b^2c^2 + 3c^2}{c^2 + 27}, \frac{a^2c^2 + 3a^2}{a^2 + 27}\right) \leq \frac{abc}{9}.$$

We received six submissions all of which were correct. We present a composite of the similar solutions by Arkady Alt and Leonard Guigiuc.

Suppose to the contrary that

$$\min\left(\frac{a^2b^2+3b^2}{b^2+27}, \frac{b^2c^2+3c^2}{c^2+27}, \frac{a^2c^2+3a^2}{a^2+27}\right) > \frac{abc}{9}.$$

Then we have $\prod_{cyc} \frac{a^2b^2+3b^2}{b^2+27} > \frac{a^3b^3c^3}{9^3}$, so $\prod_{cyc} \frac{a^2+3}{a^2+27} > \frac{abc}{9^3}$.

But since $\frac{a}{9} - \frac{a^2+3}{a^2+27} = \frac{a^3-9a^2+27a-27}{9(a^2+27)} = \frac{(a-3)^2}{9(a^2+27)} \geq 0$, we have $\frac{a^2+3}{a^2+27} \leq \frac{a}{9}$.

Similarly, $\frac{b^2+3}{b^2+27} \leq \frac{b}{9}$ and $\frac{c^2+3}{c^2+27} \leq \frac{c}{9}$.

Hence, $\prod_{cyc} \frac{a^2+3}{a^2+27} > \frac{abc}{9^3}$ is a contradiction.

4064. *Proposed by Michel Bataille.*

In the plane of a triangle ABC, let Γ be a circle whose centre O is not on the sidelines AB, BC, CA. Let A', B', C' be the poles of the lines BC, CA, AB with respect to Γ, respectively. Prove that

$$\frac{OA' \cdot B'C'}{OA \cdot BC} = \frac{OB' \cdot C'A'}{OB \cdot CA} = \frac{OC' \cdot A'B'}{OC \cdot AB}.$$

We received five solutions, all correct, and present the solution by Joel Schlosberg, slightly modified by the editor.

One way to define the pole A' of the line BC with respect to the circle Γ is by reciprocation, namely A' is the inverse in Γ of the foot of the perpendicular from O to BC. [See, for example H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited* (The Mathematical Association of America, 1967), Section 6.1.] Conversely, if D is the inverse of A in Γ, then the polar of A, namely $B'C'$, is the line through D that is perpendicular to OA. We shall use three immediate consequences of this definition. If r is the radius of Γ, then $OA \cdot OD = r^2$, or

$$OA = \frac{r^2}{OD}. \tag{1}$$

Since $OB' \perp CA$ and $OC' \perp AB$, $\angle B'OC'$ is equal to or supplementary to $\angle BAC$. Let R be the circumradius of $\triangle ABC$. By the law of sines,

$$BC = 2R \sin \angle BAC = 2R \sin \angle B'OC'. \tag{2}$$

Finally, since each is the area of $\triangle OB'C'$,

$$\frac{1}{2}B'C' \cdot OD = \frac{1}{2}OB' \cdot OC' \sin \angle B'OC'. \tag{3}$$

Using in turn (1) and (2), then (3), we get

$$\frac{OA'\cdot B'C'}{OA\cdot BC}=\frac{OA'\cdot B'C'}{(r^2/OD)\cdot 2R\sin\angle B'OC'}=\frac{OA'}{2Rr^2}\cdot\frac{B'C'\cdot OD}{\sin\angle B'OC'}=\frac{OA'\cdot OB'\cdot OC'}{2Rr^2}.$$

The same reasoning shows that $\dfrac{OB'\cdot C'A'}{OB\cdot CA}$ and $\dfrac{OC'\cdot A'B'}{OC\cdot AB}$ are also equal to $\dfrac{OA'\cdot OB'\cdot OC'}{2Rr^2}$.

4065. *Proposed by Martin Lukarevski.*

Let ABC be a triangle with a,b,c as lengths of its sides and let R,r,s denote the circumradius, inradius and semiperimeter, respectively. Prove that

$$\frac{1}{(s-a)^2}+\frac{1}{(s-b)^2}+\frac{1}{(s-c)^2}\geq\frac{2}{r}\left(\frac{1}{r}-\frac{1}{R}\right).$$

We received ten correct and complete solutions. We present the solution by the proposer.

We use the Garfunkel-Bankoff inequality (Problem 825, proposed by J. Garfunkel, solution by L. Bankoff, ***Crux*** 9 (1983), p.79 and 10 (1984), p.168) :

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \qquad (1)$$

which by the well-known identity

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R} \qquad (2)$$

is equivalent to

$$\tan^2\frac{A}{2}+\tan^2\frac{A}{2}+\tan^2\frac{A}{2}\geq 2-\frac{2r}{R}.$$

By another well-known identity, which states that

$$\frac{1}{s-a}=\frac{1}{r}\tan\frac{A}{2}, \qquad (3)$$

we have that

$$\begin{aligned}\frac{1}{(s-a)^2}+\frac{1}{(s-b)^2}+\frac{1}{(s-c)^2}&=\frac{1}{r^2}\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right)\\&\geq\frac{1}{r^2}\left(2-\frac{2r}{R}\right)\\&=\frac{2}{r}\left(\frac{1}{r}-\frac{1}{R}\right),\end{aligned}$$

with equality, as in (1), only for the equilateral triangle.

4066. *Proposed by Mihaela Berindeanu.*

Prove that for $a, b, c > 0$ and $ab + ac + bc = 2016$,

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq \frac{2019^2}{2016}.$$

We received 19 solutions all of which are correct. We present a composite of nearly identical solutions by Andrea Fanchini and Titu Zvonaru.

We prove the more general result that if $a, b, c > 0$ such that $ab+bc+ca = k$, then

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq \frac{(k+3)^2}{k}.$$

Note first that the trivial inequality $x^2 + y^2 + z^2 \geq xy + yz + zx$ implies

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}. \qquad (1)$$

Using (1) together with AM-GM and AM-HM inequalities we then have

$$(a+\frac{1}{b})^2+(b+\frac{1}{c})^2+(c+\frac{1}{a})^2 = a^2+b^2+c^2+2(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

$$\begin{aligned}
&\geq ab+bc+ca+2\cdot 3+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\\
&\geq k+6+\frac{9}{ab+bc+ca}\\
&= \frac{(k+3)^2}{k}
\end{aligned}$$

Editor's comments. Most of the other solutions used AM-GM, AM-QM and/or Cauchy Schwarz Inequalities. It is trivial to see that equality holds if and only if $a = b = c = \frac{\sqrt{3k}}{3}$.

4067. *Proposed by Mehtaab Sawhney.*

Consider a graph G such that between any three vertices in G there are either 0 or 2 edges. Classify all such graphs G.

We received seven correct and complete solutions. We present the solution by Joel Schlosberg.

We claim that a graph G satisfies the condition if and only if it is either an edgeless graph or a complete bipartite graph.

If G is edgeless, any three vertices have zero edges between them, so G trivially satisfies the condition. If G is a complete bipartite graph, the vertices of G can be partitioned into two sets S_1, S_2, such that two vertices are adjacent if and only if they are in different sets. If three vertices are in the same set, they have zero edges between them ; otherwise, two of them are in one set and one is in the other, leading to two edges between them. Thus G satisfies the condition.

Conversely, suppose that G satisfies the condition. If G is not edgeless, then there exist two vertices v_1, v_2 with an edge between them. For $k = 1, 2$, let S_k be the set of vertices that share an edge with v_k. Clearly v_1 is in S_2 but not S_1 and v_2 in S_1 but not S_2. If v is a vertex of G different from v_1 and v_2, then the three vertices v, v_1, v_2 must have exactly two edges between them, since they cannot have zero. Thus v is in exactly one of S_1 or S_2. Therefore S_1 and S_2 form a partition of the vertices of G. Suppose v, w are vertices in the same set, say S_1. Then there is no edge between v and w, as otherwise we would have three edges between v, w and v_1. Now suppose $v \in S_1$ and $w \in S_2$. If $v = v_2$ then there is an edge between v and w by the definition of S_2. Otherwise consider the three vertices v, w and v_2. There is an edge between w and v_2 by the definition of S_2 and no edge between v and v_2, as just shown. Therefore there must be an edge between v and w. Thus we have proven that G is a complete bipartite graph.

4068. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers. Prove that

$$\frac{a+2b}{2a+3b+c} + \frac{b+2c}{a+2b+3c} + \frac{c+2a}{3a+b+2c} \leq \frac{3}{2}.$$

Editor's comments. We received 25 submissions all of which are correct. However, it was pointed out by Michael Bataille, and Dionne Bailey, Elsie Campbell, and Charles Diminnie that this problem is the same as Crux problem #4016 (by the same proposer) which appeared on p. 74 of ***Crux*** 41 (2). The only difference being that in #4016, it was asked to find the maximum value of the given expression while in #4068, it becomes a proof question with the maximum value given. So, it can not be viewed as a "variation". Two different solutions to #4016 given by Arkady Alt and Šefket Arslanagić have appeared on pp. 85-86 of ***Crux*** 42 (2). It is interesting to note that both of them are also among the 25 solvers to the current problem but neither made any reference to #4016 !

4069. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let $(u_n)_{n\geq 0}$ be an arithmetic progression with a positive common difference d and with $u_1 > 0$. Let $(x_n)_{n\geq 0}$ be a sequence with $x_0 = 0, x_1 = x_2 = 1$ and

$$\sum_{k=1}^{n} u_k x_k = u_n x_{n+2} + d(x_4 - x_{n+3}) - x_2 u_1, \ \forall n \geq 0.$$

Prove that $(x_n)_{n\geq 0}$ is the Fibonacci sequence.

There were twelve correct solutions. All were variants of the ones below, with three using the closed form of the Fibonacci sums in Solution 2.

Solution 1.

When $n = 0$, the condition is that

$$0 = u_0x_2 + d(x_4 - x_3) - u_1x_2 = d(x_4 - x_3 - x_2),$$

whence $x_4 = x_3 + x_2$. When $n = 1$, we have that $u_1x_1 = u_1x_3 - u_1x_2$, whence $x_3 = x_1 + x_2$. Since $x_0 = 0 = F_0$ and $x_1 = x_2 = 1 = F_1 = F_2$, then $x_3 = F_3$ and $x_4 = F_4$.

For $n \geq 1$, we have that

$$\begin{aligned} u_{n+1}x_{n+1} &= [u_{n+1}x_{n+3} + d(x_4 - x_{n+4}) - u_1x_2] - [u_nx_{n+2} + d(x_4 - x_{n+3}) - u_1x_2] \\ &= -dx_{n+4} + (u_{n+1} + d)x_{n+3} - (u_{n+1} - d)x_{n+2}, \end{aligned}$$

so that

$$dx_{n+4} = d(x_{n+3} + x_{n+2}) + u_{n+1}(x_{n+3} - x_{n+2} - x_{n+1}).$$

We establish the result by induction. Suppose that $x_k = F_k$ for $0 \leq k \leq n + 3$. This is true for $n = 1$. The foregoing equation establishes that if $x_k = F_k$ for $k = n + 1, n + 2, n + 3$, then $dx_{n+4} = dF_{n+4} + u_{n+1}(0)$ and $x_{n+4} = F_{n+4}$.

Solution 2.

The following Fibonacci relationships are easily established by induction for $n \geq 1$:

$$F_1 + F_2, + \cdots + F_n = F_{n+2} - 1 \quad \text{and} \quad \sum_{k=1}^{n}(k-1)F_k = (n-1)F_{n+2} - F_{n+3} + 3.$$

As in Solution 1, we show that $x_k = F_k$ for $0 \leq k \leq 4$. Suppose, as an induction hypothesis, this holds for $1 \leq k \leq n + 2$. By the foregoing relationships, we have that

$$\begin{aligned} \sum_{k=1}^{n} u_kx_k &= \sum_{k=1}^{n}[u_1 + (k-1)d]F_k \\ &= u_1\sum_{k=1}^{n} F_k + d\sum_{k=1}^{n}(k-1)F_k \\ &= u_1[F_{n+2} - 1] + d(n-1)F_{n+2} - dF_{n+3} + 3d \\ &= F_{n+2}[u_1 + (n-1)d] + d(3 - F_{n+3}) - u_1 \\ &= u_nF_{n+2} + d(F_4 - F_{n+3}) - u_1F_2. \end{aligned}$$

However, the given condition provides that

$$\sum_{k=1}^{n} u_kx_k = u_nF_{n+2} + d(F_4 - x_{n+3}) - u_1F_2.$$

Therefore $x_{n+3} = F_{n+3}$, and the result holds.

4070. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Compute

$$\lim_{n\to\infty}\left[\frac{1}{\ln n}\left(\frac{\arctan 1}{n}+\frac{\arctan 2}{n-1}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right)\right].$$

We received six correct and complete solutions. We present the solution by Joel Schlosberg.

Let $H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. Since $\arctan x$ is an increasing function,

$$\begin{aligned}&\frac{1}{\ln n}\left[\frac{\arctan 1}{n}+\frac{\arctan 2}{n-1}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right]\\ &\le \frac{1}{\ln n}\left(\frac{\arctan n}{n}+\frac{\arctan n}{n-1}+\cdots+\frac{\arctan n}{2}+\arctan n\right)\\ &= \frac{H_n}{\ln n}\cdot\arctan n;\end{aligned}$$

and for any positive integer m, if $n \ge m$,

$$\begin{aligned}&\frac{1}{\ln n}\left[\frac{\arctan 1}{n}+\frac{\arctan 2}{n-1}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right]\\ &\ge \frac{1}{\ln n}\left(\frac{\arctan m}{n-m+1}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right)\\ &\ge \frac{1}{\ln n}\left(\frac{\arctan m}{n-m+1}+\cdots+\frac{\arctan m}{2}+\arctan m\right)\\ &= \frac{H_{n-m+1}}{\ln n}\cdot\arctan m.\end{aligned}$$

It is well known that H_n is asymptotic to $\ln n$ and $\lim_{x\to\infty}\arctan x = \pi/2$. Therefore,

$$\lim_{n\to\infty}\frac{H_n}{\ln n}\cdot\arctan n = \frac{\pi}{2}$$

and

$$\begin{aligned}&\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{H_{n-m+1}}{\ln n}\cdot\arctan m\right)\\ &= \lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{\ln(n-m+1)}{\ln n}\cdot\arctan m\right) = \lim_{m\to\infty}\arctan m = \frac{\pi}{2}\end{aligned}$$

so by the squeeze theorem,

$$\lim_{n\to\infty}\frac{1}{\ln n}\left(\frac{\arctan 1}{n}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right) = \frac{\pi}{2}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015 : 41(8), p. 352–354.

