RMM - Triangle Marathon 1001 - 1100

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ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
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Available online
www.ssmrmh.ro

ISSN-L 2501-0099

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$R \geq 2r;\ 5R \geq 10r;\ 8R - 3R \geq 10r;\ 8R - 10r \geq 3R;\ (8R - 10r)^3 \geq (3R)^3 = 27R^3$$

$$27R^3 \leq (8R - 10r)^3;\ 27R^3 = 27 \cdot R \cdot R \cdot R = 3\sqrt{3}R \cdot 3\sqrt{3}R \cdot R =$$

$$= 6\sqrt{3}R \cdot \frac{3\sqrt{3}}{2}R \cdot R \geq 6\sqrt{3}R \cdot s \cdot 2r = 12\sqrt{3}sRr = 3\sqrt{3} \cdot 4sRr = 3\sqrt{3}abc$$

$$(8R - 10r)^3 \geq 3\sqrt{3}abc;\ (8R - 10r)^6 \geq 27a^2b^2c^2$$

Solution 3 by Boris Colakovic-Belgrade-Serbie

$$27a^2b^2c^2 \leq (8R - 10r)^6 \Leftrightarrow (27a^2b^2c^2)^{\frac{1}{3}} \leq (8R - 10r)^2 \Leftrightarrow$$

$$\Leftrightarrow 3\sqrt[3]{a^2b^2c^2} = 3\sqrt[3]{abc} \cdot \sqrt[3]{abc} \leq (a + b + c) \cdot \frac{a + b + c}{3} = 2s \cdot \frac{2s}{3} = \frac{4}{3}s^2 \overset{Gerretsen}{\leq}$$

$$\leq \frac{4}{3}(4R^2 + 4Rr + 3r^2) \leq 4(4R - 5r)^2 \Leftrightarrow 4R^2 + 4Rr + 3r^2 \leq 3(4R - 5r)^2 \Leftrightarrow$$

$$\Leftrightarrow 11R^2 - 31Rr + 18r^2 \geq 0 \Leftrightarrow (R - 2r)(11R - 9r) \geq 0 \Rightarrow R \geq 2r \quad \textit{Euler}$$

Solution 4 by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow \sqrt{3}\sqrt[3]{abc} \overset{(1)}{\leq} 8R - 10r$. *But LHS of (1)* $\overset{G \leq A}{\leq} \frac{a+b+c}{\sqrt{3}} = \frac{2s}{\sqrt{3}} \overset{Mitrinovic}{\leq} \frac{3\sqrt{3}R}{\sqrt{3}}$

$= 3R \overset{?}{\leq} 8R - 10r \Leftrightarrow 5R \geq 10r \Leftrightarrow R \geq 2r \rightarrow$ *true (Euler) (Proved)*

1062. In ΔABC the following relationship holds:

$$\frac{aw_a^2}{h_a} + \frac{bw_b^2}{h_b} + \frac{cw_c^2}{h_c} \geq 2r^2\sqrt{\frac{486r}{R}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

We must show: $\frac{1}{2S}\left(a^2w_a^2 + b^2w_b^2 + c^2w_c^2\right) \geq 18r^2\sqrt{\frac{6r}{R}}$ *(1)*

But $r \leq \frac{R}{2} \Rightarrow 6r \leq 3R \Rightarrow \frac{6r}{R} \leq 3$ *(2)*

From (1)+(2): We must show: $a^2w_a^2 + b^2w_b^2 + c^2w_c^2 \geq 36Sr^2\sqrt{3}$ *(3)*

$$\left.\begin{array}{c} a^2w_a^2+b^2w_b^2+c^2w_c^2\ge 3\sqrt[3]{(abc)^2(w_aw_bw_c)^2} \\ But\ \sqrt[3]{w_aw_bw_c}\ge 3r \end{array}\right\}\Rightarrow$$

$$a^2w_a^2+b^2w_b^2+c^2w_c^2\ge 27r^2\sqrt[3]{(abc)^2}\quad (4)$$

From (3)+(4) we must show: $2+r^2\sqrt[3]{(abc)^2}\ge 36Sr^2\sqrt{3}\Leftrightarrow 3\sqrt[3]{(abc)^2}\ge 4S\sqrt{3}\Leftrightarrow$

$$3\sqrt[3]{(4RS)^2}\ge 4S\sqrt{3}\Leftrightarrow 27\cdot 16R^2S^2\ge 64S^3 3\sqrt{3}\Leftrightarrow$$

$$3\sqrt{3}R^2\ge 4S\Leftrightarrow 3\sqrt{3}R^2\ge 4sr\quad (5)$$

$$\left.\begin{array}{c} But\ R\ge 2r \\ r\ge \frac{2s}{3\sqrt{3}} \end{array}\right\}\Rightarrow R^2\ge \frac{4sr}{3\sqrt{3}}\Rightarrow 3\sqrt{3}R^2\ge 4sr\Rightarrow \textit{(5) it's true.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS=\sum aw_a^2\left(\frac{a}{2rs}\right)=\frac{1}{2rs}\sum a^2w_a^2$$

$$\overset{w_a\ge h_a,etc}{\ge}\frac{1}{2rs}\sum\left(a^2\left(\frac{4r^2S^2}{a^2}\right)\right)=6rs\overset{?}{\ge}2r^2\sqrt{\frac{486r}{R}}$$

$$\Leftrightarrow 36r^2S^2\overset{?}{\ge}4r^4\left(\frac{486r}{R}\right)\Leftrightarrow 9RS^2\underset{(1)}{\overset{?}{\ge}}486r^3$$

But $9R\overset{Euler}{\ge}18r$ *&* $S^2\ge 27r^2$*. Multiplying the above two,* $9RS^2\ge 486r^3$

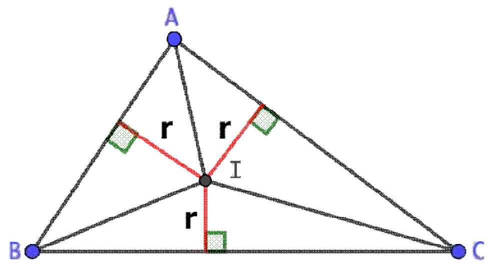
$\Rightarrow$ *(1) is true (proved)*

1063. If in ΔABC, I – incentre, R_a, R_b, R_c – circumradii in $\Delta BIC, \Delta CIA, \Delta AIB$ then:

$$\sqrt{6}\le\sqrt{\frac{R_a}{h_a}}+\sqrt{\frac{R_b}{h_b}}+\sqrt{\frac{R_c}{h_c}}\le\sqrt{\frac{6m_am_bm_c}{h_ah_bh_c}}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India



$$R_a = \frac{BI\cdot CI\cdot BC}{4\cdot\frac{1}{2}BC\cdot r} = \frac{\frac{r}{\sin\frac{B}{2}}\cdot\frac{r}{\sin\frac{C}{2}}a}{2ar} = \frac{r\sin\frac{A}{2}}{2\left(\pi\sin\frac{A}{2}\right)} = \frac{r\sin\frac{A}{2}}{2\left(\frac{r}{4R}\right)} \overset{(1)}{=} 2R\sin\frac{A}{2}$$

$$\therefore \sqrt{\frac{R_a}{h_a}} \overset{by\,(1)}{=} \sqrt{2R\sin\frac{A}{2}\cdot\frac{2R_a}{abc}} = \sqrt{\frac{4R^2}{4Rrs}a\sin\frac{A}{2}} \overset{(a)}{=} \sqrt{\frac{R}{rs}}\sqrt{a\sin\frac{A}{2}}$$

Similarly, $\sqrt{\frac{R_b}{h_b}} \overset{(b)}{=} \sqrt{\frac{R}{rs}}\sqrt{a\sin\frac{B}{2}}$ **&** $\sqrt{\frac{R_c}{h_c}} \overset{(c)}{=} \sqrt{\frac{R}{rs}}\sqrt{c\sin\frac{C}{2}}$

(a)+(b)+(c) $\Rightarrow \sum\sqrt{\frac{R_a}{h_a}} \overset{(2)}{=} \sqrt{\frac{R}{rs}}\sum\sqrt{a\sin\frac{A}{2}}$

$$\overset{A-G}{\geq} 3\sqrt{\frac{R}{rs}}\sqrt[6]{4Rrs\left(\frac{r}{4R}\right)} \overset{?}{\geq} \sqrt{6} \Leftrightarrow 27R^3 \overset{?}{\geq} 8rs^2 \to \textit{(i)}$$

Now, $R^2 \overset{Mitrinovic}{\geq} \frac{4s^2}{27}$ **&** $R \overset{Euler}{\geq} 2r$

$\therefore 27R^3 \geq 8rs^2$ ***(multiplying the above two)⇒ (i) is true*** $\therefore \sum\sqrt{\frac{R_a}{h_a}} \geq \sqrt{6}$

Also, using (2), $\sum\sqrt{\frac{R_a}{h_a}} \overset{CBS}{\leq} \sqrt{\frac{R}{rs}}\sqrt{2s}\sqrt{\sum\sin\frac{A}{2}}$

$\overset{Jensen}{\leq} \sqrt{\frac{R}{rs}}\sqrt{2S}\sqrt{3\sin\left(\frac{\pi}{6}\right)}$ ***(∵*** $f(x) = \sin\frac{x}{2}\ \forall x \in (0,\pi)$ ***is concave)***

$$= \sqrt{\frac{3R}{r}} \therefore \sum\sqrt{\frac{R_a}{h_a}} \overset{(ii)}{\leq} \sqrt{\frac{3R}{r}}$$

Now, $\sqrt{\frac{6m_am_bm_c}{h_ah_bh_c}} \overset{m_a\geq\sqrt{s(s-a)},etc}{\geq} \sqrt{\frac{6Srs}{\frac{16R^2r^2S^2}{8R^3}}} = \sqrt{\frac{3R}{r}} \overset{by\,(ii)}{\geq} \sum\sqrt{\frac{R_a}{h_a}} \Rightarrow \sum\sqrt{\frac{R_a}{h_a}} \leq \sqrt{\frac{6m_am_bm_c}{h_ah_bh_c}}$

1064. In $\Delta\ ABC$ the following relationship holds:

$$\frac{am_a}{h_a}+\frac{bm_b}{h_b}+\frac{cm_c}{h_c}\ge 2\sqrt{3\sqrt{3}S}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mustafa Tarek-Cairo-Egypt

We know that the altitude the least segment from the vertex of the triangle to the other side and coincide at the median ⇔ the triangle is isosceles then

$$m_a\ge h_a,\ m_b\ge h_b,\ m_c\ge h_c$$

$$\text{LHS}\ge a+b+c=2s\overset{??}{\ge}2\sqrt{3\sqrt{3}S}\Leftrightarrow\frac{s^2\sqrt{3}}{9}\ge\Delta\ \textit{(true)}$$

$$\text{as}\Leftrightarrow\frac{s^2\sqrt{3}}{9}\ge rs\Leftrightarrow s\overset{Mitrinovic}{\ge}3\sqrt{3}r\quad\textit{(isoperimetric inequality)}$$

Solution 2 by Marian Ursărescu-Romania

$$\frac{am_a}{h_a}+\frac{bm_b}{h_b}+\frac{cm_c}{h_c}\ge 3\sqrt[3]{\frac{abc\ m_am_bm_c}{h_ah_bh_c}}\quad (1)$$

$$\text{But } m_a\ge\frac{b+c}{2}\cos\frac{A}{2}\ge\sqrt{bc}\cos\frac{A}{2}\Rightarrow m_am_bm_c\ge abc\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\quad (2)$$

$$\text{From (1)+(2)}\Rightarrow\frac{am_a}{h_a}+\frac{bm_b}{h_b}+\frac{cm_c}{h_c}\ge 3\sqrt[3]{\frac{a^2b^2c^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{h_ah_bh_c}}\quad (3)$$

$$abc=4sRr,\ \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{s}{4R}\text{ and } h_ah_bh_c=\frac{2s^2r^2}{R}\quad (4)$$

$$\frac{am_a}{h_a}+\frac{bm_b}{h_b}+\frac{cm_c}{h_c}\ge 3\sqrt[3]{\frac{16s^2R^2r^2\cdot s\cdot R}{4R\cdot 2s^2r^2}}\Rightarrow\text{we must show:}$$

$$3\sqrt[3]{2R^2s}\ge 2\sqrt{3\sqrt{3}S}\Leftrightarrow 3^62^2R^4s^2\ge 2^63^33\sqrt{3}s^3r^3\Leftrightarrow 9R^4\ge 16\sqrt{3}sr^3\quad (5)$$

$$\left.\begin{matrix}R^3\ge 8r^3\\ R\ge\frac{2}{3\sqrt{3}}s\end{matrix}\right\}\Rightarrow R^4\ge\frac{16}{3\sqrt{3}}sr^3\Leftrightarrow 9R^4\ge 16\sqrt{3}sr^3\Rightarrow\text{(5) it's true.}$$

1065. In ΔABC the following relationship holds:

$$\frac{\sqrt{b^2+c^2}}{h_a}+\frac{\sqrt{c^2+a^2}}{h_b}+\frac{\sqrt{a^2+b^2}}{h_c}\le\frac{9R^2}{\sqrt{2}\cdot S}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution 1 by Daniel Sitaru-Romania

$$\sum_{cyc(a,b,c)} \frac{\sqrt{b^2+c^2}}{h_a} \overset{CBS}{\leq} \sqrt{\sum_{cyc(a,b,c)} (b^2+c^2) \cdot \sum_{cyc(a,b,c)} \frac{1}{h_a^2}} = \sqrt{2\sum_{cyc(a,b,c)} a^2 \cdot \sum_{cyc(a,b,c)} \frac{a^2}{4S^2}} =$$

$$= \frac{1}{\sqrt{2}\cdot S} \cdot \sum_{cyc(a,b,c)} a^2 \overset{LEIBNIZ}{\leq} \frac{9R^2}{\sqrt{2}\cdot S}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

