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SP.210. Let ABC be an acute-angled triangle. If $a+b+c=\pi$ and $A\cos a+B\cos b+C\cos c=\frac{\pi}{2}$; ($A,B,C$ – the measures in radians), then ΔABC is equilateral.

Proposed by Marian Ursărescu – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam, Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$a+b+c=\pi;\ (a,b,c>0)$$

$$\because a+b>c\Rightarrow a+b+c>2c\Rightarrow 0<c<\frac{\pi}{2}.\ \textit{Similarly: } 0<a,b<\frac{\pi}{2}.$$

$$\textit{Let } f(x)=\cos x\left(0<x<\frac{\pi}{2}\right)\Rightarrow f'(x)=-\sin x\Rightarrow f''(x)=-\cos x<0\ \left(0<x<\frac{\pi}{2}\right)$$

$$\textit{Suppose: } A\le B\le C\Rightarrow a\le b\le c\Rightarrow \cos a\ge\cos b\ge\cos c\ \left(\because f(x)=\cos x\searrow\left(0;\frac{\pi}{2}\right)\right)$$

$$\Rightarrow LHS=A\cos a+B\cos b+C\cos c\le\frac{1}{3}(A+B+C)(\cos a+\cos b+\cos c)$$

$$=\frac{\pi}{3}\cdot(\cos a+\cos b+\cos c)\overset{Jensen}{\le}\frac{\pi}{3}\cdot 3\cos\left(\frac{a+b+c}{3}\right)=\pi\cdot\cos\left(\frac{\pi}{3}\right)=\frac{\pi}{2}$$

$$\textit{Hence, } LHS=\frac{\pi}{2}\Leftrightarrow\begin{cases}A=B=C\\ a=b=c\end{cases}.\ \textit{Proved.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

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If $a \geq \frac{\pi}{2}$, then $b + c \leq \frac{\pi}{2}$ $(\because \sum a = \pi)$

$\Rightarrow b + c \leq a \Rightarrow$ violation of triangle inequality $\Rightarrow a < \frac{\pi}{2}$. Similar argument $\Rightarrow b, c < \frac{\pi}{2}$

Let $f(x) = \sin^2\frac{x}{2}, \forall x \in \left(0, \frac{\pi}{2}\right)$. Then, $f''(x) = \frac{\cos x}{2} > 0 \Rightarrow f(x)$ is strictly convex.

$$\sum A\cos a = \sum A\left(1 - 2\sin^2\frac{a}{2}\right) = \sum A - 2\pi\sum\left(\frac{A}{\pi}\sin^2\frac{a}{2}\right) = \pi - 2\pi\sum\left(\frac{A}{\pi}\sin^2\frac{a}{2}\right)$$

$$\underset{(1)}{\overset{Jensen}{\leq}} \pi - 2\pi\sin^2\left(\frac{\sum\left(\frac{A}{\pi}a\right)}{2}\right)$$ *($\because \sum\frac{A}{\pi} = 1$ and $\sin^2\frac{x}{2}\forall x \in \left(0, \frac{\pi}{2}\right)$ is strictly convex)*

Now, WLOG we may assume $a \geq b \geq c$

$$\therefore A \geq B \geq C \therefore \frac{1}{2}\sum\left(\frac{A}{\pi}a\right) \overset{Chebyshev}{\geq} \frac{1}{2\pi}\cdot\frac{1}{3}\left(\sum A\right)\left(\sum a\right) = \frac{\pi^2}{6\pi} = \frac{\pi}{6} \Rightarrow \frac{1}{2}\sum\left(\frac{A}{\pi}a\right) \overset{(i)}{\geq} \frac{\pi}{6}$$

$$\because A, B, C < \frac{\pi}{2} \text{ \& } a, b, c \text{ also } < \frac{\pi}{2} \therefore \frac{1}{2}\sum\left(\frac{A}{2}a\right) < \frac{1}{2\pi}\left(\frac{3\pi^2}{4}\right) = \frac{3\pi}{8} \Rightarrow \frac{1}{2}\left(\sum\frac{A}{\pi}a\right) \overset{(ii)}{<} \frac{3\pi}{8}$$

$$(i), (ii) \Rightarrow \frac{\pi}{6} \leq \frac{1}{2}\sum\left(\frac{A}{\pi}a\right) < \frac{3\pi}{8} \Rightarrow \sin\left(\frac{\sum\left(\frac{A}{\pi}a\right)}{2}\right) \overset{(2)}{\geq} \sin\frac{\pi}{6} = \frac{1}{2}$$

(1), (2) $\Rightarrow \sum A\cos a \leq \pi - 2\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{2}$, equality when $a = b = c$,

($\because$ the equality of Chebyshev's inequality holds at $A = B = C$ $(\&a = b = c)$ and the equality of Jensen's inequality holds at $a = b = c$, as $f(x) = \sin^2\frac{x}{2}\forall x \in \left(0, \frac{\pi}{2}\right)$ is strictly convex) and $\therefore$ equality relation holds (as $\sum A\cos a = \frac{\pi}{2}$), $\therefore a = b = c \Rightarrow \Delta ABC$ is equilateral (proved)