4071. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Prove that if $a, b, c \in (0, 1)$, then $a^{a+1}b^{b+1}c^{c+1} < e^{2(a+b+c)-6}$.

There were 14 correct solutions and one incorrect submission. We present the solution submitted by various solvers.

Let

$$f(x) = 2(x-1) - (x+1)\ln x = 2(x-1) - x\ln x - \ln x$$

for $0 < x < 1$. The derivative

$$f'(x) = 1 - x^{-1} - \ln x = \ln(x^{-1}) - (x^{-1} - 1)$$

is negative and $f(1) = 0$, so that $f(x) > 0$ on $(0, 1)$. Therefore $(x+1)\ln x < 2(x-1)$ for $0 < x < 1$.

Thus

$$(a+1)\ln a + (b+1)\ln b + (c+1)\ln c < 2(a+b+c) - 6.$$

Exponentiating yields the result.

Editor's Comments. Several solvers used the function in the solution ; another popular function studied was $\ln x - 2(x-1)(x+1)^{-1} = \ln x + 4(x+1)^{-1} - 2$ or a close relative. Three solvers noted that the function $(x+1)\ln x$ was concave and applied Jensen's inequality. One respondent took a stroll down the garden path with the following argument.

Let $L = a^{a+1}b^{b+1}c^{c+1}$. Since $\ln x < x - 1$ for $0 < x < 1$,

$$\begin{aligned}\ln L &= (a+1)\ln a + (b+1)\ln b + (c+1)\ln c < 2(\ln a + \ln b + \ln c)\\ &< 2((a-1) + (b-1) + (c-1)) = 2(a+b+c) - 6.\end{aligned}$$

Exponentiating yields the desired result.

4072. *Proposed by Michel Bataille.*

Let a, b be distinct positive real numbers and $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $L = \frac{a-b}{\ln a - \ln b}$. Prove that

$$\frac{L}{G} > \frac{4A + 5G}{A + 8G}.$$

There were five correct solutions, of which we present two.

Solution 1, by the proposer.

This inequality is a refinement of the known inequality $L > G$. Wolog, let $a > b$ and set $x = \sqrt{a}$, $y = \sqrt{b}$. The inequality can be rewritten as

$$\frac{\ln x - \ln y}{x^2 - y^2} < \frac{1}{4xy} \cdot \frac{x^2 + y^2 + 16xy}{2x^2 + 2y^2 + 5xy}.$$

Setting $t = x/y$ converts it to

$$\frac{4\ln t}{t^2 - 1} < \frac{1}{t} \cdot \frac{t^2 + 16t + 1}{2t^2 + 5t + 2}.$$

Since

$$\frac{1}{t} \cdot \frac{t^2 + 16t + 1}{2t^2 + 5t + 2} = \frac{1}{2}\left(\frac{1}{t} + \frac{27}{2t^2 + 5t + 2}\right),$$

it all boils down to proving for $t > 1$ the inequality

$$\frac{8\ln t}{t^2 - 1} < \frac{1}{t} + \frac{27}{2t^2 + 5t + 2}.$$

Recall Simpson's $\frac{3}{8}$ Rule for numerical integration of a C^4-function f on a closed interval $[u, v]$:

$$\int_u^v f(s)ds = \frac{v - u}{8}\left(f(u) + 3f\left(\frac{2u + v}{3}\right) + 3f\left(\frac{u + 2v}{3}\right) + f(v)\right) - \frac{(v - u)^5}{6480} f^{(4)}(\xi)$$

for some $\xi \in (u, v)$. Taking $f(s) = 1/s$, $u = 1$ and $v = t > 1$, we obtain

$$\ln t = \frac{t - 1}{8}\left(1 + \frac{9}{2 + t} + \frac{9}{1 + 2t} + \frac{1}{t}\right) - \frac{24(t - 1)^5}{6480\xi^2}$$

and so

$$\ln t < \frac{t - 1}{8}\left(1 + \frac{1}{t} + \frac{27(t + 1)}{2t^2 + 5t + 2}\right) = \frac{t^2 - 1}{8}\left(\frac{1}{t} + \frac{27}{2t^2 + 5t + 2}\right).$$

This leads to the desired inequality.

Solution 2, by Arkady Alt.

Assume $a > b$ and let $t = \sqrt{a/b}$. Then as in the previous solution, we have to establish that

$$4\ln t < \frac{(t^2 - 1)(t^2 + 16t + 1)}{t(t + 2)(2t + 1)}.$$

Let

$$\begin{aligned} h(t) &= \frac{(t^2 - 1)(t^2 + 16t + 1)}{t(t + 2)(2t + 1)} - 4\ln t \\ &= \frac{t}{2} - \frac{27}{2(t + 2)} - \frac{27}{4(2t + 1)} - \frac{1}{2t} + \frac{27}{4} - 4\ln t. \end{aligned}$$

Then

$$h'(t) = \frac{1}{2} + \frac{27}{2(t+2)^2} + \frac{27}{2(2t+1)^2} + \frac{1}{2t^2} - \frac{4}{t}$$
$$= \frac{2(t^2+t+1)(t-1)^4}{t^2(t+2)^2(2t+1)^2} > 0$$

for $t > 1$. Since $h(t) > h(1) = 0$ for $t > 1$, the inequality follows.

4073. *Proposed by Daniel Sitaru.*

Solve the following system :

$$\begin{cases} \sin 2x + \cos 3y = -1, \\ \sqrt{\sin^2 x + \sin^2 y} + \sqrt{\cos^2 x + \cos^2 y} = 1 + \sin(x+y). \end{cases}$$

The solution from Michel Bataille was the only one of the 2 submissions that was complete and correct. We present his solution.

We first show that the second equation is equivalent to $x + y \equiv \frac{\pi}{2} \pmod{2\pi}$.

If $x + y \equiv \frac{\pi}{2} \pmod{2\pi}$, then $\sin^2 y = \cos^2 x$ and $\cos^2 y = \sin^2 x$. It immediately follows that both sides of the equation equal 2. Conversely, if the equation holds, then squaring gives

$$2\sqrt{(\sin x \cos x - \sin y \cos y)^2 + \sin^2(x+y)} = 2\sin(x+y) - (1 - \sin^2(x+y)),$$

and therefore

$$\begin{aligned} 2\sin(x+y) \le 2\sqrt{\sin^2(x+y)} &\le 2\sqrt{(\sin x \cos x - \sin y \cos y)^2 + \sin^2(x+y)} \\ &= 2\sin(x+y) - (1 - \sin^2(x+y)) \le 2\sin(x+y). \end{aligned}$$

Thus, equality must hold throughout and in particular $\sin(x+y) \ge 0$ and $\sin^2(x+y) = 1$. We deduce that $x + y \equiv \frac{\pi}{2} \pmod{2\pi}$.

Since $\cos 3\left(\frac{\pi}{2} - x\right) = -\sin 3x$, we are led to seek the solutions to the equation $f(x) = 1$ where $f(x) = \sin 3x - \sin 2x$. Note that $f\left(-\frac{\pi}{2}\right) = 1$ so that the numbers $-\frac{\pi}{2} + 2k\pi$ ($k \in \mathbb{Z}$) are solutions. For other solutions note that f is odd and 2π-periodic ; consequently, we may restrict the study of f to the interval $[0, \pi]$ and look for x satisfying either $f(x) = 1$ or $f(x) = -1$ (the latter since then $f(-x) = 1$). Consider first the interval $[0, \frac{\pi}{2})$. We have $f(0) = 0$ and if $x \in (0, \frac{\pi}{2})$, then $\sin 2x > 0$ and so $f(x) < 1$.

- $x \in (0, \frac{\pi}{3}]$: $\sin 3x > 0$ for x between 0 and $\frac{\pi}{3}$, hence $f(x) > -1$; since $f\left(\frac{\pi}{3}\right) > -1$, *there is no* $x \in (0, \frac{\pi}{3}]$ *such that* $f(x) = -1$.

- $x \in (\frac{\pi}{3}, \frac{\pi}{2})$: $f''(x) = 4\sin 2x - 9\sin 3x > 0$, hence $f'(x) = 3\cos 3x - 2\cos 2x$ is nondecreasing on the interval $(\frac{\pi}{3}, \frac{\pi}{2})$. For some $x_1 \in (\frac{\pi}{3}, \frac{\pi}{2})$, we have $f'(x) \le 0$

for $x \in (\frac{\pi}{3}, x_1]$ and $f'(x) > 0$ for $x \in (x_1, \frac{\pi}{2})$. Since $f\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ and $f\left(\frac{\pi}{2}\right) = -1$, *we have* $f(x_1) < -1$ *and* $f(\alpha) = -1$ *for a unique* α *in* $(\frac{\pi}{3}, \frac{\pi}{2})$.

In a similar way we treat the interval $(\frac{\pi}{2}, \pi]$. We have $f(\pi) = 0$ and if $x \in (\frac{\pi}{2}, \pi)$, then $\sin 2x < 0$, hence $f(x) > -1$.

- $x \in (\frac{\pi}{2}, \frac{3\pi}{4})$: $f'(x) > 0$ and so f is increasing from -1 to $1 + \frac{\sqrt{2}}{2}$. T*hus,* $f(\beta) = 1$ *for a unique* $\beta \in (\frac{\pi}{2}, \frac{3\pi}{4})$.

- $x \in (\frac{5\pi}{6}, \pi)$: f is decreasing from $1 + \frac{\sqrt{3}}{2}$ to 0, *hence* $f(\gamma) = 1$ *for a unique* γ *of* $(\frac{5\pi}{6}, \pi)$.

- $x \in [\frac{3\pi}{4}, \frac{5\pi}{6}]$: Resorting to $f''(x)$, we see that $f'(x)$ decreases from positive to negative so that $f(x) > 1$.

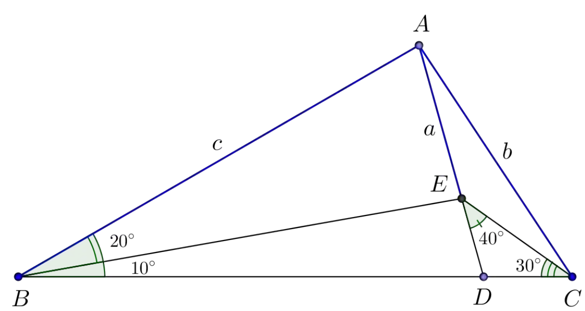
In conclusion, on the interval $[-\pi, \pi]$ the solutions (x, y) of the system are the pairs

$$\left(-\frac{\pi}{2},\ \pi\right), \left(-\alpha,\ \frac{\pi}{2} + \alpha\right), \left(\beta,\ \frac{\pi}{2} - \beta\right), \quad \text{and} \quad \left(\gamma,\ \frac{\pi}{2} - \gamma\right).$$

All other solutions are obtained by adding multiples of 2π to x or y.

4074. *Proposed by Abdilkadir Altinaş.*

Consider the triangle ABC with the following measures :



Show that $a + b = c$; that is, $|AE| + |AC| = |AB|$.

We received 16 solutions, all correct. We feature the solution of C.R. Pranesachar that is typical of the approach used by most solvers.

Because the angles of ΔBCE sum to $180°$, we see that $\angle BED = 100°$ and its supplement $\angle BEA = 80°$. It follows that in ΔBEA we also have $\angle BAE = 80°$, so that $BE = AB = c$ and

$$\frac{AE/2}{AB} = \sin\frac{20°}{2},$$

or

$$a = 2c \sin 10°.$$

Further, by the Sine Rule applied to triangle BCE,

$$CE = BE\frac{\sin 10^\circ}{\sin 30^\circ} = 2c\sin 10^\circ = a.$$

This means that ΔAEC is also isosceles (with $EA = EC = a$), and because its exterior angle at E equals 40°, its interior angles at A and C must each be 20° ; thus

$$\frac{b/2}{a} = \cos 20^\circ \quad \text{or} \quad b = 2(2c\sin 10^\circ)\cdot\cos 20^\circ.$$

The given relation $a + b = c$ now translates to

$$2\sin 10^\circ + 4\sin 10^\circ\cos 20^\circ = 1.$$

This is easy to prove :

$$\text{lhs } = 2\sin 10^\circ + 2(\sin 30^\circ - \sin 10^\circ) = 1 = \text{ rhs}.$$

4075. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Prove that in any triangle ABC with $BC = a$, $CA = b$, $AB = c$ the following inequality holds :

$$\sqrt[3]{abc}\cdot\sqrt{a^2+b^2+c^2} \geq 4[ABC],$$

where $[ABC]$ is the area of triangle ABC.

We received 16 correct solutions and we present the solution by Martin Lukarevski.