WLOG, we may assume $a \geq b \geq c$. *Then* $\sqrt{b^2+c^2} \leq \sqrt{c^2+a^2} \leq \sqrt{a^2+b^2}$ &

$$\frac{1}{h_a} \geq \frac{1}{h_b} \geq \frac{1}{h_c} \therefore \sum \frac{\sqrt{b^2+c^2}}{h_a} \overset{Chebyshev}{\leq} \frac{1}{3}\left(\sum \frac{1}{h_a}\right)\left(\sum \sqrt{b^2+c^2}\right)$$

$$\overset{CBS}{\leq} \frac{\sqrt{3}}{3r}\sqrt{2\sum a^2} \overset{LEIBNIZ}{\leq} \frac{\sqrt{3}\sqrt{2}\cdot 3R}{3r} = \frac{\sqrt{6}Rs}{rs} \overset{MITRINOVIC}{\leq} \frac{\sqrt{6}R\frac{3\sqrt{3}R}{2}}{S} = \frac{9R^2}{\sqrt{2}S}$$

1066. Find $\Omega \in \mathbb{R}$ such that in acute ΔABC holds:

$$\Omega = \left(\frac{b\cos B}{c\cos c} + \frac{c\cos c}{b\cos B}\right)\cos 2A + \left(\frac{c\cos C}{a\cos A} + \frac{a\cos A}{c\cos C}\right)\cos 2B + \left(\frac{a\cos B}{b\cos B} + \frac{b\cos B}{a\cos A}\right)\cos 2C$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Serban George Florin-Romania

$$\Omega = \sum\left(\frac{b\cos B}{c\cos C} + \frac{c\cos C}{b\cos B}\right)\cdot\cos 2A = \sum\left(\frac{2R\sin B\cos B}{2R\sin C\cos C} + \frac{2R\sin C\cos C}{2R\sin B\cos B}\right)\cdot\cos 2A$$

$$= \sum\left(\frac{\sin 2B}{\sin 2C} + \frac{\sin 2C}{\cos 2C}\right)\cdot\cos 2A$$

$$\Omega = \frac{\sin 2B\cos 2A}{\sin 2C} + \frac{\sin 2C\cos 2A}{\cos 2C} + \frac{\sin 2A\cdot\cos 2B}{\sin 2C} + \frac{\sin 2C\cos 2B}{\sin 2A} +$$

$$+\frac{\sin 2A\cos 2C}{\sin 2B} + \frac{\sin 2B\cos 2C}{\sin 2A} = \sum\left(\frac{\sin 2A\cos 2B}{\sin 2C} + \frac{\sin 2B\cos 2A}{\sin 2C}\right) =$$

$$= \sum\frac{\sin(2A+2B)}{\sin 2C} = \sum\frac{\sin(2A-2C)}{\sin 2C}$$

$$\Omega = \sum -\frac{\sin 2C}{\sin 2C} = \sum(-1) = -3$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\left(\frac{b\cos B}{c\cos C}+\frac{c\cos C}{b\cos B}\right)\cos 2A=\left(\frac{2R\sin B\cos B}{2R\sin C\cos C}+\frac{2R\sin C\cos C}{2R\sin B\cos B}\right)\cdot\cos 2A$$

$$=\left(\frac{\sin 2B}{\sin 2C}+\frac{\sin 2C}{\sin 2B}\right)\cos 2A=\left(\frac{2\sin^2 2B+2\sin^2 2C}{2\sin 2B\sin 2C}\right)\cos 2A$$

$$=\frac{(1-\cos 4B-1-\cos 4C)\cdot 2\sin 2A\cos 2A}{4\sin 2A\sin 2B\sin 2C}\overset{(1)}{=}\frac{2\sin 4A-\sin 4A\cos 4B-\sin 4A\cos 4C}{4\sin 2A\sin 2B\sin 2C}$$

Similarly, $\left(\frac{c\cos C}{a\cos A}+\frac{a\cos A}{c\cos C}\right)\cos 2B\overset{(2)}{=}\frac{2\sin 4B-\sin 4B\cos 4C-\sin 4B\cos 4A}{4\sin 2A\sin 2B\sin 2C}$ &

$$\left(\frac{a\cos A}{b\cos B}+\frac{b\cos B}{a\cos A}\right)\cos 2C\overset{(3)}{=}\frac{2\sin 4C-\sin 4C\cos 4A-\sin 4C\cos 4B}{4\sin 2A\sin 2B\sin 2C}$$

(1)+(2)+(3)⇒ $LHS=\frac{2\sum\sin 4A-\sum(\sin 4A\cos 4B+\cos 4A\sin 4B)}{4\sin 2A\sin 2B\sin 2C}$

$$=\frac{2\sum\sin 4A-\sum\sin(4A+4B)}{4\sin 2A\sin 2B\sin 2C}=\frac{2\sum\sin 4A-\sum\sin(4\pi-4C)}{4\sin 2A\sin 2B\sin 2C}$$

$$=\frac{2\sum\sin 4A+\sum\sin 4A}{4\sin 2A\sin 2B\sin 2C}\overset{(a)}{=}\frac{3\sum\sin 4A}{4\sin 2A\sin 2B\sin 2C}$$

Now, $\sum\sin 4A=2\sin(2A+2B)\cos(2A-2B)+2\sin 2C\cos 2C$

$$=-2\sin 2C\cos(2A-2B)+2\sin 2C\cos(2A+2B)$$

$$=2\sin 2C\{\cos(2A+2B)-\cos(2A-2B)\}\overset{(b)}{=}-4\sin 2C\sin 2A\sin 2B$$

(a), (b)⇒ $LHS=\frac{-12\sin 2A\sin 2B\sin 2C}{4\sin 2A\sin 2B\sin 2C}=-3$ *(answer)*

1067. In ΔABC the following relationship holds:

$$\sqrt{h_a-2r}+\sqrt{h_b-2r}+\sqrt{h_c-2r}\le\sqrt{h_a+h_b+h_c}$$

Proposed by Bogdan Fustei – Romania

Solution 1 by Mehmet Sahin-Ankara-Turkey

$$\sqrt{h_a-2r}+\sqrt{h_b-2r}+\sqrt{h_c-2r}\le\sqrt{h_a+h_b+h_c}$$

Let $T=\sqrt{h_a-2r}+\sqrt{h_b-2r}+\sqrt{h_c-2r}$. *Using* $h_a=\frac{2\Delta}{a},h_b=\frac{2\Delta}{b},h_c=\frac{2\Delta}{c}$

$$T=\sqrt{\frac{2\Delta}{a}-2r}+\sqrt{\frac{2\Delta}{b}-2r}+\sqrt{\frac{2\Delta}{c}-2r}$$

$$T=\sqrt{\frac{2r}{a}}\cdot\sqrt{(s-a)}+\sqrt{\frac{2r}{b}}\cdot\sqrt{(s-b)}+\sqrt{\frac{2r}{c}}\cdot\sqrt{(s-c)}$$

$$T^2\overset{C-S}{\leq}2r\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\cdot(s)$$

$$T^2\leq 2\Delta\left(\frac{s^2+r^2+4Rr}{4R\Delta}\right)=\frac{1}{2R}(s^2+r^2+4Rr)$$

$$T^2\leq h_a+h_b+h_c;\ T\leq\sqrt{h_a+h_b+h_c}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\Delta=\frac{a\cdot h_a}{2}=\frac{b\cdot h_b}{2}=\frac{c\cdot h_c}{2}$$

$$\sum_{cyc}\sqrt{h_a-2r}=\sum_{cyc}\sqrt{\frac{2\Delta}{a}-2r}=\sum_{cyc}\sqrt{\frac{2r}{a}(s-a)}$$

$$\overset{\substack{Cauchy\\Schwarz}}{\leq}\sqrt{\left(\sum_{cyc}\frac{2r}{a}\right)\left(\sum_{cyc}(s-a)\right)}=\sqrt{\sum_{cyc}\frac{2r}{a}}=\sqrt{\sum_{cyc}h_a}$$

1068. In $\Delta ABC,\Delta A'B'C'$ the following relationship holds:

$$(a+a')(b+b')(c+c')\geq 32\sqrt{RR'SS'}+4\left(\sqrt{RS}-\sqrt{R'S'}\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution by Lahiru Samarakoon-Sri Lanka

$$(a+a')(b+b')(c+c')\geq 24\sqrt{RR'SS'}+4RS+4R'S'\ \textbf{but},\ R=\frac{abc}{4S}$$

$$(a+a')(b+c')(c+c')\geq 6\sqrt{aa'bb'cc'}+abc+a'b'c'\Rightarrow$$

$$\Rightarrow(abc+a'bc+b'ac+c'ab+a'b'c+b'c'a+a'c'b+a'b'c')\geq 6\sqrt{aa'bb'cc'}$$

So, we have to prove, $ab'c'+bc'a'+ca'b'+abc'+bca'+acb'\geq 6\sqrt{aa'bb'cc'}$

Then, $AM\geq GM$

$$\frac{a'b'c'+b'c'a'+ca'b'+abc'+bca'+acb'}{6}\geq 6\sqrt{a^3a'^3b^3b'^3c^3c'^3}=\sqrt{aa'bb'cc'}$$

So, it's true.

1069. In ΔABC the following relationship holds:

$$\sqrt{\frac{r_b r_c}{a}}+\sqrt{\frac{r_c r_a}{b}}+\sqrt{\frac{r_a r_b}{c}}\le\sqrt{\frac{s(h_a+h_b+h_c)}{2r}}$$

Proposed by Bogdan Fustei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$LHS=\sum\sqrt{\frac{r_a r_b r_c}{as\tan\frac{A}{2}}}=\sum\sqrt{\frac{rs^2}{4Rs\sin\frac{A}{2}\cos\frac{A}{2}\tan\frac{A}{2}}}=\sum\sqrt{\frac{rs^2}{4Rs}\csc\frac{A}{2}}$$

$$=\sum\sqrt{\frac{rs^2}{4Rs}}\sqrt{\frac{bc(s-a)}{(s-b)(s-c)(s-a)}}$$

$$=\sum\sqrt{\frac{rs^2}{4RS\cdot r^2 s}}\sqrt{bc(s-a)}\overset{CBS}{\le}\sqrt{\frac{1}{4Rr}}\sqrt{\sum ab}\sqrt{\sum(s-a)}=\sqrt{\frac{2R}{4Rr}\cdot\frac{\sum ab}{2R}\cdot s}=\sqrt{\frac{s}{2r}\left(\sum h_a\right)}$$

1070. In ΔABC the following relationship holds:

$$\frac{a(s-a)}{b+c}+\frac{b(s-b)}{c+a}+\frac{c(s-c)}{a+b}\le\frac{3\sqrt{3}R}{4}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Boris Colakovic-Belgrade-Serbie

$$s-a=\frac{a+b+c}{2}-a=\frac{b+c-a}{2};\frac{a(s-a)}{b+c}=\frac{1}{2}\frac{a(b+c-a)}{b+c}=\frac{1}{2}\left(a-\frac{a^2}{b+c}\right)$$

$$s-b=\frac{a+b+c}{2}-b=\frac{a+c-b}{2};\frac{b(s-b)}{c+a}=\frac{1}{2}\frac{b(a+c-b)}{c+a}=\frac{1}{2}\left(b-\frac{b^2}{c+a}\right)$$

$$s-c=\frac{a+b+c}{2}-c=\frac{a+b-c}{2};\frac{c(s-c)}{a+b}=\frac{1}{2}\frac{c(a+b-c)}{a+b}=\frac{1}{2}\left(c-\frac{c^2}{a+b}\right)$$

$$LHS=\frac{1}{2}(a+b+c)-\frac{1}{2}\left(\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\right)\le\frac{1}{2}(a+b+c)-\frac{1}{2}\cdot\frac{(a+b+c)^2}{2(a+b+c)}=$$

$$=\frac{1}{2}\cdot 2s-\frac{1}{4}\cdot\frac{4s^2}{2s}=s-\frac{s}{2}=\frac{s}{2}\le\frac{1}{2}\cdot\frac{3\sqrt{3}}{2}R=\frac{3\sqrt{3}}{4}R$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$s-a=x;\ s-b=y;\ s-c=z;\ s=x+y+z$$

$$\frac{3\sqrt{3}R}{4}\ge\frac{s}{2}\ge\sum\frac{a(s-a)}{b+c}\ \textit{ASSURE};\ \frac{x+y+z}{2}\ge\sum\frac{(y+z)x}{2x+y+z}\ \textit{ASSURE}$$

$$\frac{1}{2}\sum\frac{(y+z)\cdot 2x}{2x+y+z}\overset{GM\le AM}{\le}\frac{1}{2}\sum\frac{\left(\frac{2x+y+z}{2}\right)^2}{2x+y+z}=$$

$$=\frac{1}{8}\sum(2x+y+z)=\frac{1}{8}\cdot 4(x+y+z)=\frac{x+y+z}{2}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$LHS=\sum\frac{a(2s-a-s)}{2s-a}=\sum a-s\sum\frac{a}{2s-a}$$

$$=2s-s\sum\frac{a-2s+2s}{2s-a}=2s-s\sum(-1)-2s^2\sum\frac{1}{b+c}$$

$$=5s-2s^2\frac{\sum(c+a)(a+b)}{2abc+\sum ab(2s-c)}=5s-2s^2\frac{\left(\sum a^2+2\sum ab\right)+\sum ab}{2s(s^2+4Rr+r^2)-4Rrs}$$

$$=5s-2s^2\cdot\frac{5s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}=s\left(5-\frac{5s^2+4Rr+r^2}{s^2+2Rr+r^2}\right)$$

$$=s\frac{6Rr+4r^2}{s^2+2Rr+r^2}\overset{Mitrinovic}{\le}\frac{3\sqrt{3}R}{2}\frac{6Rr+4r^2}{s^2+2Rr+r^2}\overset{?}{\le}\frac{3\sqrt{3}R}{4}$$

$$\Leftrightarrow\frac{4(3Rr+2r^2)}{s^2+2Rr+r^2}\overset{?}{\le}1\Leftrightarrow s^2\underset{(1)}{\overset{?}{\ge}}10Rr+7r^2$$