The inequality can be sharpened to

$$\sqrt[3]{abc}\cdot\sqrt{ab+bc+ca} \geq 4[ABC].$$

We use the inequality

$$\sqrt[3]{abc} \geq \sqrt{4[ABC]/\sqrt{3}},$$

which is equivalent to the well-known inequality

$$\sin A\sin B\sin C \leq \frac{3\sqrt{3}}{8},$$

and Gordon's inequality (4.5 in O. Bottema, R. Z. Djordjević, R. R. Janić, D. S. Mitrinovic, P. M. Vasić, *Geometric inequalities*, Groningen, Wolters-Noordhoff, 1969)

$$\sqrt{ab+bc+ca} \geq \sqrt{4[ABC]\sqrt{3}}.$$

Hence

$$\sqrt[3]{abc}\cdot\sqrt{a^2+b^2+c^2} \geq \sqrt[3]{abc}\cdot\sqrt{ab+bc+ca} \geq 4[ABC].$$

Editor's Comments. Many of the solutions were rather similar in nature as most verifications were the result of combining existing inequalities. In fact, Martin Lukarevski submitted two solutions of which his second is presented above.

4076. *Proposed by Mehtaab Sawhney.*

Prove that $(x^2+y^2+z^2)^3 \geq (x^3+y^3+z^3+3(\sqrt{3}-1)xyz)^2$ for all nonnegative reals x, y, and z.

We received seven submissions, six of which were correct. We present the solution by Michel Bataille, expanded slightly by the editor.

Note first that equality holds if $x=y=z=0$. Now suppose $x+y+z>0$. Then by homogeneity we may assume that $x+y+z=1$. Let $m=xy+yz+zx$ and $k=xyz$. Then $x^2+y^2+z^2=1-2m$ and

$$x^3+y^3+z^3=3xyz+(x+y+z)(x^2+y^2+z^2-xy-yz-zx)=1-3m+3k.$$

The given inequality is equivalent, in succession, to

$$\begin{aligned}&(1-2m)^3 \geq (1-3m+3\sqrt{3}k)^2\\&1-6m+12m^2-8m^3 \geq 1+9m^2+27k^2-6m+6\sqrt{3}k-18\sqrt{3}mk\\&3m^2+18\sqrt{3}mk-8m^3-27k^2-6\sqrt{3}k \geq 0. \qquad (1)\end{aligned}$$

Note that $1-3m \geq 0$ since $x^2+y^2+z^2 \geq xy+yz+zx$. We set $u=\sqrt{1-3m}$ so $m=\frac{1}{3}(1-u^2)$ and then (1) becomes

$$\begin{aligned}&\frac{8}{27}(1-u^2)^3+27k^2+6\sqrt{3}k \leq \frac{1}{3}(1-u^2)^2+6\sqrt{3}(1-u^2)k\\&\frac{8}{27}(1-3u^2+3u^4-u^6)+27k^2+6\sqrt{3}k \leq \frac{1}{3}(1-2u^2+u^4)+6\sqrt{3}k-6\sqrt{3}ku^2\\&27\cdot 6\sqrt{3}ku^2+(27k)^2 \leq 1+6u^2-15u^4+8u^6. \qquad (2)\end{aligned}$$

We now apply the following known result :

$$27k \leq (1-u)^2(1+2u)=1-3u^2+2u^3. \qquad (3)$$

(See the article "On a class of three-variable Inequalities" by Vo Quoc Ba Can, Mathematical Reflections, 2007, issue 2, and the proof by Paolo Perfetti in his solution to ***Crux*** Problem 3663, 38 (7), pp. 291-292.)

Using (3), we see that in order to establish (2), it suffices to prove the inequality :

$$6\sqrt{3}u^2(1-u)^2(1+2u)+(1-u)^4(1+2u)^2 \leq 1+6u^2-15u^4+8u^6. \qquad (4)$$

After straightforward computations, we see that (4) is successively equivalent to :

$$\begin{aligned}&(1-u)^2(6\sqrt{3}u^2(1+2u)+(1-u)^2(1+2u)^2) \leq (1-u)^2(1+2u+9u^2+16u^3+8u^4)\\&1+2u+9u^2+16u^3+8u^4-(6\sqrt{3}u^2+12\sqrt{3}u^3+1+2u-3u^2-4u^3+4u^4) \geq 0\\&((12-6\sqrt{3})u^2+(20-12\sqrt{3})u^3+4u^4) \geq 0\\&2u^2+(10-6\sqrt{3})u+6-3\sqrt{3} \geq 0,\end{aligned}$$

which is true since $(10-6\sqrt{3})^2-8(6-3\sqrt{3})=32(5-3\sqrt{3})<0$. Hence (4) is true and our proof is complete.

4077. *Proposed by George Apostolopoulos.*

Let ABC be a triangle. Prove that

$$\sin\frac{A}{2}\cdot\sin B\cdot\sin C+\sin A\cdot\sin\frac{B}{2}\cdot\sin C+\sin A\cdot\sin B\cdot\sin\frac{C}{2}\le\frac{9}{8}.$$

We received 15 submissions, of which 14 were correct and complete. We present the solution by Phil McCartney.

Let $a=\frac{\pi-A}{2}$, $b=\frac{\pi-B}{2}$, $c=\frac{\pi-C}{2}$. Then $a+b+c=\pi$ and a, b and c are in $\left(0,\frac{\pi}{2}\right)$. We have

$$\begin{aligned}\sum_{cyc}\sin\frac{A}{2}\sin B\sin C&=\sum_{cyc}\cos a\sin(\pi-2b)\sin(\pi-2c)\\&=\sum_{cyc}\cos a\sin(2b)\sin(2c)\\&=4\cos a\cos b\cos c\cdot\sum_{cyc}\sin a\sin b\end{aligned}$$

Hence it suffices to show that

$$\cos a\cos b\cos c\le\frac{1}{8}\quad\text{and}\quad\sum_{cyc}\sin a\sin b\le\frac{9}{4}.\qquad(\dagger)$$

Since $\cos(t)$ is a concave function on $\left(0,\frac{\pi}{2}\right)$, the AM-GM inequality followed by Jensen's inequality yields :

$$\cos a\cos b\cos c\le\left(\frac{\cos a+\cos b+\cos c}{3}\right)^3\le\cos^3\left(\frac{a+b+c}{3}\right)=\frac{1}{8},$$

proving the first of the two inequalities in ($\dagger$).

By Cauchy's inequality, $\sum\limits_{cyc}\sin a\sin b\le\sum\limits_{cyc}\sin^2 a$; so, in order to conclude the second inequality also holds, it suffices to prove that $\sum\limits_{cyc}\sin^2 a\le\frac{9}{4}$. However, one can show that $\sum\limits_{cyc}\sin^2 a=2(1+\cos a\cos b\cos c)$: using the fact that $a+b+c=\pi$, and hence $\sin(a)=\sin(b+c)$, we have

$$\begin{aligned}\sin^2 a&=(\sin b\cos c+\cos b\sin c)^2\\&=\sin^2 b\cos^2 c+2\sin b\cos b\cos c\sin c+\cos^2 b\sin^2 c\\&=(1-\cos^2 b)\cos^2 c+2\sin b\cos b\cos c\sin c+\cos^2 b(1-\cos^2 c)\\&=\cos^2 b+\cos^2 c-2\cos^2 b\cos^2 c+2\cos b\cos c\sin b\sin c\\&=\cos^2 b+\cos^2 c-2\cos b\cos c\cos(b+c)\\&=1-\sin^2 b+1-\sin^2 c+2\cos b\cos c\cos a,\end{aligned}$$

which can be rearranged to $\sin^2 a+\sin^2 b+\sin^2 c=2+2\cos a\cos b\cos c$ as claimed.

Finally, using the first inequality in (†), we can conclude that $\sum_{cyc} \sin^2 a \le 2+\frac{1}{4} = \frac{9}{8}$, showing the second inequality in (†).

Editor's Comments. The inequalities in (†) are known and can be found in O. Bottema et al., *Geometric inequalities*, Groningen, Wolters-Noordhoff, 1969.

4078. *Proposed by Michel Bataille.*

Given θ such that $\frac{\pi}{3} \le \theta \le \frac{5\pi}{3}$, let M_0 be a point of a circle with centre O and radius R and M_k its image under the counterclockwise rotation with centre O and angle $k\theta$. If M is the point diametrically opposite to M_0 and n is a positive integer, show that

$$\sum_{k=0}^{n} MM_k \ge (2n+1)\cdot\frac{R}{2}.$$

We received two submissions, both correct, and feature the solution by AN-anduud Problem Solving Group.

We can assume that $R = 1, M_0 = e^0 = 1$, and $M = -1$; then $M_k = e^{ik\theta}$, $k = 1, 2, \ldots, n$. Let $e^{i\theta} = z$, so that $M_k = z^k$, and

$$MM_k = |-1-e^{ik\theta}| = |1+z^k| \le 1+|z|^k = 2.$$

Thus we have

$$\begin{aligned}
\sum_{k=0}^{n} MM_k &= \sum_{k=0}^{n}|1+z^k| = \frac{1}{2}\sum_{k=0}^{n} 2\cdot|1+z^k| \\
&\ge \frac{1}{2}\sum_{k=0}^{n}|1+z^k|^2 = \frac{1}{2}\sum_{k=0}^{n}(1+z^k)(\overline{1+z^k}) \\
&= \frac{1}{2}\sum_{k=0}^{n}(1+z^k)(1+z^{-k}) = \frac{1}{2}\sum_{k=0}^{n}(2+(z^k+z^{-k})) \\
&= \sum_{k=0}^{n}\left(1+\frac{z^k+z^{-k}}{2}\right) = \sum_{k=0}^{n}(1+\cos k\theta) \\
&= n+\frac{1}{2}+\left(1+\frac{1}{2}+\sum_{k=1}^{n}\cos k\theta\right) \\
&= \frac{2n+1}{2}+\left(1+\frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}\right) \\
&= \frac{2n+1}{2}+\frac{2\sin\frac{\theta}{2}+\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} \\
&\quad \left(\frac{\pi}{6} \le \frac{\theta}{2} \le \frac{5\pi}{6} \Rightarrow \sin\frac{\theta}{2} \ge \frac{1}{2} \Rightarrow 2\sin\frac{\theta}{2} \ge 1\right) \\
&\ge \frac{2n+1}{2}+\frac{1+\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} \ge \frac{2n+1}{2}.
\end{aligned}$$

4079. *Proposed by Mihaela Berindeanu.*

Let $x, y, z > 0$ and $x + y + z = 2016$. Prove that :

$$x\sqrt{\frac{yz}{y+2015z}} + y\sqrt{\frac{xz}{z+2015x}} + z\sqrt{\frac{xy}{x+2015y}} \leq \frac{2016}{\sqrt{3}}.$$

We received eleven solutions. We present 2 solutions.

Solution 1, by Titu Zvonaru.

We prove the general case. Let $x + y + z = t$, where $t \geq 1$. We have

$$\frac{yz}{y+(t-1)z} \leq \frac{(t-1)y+z}{t^2}. \tag{1}$$

Indeed, the inequality (1) is equivalent to

$$(t-1)y^2 + (t-1)z^2 + ((t-1)^2+1)yz \geq t^2yz \iff (t-1)(y-z)^2 \geq 0.$$

Using (1), the Cauchy-Schwarz Inequality and the known inequality

$$(x+y+z)^2 \geq 3(xy+yz+zx),$$

we obtain

$$\begin{aligned}
& x\sqrt{\frac{yz}{y+(t-1)z}} + y\sqrt{\frac{xz}{z+(t-1)x}} + z\sqrt{\frac{xy}{x+(t-1)z}} \\
\leq\; & \frac{x}{t}\sqrt{(t-1)y+z} + \frac{y}{t}\sqrt{(t-1)z+x} + \frac{z}{t}\sqrt{(t-1)x+y} \\
=\; & \frac{1}{t}\left(\sqrt{x}\sqrt{(t-1)xy+zx} + \sqrt{y}\sqrt{(t-1)yz+xy} + \sqrt{z}\sqrt{(t-1)zx+yz}\right) \\
\leq\; & \frac{1}{t}\sqrt{(x+y+z)[(t-1)xy+zx+(t-1)yz+xy+(t-1)zx+yz]} \\
=\; & \frac{1}{t}\sqrt{t^2(xy+yz+zx)} \\
\leq\; & \frac{1}{t}\cdot t\sqrt{\frac{(x+y+z)^2}{3}} = \frac{t}{\sqrt{3}}
\end{aligned}$$

The equality holds if and only if $x = y = z = t/3$.

Solution 2, by Dionne Bailey, Elsie Campbell, and Charles Diminnie.

We will prove the following slight generalization of the given problem :

If $x, y, z > 0$ and $x + y + z = S + 1$, then

$$x\sqrt{\frac{yz}{y+Sz}} + y\sqrt{\frac{zx}{z+Sx}} + z\sqrt{\frac{xy}{x+Sy}} \leq \frac{S+1}{\sqrt{3}}.$$

To begin, we invoke the general form of the Arithmetic - Geometric Mean Inequality which states that if $a, b, \alpha, \beta > 0$ and $\alpha + \beta = 1$, then

$$a^{\alpha} \cdot b^{\beta} \leq \alpha a + \beta b, \tag{2}$$

with equality if and only if $a = b$. It follows from (2) that

$$\begin{aligned}
(y + Sz)(Sy + z) &= (S+1)^2 \left(\frac{1}{S+1} y + \frac{S}{S+1} z \right) \left(\frac{S}{S+1} y + \frac{1}{S+1} z \right) \\
&\geq (S+1)^2 \left(y^{\frac{1}{S+1}} z^{\frac{S}{S+1}} \right) \left(y^{\frac{S}{S+1}} z^{\frac{1}{S+1}} \right) \\
&= (S+1)^2 yz,
\end{aligned}$$

and hence,

$$x \sqrt{\frac{yz}{y + Sz}} \leq \frac{x}{S+1} \sqrt{Sy + z}. \tag{3}$$

Further, equality is attained in (3) if and only if $y = z$.

Similar arguments show that

$$y \sqrt{\frac{zx}{z + Sx}} \leq \frac{y}{S+1} \sqrt{Sz + x}, \tag{4}$$

with equality if and only if $z = x$, and

$$z \sqrt{\frac{xy}{x + Sy}} \leq \frac{z}{S+1} \sqrt{Sx + y}, \tag{5}$$

with equality if and only if $x = y$.

Since $f(t) = \sqrt{t}$ is strictly concave on $(0, \infty)$, we utilize conditions (3), (4), (5), the constraint equation $x + y + z = S + 1$, and Jensen's Inequality to obtain

$$\begin{aligned}
& x \sqrt{\frac{yz}{y + Sz}} + y \sqrt{\frac{zx}{z + Sx}} + z \sqrt{\frac{xy}{x + Sy}} \\
& \leq \frac{x}{S+1} \sqrt{Sy + z} + \frac{y}{S+1} \sqrt{Sz + x} + \frac{z}{S+1} \sqrt{Sx + y} \\
& \leq \sqrt{\frac{x(Sy + z) + y(Sz + x) + z(Sx + y)}{S+1}} \\
& = \sqrt{xy + yz + zx} \\
& \leq \sqrt{\frac{(x + y + z)^2}{3}} \\
& = \frac{S+1}{\sqrt{3}},
\end{aligned}$$

with equality if and only if $x = y = z = \dfrac{S+1}{3}$.

4080. *Proposed by Alina Sîntămărian and Ovidiu Furdui.*

Let $a, b \in \mathbb{R}$, with $ab > 0$. Calculate

$$\int_0^\infty x^2 \mathrm{e}^{-\left(ax-\frac{b}{x}\right)^2} \mathrm{d}x.$$

We received nine submissions of which five were correct and complete solutions. We present the solution by Michel Bataille.

We show that the value of the given integral I is

$$I = \frac{\sqrt{\pi}(1+2ab)}{4|a|^3}.$$

Recall that $\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$. For later use, we also calculate $J = \int_0^\infty t^2 e^{-t^2}\,dt$. For $X > 0$, integrating by parts, we obtain

$$\int_0^X t^2 e^{-t^2}\,dt = \frac{1}{2}\left(\left[-te^{-t^2}\right]_0^X + \int_0^X e^{-t^2}\,dt\right) = \frac{1}{2}\left(-Xe^{-X^2} + \int_0^X e^{-t^2}\,dt\right)$$

and letting $X \to \infty$,

$$J = \frac{1}{2}\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{4}.$$

The equation $ax - \frac{b}{x} = y$ has a unique positive solution for $x = \frac{1}{2a}(y + \varepsilon\sqrt{y^2+4ab})$, where $\varepsilon = 1$ if $a, b > 0$ and $\varepsilon = -1$ if $a, b < 0$. The change of variables

$$x = \frac{1}{2a}(y + \varepsilon\sqrt{y^2+4ab}), \qquad dx = \frac{\varepsilon}{2a} \cdot \frac{y + \varepsilon\sqrt{y^2+4ab}}{\sqrt{y^2+4ab}}\,dy$$

yields

$$I = \frac{1}{8a^3}\int_{-\infty}^\infty \frac{(y+\varepsilon\sqrt{y^2+4ab})^3}{\sqrt{y^2+4ab}}\,e^{-y^2}\,dy.$$

We expand the non-exponential factor in the integrand as

$$\frac{(y+\varepsilon\sqrt{y^2+4ab})^3}{\sqrt{y^2+4ab}} = \frac{y^3}{\sqrt{y^2+4ab}} + 3\varepsilon y^2 + 3y\sqrt{y^2+4ab} + \varepsilon(y^2+4ab).$$

Note the behaviour of the first and third terms on the right-hand side :

$$|y|\sqrt{y^2+4ab}e^{-y^2} \sim \frac{|y|^3}{\sqrt{y^2+4ab}}\,e^{-y^2} \sim y^2 e^{-y^2} \quad \text{as} \quad y \to \infty$$

It follows that the integrals

$$\int_{-\infty}^\infty \frac{y^3}{\sqrt{y^2+4ab}}\,e^{-y^2}\,dy \quad \text{and} \quad \int_{-\infty}^\infty 3y\sqrt{y^2+4ab}\,e^{-y^2}\,dy$$

exist and are therefore zero, since each integrand is odd. Thus,

$$I = \frac{1}{8a^3}\left(6\varepsilon J + 2\varepsilon J + 8\varepsilon ab\int_0^\infty e^{-y^2}\,dy\right) = \frac{\sqrt{\pi}(1+2ab)}{4|a|^3}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(9), p. 397–400.

4081. *Proposed by Daniel Sitaru.*

Determine all $A, B \in M_2(\mathbb{R})$ such that:

$$\begin{cases} A^2 + B^2 = \begin{pmatrix} 22 & 44 \\ 14 & 28 \end{pmatrix}, \\ AB + BA = \begin{pmatrix} 10 & 20 \\ 2 & 4 \end{pmatrix}. \end{cases}$$

We received 17 correct solutions and will feature the solution by Joseph DiMuro.

Summing the two equations, we obtain:

$$(A+B)^2 = A^2 + AB + BA + B^2 = \begin{pmatrix} 32 & 64 \\ 16 & 32 \end{pmatrix}.$$

We can diagonalize this matrix in order to find its square roots:

$$(A+B)^2 = PDP^{-1} = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 64 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/4 & 1/2 \\ 1/4 & -1/2 \end{pmatrix},$$

$$A + B = PD^{1/2}P^{-1} = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \pm 8 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/4 & 1/2 \\ 1/4 & -1/2 \end{pmatrix} = \pm \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix}.$$

We can also subtract the original two equations to obtain:

$$(A-B)^2 = A^2 - AB - BA + B^2 = \begin{pmatrix} 12 & 24 \\ 12 & 24 \end{pmatrix}.$$

As before, we diagonalize this matrix:

$$(A-B)^2 = PDP^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 36 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix},$$

$$A - B = PD^{1/2}P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \pm 6 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix} = \pm \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}.$$

Now we have the two equations

$$A+B=\pm\begin{pmatrix}4 & 8\\ 2 & 4\end{pmatrix},\, A-B=\pm\begin{pmatrix}2 & 4\\ 2 & 4\end{pmatrix},$$

which can easily be solved to produce four possible pairs of matrices for A and B. One solution is

$$A=\begin{pmatrix}3 & 6\\ 2 & 4\end{pmatrix},\, B=\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}.$$

The other solutions may be obtained by interchanging A and B, and/or replacing A and B with their negatives.

4082. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let ABC be a right-angle triangle with $\angle A = 90^\circ$ and $BC = a$, $AC = b$ and $AB = c$. Consider the Fibonacci sequence F_n with $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all non-negative integers n. Prove that

$$\frac{F_m^2}{(bF_n+cF_p)^2}+\frac{F_n^2}{(bF_p+cF_m)^2}+\frac{F_p^2}{(bF_m+cF_n)^2}\geq\frac{3}{2a^2}$$

for all non-negative integers m, n, p.

We received 8 correct solutions and present the solution by Adnan Ali.

From the Cauchy-Schwarz Inequality,
$(b^2+c^2)(F_k^2+F_\ell^2) = a^2(F_k^2+F_\ell^2) \geq (bF_k+cF_\ell)^2$, for all $k,\ell \geq 0$. Thus,

$$\frac{F_m^2}{(bF_n+cF_p)^2}+\frac{F_n^2}{(bF_p+cF_m)^2}+\frac{F_p^2}{(bF_m+cF_n)^2}\geq$$
$$\frac{F_m^2}{a^2(F_n^2+F_p^2)}+\frac{F_n^2}{a^2(F_p^2+F_m^2)}+\frac{F_p^2}{a^2(F_m+F_n^2)}\geq\frac{3}{2a^2},$$

where the last inequality follows from Nesbitt's Inequality. Equality holds iff $F_m = F_n = F_p$ and $b = c$.

Editor's Comments. As solvers pointed out, the fact that the F_n's were Fibonacci numbers was irrelevant; it was only necessary that they were nonnegative.

4083. *Proposed by Ovidiu Furdui.*

Calculate

$$\lim_{n\to\infty}\frac{1}{n\sqrt{n}}\int_0^n\frac{x}{1+n\cos^2 x}\mathrm{d}x.$$

We received 10 solutions, of which 6 were correct and complete. We present the solution by Michel Bataille.

We show that the required limit is $\dfrac{1}{2}$.

Let $f_n(x) = \frac{1}{1+n\cos^2 x}$.

The π-periodicity of f_n and the change of variables $x = \tan^{-1}(t),\ dx = \frac{dt}{1+t^2}$ easily yield

$$\int_{(2k-1)\pi/2}^{(2k+1)\pi/2} f_n(x)\,dx = \int_{-\pi/2}^{\pi/2} f_n(x)\,dx = \int_{-\infty}^{\infty} \frac{dt}{n+1+t^2} = \frac{\pi}{\sqrt{n+1}}$$

for any $k, n \in \mathbb{N}$.

This said, for every $n \in \mathbb{N}$ with $n \geq 2$, let $p_n = \left\lfloor \frac{n}{\pi} + \frac{1}{2} \right\rfloor$ and $I_n = \int_0^n x f_n(x)\,dx$. Then, $(2p_n - 1)\frac{\pi}{2} \leq n < (2p_n + 1)\frac{\pi}{2}$ and

$$I_n = \int_0^{\pi/2} x f_n(x)\,dx + \sum_{k=1}^{p_n - 1} \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} x f_n(x)\,dx + \int_{(2p_n-1)\frac{\pi}{2}}^{n} x f_n(x)\,dx.$$

Clearly,

$$0 \leq \int_0^{\pi/2} x f_n(x)\,dx \leq \frac{\pi}{2} \cdot \frac{1}{2} \int_{-\pi/2}^{\pi/2} f_n(x)\,dx = \frac{\pi^2}{4\sqrt{n+1}}$$

and for $k \in \{1, 2, \ldots, p_n - 1\}$,

$$(2k-1)\frac{\pi}{2} \cdot \frac{\pi}{\sqrt{n+1}} \leq \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} x f_n(x)\,dx \leq (2k+1)\frac{\pi}{2} \cdot \frac{\pi}{\sqrt{n+1}}.$$

Similarly,

$$0 \leq \int_{(2p_n-1)\frac{\pi}{2}}^{n} x f_n(x)\,dx \leq n \int_{(2p_n-1)\frac{\pi}{2}}^{n} f_n(x)\,dx \leq \frac{n\pi}{\sqrt{n+1}}.$$

Thus,

$$\frac{\pi^2}{2\sqrt{n+1}} \sum_{k=1}^{p_n-1} (2k-1) \leq I_n \leq \frac{\pi^2}{4\sqrt{n+1}} + \frac{\pi^2}{2\sqrt{n+1}} \sum_{k=1}^{p_n-1} (2k+1) + \frac{n\pi}{\sqrt{n+1}}$$

so that

$$\frac{\pi^2 (p_n - 1)^2}{2\sqrt{n+1}} \leq I_n \leq \frac{\pi}{\sqrt{n+1}} \left(\frac{\pi}{4} + \frac{\pi}{2} \cdot p_n^2 + n \right) = \frac{\pi p_n^2}{\sqrt{n+1}} \left(\frac{\pi}{2} + \frac{\pi}{4p_n^2} + \frac{n}{p_n^2} \right).$$

Since $p_n \sim \frac{n}{\pi}$ as $n \to \infty$, we obtain

$$I_n \sim \frac{\pi^2 p_n^2}{2\sqrt{n+1}} \sim \frac{n\sqrt{n}}{2}$$

as $n \to \infty$. The result follows.

4084. *Proposed by Michel Bataille.*

In the plane, let Γ be a circle and A, B be two points not on Γ. Determine when $\frac{MA}{MB}$ is not independent of M on Γ and, in these cases, construct with ruler and compass I and S on Γ such that

$$\frac{IA}{IB} = \inf\left\{\frac{MA}{MB} : M \in \Gamma\right\} \quad \text{and} \quad \frac{SA}{SB} = \sup\left\{\frac{MA}{MB} : M \in \Gamma\right\}.$$

We feature the proposer's solution; we received no others.

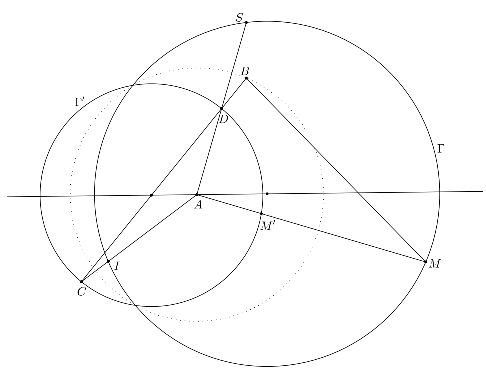
Because A is not on Γ, inversion in the circle with centre A and radius AB takes Γ to another circle, call it Γ'. For any point M on Γ, this inversion takes the pair of points M, B to another pair M', B, whose distances satisfy

$$MB = \frac{AB^2 \cdot M'B}{AM' \cdot AB} = \frac{AM \cdot AM' \cdot M'B}{AM' \cdot AB} = MA\frac{M'B}{AB};$$

consequently,

$$\frac{MA}{MB} = \frac{AB}{BM'}. \tag{1}$$

From (1), $\frac{MA}{MB}$ is independent of M on Γ if and only if BM' is constant. This occurs if and only if B is the centre of Γ'; that is, if and only if A and B are an inverse pair with respect to Γ. Otherwise, let the diameter of Γ' through B intersect Γ' at C and D with $BC > BD$. Then $\frac{MA}{MB}$ is minimal when BM' is maximal; that is, when $M' = C$; $\frac{MA}{MB}$ is maximal when BM' is minimal, in which case $M' = D$. Thus, I coincides with C' (the image of C under our inversion), and S coincides with D'. The construction of I and S is immediate once Γ' has been drawn. The circle Γ' can be readily constructed from the inverses of three points of Γ (as in the figure):



Editor's Comments. For any two fixed points A and B, the locus of points M for which $\frac{MA}{MB}$ is constant is called *the circle of Apollonius*; inversion in that circle interchanges A and B. See, for example H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited* (Mathematical Association of America, 1967), exercise 5.4.1, pages 114 and 172. Also, Theorem 5.41 there provides the distance formula used above to obtain (1).