But, LHS of (1) $\overset{Gerretsen}{\ge}16Rr-5r^2\overset{?}{\ge}10Rr+7r^2$

$\Leftrightarrow 6Rr\overset{?}{\ge}12r^2\Leftrightarrow R^2\overset{?}{\ge}2r\rightarrow$ ***true (Euler) (proved)***

1071. In ΔABC the following relationship holds:

$$\left(\frac{h_a}{aw_a^2}\right)^2+\left(\frac{h_b}{bw_b^2}\right)^2+\left(\frac{h_c}{cw_c^2}\right)^2\ge\frac{1}{R^2(2R^2+r^2)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\left(\frac{h_a}{aw_a^2}\right)^2+\left(\frac{h_b}{bw_b^2}\right)^2+\left(\frac{h_c}{cw_c}\right)^2\geq 3\sqrt[3]{\frac{(h_ah_bh_c)^2}{a^2b^2c^2(w_aw_bw_c)^4}}\ \textbf{(1)}$$

But $w_a\leq\sqrt{s(s-a)}\Rightarrow w_a^4\leq s^2(s-a)^2\Rightarrow\frac{1}{w_a^4}\geq\frac{1}{s^2(s-a)^2}$ **(2)**

From (1)+(2)$\Rightarrow\sum\left(\frac{h_a}{aw_a^2}\right)^2\geq 3\sqrt[3]{\frac{(h_ah_bh_c)^2}{a^2b^2c^2s^6(s-a)^2(s-b)^2(s-c)^2}}$ **(3)**

$$(h_ah_bh_c)^2=\frac{4s^4r^4}{R^2}\ \textbf{(4)}$$

$(abc)^2=16s^2R^2r^2$ **(5) and** $\left((s-a)(s-b)(s-c)\right)^2=s^2r^4$ **(6)**

From (3)+(4)+(5) +(6)$\Rightarrow\sum\left(\frac{h_a}{aw_a^2}\right)^2\geq\frac{3}{\sqrt[3]{4R^4r^2s^6}}$ **(7)**

From (7) we must show this: $\frac{3}{\sqrt[3]{4R^4s^2s^6}}\geq\frac{1}{R^2(2R^2+r^2)}\Leftrightarrow\frac{27}{4R^4r^2s^6}\geq\frac{1}{R^6(2R^2+r^2)^3}\Leftrightarrow$

$$27R^2(2R^2+r^2)^3\geq 4r^2s^6\ \textbf{(8)}$$

But $R\geq 2r\Rightarrow R^2\geq 4r^2$ **(9)**

Form (8)+(9) we must show this: $27(2R^2+r^2)^3\geq s^6\Leftrightarrow 3(2R^2+r^2)\geq s^2$ **(10)**

But from Gerretsen we have: $s^2\leq 4R^2+4Rr+3r^2\Rightarrow$

$s^2\leq 4R^2+4Rr+3r^2\leq 6R^2+3r^2\Leftrightarrow 4Rr\leq 2R^2\Leftrightarrow 2r\leq R$ **true.**

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum\left(\frac{h_a}{aw_a^2}\right)^2\overset{(1)}{\geq}\frac{1}{3}\left(\sum\frac{h_a}{aw_a^2}\right)^2$$

$$\sum\frac{h_a}{aw_a^2}=\sum\frac{2rs}{a}\cdot\frac{1}{a\cdot\frac{4b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc}}$$

$$=\sum\frac{2rs}{a}\cdot\frac{(b+c)^2}{4s(s-a)\cdot 4Rrs}=\sum\frac{(b+c)^2}{8Rsa(s-a)}$$

$$=\frac{1}{8Rs}\sum\frac{(s+s-a)^2}{a(s-a)}=\frac{1}{8Rs}\sum\frac{s^2+(s-a)^2+2s(s-a)}{a(s-a)}$$

$$=\frac{1}{8R}\sum\frac{(s-a)+a}{a(s-a)}+\frac{1}{8Rs}\sum\frac{s-a}{a}+\frac{2s}{8Rs}\sum\frac{1}{a}$$

$$=\frac{1}{8R}\sum\frac{1}{a}+\frac{1}{8R}\sum\frac{1}{s-a}+\frac{1}{8R}\sum\frac{1}{a}+\frac{1}{4R}\sum\frac{1}{a}-\frac{3}{8Rs}$$

$$=\frac{1}{2R}\sum\frac{1}{a}+\frac{1}{8R}\sum\frac{1}{s-a}-\frac{3}{8Rs}=\left(\frac{\sum ab}{2R}\right)\left(\frac{1}{4Rrs}\right)+\frac{4Rr+r^2}{8Rr^2s}-\frac{3}{8Rrs}$$

$$\left(\because\sum(s-a)(s-c)=\sum\left(s^2-s(b+c)+bc\right)=3s^2-4s^2+s^2+4Rr+r^2=4Rr+r^2\right)$$

$$=\frac{\sum h_a}{4Rrs}+\frac{4R+r}{8Rrs}-\frac{3}{8Rs}\overset{\sum h_a\geq 9r}{\geq}\frac{9}{4Rs}-\frac{3}{8Rs}+\frac{4R+r}{8Rrs}$$

$$=\frac{18r-3r+4R+r}{8Rrs}=\frac{4R+16r}{8Rrs}=\frac{R+4r}{2Rrs}\therefore\sum\frac{h_a}{aw_a^2}\overset{(2)}{\geq}\frac{R+4r}{2Rrs}$$

(1), (2)$\Rightarrow LHS\geq\frac{1}{3}\frac{(R+4r)^2}{4R^2r^2s^2}\overset{?}{\geq}\frac{1}{R^2(2R^2+r^2)}\Leftrightarrow(2R^2+r^2)(R+4r)^2\underset{(3)}{\overset{?}{\geq}}12r^2s^2$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\left(\frac{h_a}{a\cdot w_a^2}\right)^2\geq\sum\left(\frac{h_a}{a\cdot s(s-a)}\right)^2=\frac{1}{s^2}\cdot\sum\frac{1^3}{\left(\frac{a(s-a)}{h_a}\right)^2}\geq$$

$$\geq\frac{1}{s^2}\cdot\frac{(1+1+1)^3}{\left(\sum\frac{a(s-a)}{h_a}\right)^2}=\frac{27}{s^2}\cdot\frac{4\Delta^2}{(s\sum a^2-\sum a^3)^2}=$$

$$=\frac{27}{s^2}\cdot 4\Delta^2\cdot\frac{1}{4s^2(s^2-4Rr-r^2-s^2+6Rr+3r^2)^2}$$

$$=\frac{27r^2}{s^2}\cdot\frac{1}{(2Rr+2r^2)^2}=\frac{27r^2}{s^2}\cdot\frac{1}{4r^2(R+r)^2}=$$

$$=\frac{27}{4s^2}\cdot\frac{1}{(R+r)^2}\geq\frac{1}{R^2}\cdot\frac{1}{(R+r)^2}=\frac{1}{R^2}\left(\frac{1}{R^2+2Rr+r^2}\right)\geq\frac{1}{R^2}\cdot\frac{1}{2R^2+r^2}$$

Now, RHS of (3) $\overset{Gerretsen}{\leq}12r^2(4R^2+4Rr+3r^2)\overset{?}{\leq}(2R^2+r^2)(R+4r)^2$

$$\Leftrightarrow 2t^5+16t^3-15t^2-40t-20\overset{?}{\geq}0\Leftrightarrow(t-2)(2t^3+20t^2+25t+10)\overset{?}{\geq}0$$

$\rightarrow$ ***true*** $\because t\overset{Euler}{\geq}2$ ***(Proved)***

1072. **In ΔABC the following relationship holds:**

$$4\sqrt{3}\leq\frac{b^2+c^2}{ar_a}+\frac{c^2+a^2}{br_b}+\frac{a^2+b^2}{cr_c}\leq\frac{3\sqrt{3}}{2}\left(\frac{R}{r}\right)^3-8\sqrt{3}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution by Soumava Chakraborty-Kolkata-India

$$4\sqrt{3}\overset{(a)}{\leq}\frac{b^2+c^2}{ar_a}+\frac{c^2+a^2}{br_b}+\frac{a^2+b^2}{cr_c}\overset{(b)}{\leq}\frac{3\sqrt{3}}{2}\left(\frac{R}{r}\right)^3-8\sqrt{3}$$

$$\frac{b^2+c^2}{ar_a}+\frac{c^2+a^2}{br_b}+\frac{a^2+b^2}{cr_c}$$

$$=\left(\sum a^2\right)\left(\sum\frac{1}{ar_a}\right)-\sum\frac{a}{r_a}=\left(\sum a^2\right)\left(\sum\frac{s-a}{a\Delta}\right)-\sum\frac{a(s-a)}{\Delta}$$

$$=\frac{\sum a^2}{\Delta}\left(s\sum\frac{1}{a}-3\right)-\frac{s(2s)-2(s^2-4Rr-r^2)}{\Delta}$$

$$=\frac{\sum a^2}{\Delta}\left\{\frac{S(S^2+4Rr+r^2)}{4Rrs}-3\right\}-\frac{2(4Rr+r^2)}{\Delta}$$

$$=\frac{(s^2-4Rr-r^2)(s^2-8Rr+r^2)}{2Rr\Delta}-\frac{2(4Rr+r^2)}{\Delta}$$

$$=\frac{s^4-12Rrs^2+r^2(4R+r)(8R-r)-4R(4R+r)r^2}{2Rr\Delta}$$

$$\overset{(c)}{=}\frac{s^4-12Rrs^2+r^2(4R+r)(4R-r)}{2sRr^2}$$

$$\overset{Mitrinovic}{\leq}\frac{S^4-12Rrs^2+r^2(16R^2-r^2)}{2sr^2\frac{2S}{3\sqrt{3}}}\overset{?}{\leq}\frac{3\sqrt{3}}{2}\left(\frac{R}{r}\right)^3-8\sqrt{3}$$

$$\Leftrightarrow\frac{3\{S^4-2Rrs^2+r^2(16R^2-r^2)\}}{4S^2r^2}\overset{?}{\leq}\frac{3R^3}{2r^3}-8=\frac{3R^3-16r^3}{2r^3}$$

$$\Leftrightarrow 3r\{S^4-12Rrs^2+r^2(16R^2-r^2)\}\underset{(1)}{\overset{?}{\leq}}2S^2(3R^3-16r^3)$$

Now, LHS of (1) $\overset{Gerretsen}{\leq}3r\{S^2(4R^2-8Rr+3r^2)+r^2(16R^2-r^2)\}$

$$\overset{?}{\leq}2S^2(3R^3-16r^3)$$

$$\Leftrightarrow S^2\left(6R^3-12R^2r+24Rr^2-\frac{4}{r^3}\right)\underset{(2)}{\overset{?}{\geq}}3r^3(16R^2-r^2)\because 6R^3-12R^2r+24Rr^2-\frac{4}{r^3}$$

$$=(R-2r)(6R^2+24r^2)+7r^3>0\left(\because R\overset{Euler}{\geq}2r\right)$$

$\therefore$ *LHS of (2)* $\overset{Gerretsen}{\geq}(16Rr-5r^2)\left(6R^3-12R^2r+24Rr^2-\frac{4}{r^3}\right)\overset{?}{\geq}3r^3(16R^2-r^2)$

$$\Leftrightarrow 48t^3 - 111t^3 + 198t^2 - 388t + 104 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\{(t-2)(48t^2 + 81t + 330) + 608\} \overset{?}{\geq} 0 \to \textit{true} \because t \overset{Euler}{\geq} 2 \Rightarrow \textit{(b) is true}$$

Also, using (c) & $2s \overset{Mitrinovic}{\leq} 3\sqrt{3}R$*:* $\sum \frac{b^2+c^2}{ar_a} \geq \frac{S^4 - 12Rrs^2 + r^2(16R^2 - r^2)}{3\sqrt{3}R^2r^2} \overset{?}{\geq} 4\sqrt{3}$

$$\Leftrightarrow S^4 - 12Rrs^2 + r^2(16R^2 - r^2) \overset{?}{\geq} 36R^2r^2 \Leftrightarrow S^4 - 12Rrs^2 \underset{(3)}{\overset{?}{\geq}} r^2(20R^2 + r^2)$$