4085. *Proposed by José Luis Díaz-Barrero. Correction.*

Let ABC be an acute triangle. Prove that

$$\sqrt[4]{\sin(\cos A)\cdot\cos B}+\sqrt[4]{\sin(\cos B)\cdot\cos C}+\sqrt[4]{\sin(\cos C)\cdot\cos A}<\frac{3\sqrt{2}}{2}.$$

We received eight submissions, six of which are correct. We present the solution by Titu Zvonaru.

It is well known that $\cos A+\cos B+\cos C\le\frac{3}{2}$ [Item 2.16 on p.22 of the book *Geometric Inequalities* by O. Bottema et al; Groningen, 1969]. Using this, together with the facts that $\sin x<x$ for $0<x<\frac{\pi}{2}$, $xy+yz+zx\le x^2+y^2+z^2$, and $(x+y+z)^2\le 3(x^2+y^2+z^2)$ we then have

$$\begin{aligned}\sum_{\text{cyc}}\sqrt[4]{\sin(\cos A)\cdot\cos B}&<\sum_{\text{cyc}}\sqrt[4]{\cos A\cdot\cos B}\le\sum_{\text{cyc}}\sqrt{\cos A}\\&\le\sqrt{3(\cos A+\cos B+\cos C)}\le\sqrt{3(\frac{3}{2})}=\frac{3\sqrt{3}}{2}.\end{aligned}$$

Editor's comments. Arkady Alt proved the stronger result that the given upper bound could be replaced by $3\sqrt[4]{\frac{1}{2}\sin\frac{1}{2}}$ which is less than $\frac{3\sqrt{3}}{2}$ since $\sin\frac{1}{2}<\frac{1}{2}$. This new upper bound is attained if and only if the triangle is equilateral. His proof used the Cauchy-Schwarz Inequality, concavity of the functions $\sqrt{\sin x}$ and $\sqrt{\cos x}$, Jensen's Inequality as well as the fact that $\sum\cos A=1+\frac{r}{R}$ and the Euler's Inequality $2r\le R$.

4086. *Proposed by Daniel Sitaru.*

Let be $f:[0,1]\to\mathbb{R}$; f twice differentiable on $[0,1]$ and $f''(x)<0$ for all $x\in[0,1]$. Prove that

$$25\int_{\frac{1}{5}}^{1}f(x)dx\ge 16\int_0^1 f(x)dx+4f(1).$$

We received seven solutions and present two of them.

Solution 1, by AN-anduud Problem Solving Group.

From the given conditions, f is concave on $[0,1]$. Using Hermite-Hadamard's inequality we get

$$\begin{aligned}16\int_{\frac{1}{5}}^{1}f(x)dx+9\int_{\frac{1}{5}}^{1}f(x)dx&\ge 16\cdot\int_{\frac{1}{5}}^{1}f(x)dx+9\cdot\frac{1-\frac{1}{5}}{2}\cdot\left(f(1)+f\left(\frac{1}{5}\right)\right)\\&=16\int_{\frac{1}{5}}^{1}f(x)dx+\frac{18}{5}f(1)+\frac{18}{5}f\left(\frac{1}{5}\right).\end{aligned}$$

On the other hand, we have

$$f\left(\frac{1}{5}\right)=f\left(\frac{1}{9}\cdot 1+\frac{8}{9}\cdot\frac{1}{10}\right)\ge\frac{1}{9}f(1)+\frac{8}{9}\cdot f\left(\frac{1}{10}\right),$$

so
$$\frac{18}{5} f\left(\frac{1}{5}\right) \geq \frac{2}{5} f(1)+\frac{16}{5} f\left(\frac{1}{10}\right).$$

From here, we get
$$25 \int_{\frac{1}{5}}^{1} f(x) dx \geq 16 \int_{\frac{1}{5}}^{1} f(x) dx+4 f(1)+\frac{16}{5} f\left(\frac{1}{10}\right).$$

Using Hermite-Hadamard's inequality, we get
$$f\left(\frac{1}{10}\right)=f\left(\frac{\frac{1}{5}+0}{2}\right) \geq \frac{1}{\frac{1}{5}-0} \int_{0}^{\frac{1}{5}} f(x) dx \quad \Longleftrightarrow \quad \frac{1}{5} f\left(\frac{1}{10}\right) \geq \int_{0}^{\frac{1}{5}} f(x) dx.$$

Hence, we get
$$25 \int_{\frac{1}{5}}^{1} f(x) dx \geq 16 \int_{0}^{1} f(x) dx+4 f(1).$$

Solution 2, by Leonard Giugiuc.

In $\displaystyle\int_{\frac{1}{5}}^{1} f(x)\ dx$, we make the substitution $x \rightarrow \dfrac{5x-1}{4}$ and clear fractions to get
$$25 \int_{\frac{1}{5}}^{1} f(x)\ dx=20 \int_{0}^{1} f\left(\frac{4x+1}{5}\right)\ dx.$$

We need to prove
$$\begin{aligned}
20 \int_{0}^{1} f\left(\frac{4x+1}{5}\right)\ dx \geq 16 \int_{0}^{1} f(x)\ dx+4 f(1) &\Longleftrightarrow \\
\int_{0}^{1} f\left(\frac{4x+1}{5}\right)\ dx \geq \frac{4}{5} \int_{0}^{1} f(x)\ dx+\frac{1}{5} f(1) &\Longleftrightarrow \\
\int_{0}^{1} f\left(\frac{4x+1}{5}\right)\ dx \geq \int_{0}^{1}\left[\frac{4}{5} f(x)+\frac{1}{5} f(1)\right]\ dx.&
\end{aligned}$$

But $f''(x)<0\ \forall x \in[0,1]$, so f is concave on $[0,1]$ and from here
$$f\left(\frac{4x+1}{5}\right) \geq \frac{4}{5} f(x)+\frac{1}{5} f(1).$$

Integrating, we conclude that
$$\int_{0}^{1} f\left(\frac{4x+1}{5}\right)\ dx \geq \int_{0}^{1}\left[\frac{4}{5} f(x)+\frac{1}{5} f(1)\right]\ dx.$$

Editor's Comments. Henry Ricardo observed that this problem appears as problem MA 110 (with solution) in the Daniel Sitaru's book *Math Phenomenon*, published in English by the Romanian publisher Editura Paralela 45 in 2016.

4087. *Proposed by Lorian Saceanu.*

If S is the area of triangle ABC, prove that

$$m_a(b+c)+2m_a^2 \geq 4S\sin A,$$

where b and c are the lengths of sides that meet in vertex A, and m_a is the length of the median from that vertex; furthermore, equality holds if and only if $b=c$ and $\angle A = 120^\circ$.

We received seven correct submissions and present the solution by Leonard Giugiuc.

Let A' be the reflection of A in the midpoint M of BC. Because $\Delta AMC \cong \Delta A'MB$, we have

$$AA' = 2m_a,\ A'B = b,\ \angle A'BA = \pi - A, \text{ and } [A'AB] = [ABC] = S$$

(where the square brackets denote area). Let $m = 2m_a$ and denote by r', R', and s' $(= \frac{m+b+c}{2})$ the inradius, circumradius, and semiperimeter, respectively, of $\Delta A'AB$. We need to prove that

$$m(b+c)+m^2 \geq 8S\sin\angle A'BA,$$

which is equivalent, in turn, to

$$\begin{aligned} m(m+b+c) &\geq 8S\sin(\pi - A) \\ \frac{2mS}{r'} &\geq 8S\sin A \\ \frac{m}{\sin A} &\geq 4r' \\ R' &\geq 2r'. \end{aligned}$$

But the final line is Euler's inequality applied to $\Delta A'AB$, which completes the proof. Equality holds for Euler's inequality if and only if $\Delta A'AB$ is equilateral, which implies that $b=c$ and $\angle A = 120^\circ$, as desired.

4088. *Proposed by Ardak Mirzakhmedov.*

Let a, b and c be positive real numbers such that $a^2b+b^2c+c^2a+a^2b^2c^2 = 4$. Prove that

$$a^2+b^2+c^2+abc(a+b+c) \geq 2(ab+bc+ca).$$

We received four submissions all of which are correct. We present the solution by the proposer, expanded by the editor with some details.

We first show that the given condition implies

$$\frac{a}{2a+bc^2}+\frac{b}{2b+ca^2}+\frac{c}{2c+ab^2} = 1 \tag{1}$$

or

$$\begin{aligned} a(2b+ca^2)(2c+ab^2)+b(2c+ab^2)(2a+bc^2)+c(2a+bc^2)(2b+ca^2)& \\ = (2a+bc^2)(2b+ca^2)(2c+ab^2).& \end{aligned} \tag{2}$$

Let S and P denote the left side and the right side of (2), respectively. Then by straightforward computations, we find

$$\begin{aligned} S &= \sum_{\text{cyc}} a(4bc + 2ab^3 + 2c^2a^2 + a^3b^2c) \\ &= 12abc + 4(a^2b^3 + b^2c^3 + c^2a^3) + abc(a^3b + b^3c + c^3a) \\ &= 12abc + 4(a^2b^3 + b^2c^3 + c^2a^3) + abc(4 - a^2b^2c^2) \\ &= 16abc + 4(a^2b^3 + b^2c^3 + c^2a^3) - a^3b^3c^3 \end{aligned}$$

and

$$\begin{aligned} P &= (4ab + 2ca^3 + 2b^2c^2 + a^2bc^3)(2c + ab^2) \\ &= 8abc + 4(c^2a^3 + a^2b^3 + b^2c^3) + 2abc(a^3b + b^3c + c^3a) + a^3b^3c^3 \\ &= 8abc + 4(a^2b^3 + b^2c^3 + c^2a^3) + 2abc(4 - a^2b^2c^2) + a^3b^3c^3 \\ &= 16abc + 4(a^2b^3 + b^2c^3 + c^2a^3) - a^3b^3c^3. \end{aligned}$$

Hence, $S = P$ which establishes (2).

Now, for all $u, v, w > 0$, we have by the Cauchy-Schwarz Inequality that

$$(u + v + w)\left(\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w}\right) \geq (a + b + c)^2. \tag{3}$$

Setting

$$u = 2a^2 + bc^2a, \quad v = 2b^2 + ca^2b \quad \text{and} \quad w = 2c^2 + ab^2c,$$

we then have by (1) and (3) that

$$1 = \frac{a^2}{2a^2 + bc^2a} + \frac{b^2}{2b^2 + ca^2b} + \frac{c^2}{2c^2 + ab^2c} \geq \frac{(a + b + c)^2}{u + v + w},$$

so

$$(2a^2 + bc^2a) + (2b^2 + ca^2b) + (2c^2 + ab^2c) = u + v + w \geq (a + b + c)^2$$

from which it follows that

$$a^2 + b^2 + c^2 + abc(a + b + c) \geq 2(ab + bc + ca).$$

4089. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let a, b, c and d be real numbers with $0 < a < b < c < d$. Prove that

$$\frac{b}{a} + \frac{c}{b} + \frac{d}{c} > 3 + \ln\frac{d}{a}.$$

There were 14 correct solutions. We present four of them here. Most of the solvers approached the problem along the lines of one of the first two solutions.

Solution 1.

Since $x > 1 + \ln x$ for $x \neq 1$,

$$\frac{b}{a} + \frac{c}{b} + \frac{d}{c} > \left(1 + \ln\frac{b}{a}\right) + \left(1 + \ln\frac{c}{b}\right) + \left(1 + \ln\frac{d}{c}\right) = 3 + \ln\frac{d}{a}$$

as desired.

Solution 2.

Applying the AM-GM Inequality, we find that the left side is not less than

$$3\sqrt[3]{\frac{d}{a}} > 3\left(1 + \ln\sqrt[3]{\frac{d}{a}}\right) = 3 + \ln\frac{d}{a}.$$

Solution 3, by Kee-Wai Lau.

For $0 < a < b < c < d$, let

$$\begin{aligned} f(a,b,c,d) &= \frac{b}{a} + \frac{c}{b} + \frac{d}{c} - \ln\frac{d}{a}, \\ g(a,b,c) &= \frac{b}{a} + \frac{c}{b} + 1 - \ln\frac{c}{a}, \\ h(a,b) &= \frac{b}{a} + 2 - \ln\frac{b}{a}. \end{aligned}$$

An analysis of the partial derivatives reveals that each of its functions strictly increases in its final variable, so that

$$f(a,b,c,d) > f(a,b,c,c) = g(a,b,c) > g(a,b,b) = h(a,b) > h(a,a) = 3,$$

which yields the desired result.

Solution 4, by the proposers.

Let $f(x) = 1/x$. A diagram shows that

$$(b-a)f(a) + (c-b)f(b) + (d-c)f(c) > \int_a^d \frac{dx}{x},$$

whence

$$\frac{b}{a} - 1 + \frac{c}{b} - 1 + \frac{d}{c} - 1 > \ln d - \ln a,$$

as desired.

Editor's Comments. Two solvers provided a straightforward generalization for an increasing sequence $\{a_k\}$ of $n+1$ positive reals:

$$\sum_{k=1}^{n} \frac{a_{k+1}}{a_k} > n + \ln\frac{a_{n+1}}{a_1}.$$

4090. *Proposed by Nermin Hodžić and Salem Malikić.*

Let a, b and c be non-negative real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{a}{3b^2+6c-bc}+\frac{b}{3c^2+6a-ca}+\frac{c}{3a^2+6b-ab}\geq\frac{3}{8}.$$

We received two correct solutions. We present the solution of the proposers, slightly modified by the editor.

Using Jensen's inequality for $f(x)=\frac{1}{x}$ (which is convex on $(0,\infty)$), we have

$$\frac{a}{a+b+c}\cdot\frac{1}{3b^2+6c-bc}+\frac{b}{a+b+c}\cdot\frac{1}{3c^2+6a-ca}+\frac{c}{a+b+c}\cdot\frac{1}{3a^2+6b-ab}$$
$$\geq\left(\frac{a(3b^2+6c-bc)}{a+b+c}+\frac{b(3c^2+6a-ca)}{a+b+c}+\frac{c(3a^2+6b-ab)}{a+b+c}\right)^{-1},$$

which we can rearrange to

$$\frac{a}{3b^2+6c-bc}+\frac{b}{3c^2+6a-ca}+\frac{c}{3a^2+6b-ab}\geq\frac{(a+b+c)^2}{3(ab^2+bc^2+ca^2)+6(ab+bc+ca)-3abc}.$$

In order to prove the inequality given in the question, it thus suffices to show

$$\frac{(a+b+c)^2}{3(ab^2+bc^2+ca^2)+6(ab+bc+ca)-3abc}\geq\frac{3}{8},$$

which holds (by cross multiplying and rearranging) if and only if

$$8(a+b+c)^2\geq 9(ab^2+bc^2+ca^2)+18(ab+bc+ca)-9abc\iff$$
$$8(a^2+b^2+c^2)\geq 9(ab^2+bc^2+ca^2)+2(ab+bc+ca)-9abc.$$

By the Cauchy-Schwarz inequality, $ab+bc+ca\leq a^2+b^2+c^2$. Note for later that equality holds if and only if $a=b=c=1$. Hence, it suffices to show that

$$6(a^2+b^2+c^2)\geq 9(ab^2+bc^2+ca^2)-9abc.$$

Finally, since $a^2+b^2+c^2=3$, this reduces to

$$2\geq(ab^2+bc^2+ca^2)-abc. \tag{1}$$

Assume that $a\geq b\geq c$. Then $(a-b)(b-c)\geq 0$, equivalent to $ab+bc\geq b^2+ac$. Multiply both sides by $a>0$ and rearrange to get $abc\geq ab^2+a^2c-a^2b$. Note that the cubic $g(b)=3b-b^3$ has a local maximum at $b=1$, and in fact for all $b\geq 0$ we have $3b-b^3\leq g(1)=2$. Hence

$$abc+2\geq ab^2+a^2c-a^2b+3b-b^3,$$

which is equivalent to

$$abc+2\geq ab^2+a^2c+c^2b-b(a^2+b^2+c^2)+3b.$$

Since $a^2+b^2+c^2=3$, this shows that $abc+2\geq ab^2+a^2c+c^2b$, which is equivalent to (1), concluding the proof.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(10), p. 441–445.

4091. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Find the greatest positive number k such that

$$a+b+c+3k-3 \geq k\left(\sqrt[3]{\frac{b}{a}}+\sqrt[3]{\frac{c}{b}}+\sqrt[3]{\frac{a}{c}}\right)$$

for any positive numbers a, b and c with $abc = 1$.

We received three submissions, two of which were correct. We present the solution of the proposers, modified by the editor.

Using $abc = 1$ we can rewrite the inequality as

$$a+b+c+3(k-1)-k(\sqrt[3]{a^2b}+\sqrt[3]{b^2c}+\sqrt[3]{c^2a}) \geq 0.$$

By substituting $a = x^3$, $b = y^3$, and $c = z^3$ (and thus $xyz = 1$) and defining the homogeneous cyclic polynomial $f : [0,\infty)^3 \to \mathbb{R}$ by

$$f(x,y,z) = x^3+y^3+z^3+3(k-1)xyz-k(x^2y+y^2z+z^2x),$$

we thus need to find the greatest positive number k such that

$$f(x,y,z) \geq 0$$

for all positive $x, y, z \geq 0$ with $xyz = 1$. We will proceed by first finding the greatest k such that $f(x,y,z) \geq 0$ for all $x,y,z \geq 0$ and then showing that this k is optimal for the case of $xyz = 1$ as well. To accomplish the first part we will use a lemma proven by Pham Kim Hung in his book *Secrets in Inequalities (Volume 2)*. As the lemma might be of interest to the reader but is not a common reference we will give a proof here.

Lemma. *Let $P(x,y,z)$ be a cyclic homogeneous polynomial of degree* 3*. Then $P(x,y,z) \geq 0$ for all $x,y,z \geq 0$ if and only if $P(1,1,1) \geq 0$ and $P(u,1,0) \geq 0$ for all $u \geq 0$.*

Proof of the lemma: Clearly the given conditions are necessary. So assume that $P(1,1,1)$ and $P(u,1,0)$ are nonnegative for all $u \geq 0$. We can write

$$P(x,y,z) = m\sum_{cyc} x^3 + n\sum_{cyc} x^2y + p\sum_{cyc} xy^2 + qxyz$$

for some $m, n, p, q \in \mathbb{R}$. Then $P(1,1,1)$ gives us

$$3m + 3n + 3p + q \geq 0,$$

while setting $u = 0$ and $u = 1$ respectively yields $m \geq 0$ and $2m + n + p \geq 0$. The derivative of $P(x, y, z)$ in the direction $(1, 1, 1)$ is

$$\begin{aligned} &3m \sum_{cyc} x^2 + n\Big(\sum_{cyc} x^2 + 2\sum_{cyc} xy\Big) + p\Big(\sum_{cyc} x^2 + 2\sum_{cyc} xy\Big) + q\sum_{cyc} xy \\ &\quad = (3m + n + p)\sum_{cyc} x^2 + (2n + 2p + q)\sum_{cyc} xy \end{aligned}$$

Note that

$$3m + n + p = m + (2m + n + p) \geq 0$$

and

$$(3m + n + p) + (2n + 2p + q) = 3m + 3n + 3p + q \geq 0,$$

thus

$$(3m + n + p)\sum_{cyc} x^2 + (2n + 2p + q)\sum_{cyc} xy \geq (3m + n + p)\Big(\sum_{cyc} x^2 - \sum_{cyc} xy\Big) \geq 0$$

by the rearrangement inequality. Since the derivative in the direction $(1, 1, 1)$ is nonnegative for all $x, y, z \geq 0$ we only need to show that $P(x, y, z) \geq 0$ on the boundary, that is where at least one variable is equal to 0. This now follows from the cyclicity of $P(x, y, z)$ and $P(a, b, 0) = b^3 P(\frac{a}{b}, 1, 0)$ for $b \neq 0$ which finishes the proof of the lemma.

We can now apply the lemma to show that $f(x, y, z) \geq 0$ for all $x, y, z \geq 0$ if and only if $f(1, 1, 1) \geq 0$ (which clearly holds), and $f(u, 1, 0) \geq 0$. Defining

$$g(u) = f(u, 1, 0) = u^3 - ku^2 + 1,$$

we obtain $g'(u) = u(3u - 2k)$. Thus the minimum of $g(u)$ occurs at $u = \frac{2k}{3}$ and $g(u)$ is positive for all $u \geq 0$ if and only if $g(\frac{2k}{3}) \geq 0$, which occurs for $k \leq \frac{3}{\sqrt[3]{4}}$.

It remains to be shown that this is best possible for the case of $xyz = 1$ as well. Suppose $k > \frac{3}{\sqrt[3]{4}}$ and set $s = \frac{2k}{3}$. Consider $h(w) = f(s, 1, w)$. Since $h(w)$ is continuous and $h(0) = g(s) < 0$ there exists a positive t such that $h(t) < 0$. Set

$$x = \frac{s}{\sqrt[3]{st}}, y = \frac{1}{\sqrt[3]{st}} \quad \text{and} \quad z = \frac{t}{\sqrt[3]{st}}.$$

Then $x, y, z \geq 0$ with $xyz = 1$ and

$$f(x, y, z) = stf(s, 1, t) = sth(t) < 0.$$

It follows that the answer to the question is $k = \frac{3}{\sqrt[3]{4}}$.

4092. *Proposed by Mihaela Berindeanu.*

Show that

$$\left[\frac{a^2+16a+80}{16\,(a+4)}+\frac{2}{\sqrt{2\,(b^2+16)}}\right]\left[\frac{b^2+16b+80}{16\,(b+4)}+\frac{2}{\sqrt{2\,(a^2+16)}}\right]\geq\frac{9}{4}$$

for all $a, b > 0$. When does equality hold?

We received ten correct submissions. We present two solutions.

Solution 1, by Arkady Alt.

Since

$$\frac{a^2+16a+80}{16\,(a+4)}=1+\frac{a^2+4^2}{16\,(a+4)},$$

we have

$$\left(\frac{a^2+16a+80}{16\,(a+4)}+\frac{2}{\sqrt{2\,(b^2+16)}}\right)\left(\frac{b^2+16b+80}{16(b+4)}+\frac{2}{\sqrt{2\,(a^2+16)}}\right)$$
$$=\left(1+\frac{a^2+4^2}{16\,(a+4)}+\frac{2}{\sqrt{2\,(b^2+4^2)}}\right)\left(1+\frac{b^2+4^2}{16\,(b+4)}+\frac{2}{\sqrt{2\,(a^2+4^2)}}\right)$$

and, combining Cauchy-Schwarz Inequality and inequality $\sqrt{2\,(u^2+v^2)}\geq u+v$, we obtain

$$\left(1+\frac{a^2+4^2}{16\,(a+4)}+\frac{2}{\sqrt{2\,(b^2+4^2)}}\right)\left(1+\frac{2}{\sqrt{2\,(a^2+4^2)}}+\frac{b^2+4^2}{16\,(b+4)}\right)$$
$$\geq\left(1\cdot 1+\sqrt{\frac{a^2+4^2}{16\,(a+4)}}\cdot\sqrt{\frac{2}{\sqrt{2\,(a^2+4^2)}}}+\sqrt{\frac{2}{\sqrt{2\,(b^2+4^2)}}}\cdot\sqrt{\frac{b^2+4^2}{16\,(b+4)}}\right)^2$$
$$=\left(1+\frac{1}{4}\sqrt{\frac{\sqrt{2\,(a^2+4^2)}}{a+4}}+\frac{1}{4}\sqrt{\frac{\sqrt{2\,(b^2+4^2)}}{b+4}}\right)^2$$
$$\geq\left(1+\frac{1}{4}+\frac{1}{4}\right)^2=\frac{9}{4}.$$

Since in inequality $\sqrt{2\,(u^2+v^2)}\geq u+v$ equality occurs if and only if $u=v$, it is easy to see that the equality holds if and only if $a=b=4$.

Solution 2, by AN-anduud Problem Solving Group.

Using AM-GM inequality, we get

$$\sqrt{2(b^2+16)}=\sqrt{8\cdot\frac{b^2+16}{4}}\leq\frac{1}{2}\left(8+\frac{b^2+16}{4}\right)=\frac{b^2+48}{8}. \qquad (1)$$

Applying AM-GM inequality and using (1), we have

$$\begin{aligned}\frac{a^2+16a+80}{16(a+4)}+\frac{2}{\sqrt{2(b^2+16)}} &\ge \frac{a^2+16a+80}{16(a+4)}+\frac{16}{b^2+48}\\ &= \frac{(a^2+48)+(a+4)^2+24(a+4)}{32(a+4)}+\frac{16}{b^2+48}\\ &= \frac{a^2+48}{32(a+4)}+\frac{a+4}{32}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{16}{b^2+48}\\ &\ge 6\sqrt[6]{\frac{a^2+48}{32(a+4)}\cdot\frac{a+4}{32}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{16}{b^2+48}}\\ &= \frac{3}{2}\cdot\sqrt[6]{\frac{a^2+48}{b^2+48}}. \qquad (2)\end{aligned}$$

Similarly,
$$\frac{b^2+16b+80}{16(b+4)}+\frac{2}{\sqrt{2(a^2+16)}} \ge \frac{3}{2}\cdot\sqrt[6]{\frac{b^2+48}{a^2+48}}. \qquad (3)$$

Multiplying (2) and (3), we obtain the desired inequality. Equality holds only when $a=b=4$.

4093. *Proposed by Dragolijub Milošević.*

Let ABC be an arbitrary triangle. Let r and R be the inradius and the circumradius of ABC, respectively. Let m_a be the length of the median from vertex A to side BC and let w_a be the length of the internal bisector of $\angle A$ to side BC. Define m_b, m_c, w_b and w_c similarly. Prove that

$$\frac{a^2}{m_a w_a}+\frac{b^2}{m_b w_b}+\frac{c^2}{m_c w_c} \le 4\left(\frac{R}{r}-1\right).$$

We received five correct solutions and present the solution by Andrea Fanchini.

We have that
$$4\left(\frac{R}{r}-1\right)=\frac{abcs-4K^2}{K^2}$$
where s is the semiperimeter and K is the area of the triangle. We know (see Wei-Dong Jiang and Mihály Bencze, *JMI Volume 5 Number 3*, 2011, p. 365) that $m_a w_a \ge s(s-a)$, so we have to prove

$$\frac{a^2}{s(s-a)}+\frac{b^2}{s(s-b)}+\frac{c^2}{s(s-c)} \le \frac{abcs-4K^2}{K^2},$$

that is
$$a^2(s-b)(s-c)+b^2(s-a)(s-c)+c^2(s-a)(s-b) \le s\left(abc-4(s-a)(s-b)(s-c)\right).$$

In fact, the inequality can be replaced by equality since both sides equal:

$$\begin{aligned}&s\left(abc-4(s-a)(s-b)(s-c)\right)\\ &=\frac{1}{4}\left(a^4+b^4+c^4+2a^2bc+2ab^2c+2abc^2-2a^2b^2-2b^2c^2-2c^2a^2\right).\end{aligned}$$

4094. *Proposed by Michel Bataille.*

Let $x_1, x_2, \ldots, x_n$ be real numbers such that $0 \le x_1 \le x_2 \le \cdots \le x_n$. Prove that

$$n-1+\cosh\left(\sum_{k=1}^{n}(-1)^{k-1}x_k\right) \le \sum_{k=1}^{n}\cosh x_k \le n-1+\cosh\left(\sum_{k=1}^{n}x_k\right).$$

We received four correct and complete solutions. We present the solution by the proposer.