Now, LHS of (3) $\overset{Gerretsen}{\geq} S^2(4Rr - 5r^2)$

$$\overset{Gerretsen}{\geq} r^2(16R - 5r)(4R - 5r) \overset{?}{\geq} r^2(20R^2 + r^2) \Leftrightarrow 11R^2 - 25Rr + 6r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R - 2r)(11R - 2r) \overset{?}{\geq} 0 \to \textit{true} \Rightarrow \textit{(a) is true (Done).}$$

1073. In ΔABC the following relationship holds:

$$\frac{r_a h_a}{a} + \frac{r_b h_b}{b} + \frac{r_c h_c}{c} \leq \frac{3(a+b+c)}{4}$$

Proposed by Bodgan Fustei – Romania

Solution 1 by Marian Ursărescu-Romania

$r_a = \frac{S}{s-a}, h_a = \frac{2S}{a} \Rightarrow$ *inequality becomes:* $2S^2 \sum \frac{1}{a^2(s-a)} \leq \frac{3 \cdot 2s}{4} \Leftrightarrow$

$$s^2r^2 \sum \frac{1}{a^2(s-a)} \leq \frac{3s}{4} \quad (1)$$

But $\sum \frac{1}{a^2(s-a)} = \frac{s^4 - 2s^2(2Rr - r^2) + (4R+r)^3}{16R^2r^2s^3} \quad (2)$

From (1)+(2) we must show: $s^2r^2 \frac{s^4 - 2s^2(2Rr - r^2) + r(4R+r)^3}{16R^2r^2s^3} \leq \frac{3s}{4} \Leftrightarrow$

$$s^4 - 2s^2(2Rr - r^2) + r(4R + r)^3 \leq 12s^2R^2 \Leftrightarrow$$

$$s^2(12R^2 - s^2 + 4Rr - 2r^2) \geq r(4R + r)^3 \quad (3)$$

Now, from Doucet's inequality, we have: $s^2 \geq 3r(4R + r)$ *(4)*

From (3)+(4) we must show this:

$$3r(4R + r)(12R^2 - s^2 + 4Rr - 2r^2) \geq r(4R + r)^3 \Leftrightarrow$$

$$3(12R^2 - s^2 + 4Rr - 2r^2) \geq (4R + r)^2 \Leftrightarrow$$

$$36R^2 - 3s^2 + 12Rr - 6r^2 \geq 16R^2 + 8Rr + r^2 \Leftrightarrow$$

$$20R^2+4Rr\ge 3s^2+7r^2 \quad \textbf{(5)}$$

Now, form Doucet's inequality we have:

$$3s^2\le(4R+r)^2 \quad \textbf{(6)} \Leftrightarrow 3s^2\le 16R^2+8Rr+r^2\Rightarrow$$

$$3s^2+7r^2\le 16R^2+8Rr+8r^2 \quad \textbf{(7)}$$

From (5)+(6) +(7) we must show this: $20R^2+4Rr\ge 16R^2+8Rr+8r^2\Leftrightarrow$

$$4R^2\ge 4Rr+8r^2\Leftrightarrow R^2\ge r(R+2r) \quad \textbf{(8)}$$

But from Euler's inequality we have $R\ge 2r\Rightarrow$

$$R^2\ge 2Rr \quad \textbf{(9)}$$

From (8)+(9) we must show: $2R\ge r(R+2r)\Leftrightarrow 2R\ge R+2r\Leftrightarrow R\ge 2r$ ***(true)***

Observation: Relationship (2) it's from Viète and Newton relations from the equation with the roots a,b,c.

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\frac{r_a\cdot h_a}{a}=\frac{1}{2\Delta}\cdot\sum\frac{2\Delta}{a}r_ah_a=\frac{1}{2\Delta}\cdot\sum r_ah_a^2=$$

$$=\frac{1}{2\Delta}\cdot\Delta\cdot\sum\frac{1}{s-a}\cdot h_a^2=\frac{1}{2}\cdot\sum\frac{s}{s(s-a)}\cdot h_a^2\le$$

$$\left(h_a\le l_a\le\sqrt{s(s-a)}\right)$$

$$\le\frac{1}{2}\sum\frac{s}{s(s-a)}\cdot s(s-a)=\frac{3}{4}(a+b+c)$$

$$\left.\begin{array}{c}s-a=x\\ s-b=y\\ s-c=z\end{array}\right\}$$

$$r_a=\frac{\sqrt{(x+y+z)\cdot xyz}}{x},\dots,r_b,r_c;\ h_a=\frac{2\sqrt{(x+y+z)xyz}}{y+z},\dots,h_b,h_c$$

$$a+b+c=2(x+y+z)$$

$$\sum\frac{r_ah_a}{a}=\sum\frac{\sqrt{(x+y+z)\cdot xyz}}{x}\cdot\frac{2\sqrt{(x+y+z)\cdot xyz}}{y+z}\cdot\frac{1}{\underbrace{y+z}_{a}}=$$

$$=\sum\frac{2(x+y+z)xyz}{x(y+z)^2}=2(x+y+z)\sum\frac{yz}{(y+z)^2}\overset{AM\ge GM}{\le}$$

$$\le 2(x+y+z)\sum\frac{yz}{4yz}=2(x+y+z)\cdot\frac{3}{4}=\left(\frac{x+y+z}{2}\right)\cdot 3=(a+b+c)\cdot 3$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$LHS=\sum\frac{\Delta}{s-a}\cdot\frac{2\Delta}{a}\cdot\frac{1}{a}=\frac{2\Delta^2}{s}\sum\frac{s-a+a}{a^2(s-a)}=\frac{2\Delta^2}{s}\sum\frac{1}{a^2}+\frac{2\Delta^2}{s^2}\sum\frac{s-a+a}{a(s-a)}$$

$$\overset{Goldstone}{\leq}\frac{2r^2s^2}{s}\cdot\frac{1}{4r^2}+\frac{2r^2s^2}{s^2}\sum\frac{1}{a}+\frac{2r^2s^2}{s^2}\sum\frac{1}{s-a}$$

$$=\frac{s}{2}+\frac{2r^2(\sum ab)}{4Rrs}+2r^2\cdot\frac{\sum(s-b)(s-c)}{r^2s}$$

$$=\frac{s}{2}+\frac{r(s^2+4Rr+r^2)}{2Rs}+\frac{2}{s}(3s^2-4s^2+s^2+4Rr+r^2)$$

$$=\frac{s}{2}+\frac{r(s^2+4Rr+r^2)}{2Rs}+\frac{2(4Rr+r^2)}{s}\overset{?}{\leq}\frac{3\cdot 2s}{4}=\frac{3s}{2}$$

$$\Leftrightarrow\frac{r(s^2+4Rr+r^2)+4R(4Rr+r^2)}{2Rs}\overset{?}{\leq}s\Leftrightarrow(2R-r)s^2\underset{(1)}{\overset{?}{\geq}}r(4R+r)^2$$

$$\text{LHS of (1)}\overset{Gerretsen}{\geq}r(2R-r)(16R-5r)\overset{?}{\geq}r(4R+r)^2$$

$$\Leftrightarrow 8R^2-17Rr+2r^2\overset{?}{\geq}0\Leftrightarrow(R-2r)(8R-r)\overset{?}{\geq}0\to\textbf{true}\because R\overset{Euler}{\geq}2r\ \textbf{(Proved).}$$

1074. In ΔABC the following relationship holds:

$$\frac{2m_am_bm_c}{h_ah_bh_c}\geq 1+\frac{r_a^2+r_b^2+r_c^2}{r_ar_b+r_br_c+r_cr_a}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$LHS=\frac{2\prod m_a}{\prod h_a}+1\geq 2+\frac{\sum r_a^2}{\sum r_br_c}=RHS$$

$$\textit{1) LHS: }\frac{2\prod m_a}{\prod h_a}+1\geq\frac{2\cdot\prod\left(\frac{b+c}{2}\right)\cdot\cos\frac{A}{2}}{\prod\frac{bc}{2R}}+1=$$

$$=\frac{2R^2\cdot\prod(b+c)\cdot\sqrt{\frac{s(s-a)}{bc}}}{(abc)^2}+1=\frac{\frac{2R^2s\cdot\Delta}{abc}\cdot\prod(b+c)}{(abc)^2}+1=$$

$$=\frac{2R^3\cdot s\cdot\Delta}{(abc)^3}\left(\sum a\sum ab-abc\right)+1=\frac{2R^3\cdot s\cdot\Delta\cdot 2s(s^2+2Rr+r^2)}{64R^3s^3r^3}+1=$$

$$= \frac{1}{16r^2}(s^2 + 2Rr + r^2) + 1 = \frac{s^2+2Rr+17r^2}{16r^2} \quad (*)$$

$$2)\ 2 + \frac{\sum r_a^2}{\sum r_b r_c} = \frac{\sum r_a^2 + 2\cdot\sum r_b r_c}{\sum r_b r_c} = \frac{(\sum r_a)^2}{\sum r_b r_c} =$$

$$= \frac{(4R+r)^2}{\Delta^2\cdot\sum\frac{1}{(s-b)(s-c)}} = \left(\frac{4R+r}{s}\right)^2 \quad (**)$$

$$(*), (**) \Rightarrow \frac{s^2+2Rr+17r^2}{16r^2} \ge \frac{(4R+r)^2}{s^2} \quad (ASSURE)$$

$$s^2(s^2 + 2Rr + 17r^2) \ge 16r^2(4R + r)^2 \quad (s^2 \ge 16Rr - 5r)^2$$

$$(16Rr - 5r^2)(16Rr - 5r^2 + 2Rr + 17r^2) \ge 16r^2(4R + r)^2$$

$$2r^2(16R - 5r)(9R + 6r) \ge 16r^2\,(4R + r)^2$$

$$(16R - 5r)(9R + 6r) \ge 8(4R + r)^2 \quad \left(\frac{R}{r} = t\right)$$

$$(16t - 5)(9t + 6) \ge 8(4t + 1)^2$$

$$144t^2 - 45t + 96t - 30 \ge 128t^2 + 64t + 8$$

$$16t^2 - 13t - 38 \ge 0;\ (t - 2)(16t + 19) \ge 0$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\because m_a \ge \sqrt{s(s-a)}, \text{etc} \therefore LHS \ge \frac{2\sqrt{s(s-a)s(s-b)s(s-c)}}{\frac{16R^2r^2s^2}{8R^3}}$$

$$= \frac{16R^3rs^2}{16R^2r^2s^2} = \frac{R}{r} \therefore \text{it suffices to prove: } \frac{R}{r} \ge 1 + \frac{\sum r_a^2}{\sum r_a r_b}$$

$$\Leftrightarrow \frac{R-r}{r} \ge \frac{(4R+r)^2 - 2s^2}{s^2} \Leftrightarrow (R-r)s^2 + 2rs^2 \ge r(4R+r)^2$$

$$\Leftrightarrow (R+r)s^2 \overset{(1)}{\ge} r(4R+r)^2$$

$$\text{Now, LHS of (1)} \overset{Gerretsen}{\ge} (R+r)(16Rr - 5r^2) \overset{?}{\ge} r(4R+r)^2$$

$$\Leftrightarrow 16R^2 + 11Rr - 5r^2 \overset{?}{\ge} 16R^2 + 8Rr + r^2 \Leftrightarrow 3Rr \overset{?}{\ge} 6r^2 \to \text{true (Euler) (Done)}$$

1075. In ΔABC the following relationship holds:

$$4\left(\sum_{cyc} m_a(h_b - h_c)\right)^2 < 9\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} h_a^2\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Vietnam

$$\left[\sum m_a(h_b-h_c)\right]^2 \le \left[\sum m_a|h_b-h_c|\right]^2 \overset{BCS}{\le} \sum m_a^2 \cdot \sum (h_b-h_c)^2$$

$$= \frac{9}{4}\left(\sum a^2\right)\sum(h_b-h_c)^2 = \frac{3}{4}\left(\sum a^2\right)\left\{2\left(\sum h_a^2 - \sum h_a h_b\right)\right\} \ (*)$$

We must show that: $2\left(\sum h_a^2 - \sum h_a h_b\right) < 3\sum h_a^2 \Leftrightarrow -2\sum h_a h_b < \sum h_a^2$

(It is true because: $h_a, h_b, h_c > 0$***)*** $\Rightarrow$ ***(*)*** $< \frac{9}{4}\left(\sum a^2\right)\sum h_a^2$

$$\Rightarrow 4\left[\sum m_a(h_b-h_c)\right]^2 < 9\left(\sum a^2\right)\left(\sum h_a^2\right)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\because \left(\sum x\right)^2 \le 3\sum x^2, \therefore LHS;\ m_a < \frac{b+c}{2} \text{ etc}$$

$$\le 12\sum m_a^2(h_b-h_c)^2 \le \frac{12}{4}\sum (b+c)^2(h_b-h_c)^2$$

$$= 3\sum (b+c)^2\frac{(ca-ab)^2}{4R^2} \overset{?}{<} 9\left(\sum a^2\right)\left(\frac{\sum b^2c^2}{4R^2}\right)$$

$$\Leftrightarrow \sum a^2(b^2-c^2)^2 \overset{?}{<} 3\left(\sum a^2\right)\left(\sum a^2b^2\right)$$

$$\Leftrightarrow 2\sum a^4b^2 + 2\sum a^2b^4 + 15a^2b^2c^2 \overset{?}{>} 0$$

$\rightarrow$ ***true*** $\Rightarrow$ ***given inequality is true (proved)***

1076. In ΔABC the following relationship holds:

$$\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \ge \frac{1}{2}\left(\frac{h_b+h_c}{h_a} + \frac{h_c+h_a}{h_b} + \frac{h_a+h_b}{h_c}\right)$$