Let f be the function defined on $\mathbb{R}$ by $f(x) = (\cosh x) - 1$. As a lemma, we show that

$$f(a) + f(b) \le f(a+b) \text{ if } ab \ge 0 \text{ and } f(a) + f(b) \ge f(a+b) \text{ if } ab \le 0.$$

The result clearly holds if $b = 0$. Fix $b > 0$ and consider the function $\phi : x \mapsto f(x+b) - f(x)$. This function ϕ is differentiable and its derivative, defined by $\phi'(x) = f'(x+b) - f'(x)$, is nonnegative (since $f' = \sinh$ is nondecreasing). Thus $\phi(a) \ge \phi(0) = f(b)$ if $a \ge 0$ and $\phi(a) \le f(b)$ if $a \le 0$. The result follows when $b > 0$. In a similar way, we see that it also holds when $b < 0$.

This lemma and an easy induction lead to

$$f(x_1) + f(x_2) + \cdots + f(x_n) \le f(x_1 + x_2 + \cdots + x_n)$$

if $x_1, x_2, \ldots, x_n \ge 0$. The right-hand inequality immediately follows.

We also prove the left-hand inequality by induction. The case $n = 1$ is obvious. Consider the case $n = 2$: Since $x_1(-x_2) \le 0$, we have

$$f(x_1) + f(-x_2) \ge f(x_1 - x_2) \quad \text{or} \quad \cosh(x_1) + \cosh(x_2) - \cosh(x_1 - x_2) \ge 1,$$

the desired inequality. Now, assume that for some $n \ge 2$

$$\left(\sum_{k=1}^{n}\cosh x_k\right) \ge n-1+\cosh\left(\sum_{k=1}^{n}(-1)^{k-1}x_k\right) \tag{1}$$

whenever $0 \le x_1 \le x_2 \le \cdots \le x_n$.

Let $x_1, x_2, \ldots, x_n, x_{n+1}$ be such that $0 \le x_1 \le x_2 \le \cdots \le x_n \le x_{n+1}$. Then we observe that if n is even, then

$$x_{n+1}(x_1 - x_2 + \cdots + x_{n-1} - x_n) \le 0$$

and if n is odd, then

$$(-x_{n+1})(x_1 - x_2 + \cdots - x_{n-1} + x_n) \le 0.$$

Applying the lemma and using $\cosh(-x) = \cosh(x)$, we obtain

$$\cosh x_{n+1} - 1 + \cosh\left(\sum_{k=1}^{n}(-1)^{k-1}x_k\right) - 1 \ge \cosh\left(\sum_{k=1}^{n+1}(-1)^{k-1}x_k\right) - 1$$

in either case. With the help of (1), we are led to

$$\left(\sum_{k=1}^{n+1}\cosh x_k\right) \geq \cosh x_{n+1} + n - 1 + \cosh\left(\sum_{k=1}^{n}(-1)^{k-1}x_k\right)$$
$$\geq n + \cosh\left(\sum_{k=1}^{n+1}(-1)^{k-1}x_k\right).$$

This completes the induction step and the proof.

4095. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$. Prove that

$$ab(a+b) + bc(b+c) + ac(a+c) \geq \frac{2}{3}(a^2+b^2+c^2) + 4abc.$$

There were 14 correct solutions. We present six different ones here.

Solution 1, by Arkady Alt.

With $1 = ax = by = cz$ and $x + y + z = 3$, the inequality is equivalent to

$$\begin{aligned}&(x+y+z)[z^2(x+y) + x^2(y+z) + y^2(z+x)]\\ &\geq 2(x^2y^2 + y^2z^2 + z^2x^2) + 4(x+y+z)(xyz)\\ &= 2(xy+yz+zx)^2.\end{aligned}$$

The difference between the two sides is

$$\begin{aligned}&(x+y+z)[(x+y+z)(xy+yz+zx) - 3xyz] - 2(xy+yz+zx)^2\\ &= (x+y+z)^2(xy+yz+zx) - 2(xy+yz+zx)^2 - 3xyz(x+y+z)\\ &\geq 3(xy+yz+zx)(xy+yz+zx) - 2(xy+yz+zx)^2 - 3xyz(x+y+z)\\ &= (xy+yz+zx)^2 - 3xyz(x+y+z) \geq 0\end{aligned}$$

(from two applications of the inequality $(u+v+w)^2 \geq 3(uv+vw+wu)$). Equality occurs iff $1 = x = y = z = a = b = c$.

Solution 2, by Andrew Siefker and Digby Smith (done independently).

When a, b, c, x, y, z are all positive, we have

$$\begin{aligned}&(a^2yz + b^2zx + c^2xy)(x+y+z) - (a+b+c)^2xyz\\ &= z[(a^2y^2 + b^2x^2 - 2abxy] + y[a^2z^2 + c^2x^2 - 2acxz] + x[b^2z^2 + c^2y^2 - 2bcyz] \geq 0,\end{aligned}$$

so that

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}.$$

Equality occurs iff $a : b : c = x : y : z$. (This also results from Cauchy's Inequality.)

Let $bx = cy = az = 1$. Then (since $ab + bc + ca = 3abc$),

$$a^2b + b^2c + c^2a \geq \frac{abc(a+b+c)^2}{ab+bc+ca} = \frac{(a+b+c)^2}{3} = \frac{a^2+b^2+c^2}{3} + 2abc.$$

Let $cx = ay = bz = 1$. Then,

$$a^2c + b^2a + c^2b = \frac{(a+b+c)^2}{3} = \frac{a^2+b^2+c^2}{3} + 2abc.$$

Adding these two inequalities yields the result. Equality occurs iff $a = b = c = 1$.

Solution 3, by Prithwijit De.

Since $ab + bc + ca = 3abc$, it follows that

$$2(a^2 + b^2 + c^2) = (b-c)^2 + (c-a)^2 + (a-b)^2 + 6abc.$$

Also,

$$ab(a+b) + bc(b+c) + ac(a+c) = a(b-c)^2 + b(c-a)^2 + c(a-b)^2 + 6abc$$

and

$$a\left(\frac{1}{b} + \frac{1}{c}\right) = a\left(3 - \frac{1}{a}\right) = 3a - 1,$$

$$b\left(\frac{1}{c} + \frac{1}{a}\right) = 3b - 1 \qquad \text{and} \qquad c\left(\frac{1}{a} + \frac{1}{b}\right) = 3c - 1.$$

Therefore,

$$\begin{aligned}
&3[ab(a+b) + bc(c+a) + ca(a+c)] - 2(a^2+b^2+c^2) - 12abc \\
&= 3[a(b-c)^2 + b(c-a)^2 + c(a-b)^2] + 18abc \\
&\quad - [(b-c)^2 + (c-a)^2 + (a-b)^2 + 6abc] - 12abc \\
&= (3a-1)(b-c)^2 + (3b-1)(c-a)^2 + (3c-1)(a-b)^2 \\
&= a\left(\frac{1}{b} + \frac{1}{c}\right)(b-c)^2 + b\left(\frac{1}{c} + \frac{1}{a}\right)(c-a)^2 + c\left(\frac{1}{a} + \frac{1}{b}\right)(a-b)^2 \\
&\geq 0,
\end{aligned}$$

with equality iff $a = b = c = 1$. The desired result follows.

Solution 4, by Titu Zvonaru.

Replacing the 3 in the denominator by $(ab + bc + ca)/abc$ and multiplying by $ab + bc + ca$, we obtain the equivalent homogeneous inequality

$$\begin{aligned}
&a^2b^2(a+b) + b^2c^2(b+c) + c^2a^2(c+a) + abc[(a+b)^2 + (b+c)^2 + (c+a)^2] \\
&\geq 2(a^3bc + b^3ca + c^3ab) + 4(ab^2c^2 + bc^2a^2 + ca^2b^2)
\end{aligned}$$

which in turn is equivalent to

$$a^3b^2 + a^2b^3 + b^3c^2 + b^2c^3 + c^3a^2 + c^2a^3 \geq 2(ab^2c^2 + bc^2a^2 + ca^2b^2).$$

By the arithmetic-geometric means inequality, we have that

$$2a^3b^2 + b^2c^3 \geq 3ca^2b^2; \quad 2b^3c^2 + c^2a^3 \geq 3ab^2c^2; \quad 2c^3a^2 + a^2b^3 \geq 3bc^2a^2;$$

$$a^3b^2 + 2b^2c^3 \geq 3ab^2c^2; \quad b^3c^2 + 2c^2a^3 \geq 3bc^2a^2; \quad c^3a^2 + 2a^2b^3 \geq 3ca^2b^2.$$

Adding these inequalities yields the desired result, with equality iff $a = b = c = 1$.

Solution 5, by AN-Anduud Problem Solving Group; and Dionne Bailey, Elsie Campbell and Charles R. Diminnie (independently).

Since $a^2 + b^2 + c^2 \geq ab + bc + ca = 3abc$ and $x + (1/x) \geq 2$ for $x > 0$, we have that

$$\begin{aligned}
&3[ab(a+b) + bc(b+c) + ca(c+a)] \\
&= \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)[ab(a+b) + bc(b+c) + ca(c+a)] \\
&= 2(a^2 + b^2 + c^2) + 2(ab + bc + ca) + \frac{bc}{a}(b+c) + \frac{ca}{b}(c+a) + \frac{ab}{c}(a+b) \\
&= 2(a^2 + b^2 + c^2) + 6abc + a^2\left(\frac{b}{c} + \frac{c}{b}\right) + b^2\left(\frac{c}{a} + \frac{a}{c}\right) + c^2\left(\frac{a}{b} + \frac{b}{a}\right) \\
&\geq 6abc + 6abc + 2(a^2 + b^2 + c^2),
\end{aligned}$$

yielding the desired result. Equality holds iff $a = b = c = 1$.

Solution 6, by the proposer.

Applying the arithmetic-harmonic means inequality to the three pairs $(1/a, 1/b)$, $(1/b, 1/c)$ and $(1/c, 1/a)$, we find that

$$3 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a}.$$

This is equivalent to

$$\begin{aligned}
&3(a+b)(b+c)(c+a) \geq 2[(b+c)(c+a) + (c+a)(a+b) + (a+b)(b+c)] \iff \\
&6abc + 3[ab(a+b) + bc(b+c) + ca(c+a)] \geq 2(a^2 + b^2 + c^2) + 6(ab + bc + ca) \\
&\qquad = 2(a^2 + b^2 + c^2) + 18abc \iff \\
&ab(a+b) + bc(b+c) + ca(c+a) \geq \frac{2}{3}(a^2 + b^2 + c^2) + 4abc.
\end{aligned}$$

Equality holds iff $a = b = c = 1$.

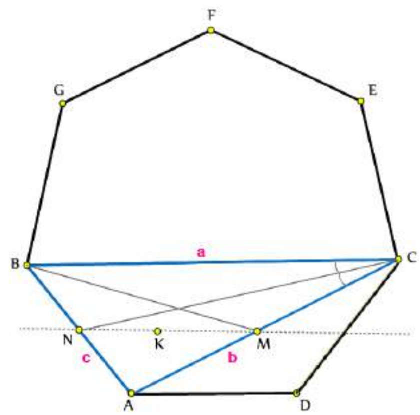
Editor's Comments. Four solvers reformulated the inequality as in Solution 4 and applied Muirhead's Inequality $[3, 2, 0] \geq [2, 2, 1]$, where

$$[p, q, r] = a^p b^q c^r + a^p b^r c^q + a^q b^p c^r + a^q b^r c^p + a^r b^p c^q + a^r b^q c^p.$$

Students Ahmad Talafha and Kevin Wunderlich gave a variant of Solution 2.

4096. *Proposed by Abdilkadir Altintaş.*

Let ABC be a heptagonal triangle with $BC = a$, $AC = b$ and $AB = c$. Suppose CN is the internal angle bisector of $\angle BCA$, BM is the median of triangle ABC and K is the symmedian point of ABC. Show that N, K and M are collinear.



We received six correct submissions. We present a combined solution based on those received from Michel Bataille and Titu Zvonaru.

In barycentric coordinates with respect to the vertices A, B, C of the triangle we have $N = (a : b : 0)$, $K = (a^2 : b^2 : c^2)$, $M = (1 : 0 : 1)$. The points N, K and M are collinear if and only if

$$\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 0 \\ 0 & c^2 & 1 \end{vmatrix} = 0,$$

which is equivalent to

$$c^2 + ab = a^2. \qquad (1)$$

Consider the cyclic quadrilateral $ABEC$. Since $ADCEFGB$ is a regular heptagon, $|CE| = |AB| = c$ and $|EB| = |EA| = |BC| = a$. Applying Ptolemy's theorem to quadrilateral $ABEC$ we have

$$|AB| \cdot |CE| + |AC| \cdot |BE| = |BC| \cdot |EA|,$$

which gives us (1). Therefore the points N, K and M are collinear.

4097. *Proposed by Leonard Giugiuc.*

Let $a_i, 1 \leq i \leq 6$ be real numbers such that

$$\sum_{i=1}^{6} a_i = \frac{15}{2} \quad \text{and} \quad \sum_{i=1}^{6} a_i^2 = \frac{45}{4}.$$

Prove that $\prod_{i=1}^{6} a_i \leq \dfrac{5}{2}$.

We received five submissions, four of which were incorrect for various reasons. We present the proposer's solution, modified by the editor.