Proposed by Bogdan Fustei-Romania

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum \frac{m_a}{h_a} \ge \sum \frac{\frac{b^2+c^2}{4R}}{\frac{bc}{2R}} = \frac{1}{2}\sum \frac{b^2+c^2}{bc} = \frac{1}{2}\sum \frac{ab^2+ac^2}{abc} =$$

$$=\frac{1}{2}\sum\frac{bc^2+c^2a}{abc}=\frac{1}{2}\sum\frac{bc+ca}{ab}=\frac{1}{2}\sum\frac{\frac{bc}{2R}+\frac{ca}{2R}}{\frac{ab}{2R}}=\frac{1}{2}\sum\frac{h_a+h_b}{h_c}$$

1077. In acute ΔABC the following relationship holds:

$$a\cos A+b\cos B+c\cos C\le\frac{3\sqrt{3}R}{2}$$

Proposed by Daniel Sitaru – Romania

Solution by Lahiru Samarakoon-Sri Lanka

$$\sum 2R\sin A\sin B\le\frac{3\sqrt{3}}{2}R$$

$$R\sum\sin 2A\le\frac{3\sqrt{3}}{2}R\Rightarrow 4R\sin A\cos B\cos C\le\frac{3\sqrt{3}}{2}R$$

We have to prove, $\sin A\cos B\cos C\le\frac{3\sqrt{3}}{8}$. *But,* $\frac{\sum\sin A}{3}\le\cos\left(\frac{A+B+C}{3}\right)=\frac{\sqrt{3}}{2}$

GM ≤ AM: $\frac{\sum\cos A}{3}\ge\sqrt[3]{\sin A\sin B\cos C}$. *So,* $\sin A\sin B\cos C\le\left(\frac{\sqrt{3}}{2}\right)^3=\frac{3\sqrt{3}}{8}$. *So, it's true.*

1078. In ΔABC the following relationship holds:

$$\frac{am_a^5+bm_b^5+cm_c^5}{(am_a+bm_b+cm_c)^5}\ge\frac{1}{729R^4}$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Dong Thap-Vietnam

$\because f(x)=x^5\,(x>0)\Rightarrow f''(x)=20x^3>0\,(x>0)$. *Using Jensen's inequality:*

$$\sum am_a^5=2s\sum\frac{a}{2s}m_a^5\ge 2s\sum\left(\frac{a}{2s}\cdot m_a\right)^5=\frac{1}{(2s)^4}\sum(am_a)^5\Leftrightarrow\frac{\sum am_a^5}{\sum(am_a)^5}\ge\frac{1}{16s^4}$$

Must show that: $\frac{1}{16s^4}\ge\frac{1}{729R^4}\Leftrightarrow 729R^4\ge 16s^4$. *It is true because:*

$$\because s\le\frac{3\sqrt{3}}{2}R\Rightarrow s^4\le\frac{729}{16}R^4\Leftrightarrow 729R^4\ge 16s^4$$

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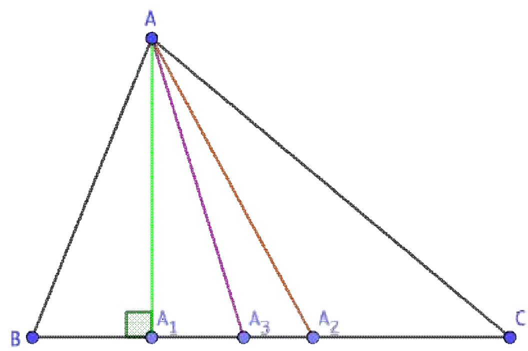
1079. In acute ΔABC with sides different in pairs, AA_1, BB_1, CC_1 – altitudes, AA_2, BB_2, CC_2 – medians, AA_3, BB_3, CC_3 – symedians. Prove that:

$$\frac{A_2A_3}{A_2A_1}+\frac{B_2B_3}{B_2B_1}+\frac{C_2C_3}{C_2C_1}>\frac{108r^2}{a^2+b^2+c^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India



Let $BA_3=m$ *&* $CA_3=n$*. Then,* $\frac{m}{n}=\frac{c^2}{b^2}$ *(&* $m+n=a$*)* $\therefore \frac{m+n}{n}=\frac{c^2+b^2}{b^2}$

$$\Rightarrow \frac{a}{n}=\frac{c^2+b^2}{b^2}\Rightarrow n=\frac{ab^2}{c^2+b^2}\Rightarrow m=\frac{c^2}{b^2}n=\frac{c^2}{b^2}\cdot\frac{ab^2}{b^2+c^2}=\frac{ac^2}{b^2+c^2}$$

$$\Rightarrow BA_3\overset{(i)}{=}\frac{ai^2}{b^2+c^2}\therefore A_2A_3=BA_1-BA_3$$

$$\overset{by\,(i)}{=}\frac{a}{2}-\frac{ai^2}{b^2+c^2}=\frac{a(b^2+c^2)-2ai^2}{2(b^2+c^2)}\overset{(1)}{=}\frac{a(b^2-c^2)}{2(b^2+c^2)}$$

From $\Delta ABA,\frac{BA_1}{c}=\cos B\Rightarrow BA_1=c\cos B=\frac{c(c^2+a^2-b^2)}{2ca}\overset{(ii)}{=}\frac{c^2+a^2-b^2}{2a}$

$$\therefore A_2A_1=BA_2-BA_1\overset{by\,(ii)}{=}\frac{a}{2}-\frac{c^2+a^2-b^2}{2a}=\frac{a^2-(c^2+a^2-b^2)}{2a}\overset{(2)}{=}\frac{b^2-c^2}{2a}$$

(1), (2) $\Rightarrow \frac{A_2A_3}{A_2A_1}\overset{(a)}{=}\frac{a^2}{b^2+c^2}$*. Similarly,* $\frac{B_2B_3}{B_2B_1}\overset{(b)}{=}\frac{b^2}{c^2+a^2}$ *&* $\frac{c_2c_3}{c_2c_1}\overset{(c)}{=}\frac{c^2}{a^2+b^2}$

(a)+(b)+(c)$\Rightarrow LHS=\sum\frac{a^2}{b^2+c^2}\overset{Nesbitt}{>}3\overset{?}{>}\frac{108r^2}{\sum a^2}\Leftrightarrow\sum a^2\underset{(3)}{\overset{?}{>}}36r^2$

But $\sum a^2\overset{Ionescu-Weitzenbock}{>}4\sqrt{3}rs\overset{Mitrinovic}{>}4\sqrt{3}r(3\sqrt{3}r)=36r^2\Rightarrow$ ***(3) is true (Proved)***

Solution 2 by Tran Hong-Dong Thap-Vietnam

Let $S=[ABC]\Rightarrow[ABA_2]=[ACA_2]=\frac{S}{2}\because S_1=[ABA_3],S_2=[ACA_3]=S_2,S_1+S_2=S$;

$$\text{More: }\frac{S_1}{S_2}=\frac{c^2}{b^2}\overset{S_1+S_2=S}{\Rightarrow}\begin{cases}S_1=\frac{c^2}{b^2+c^2}S\\ S_2=\frac{b^2}{b^2+c^2}S\end{cases}\therefore\frac{A_2A_3}{A_2A_1}=\frac{\left(\frac{1}{2}\right)\cdot A_2A_3\cdot AA_1}{\left(\frac{1}{2}\right)A_2A_1\cdot AA_1}=\frac{[AA_2A_3]}{[AA_1A_2]}$$

$$[AA_2A_3]=S_2-\frac{S}{2}=\frac{b^2}{b^2+c^2}\cdot S-\frac{S}{2}=\frac{b^2-c^2}{b^2+c^2}\cdot\frac{S}{2};$$

$$[AA_1A_2]=\frac{S}{2}-[ABA_1]=\frac{S}{2}-\frac{c^2+a^2-b^2}{2a^2}S=\frac{b^2-c^2}{a^2}\cdot\frac{S}{2}$$

$$\text{Because: }\because[ABA_1]=\frac{1}{2}\cdot AA_1\cdot BA_1=\frac{c^2+a^2-b^2}{2a^2}\cdot S$$

$$\text{With: }AA_1=\frac{2S}{a},\text{ and }BA_1=\sqrt{c^2-\frac{4S^2}{a^2}}=\sqrt{\frac{a^2c^2-4S^2}{a^2}}=\sqrt{\frac{a^2c^2-\frac{1}{4}\{2\sum a^2b^2-\sum a^4\}}{a^2}}$$

$$=\sqrt{\frac{(a^2+c^2-b^2)^2}{4a^2}}=\frac{a^2+c^2-b^2}{2a}.\text{ Hence, }\frac{A_2A_3}{A_2A_1}=\frac{[AA_2A_3]}{[AA_2A_1]}=\frac{a^2}{b^2+c^2};\text{ (etc)}$$

$$\Rightarrow LHS=2\sum\frac{a^2}{b^2+c^2}\overset{(Schwarz)}{>}\frac{(a+b+c)^2}{\sum a^2}.\text{ Must show that: }(a+b+c)^2>108r^2$$

$$\Leftrightarrow 4s^2>108r^2\Leftrightarrow s^2>27r^2\Leftrightarrow s>3\sqrt{3}r\ \text{(true) Proved.}$$

1080. If in ΔABC, $a\le b\le c$ then:

$$h_a^{20}-h_b^{20}+h_c^{20}\ge(h_a-h_b+h_c)^{20}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$(h_a-h_b+h_c)^{20}\le h_a^{20}-h_b^{20}+h_c^{20}\quad (*)$$

$$a\le b\le c\Rightarrow h_a\ge h_b\ge h_c.\text{ Let }h_a=kh_c;h_b=mh_c(k\ge m\ge 1)$$

$$(*)\Leftrightarrow(k-m+1)^{20}\le k^{20}-m^{20}+1.\text{ Let }f(x)=k^{20}-m^{20}+1-(k-m+1)^{20}$$

$$(\text{with }k\ge m\ge 1)\Rightarrow f'(k)=20k^{19}-20(k-m+1)^{19}$$

$$k^{19}\ge(k-m+1)^{19}\Leftrightarrow k\ge k-m+1\Leftrightarrow m\ge 1\ \text{(true)}$$

$$\Rightarrow f'(k)\ge 0\Rightarrow f(k)\nearrow[1;+\infty)$$

$$\text{Then: }k\ge m\ge 1\Rightarrow f(k)\ge f(m)=m^{20}-m^{20}+1-(m-m+1)^{20}=0\Rightarrow(*)\text{ true.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$h_a^{20} - h_b^{20} + h_c^{20} \overset{(1)}{\geq} (h_a - h_b + h_c)^{20}$$

$$(1) \Leftrightarrow h_a^{20} - h_b^{20} \geq (h_a - h_b + h_c)^{20} - h_c^{20}$$

$$\Leftrightarrow (h_a - h_b)(h_a^{19} + h_a^{18}h_b + h_a^{17}h_b^2 + \cdots + h_a^2h_b^{17} + h_ah_b^{18} + h_b^{19})$$

$$\geq (h_a - h_b)\begin{bmatrix}(h_a - h_b + h_c)^{19} + (h_a - h_b + h_c)^{18}h_c + (h_a - h_b + h_c)^{17}h_c^2 + \cdots + \\ +(h_a - h_b + h_c)^2h_c^{17} + (h_a - h_b + h_c)^{18} + h_c^{19}\end{bmatrix}$$

$$\Leftrightarrow (h_a - h_b)\begin{bmatrix}\{h_a^{19} - (h_a - h_b + h_c)^{19}\} + \{h_a^{18}h_b - (h_a - h_b + h_c)^{18}h_c\} + \\ + \cdots + \{h_ah_b^{18} - (h_a - h_b + h_c)h_c^{18}\} + \{h_b^{19} - h_c^{19}\}\end{bmatrix} \geq 0$$

$$\Leftrightarrow (h_a - h_b) \overset{(2)}{\geq} 0 \text{ (say). Now, } h_a - h_b = \frac{bc - ca}{2R} = \frac{c(b-a)}{2R} \overset{(i)}{\geq} 0\ (\because b \geq a)$$

Also, $h_a \geq h_a - h_b + h_c \Leftrightarrow h_b \geq h_c \Leftrightarrow ca \geq ab \Leftrightarrow c \geq b \rightarrow$ *true* $\Rightarrow h_a \overset{(ii)}{\geq} h_a - h_b + h_c$

$$\text{Also, } \because ca \geq ab, \therefore h_b \overset{(iii)}{\geq} h_c$$

$$\text{(ii), (iii)} \Rightarrow h_a^{18}h_b \geq (h_a - h_b + h_c)^{18}h_c \Rightarrow h_a^{18}h_b - (h_a - h_b + h_c)^{18}h_c \overset{(a)}{\geq} 0$$

$$\text{Similarly, } h_ah_b^{18} \geq (h_a - h_b + h_c)h_c^{18} \quad \text{(by (ii), (iii))}$$

$$\Rightarrow h_ah_b^{18} - (h_a - h_b + h_c)h_c^{18} \overset{(b)}{\geq} 0$$

Similarly, for the other terms. Also, $h_a \underset{(i)}{\overset{by\,(ii)}{\geq}} (h_a - h_b + h_c)^{19}$ & $h_b^{19} \underset{(d)}{\overset{by\,(iii)}{\geq}} h_c^{19}$

(a),(b),(c),(d), etc $\Rightarrow Q \overset{(iv)}{\geq} 0$; *(iv)·(i)⇒ (2) ⇒ (1) is true (Proved)*

1081. In ΔABC the following relationship holds:

$$\sqrt[5]{\frac{2(s-a)}{c}} + \sqrt[5]{\frac{2(s-b)}{a}} + \sqrt[5]{\frac{2(s-c)}{b}} \leq 3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\text{Let } f(t) = \sqrt[5]{t}(t > 0) \Rightarrow f''(t) = -\frac{4}{25}t^{-\frac{9}{5}} < 0(t > 0);$$

Using Jensen's inequality, we have:

$$LHS \le 3\sqrt[5]{\frac{2\left(\frac{s-a}{c}+\frac{s-b}{a}+\frac{s-c}{b}\right)}{3}} = \Phi$$

WLOG, suppose: $a \ge b \ge c$***. We must show that:*** $\Phi \le 3 \Leftrightarrow \frac{s-a}{c}+\frac{s-b}{a}+\frac{s-c}{b} \le \frac{3}{2}$

$$\Leftrightarrow \frac{b+c-a}{2c}+\frac{a+c-b}{2a}+\frac{a+b-c}{2b} \le \frac{3}{2} \Leftrightarrow \frac{b-a}{c}+\frac{c-b}{a}+\frac{a-c}{b} \le 0$$

$$\Leftrightarrow \frac{a}{b}-\frac{b}{a}+\frac{b}{c}-\frac{c}{b}+\frac{c}{a}-\frac{a}{c} \le 0 \Leftrightarrow \frac{a^2-b^2}{ab}+\frac{b^2-c^2}{cb}+\frac{c^2-a^2}{ac} \le 0$$

$$\Leftrightarrow c(a^2-b^2)+a(b^2-c^2)+b(c^2-a^2) \le 0$$

$$\Leftrightarrow ca^2-cb^2+ab^2-ac^2+bc^2-ba^2 \le 0 \Leftrightarrow (a-c)[b(b-a)-c(b-a)] \le 0$$

$\Leftrightarrow (a-c)(b-c)(b-a) \le 0$ ***(True:*** $a-c \ge 0; b-c \ge 0, b-a \le 0$***) Proved.***

Solution 2 by Boris Colakovic-Belgrade-Serbie

Yet another approach WLOG $b \ge a \ge c$

$$\sqrt[5]{\frac{2(s-a)}{c}} = \sqrt[5]{\frac{2(s-a)c^4}{c^5}} = \frac{1}{c}\sqrt[5]{2(s-a)c^4} \le \frac{1}{c}\cdot\frac{4c+2(s-a)}{5} = \frac{4}{5}+\frac{2(s-a)}{5c}$$

Similarly, $\sqrt[5]{\frac{2(s-b)}{a}} \le \frac{4}{5}+\frac{2(s-b)}{5a}, \sqrt[5]{\frac{2(s-c)}{b}} = \frac{4}{5}+\frac{2(s-c)}{5b}$

LHS$\le \frac{12}{5}+\frac{2(s-a)}{5c}+\frac{2(s-b)}{5a}+\frac{2(s-c)}{5b} \le 3 \Rightarrow \frac{2(s-a)}{5c}+\frac{2(s-b)}{5a}+\frac{2(s-c)}{5b} \le \frac{3}{5} \Leftrightarrow$

$\Leftrightarrow \frac{2(s-a)}{c}+\frac{2(s-b)}{a}+\frac{2(s-c)}{b} \le 3$ ***(1)***

$\left.\begin{array}{l} a = x+y \\ b = y+z \\ c = z+x \end{array}\right\} \Rightarrow x = \frac{a+c-b}{2}; y = \frac{a+b-c}{2}; z = \frac{c+b-a}{2}$ ***(2)***

$$2s = a+b+c = 2(x+y+z)$$

From (1)$\Rightarrow \frac{2z}{z+x}+\frac{2x}{x+y}+\frac{2y}{y+z} \le 3 \Leftrightarrow \frac{x}{x+y}+\frac{y}{y+z}+\frac{z}{z+x} \le \frac{3}{2} \Leftrightarrow$

$$\Leftrightarrow x^2y+y^2z+z^2x-xy^2-yz^2-zx^2 \le 0 \Leftrightarrow \frac{(x-y)^3+(y-z)^3+(z-x)^3}{3} \le 0 \Leftrightarrow$$

$$\Leftrightarrow (x-y)^3+(y-z)^3+(z-x)^3 \le 0 \Leftrightarrow (x-y)(y-z)(z-x) \le 0 \Rightarrow$$

$$\Rightarrow \begin{array}{ccccc} x-y \le 0 & & y-z \ge 0 & & z-x \ge 0 \\ \Downarrow & ; & \Downarrow & ; & \Downarrow \\ y \ge x & & y \ge z & & z \ge x \end{array}$$

$y \ge z \ge x \Rightarrow$ ***From (2)*** $b \ge a \ge c$

1082. In ΔABC the following relationship holds:

$$\frac{b^2+c^2-a^2}{\sqrt{r_br_c}}+\frac{c^2+a^2-b^2}{\sqrt{r_cr_a}}+\frac{a^2+b^2-c^2}{\sqrt{r_ar_b}}\le 4(R+r)$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\frac{b^2+c^2-a^2}{\sqrt{r_br_c}}=\frac{b^2+c^2-a^2}{\sqrt{\frac{S}{s-b}\cdot\frac{S}{s-c}}}=\frac{(b^2+c^2-a^2)\sqrt{(s-b)(s-c)}}{S}$$

$$\overset{AM-GM}{\le}\frac{(b^2+c^2-a^2)}{S}\cdot\frac{s-b+s-c}{2}=\frac{(b^2+c^2-a^2)a}{2S}=\frac{2abc\cos A}{2S}=\frac{abc\cos A}{S}$$

Similarly: $\frac{c^2+a^2-b^2}{\sqrt{r_cr_a}}\le\frac{abc\cos B}{S}$ and $\frac{a^2+b^2-c^2}{\sqrt{r_ar_b}}\le\frac{abc\cos C}{S}$

$$\Rightarrow LHS\le\frac{abc(\cos A+\cos B+\cos C)}{S}=\frac{4RS}{S}\left(1+\frac{r}{R}\right)=4R\left(1+\frac{r}{R}\right)=4(R+r).$$ *(Proved).*

1083. In ΔABC the following relationship holds:

$$\frac{2r}{h_a}\left(\frac{1}{h_a^2}+\frac{1}{h_c^2}\right)\le\left(\frac{R}{S}\right)^2$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Marian Ursărescu-Romania

$$h_a=\frac{2S}{a}\Rightarrow \text{inequality}\Leftrightarrow\frac{\frac{2S}{s}}{\frac{2S}{a}}\left(\frac{b^2+c^2}{4S^2}\right)\le\frac{R^2}{S^2}\Leftrightarrow r=\frac{S}{s}, s=a+b+c$$

$$\frac{a}{s}\left(\frac{b^2+c^2}{4}\right)\le R^2\Leftrightarrow a(b^2+c^2)\le 4sR^2\quad (1)$$

But in any ΔABC we have: $\frac{b}{c}+\frac{c}{b}\le\frac{R}{r}$ (2) $\Leftrightarrow$

$$\Leftrightarrow b^2+c^2\le\frac{R}{r}bc\Rightarrow a(b^2+c^2)\le\frac{R}{r}\cdot abc\quad (3)$$

But in any ΔABC we have $abc=4sRr$ (4)

From (3)+(4)$\Rightarrow a(b^2+c^2)=4sR^2\Rightarrow$ (1) is true.

Observation: For relationship (2) we use Ravi substitution

(2) $\Leftrightarrow \frac{(x+y)(y+z)(z+x)}{4xyz} \geq \frac{x+z}{x+y} + \frac{x+y}{x+z} \Rightarrow$

$$\frac{y+z}{4xyz} \geq \frac{1}{(x+y)^2} + \frac{1}{(x+z)^2} \quad (5)$$

But $\frac{1}{(x+y)^2} \leq \frac{1}{4xy}$ **(6)** $\Leftrightarrow (x-y)^2 \geq 0$; $\frac{1}{(x+z)^2} \leq \frac{1}{4xz} \Leftrightarrow$ **(7)** $(x-z)^2 \geq 0$

From (6) + (7) ⇒ (5) it is true.

Solution 2 by Lahiru-Samarakoon-Sri Lanka

For $\Delta ABC, \frac{2r}{h_a}\left(\frac{1}{h_b^2} + \frac{1}{h_c^2}\right) \leq \left(\frac{R}{S}\right)^2$; $LHS = \frac{2r}{h_a}\left(\frac{b^2}{4S^2} + \frac{c^2}{4S^2}\right) = \frac{2R}{4S^2 h_a}(b^2 + c^2)$

But, $m_a \geq \frac{(b^2+c^2)}{4R} \leq \frac{2r}{4S^2 H_a} 4Rm_a = \frac{2rR}{S^2} \times \left(\frac{m_a}{h_a}\right)$. **So, then** $\frac{m_a}{h_a} \leq \frac{R}{2r}$ **therefore**

$$= \frac{2rR}{S^2} \times \frac{R}{2r} = \left(\frac{R}{S}\right)^2 \textit{ (proved)}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{2r}{h_a}\left(\frac{1}{h_b^2} + \frac{1}{h_c^2}\right) \overset{?}{\leq} \left(\frac{R}{S}\right)^2$$

$$(1) \Leftrightarrow \frac{2r}{\frac{2rs}{a}}\left(\frac{b^2+c^2}{4S^2}\right) \leq \frac{R^2}{S^2} \Leftrightarrow \frac{a}{s}\left(\frac{b^2+c^2}{4}\right) \leq \frac{a^2b^2c^2}{16s(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow ab^2c^2 \overset{(2)}{\geq} 4(b^2 + c^2)(s-a)(s-b)(s-c)$$

Let $s - a = x, s - b = y, s - c = z$ **of course,** $x, y, z > 0$

Then $a = y + z, b = z + x, c = x + y$

Using above substitution, (2) $\Leftrightarrow$

$$(y + z)(z + x)^2(x + y)^2 - 4xyz\{(z + x)^2 + (x + y)^2\} \geq 0$$

$$\Leftrightarrow x^4y + x^4z + 2x^3y^2 + 2x^3z^2 + x^2y^3 + x^2z^3 + 4xy^2z^2 + y^3z^2 + y^2z^3 \overset{(3)}{\geq}$$

$$\geq 4x^3yz + 3x^2y^2z + 3x^2yz^2 + 2xy^3z + 2xyz^3$$

Now, $x^3y + x^4z + xy^2z^2 \overset{A-G}{\geq} 3x^3yz \quad (a)$

Also, $\frac{x^3y^2+x^3z^2}{2} \overset{A-G}{\geq} x^3yz \quad (b)$

(a), (b) ⇒ in order to prove (3), it suffices to prove:

$$3x^3y^2 + 3x^3z^2 + 2x^2y^3 + 2x^2z^3 + 6xy^2z^2 + 2y^3z^2 + 2y^2z^3 \overset{(4)}{\geq} 6x^2y^2z + 6x^2yz^2 +$$

$$+4xy^3z+4xyz^3$$

Now, $3x^3y^2+3xy^2z^2\overset{A-G}{\geq}6x^2y^2z$ *(i)*

Also, $3x^3z^2+3xy^2z^2\overset{A-G}{\geq}6x^2yz^2$ *(ii)*

Again, $2x^2y^3+2y^3z^2\overset{A-G}{\geq}4xy^3z$ *(iii)*

$2x^2z^3+2y^2z^3\overset{A-G}{\geq}4xyz^3$ *(iv)*

(i)+(ii)+(iii)+(iv)⇒ (4) is true (proved)

Solution 4 by Bogdan Fustei-Romania

$h_a=\frac{2S}{a}$ *(and the analogs)* $\Rightarrow \frac{\frac{2S}{s}}{\frac{2S}{a}}\left(\frac{b^2+c^2}{4S^2}\right)\leq\left(\frac{R}{S}\right)^2$

$$r=\frac{S}{s};s=\frac{a+b+c}{2}\Rightarrow\frac{a}{s}\frac{(b^2+c^2)}{4}\leq R^2$$

$$\left.\begin{array}{r}a(b^2+c^2)\leq 4R^2s=R\cdot 4Rs\\ abc=4RS=4Rrs\end{array}\right\}\Rightarrow a(b^2+c^2)\leq\frac{R}{r}abc\Rightarrow\frac{b^2+c^2}{bc}\leq\frac{R}{r}$$

$\frac{b}{c}+\frac{c}{b}\leq\frac{R}{r}$ *(and the analogs)*

We will prove that $\frac{b}{c}+\frac{c}{b}\leq\frac{R}{r}$ *(and the analogs)*

Method I: $l_a^2\leq s(s-a)$ *(and the analogs)*

$h_a\leq l_a$ *(and the analogs)*

$$l_b^2+l_c^2\leq s(s-b)+s(s-c)=s(2s-b-c)=as$$

$h_b^2+h_c^2\leq l_b^2+l_c^2\Rightarrow h_b^2+h_c^2\leq as$ *(and the analogs)*

$$\left.\begin{array}{r}h_b=\dfrac{2S}{b}\\ h_c=\dfrac{2S}{c}\end{array}\right\}\Rightarrow\frac{4S^2}{b^2}+\frac{4S^2}{c^2}\leq as\Leftrightarrow 4S^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)\leq as|\cdot\frac{bc}{S}$$

$$4Sbc\left(\frac{1}{b^2}+\frac{1}{c^2}\right)\leq\frac{abc}{S}\cdot s=\frac{4RS}{S}\cdot s=4Rs$$

$$r\left(\frac{1}{b^2}+\frac{1}{c^2}\right)\leq 4Rs\Rightarrow bc\left(\frac{1}{b^2}+\frac{1}{c^2}\right)\leq\frac{R}{r}$$