Using Jensen's inequality with the function $g(x) = x^2$ we have

$$5\left(\sum_{i=1}^{5} a_i^2\right) \geq \left(\sum_{i=1}^{5} a_i\right)^2$$

and thus using the assumptions from the question,

$$5\left(\frac{45}{4} - a_6^2\right) \geq \left(\frac{15}{2} - a_6\right)^2,$$

which lets us conclude $0 \leq a_6 \leq \frac{5}{2}$. By symmetry we obtain $0 \leq a_i \leq \frac{5}{2}$ for $i = 1, \dots, 6$. If any of the six variables is zero, then $\prod_{i=1}^{6} a_i = 0 \leq \frac{5}{2}$, and we are done. Thus we assume that all a_i are positive and at most $\frac{5}{2}$.

Note that

$$\sum_{1 \leq i < j \leq 6} a_i a_j = \frac{1}{2}\left[\left(\sum_{i=1}^{6} a_i\right)^2 - \sum_{i=1}^{6} a_i^2\right] = \frac{1}{2}\left[\left(\frac{15}{2}\right)^2 - \frac{45}{4}\right] = \frac{45}{2}.$$

Define the polynomial $P(x) = \prod_{i=1}^{6}(x - a_i)$ which can be written as

$$P(x) = x^6 - \frac{15}{2}x^5 + \frac{45}{2}x^4 - mx^3 + nx^2 - qx + p$$

for $p = \prod_{i=1}^{6} a_i$ and suitable $m, n, q \in \mathbb{R}$. Set $f(x) = \frac{P(x)}{x}$ and define the sequence of polynomials

$$P(x) = P_0(x), P_1(x), P_2(x), P_3(x) = Q(x),$$

where $P_i(x) = x^{i+1} f^{(i)}(x)$ is the numerator of the i-th derivative of $f(x)$. Note that we can calculate

$$P_{i+1}(x) = x^{i+2} f^{(i+1)}(x) = x^{i+2} \frac{d}{dx} \frac{P_i(x)}{x^{i+1}} = xP_i'(x) - (i+1)P_i(x).$$

Now consider the positive roots of $P_{i+1}(x)$. If $P_i(x)$ has a root at α with multiplicity k, then $P_i'(x)$ has a root at α with multiplicity $k-1$, and therefore so does $P_{i+1}(x)$. On the other hand suppose we have two distinct positive roots of $P_i(x)$. Then they are also roots of $f^{(i)}(x)$. By Rolle's theorem (note that $f^{(i)}(x)$ is continuous for $x > 0$), there exists a root of $f^{(i+1)}(x)$ (and thus of $P_{i+1}(x)$) between these two roots. Using those two facts we can conclude that if $P_i(x)$ has n positive roots then $P_{i+1}(x)$ has at least $n-1$ positive roots. As $P(x)$ has six positive roots, $Q(x)$ must have at least three positive roots.

We can calculate

$$Q(x) = 60x^6 - 180x^5 + 135x^4 - 6p$$

and from there

$$Q'(x) = 180x^3(x-1)(2x-3).$$

If $Q(x)$ had two roots in the interval $(0,1)$, then $Q'(x)$ would have a root in the interval $(0,1)$ (by Rolle's theorem and our remarks about roots with multiplicity), which is not the case. Similarly $Q(x)$ cannot have more than one root greater than $\frac{3}{2}$. Since $Q(x)$ has at least three positive roots, though, it has a root in the interval $[1,\frac{3}{2}]$. Looking at $Q'(x)$ we can see that $Q(x)$ is decreasing on this interval and we obtain $Q(1)\geq 0$ and therefore $p\leq \frac{5}{2}$, which finishes the proof.

The bound can be obtained by setting one variable to $\frac{5}{2}$ and the others to 1.

4098. *Proposed by Ardak Mirzakhmedov.*

Let α,β and γ be acute angles such that $\alpha+\beta=\gamma$. Show that

$$\cos\alpha+\cos\beta+\cos\gamma-1\geq 2\sqrt{\cos\alpha\cdot\cos\beta\cdot\cos\gamma}.$$

We received six correct submissions. We present the solution by Arkady Alt.

Note first that for all $a,b,c,d\in\mathbb{R}$, we have $(ac-bd)^2-(ad-bc)^2=(a^2-b^2)(c^2-d^2)$, so $(a^2-b^2)(c^2-d^2)\leq(ac-bd)^2$. In particular, if $a>b>0$ and $c>d>0$, then

$$ac-bd\geq\sqrt{a^2-b^2}\cdot\sqrt{c^2-d^2}. \qquad (1)$$

Next,

$$\begin{aligned}\cos\alpha+\cos\beta+\cos\gamma-1&=2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}-2\sin^2\frac{\gamma}{2}\\&=2\left(\cos\frac{\gamma}{2}\cdot\cos\frac{\alpha-\beta}{2}-\sin\frac{\gamma}{2}\cdot\sin\frac{\alpha+\beta}{2}\right). \qquad (2)\end{aligned}$$

Since $\frac{\alpha+\beta}{2}=\frac{\gamma}{2}\in(0,\frac{\pi}{4})$ we have

$$\cos\frac{\alpha-\beta}{2}>\cos\frac{\alpha+\beta}{2}=\cos\frac{\gamma}{2}>\sin\frac{\gamma}{2}.$$

Hence, if we let

$$a=\cos\frac{\gamma}{2}, b=d=\sin\frac{\gamma}{2}\quad\text{and}\quad c=\cos\frac{\alpha-\beta}{2},$$

then $a>b>0$ and $c>d>0$ so applying (1) we obtain

$$\begin{aligned}\cos\frac{\gamma}{2}\cdot\cos\frac{\alpha-\beta}{2}-\sin\frac{\gamma}{2}\cdot\sin\frac{\alpha+\beta}{2}&\geq\sqrt{\cos^2\frac{\gamma}{2}-\sin^2\frac{\gamma}{2}}\cdot\sqrt{\cos^2\frac{\alpha-\beta}{2}-\sin^2\frac{\alpha+\beta}{2}}\\&=\sqrt{\cos\gamma}\sqrt{\frac{1}{2}\left(1+\cos(\alpha-\beta)-(1-\cos(\alpha+\beta)\right)}\\&=\sqrt{\cos\gamma}\sqrt{\cos\alpha\cdot\cos\beta}\\&=\sqrt{\cos\alpha\cdot\cos\beta\cdot\cos\gamma}. \qquad (3)\end{aligned}$$

Substituting (3) into (2), we then have

$$\cos\alpha+\cos\beta+\cos\gamma-1\geq 2\sqrt{\cos\alpha\cdot\cos\beta\cdot\cos\gamma},$$

thus completing the proof.

4099. *Proposed by Lorian Saceanu.*

Let ABC be an acute angle triangle. Suppose the internal bisectors of angles A, B and C intersect the sides of ABC in points A', B' and C' and they intersect the circumcircle of ABC in points L, M and N respectively. Let I be the point of intersection of all internal bisectors. Show that:

a) $\dfrac{AI}{IL} = \dfrac{IA'}{A'L}$,

b) $\sqrt{\dfrac{AI}{IL}} + \sqrt{\dfrac{BI}{IM}} + \sqrt{\dfrac{CI}{IN}} \geq 3$.

We received seven submissions, all correct, and will feature parts from two of them.

Solution to part a), by Prithwimit De and B.J. Venkatachala (together).

In triangle IBL, $\angle IBL = \angle BIL = \frac{A+B}{2}$. Thus, $IL = BL = 2R\sin\frac{A}{2}$, where R is the circumradius of ΔABC. Moreover, since $AI = \frac{r}{\sin(A/2)}$, where r is the inradius of ΔABC,

$$\frac{AI}{IL} = \frac{r}{BL\sin\frac{A}{2}}.$$

Observe that

$$\frac{IA'}{A'L} = \frac{[BIA']}{[BLA']} = \frac{BI\sin\frac{B}{2}}{BL\sin\frac{A}{2}} = \frac{r}{BL\sin\frac{A}{2}} = \frac{AI}{IL}.$$

This settles part a).

Solution to part b), by Salem Malikić, modified by the editor.

We first transform the form of $\frac{AI}{IL}$ obtained above into something more helpful. In particular, we use area formulas

$$r = \frac{[ABC]}{s} = \frac{bc\sin^2 A}{2s},$$

the sine law $2R = \dfrac{a}{\sin A}$, and the half-angle formulas together with the cosine law

$$\frac{\sin^2 A}{\sin^2\frac{A}{2}} = 4\cos^2\frac{A}{2} = \frac{2s(b+c-a)}{bc}$$

in turn to get

$$\frac{AI}{IL} = \frac{r}{BL\sin\frac{A}{2}} = \frac{r}{2R\sin^2\frac{A}{2}} = \frac{bc\sin^2 A}{2sa\sin^2\frac{A}{2}} = \frac{b+c-a}{a}.$$

This, with analogous expressions for $\frac{BI}{IM}$ and $\frac{CI}{IN}$, reduces the inequality of part b) to

$$\sqrt{\frac{b+c-a}{a}} + \sqrt{\frac{c+a-b}{b}} + \sqrt{\frac{a+b-c}{c}} \geq 3, \tag{1}$$

which is an inequality due to Sorin Rădulescu that holds for all acute triangles. [*Editor's comment.* Malikić found a proof on the web page
`www.artofproblemsolving.com/community/c6h1102804p5008234`,
which he reproduced while adding some helpful details as follows.] After squaring both sides of (1) we need to prove that

$$\sum_{cyclic} \frac{b+c-a}{a} + 4\sum_{cyclic} \sqrt{\frac{(a+b-c)(a+c-b)}{4bc}} \geq 9,$$

which is

$$\sum_{cyclic} \frac{b+c-a}{a} + 4\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \geq 9.$$

Since cosine is a concave function on $[0, \frac{\pi}{2}]$, Popoviciu's inequality tells us that

$$\cos A + \cos B + \cos C + 3\cos\left(\frac{A+B+C}{3}\right) \leq 2\left(\cos\frac{A+B}{2} + \cos\frac{B+C}{2} + \cos\frac{C+A}{2}\right),$$

or equivalently,

$$2\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \geq \sum_{cyclic} \frac{b^2+c^2-a^2}{2bc} + \frac{3}{2}. \qquad (2)$$

With the help of inequality (2) it remains to prove that

$$\sum_{cyclic} \frac{b+c-a}{a} + \sum_{cyclic} \frac{b^2+c^2-a^2}{bc} \geq 6.$$

Setting $a = y+z, b = x+z$ and $c = x+y$ reduces the last inequality to the equivalent

$$\sum_{cyclic} (x^3 - x^2y - x^2z + xyz) \geq 0,$$

which directly follows from Schur's inequality, thus completing the proof. Moreover, from the last step we see that for equality one must have $x = y = z$, which implies that ΔABC must be equilateral. It is easily seen that equality is indeed achieved for an equilateral triangle.

Editor's Comments. Barroso Campos observed that the inequality of part b) might fail when ΔABC has an obtuse angle. One can easily construct counterexamples using the left-hand-side of (1); for example, with sides $a = 9, b = 10, c = 18$ we get a sum slightly smaller than 2.993.

4100. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let ABC be an arbitrary triangle with area S, $\angle A < 90^\circ$ and sides $BC = a$, $AC = b$ and $AB = c$. Show that

$$\frac{c\cos B}{ac+2S} + \frac{b\cos C}{ab+2S} < \frac{a}{2S}.$$

We received nine submissions, of which eight were correct. We present the solution by Salem Malikić, slightly modified by the editor.

Using the area formula $S = \frac{1}{2}ac\sin B$ we get

$$\frac{c\cos B}{ac+2S} = \frac{c\cos B}{ac+ac\sin B} = \frac{\cos B}{a(1+\sin B)};$$

treating the second term similarly, the left hand side of the inequality can be re-written as

$$\frac{1}{a}\cdot\left(\frac{\cos B}{1+\sin B}+\frac{\cos C}{1+\sin C}\right).$$

On the right hand side of the given inequality we have

$$\begin{aligned}
\frac{a}{2S} &= \frac{a}{ac\sin B} && \text{(area formula)}\\
&= \frac{a}{a\cdot\frac{a\sin C}{\sin A}\cdot\sin B} && \text{(sine law)}\\
&= \frac{\sin(B+C)}{a\sin C\sin B} && (A+B+C=\pi,\text{ so }\sin(A)=\sin(B+C))\\
&= \frac{1}{a}\cdot\left(\frac{\cos C}{\sin C}+\frac{\cos B}{\sin B}\right) && \text{(sum of angles formula for sine)}
\end{aligned}$$

Hence, since $a>0$, the given inequality is equivalent to

$$\frac{\cos B}{1+\sin B}+\frac{\cos C}{1+\sin C} < \frac{\cos B}{\sin B}+\frac{\cos C}{\sin C},$$

and, by moving all the terms to one side, this is in turn equivalent to

$$\frac{\cos B}{\sin B(1+\sin B)}+\frac{\cos C}{\sin C(1+\sin C)} > 0. \qquad (\dagger)$$

If neither of $\angle B$ and $\angle C$ is obtuse then all the trigonometric functions are non-negative and the last inequality clearly holds (note that both terms cannot be simultaneously zero, as that would imply $\angle B = \angle C = 90^\circ$).

Assume now that one of the angles B and C, say without loss of generality C, is obtuse. As $\cos(C) = -\cos(\pi - C)$ and $\sin(C) = \sin(\pi - C)$, the inequality $(\dagger)$ holds if and only if

$$\frac{\cos B}{\sin B(1+\sin B)} > \frac{\cos(\pi-C)}{\sin(\pi-C)(1+\sin(\pi-C))}.$$

However, $A+B+C=\pi$ and C obtuse imply $0<B<\pi-C<\frac{\pi}{2}$; since on the interval $(0,\pi/2)$ cosine is decreasing and sine is increasing, this last inequality follows, concluding the proof of $(\dagger)$ and thus also of the desired inequality.