$\frac{bc}{b^2}+\frac{bc}{c^2}\leq\frac{R}{r}\Rightarrow\frac{c}{b}+\frac{b}{c}\leq\frac{R}{r}$ *(and the analogs)*

Method II: $\frac{m_a}{s_a}=\frac{b^2+c^2}{2bc}$ *(and the analogs)*

$\frac{m_a}{s_a}=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$ *(and analogs). From* $h_a\le s_a$ *(and the analogs)*

$$\frac{m_a}{s_a}\le\frac{m_a}{h_a}\le\frac{R}{2r}\Rightarrow\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)\le\frac{R}{2r}\Rightarrow\frac{c}{b}+\frac{b}{c}\le\frac{R}{r}$$

1084. In ΔABC the following relationship holds:

$$\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C}\le 3$$

Proposed by Mustafa Tarek-Cairo-Egypt

Solution 1 by Marian Ursărescu-Romania

In any ΔABC, *we have:* $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s}{r}$ *(1)*

and $\cot A+\cot B+\cot C=\frac{s^2-r(4R+r)}{2sr}$ *(2)* $s=\frac{a+b+c}{2}$

From (1)+ (2), we must show: $\frac{2s^2}{s^2-r(4R+r)}\le 3\Leftrightarrow 2s^2\le 3s^2-3r(4R+r)\Leftrightarrow$

$12Rr+3r^2\le s^2$ *(3)*

From Gerretsen's inequality, we have: $s^2\ge 16Rr-5r^2$ *(4). From (3) + (4) we must show:* $16Rr-5r^2\ge 12Rr+3r^2\Leftrightarrow 4Rr\ge 8r^2\Leftrightarrow R\ge 2r$ *true*

Solution 2 by Tran Hong-Dong Thap-Vietnam

We have: $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\frac{s}{r}$; $\cot A+\cot B+\cot C=\frac{s^2-r^2-4Rr}{2sr}$

We have shown that: $\frac{\frac{s}{r}}{\frac{s^2-r^2-4Rr}{2sr}}=\frac{2s^2}{s^2-r^2-4Rr}\le 3\Leftrightarrow 2s^2\le 3s^2-3r^2-12Rr$

$\Leftrightarrow 3r^2+12Rr\le s^2$ *(*). But* $s^2\ge 16Rr-5r^2$*. Must show that:*

$16Rr-5r^2\ge 12Rr+3r^2\Leftrightarrow 4Rr\ge 8r^2\Leftrightarrow R\ge 2r$ *(Euler) (Proved)*

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sum\cot\frac{A}{2}=\sum\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}=\sum\sqrt{\frac{s(s-a)^2}{(s-a)(s-b)(s-c)}}=\sqrt{\frac{s}{r^2s}}\sum(s-a)$$

$$=\frac{3s-2s}{r}\overset{(1)}{=}\frac{s}{r}$$

Also, $\sum \cot A \overset{(2)}{=} \frac{\sum a^2}{4rs}$

(1), (2) $\Rightarrow$ *given inequality* $\Leftrightarrow \frac{3\sum a^2}{4rs} \geq \frac{s}{r} \Leftrightarrow 3\sum a^2 \geq (\sum a)^2 \rightarrow$ *true (Proved)*

1085. In ΔABC the following relationship holds:

$$\frac{\csc\frac{A}{2}}{b^2} + \frac{\csc\frac{B}{2}}{c^2} + \frac{\csc\frac{C}{2}}{a^2} \geq \frac{1}{Rr}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

We must show: $\frac{1}{\sin\frac{A}{2}b^2} + \frac{1}{\sin\frac{B}{2}c^2} + \frac{1}{\sin\frac{C}{2}a^2} \geq \frac{1}{Rr}$ *(1)*

But $\frac{1}{\sin\frac{A}{2}b^2} + \frac{1}{\sin\frac{B}{2}c^2} + \frac{1}{\sin\frac{C}{2}a^2} \geq 3\sqrt[3]{\frac{1}{(abc)^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}}$ *(2)*

From (1)+(2) we must show: $\frac{27}{(abc)^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \geq \frac{1}{R^3r^3}$ *(3)*

But $abc = 4sRr$ *and* $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R}$ *(4)*

From (3)+(4) we must show: $\frac{27}{16s^2R^2r^2\cdot\frac{r}{4R}} \geq \frac{1}{R^3r^3} \Leftrightarrow \frac{27}{4s^2Rr^3} \geq \frac{1}{R^3r^3} \Leftrightarrow$

$27R^2 \geq 4s^2$ *(true, because it it's Mitrinovic inequality)*

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\frac{\csc\frac{A}{2}}{b^2} + \frac{\csc\frac{B}{2}}{c^2} + \frac{\csc\frac{C}{2}}{a^2} = \frac{1}{b\sin\frac{A}{2}} + \frac{1}{c^2\sin\frac{B}{2}} + \frac{1}{a^2\sin\frac{C}{2}}$$

$$= \sum \frac{1}{(2R\sin B)^2\sin\frac{A}{2}} = \frac{1}{16R^2}\sum \frac{1}{\sin^2\frac{B}{2}\cos^2\frac{B}{2}\sin\frac{A}{2}}$$

$r = 4R\prod \sin\frac{A}{2} \Rightarrow \frac{1}{Rr} = \frac{1}{4R^2\prod\sin\frac{A}{2}}$. *We need to prove:* $\sum \frac{1}{\sin^2\frac{B}{2}\cos^2\frac{B}{2}\sin\frac{A}{2}} \geq \frac{4}{\prod\sin\frac{A}{2}}$

By AM-GM we have: $\sum \frac{1}{\sin^2\frac{B}{2}\cos^2\frac{B}{2}\sin\frac{A}{2}} \geq \frac{3}{\left(\prod\sin\frac{A}{2}\right)\left(\sqrt[3]{\prod\cos^2\frac{B}{2}}\right)}$. *We must show that:*

$\frac{3}{\sqrt[3]{\prod\cos^2\frac{B}{2}}} \geq 4 \Leftrightarrow \prod\cos^2\frac{B}{2} \leq \frac{27}{64}$. *It is true because:*

$$\prod\cos^2\frac{B}{2} \leq \left(\frac{\sin A + \sin B + \sin C}{4}\right)^2 \leq \frac{\left(\frac{3\sqrt{3}}{2}\right)^2}{16} = \frac{27}{64}$$

Proved

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{\csc\frac{A}{2}}{b^2} + \frac{\csc\frac{B}{2}}{c^2} + \frac{\csc\frac{C}{2}}{a^2} \geq \frac{1}{Rr}$$

$$LHS = \frac{\left(\frac{1}{b}\right)^2}{\sin\frac{A}{2}} + \frac{\left(\frac{1}{c}\right)^2}{\sin\frac{B}{2}} + \frac{\left(\frac{1}{a}\right)^2}{\sin\frac{C}{2}} \overset{Bergstrom}{\geq} \frac{\left(\sum\left(\frac{1}{a}\right)\right)^2}{\sum\sin\frac{A}{2}} \overset{Jensen}{\geq} \frac{(\sum ab)^2}{\left(\frac{3}{2}\right)16R^2r^2s^2}$$

$$\left(\because f(x) = \sin\frac{x}{2} \text{ is concave } \forall x \in (0,\pi)\right) = \frac{(s^2 + 4Rr + r^2)^2}{24R^2r^2s^2} \overset{?}{\geq} \frac{1}{Rr}$$

$$\Leftrightarrow s^4 + r^2(4R + r)^2 + 2s^2(4Rr + r^2) \overset{?}{\geq} 24Rrs^2 \Leftrightarrow s^4 + r^2(4R + r)^2 \underset{(1)}{\overset{?}{\geq}} s^2(16Rr - 2r^2)$$

Now, LHS of (1) $\overset{Gerretsen}{\geq} s^2(16Rr - 5r^2) + r^2(4R + r)^2 \overset{?}{\geq} s^2(16Rr - 2r^2)$

$\Leftrightarrow r^2(4R + r)^2 \overset{?}{\geq} 3r^2s^2 \Leftrightarrow 4R + r \overset{?}{\geq} \sqrt{3}s \to$ ***true (Trucht)*** $\Rightarrow$ ***(1) is true (proved)***

1086. In scalene ΔABC the following relationship holds:

$$\frac{(r_a + r_b)(r_b + r_c)(r_c + r_a)}{(r_a - r)(r_b - r)(r_c - r)} > 25$$

Proposed by Mustafa Tarek-Cairo-Egypt

Solution 1 by Daniel Sitaru – Romania

$$\prod_{cyc}\left(\frac{r_a + r_b}{r_a - r}\right) = \prod_{cyc}\left(\frac{\frac{S}{s-a} + \frac{S}{s-b}}{\frac{S}{s-a} - \frac{S}{s}}\right) = \prod_{cyc}\left(\frac{\frac{s-b+s-a}{(s-a)(s-b)}}{\frac{s-s+a}{s(s-a)}}\right) =$$

$$= \prod_{cyc}\left(\frac{c}{s-b}\cdot\frac{s}{a}\right) = \frac{s^3}{(s-a)(s-b)(s-c)} =$$

$$= \frac{8s^3}{(b+c-a)(c+a-b)(a+b-c)} \overset{PADOA}{\geq} \frac{8s^3}{abc} = \frac{8s^3}{4Rrs} = \frac{2s^2}{Rr} >$$

$$\overset{GERRETSEN}{\geq} \frac{2(16Rr-5r^2)}{Rr} = \frac{32R-5r}{R} = 32 - \frac{5r}{R} \overset{EULER}{\geq} 32 - \frac{5}{2} = 29.5 > 25$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$(r_a + r_b)(r_b + r_c)(r_c + r_a) = 4s^2R$$

$$(r_a - r)(r_b - r)(r_c - r) = \left(4R\sin^2\frac{B}{2}\right)\left(4R\sin^2\frac{C}{2}\right)$$

$$= 64R^3\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2 = 64R^3\left(\frac{r}{4R}\right)^2 = 4Rr^2$$

Must show that: $4s^2R > 25 \cdot 4 \cdot Rr^2 \Leftrightarrow s^2 > 25r^2$

$$\therefore s^2 \geq 16Rr - 5r^2 \Rightarrow 16Rr - 5r^2 > 25r^2 \Leftrightarrow 16Rr > 30r^2 \Leftrightarrow 8R > 15r \Leftrightarrow R > \frac{15}{8}r$$

It is true, because: $R \geq 2r > \frac{15}{8}r$

1087. In ΔABC the following relationship holds:

$$\frac{m_a}{\sqrt{b}} + \frac{m_b}{\sqrt{c}} + \frac{m_c}{\sqrt{a}} \geq \frac{h_a}{\sqrt[4]{bc}} + \frac{h_b}{\sqrt[4]{ca}} + \frac{h_c}{\sqrt[4]{ab}}$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\sum \frac{h_a}{\sqrt[4]{bc}} = \frac{2S}{a\sqrt[4]{bc}} + \frac{2S}{b\sqrt[4]{ca}} + \frac{2S}{c\sqrt[4]{ab}} = 2S\left(\frac{bc\sqrt[4]{a^2bc} + ac\sqrt[4]{b^2ca} + ab\sqrt[4]{abc^2}}{abc\sqrt{abc}}\right)$$

$$\sum \frac{m_a}{\sqrt{b}} \geq \sum \frac{h_a}{\sqrt{b}} = 2S\sum \frac{1}{a\sqrt{b}} = 2S\left(\frac{bc\sqrt{ac} + ac\sqrt{ab} + ab\sqrt{bc}}{abc\sqrt{abc}}\right)$$

We must show that:

$$bc\sqrt{ac} + ac\sqrt{ab} + ab\sqrt{bc} \geq bc\sqrt[4]{a^2bc} + ac\sqrt[4]{b^2ca} + ab\sqrt[4]{abc^2} \quad (*)$$

(Let $x = \sqrt[4]{a^2bc}; y = \sqrt[4]{b^2ca}; z = \sqrt[4]{abc^2} \Rightarrow x^4 = a^2bc; y^4 = b^2ca; z^4 = abc^2$

$$\Rightarrow (xyz)^4 = (abc)^4 \Rightarrow xyz = abc; a = \frac{x^3}{yz}; b = \frac{y^3}{xz}; c = \frac{z^3}{xy}$$

Suppose: $a \le b \le c \Rightarrow x \le y \le z$***).***

$$(*) \Leftrightarrow \left(\frac{yz}{x}\right)^2 \cdot \frac{xz}{y} + \left(\frac{xz}{y}\right)^2 \cdot \frac{xy}{z} + \left(\frac{xy}{z}\right)^2 \cdot \frac{yz}{x} \ge \left(\frac{yz}{x}\right)^2 x + \left(\frac{xz}{y}\right)^2 y + \left(\frac{xy}{z}\right)^2 z$$

$$\Leftrightarrow \frac{z^3y}{x} + \frac{x^3z}{y} + \frac{y^3x}{z} \ge \frac{(yz)^2}{x} + \frac{(xz)^2}{y} + \frac{(xy)^2}{z}$$

$$\Leftrightarrow y^2z^4 + z^2x^4 + x^2y^4 \ge (yz)^3 + (xz)^3 + (xy)^3 \quad (1)$$

$$y^2z^4 + z^2y^4 \ge 2(yz)^3 \quad (2)$$

$$z^2x^4 + x^2z^4 \ge 2(xz)^3 \quad (3)$$

$$x^2y^4 + y^2x^4 \ge 2(xy)^3 \quad (4)$$

$$\overset{(2)+(3)+(4)}{\Rightarrow} (y^2z^4 + z^2x^4 + x^2y^4) + (y^4z^2 + z^4x^2 + x^4z^2) \ge 2[(yz)^3 + (xz)^3 + (xy)^3]$$

But: $y^4z^2 + z^4x^2 + x^4y^2 \le x^2y^4 + y^2z^4 + z^2x^4$

$\Leftrightarrow (x^2 - y^2)(y^2 - z^2)(x^2 - z^2) \le 0$ ***(true because:*** $x \le y \le z$***)***

So, $2[(yz)^3 + (zx)^3 + (xy)^3] \le 2[x^2y^4 + y^2z^4 + z^2x^4] \Rightarrow$ ***(1) true. Proved.***

1088. In $\Delta ABC, K$ – Lemoines' point, the following relationship holds:

$$\frac{m_bm_c}{h_a} + \frac{m_cm_a}{h_b} + \frac{m_am_b}{h_c} \ge \sqrt{3}(\sin A \cdot AK + \sin B \cdot BK + \sin C \cdot CK)$$

Proposed by Mustafa Tarek-Cairo-Egypt

Solution 1 by Tran Hong-Dong Thap-Vietnam

We have: $AK = m_a \cdot \tan\omega \cdot \csc A = m_a \cdot \tan\omega \cdot \frac{1}{\sin A}$

(with w: ***Brocard angle:*** $\omega \le \frac{\pi}{6} \Rightarrow \tan\omega \le \frac{\sqrt{3}}{3}$***)***$\Rightarrow AK \le m_a \cdot \frac{\sqrt{3}}{3} \cdot \frac{1}{\sin A}$***; similarly:***

$$BK \le m_b \cdot \frac{\sqrt{3}}{3} \cdot \frac{1}{\sin B}; CK \le m_c \cdot \frac{\sqrt{3}}{3} \cdot \frac{1}{\sin C} \Rightarrow RHS \le m_a + m_b + m_c$$

$LHS \ge \frac{m_bm_c}{m_a} + \frac{m_cm_a}{m_b} + \frac{m_am_b}{m_c}$ ***(∵ Because:*** $h_a \le m_a \Rightarrow \frac{1}{h_a} \ge \frac{1}{m_a}$ ***(etc))***

We must show that: $\frac{yz}{x} + \frac{xz}{y} + \frac{xy}{z} \ge x + y + z \quad (x = m_a; y = m_b; z = m_c)$

$\Leftrightarrow (yz)^2 + (xz)^2 + (xy)^2 \ge xyz(x + y + z)$. ***It is true because we are using the inequality:*** $X^2 + Y^2 + Z^2 \ge XY + YZ + ZX$ ***with*** $X = yz; Y = xz; Z = xy$

Proved.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\Delta ABC, \sum \frac{m_b m_c}{h_a} \geq \sqrt{3}(AK \sin A + BK \sin B + CK \sin C)$$

We shall first prove: $(\sum a^2)(\sum bcm_a) \overset{(1)}{\geq} 16\sqrt{3}r^2s^3$

LHS of (1) $\overset{m_a \geq h_a \text{ etc}}{\geq} (\sum a^2)(\sum bch_a) \overset{Ionescu-Weitzenbock}{\geq} 4\sqrt{3}rs(\sum bch_a) \overset{?}{\geq} 16\sqrt{3}r^2s^3$

$$\Leftrightarrow \sum bch_a \overset{?}{\geq} 4rs^2 \Leftrightarrow \sum b^2c^2 \underset{(2)}{\overset{?}{\geq}} 8Rs^2$$

But, $\sum b^2c^2 \geq abc(\sum a) = 4Rrs \cdot 2s = 8Rrs^2 \Rightarrow$ ***(2)*** $\Rightarrow$ ***(1) is true.***

$$\Rightarrow \frac{4}{3}\left(\sum m_a^2\right)\left(\sum bc\, m_a\right) \geq 16\sqrt{3}s\Delta^2 \Rightarrow \left(\sum m_a^2\right)\left(\sum bcm_a\right) \overset{(3)}{\geq} 12\sqrt{3}s\Delta^2$$

Applying (3) on a triangle with sides $\frac{2}{3}m_a, \frac{2}{3}m_b, \frac{2}{3}m_c$ ***whose medians are obviously*** $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ ***respectively and area of course*** $= \frac{\Delta}{3}$***, we get:***

$$\left(\sum\left(\frac{1}{4}a^2\right)\right)\left(\sum\left(\frac{4}{9}\cdot\frac{1}{2}\right)m_bm_c\, a\right) \geq 12\sqrt{3}\left(\left(\frac{1}{2}\cdot\frac{2}{3}\right)\sum m_a\right)\frac{\Delta^2}{9}$$

$$\Rightarrow \left(\sum a^2\right)\sum m_bm_ca \geq 8\sqrt{3}r^2s^2\left(\sum m_a\right) \Rightarrow \sum m_bm_c\frac{a}{2rs} \geq \frac{4\sqrt{3}Rrs}{R}\left(\sum \frac{m_a}{\sum a^2}\right)$$

$$\Rightarrow \sum \frac{m_bm_c}{h_a} \geq \sqrt{3}\sum\left(\frac{abcm_a}{R\sum a^2}\right) = \sqrt{3}\sum\left(\frac{a}{2R}\cdot\frac{2bc}{\sum a^2}m_a\right) = \sqrt{3}\sum(\sin A \cdot AK)$$

$$\Rightarrow \sum \frac{m_bm_c}{h_a} \geq \sqrt{3}(AK \sin A + BK \sin B + CK \sin C) \textit{ (proved)}$$

1089. **In $\Delta ABC, n_a, n_b, n_c$ – Nagel's cevians, g_a, g_b, g_c – Gergonne's cevians.**

Find: $\min \Omega$

$$\Omega = \frac{n_a^2 + n_b^2 + n_c^2}{ag_a + bg_b + cg_c}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

We know: $n_a \geq m_a \geq g_a$ $(n_a \geq m_a - Tarek\ Lemma)$

$$n_a^2+n_b^2+n_c^2 \geq m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$$

$$ag_a+bg_b+cg_c \leq am_a+bm_b+cm_c \overset{BCS}{\leq}$$

$$\sqrt{(a^2+b^2+c^2)}\cdot\sqrt{(m_a^2+m_b^2+m_c^2)}=\frac{\sqrt{3}}{2}(a^2+b^2+c^2)$$

$$\Rightarrow \Omega \geq \frac{3}{4}(a^2+b^2+c^2)\cdot\frac{2}{\sqrt{3}}\cdot\frac{1}{a^2+b^2+c^2}=\frac{\sqrt{3}}{2}\Rightarrow \Omega_{\min}=\frac{\sqrt{3}}{2}\Leftrightarrow a=b=c.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Let g_a intersect BC at D. Then $BD=s-b, CD=s-c$

By Stewarts' theorem, $b^2(s-b)+c^2(s-c)=ag_a^2+a(s-b)(s-c)$

$$\Rightarrow ag_a^2=b^2(s-b)+c^2(s-c)-a(s-b)(s-c)\leq as(s-a)$$

$$\Leftrightarrow a(b+c-a)(b+c+a)+a(c+a-b)(a+b-c)-2b^2(c+a-b)-$$

$$-2c^2(a+b-c)\geq 0$$

$$\Leftrightarrow b^3+c^3-bc(b+c)\geq a(b^2+c^2-2bc)\Leftrightarrow (b+c)(b-c)^2-a(b-c)^2\geq 0$$

$$\Leftrightarrow (b+c-a)(b-c)^2\geq 0\rightarrow \textit{true} \therefore ag_a^2\leq as(s-a)\Rightarrow g_a \overset{(a)}{\leq}\sqrt{s(s-a)}$$

Similarly, $g_b \overset{(b)}{\leq}\sqrt{s(s-b)}$ *and,* $g_c \overset{(c)}{\leq}\sqrt{s(s-c)}$

Also, by Mustafa Tarek, $n_a\geq m_a$*, etc* $\Rightarrow \sum n_a^2 \overset{(1)}{\geq}\sum m_a^2=\frac{3}{4}\sum a^2$

Again, by (a), (b), (c):

$$\sum ag_a \leq \sum a\sqrt{s(s-a)}=\sqrt{s}\sum\sqrt{a(s-a)}\sqrt{a}\overset{CBS}{\leq}\sqrt{s}\sqrt{2s}\sqrt{\sum a(s-a)}$$

$$=\sqrt{2}s\sqrt{s(2s)-2(s^2-4Rr-r^2)}=2s\sqrt{4Rr+r^2}\Rightarrow \frac{1}{\sum ag_a}\overset{(2)}{\geq}\frac{1}{2s\sqrt{4Rr+r^2}}$$

(1), (2)$\Rightarrow \frac{\sum n_a^2}{\sum ag_a}\overset{(3)}{\geq}\frac{6(s^2-4Rr-r^2)}{8s\sqrt{4Rr+r^2}}=\frac{3}{4}\cdot\frac{s}{\sqrt{4Rr+r^2}}-\frac{3}{4s}\sqrt{4Rr+r^2}$

Now, $s^2\geq 12Rr+3r^2\Leftrightarrow s^2-16Rr+5r^2+4r(R-2r)\geq 0\rightarrow \textit{true}$

$$\because s^2-16Rr+5r^2 \overset{Gerretsen}{\geq} 0$$

and, $R-2r\overset{Euler}{\geq}0\Rightarrow s\geq\sqrt{3}\sqrt{4Rr+r^2}\Rightarrow\frac{s}{\sqrt{4Rr+r^2}}\overset{(i)}{\geq}\sqrt{3}\therefore -\frac{3}{4s}\sqrt{4Rr+r^2}\underset{(4)}{\overset{by\ (i)}{\geq}}-\frac{3}{4\sqrt{3}}$

(4), (i), (3)$\Rightarrow\frac{\sum n_a^2}{\sum ag_a}\geq\frac{3}{4}\sqrt{3}-\frac{3}{4\sqrt{3}}=\frac{6}{4\sqrt{3}}=\frac{\sqrt{3}}{2}\Rightarrow\Omega\geq\frac{\sqrt{3}}{2}\Rightarrow\Omega_{\min}=\frac{\sqrt{3}}{2}$ *(answer)*

1090. In ΔABC the following relationship holds:

$$3+\cos(A-B)+\cos(B-C)+\cos(C-A)\ge\frac{6h_ah_bh_c}{m_am_bm_c}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\cos(A-B)=\frac{2\sin(A+B)\cos(A-B)}{2\sin C}=\frac{\sin 2A+\sin 2B}{2\sin C}=\frac{\sum\sin 2A-\sin 2C}{2\sin C}$$

$$\stackrel{(1)}{=}\frac{\left(\sum\sin 2A\right)}{2}\left(\frac{1}{\sin C}\right)-\cos C$$

Similarly, $\cos(B-C)\stackrel{(1)}{=}\frac{\sum\sin 2A}{2}\left(\frac{1}{\sin A}\right)-\cos A$ & $\cos(C-A)\stackrel{(3)}{=}\frac{\sum\sin 2A}{2}\left(\frac{1}{\sin B}\right)-\cos B$

(1) + (2) + (3) $\Rightarrow LHS=3+\frac{\sum\sin 2A}{2}\left(\sum\frac{1}{\sin A}\right)-\sum\cos A$

$$=3-1-\frac{r}{R}+\frac{4\sin A\sin B\sin C}{2}\left(\sum\frac{2R}{a}\right)=\frac{2R-r}{R}+4R\left(\frac{abc}{8R^3}\right)\left(\frac{\sum ab}{abc}\right)$$

$$=\frac{2R-r}{R}+\frac{\sum ab}{2R^2}=\frac{4R^2-2Rr+s^2+4Rr+r^2}{2R^2}\stackrel{(a)}{=}\frac{s^2+4R^2+2Rr+r^2}{2R^2}$$

Also, $\frac{\prod m_a}{\prod h_a}\underset{(b)}{\overset{m_a\ge\sqrt{s(s-a)}}{\ge}}\frac{s\cdot rs}{\frac{a^2b^2c^2}{8R^3}}=\frac{rs^2\cdot 8R^3}{16R^2r^2s^2}=\frac{R}{2r}$

(a), (b)⇒ it suffices to prove:

$$\frac{s^2+4R^2+2Rr+r^2}{2R^2}\cdot\frac{R}{2r}\ge 6\Leftrightarrow s^2+4R^2+2Rr+r^2\stackrel{(4)}{\ge}24Rr$$

Now, LHS of (4) $\stackrel{Gerretsen}{\ge}4R^2+18Rr-4r^2\stackrel{?}{\ge}24Rr$

$\Leftrightarrow 2R^2-3Rr-2r^2\stackrel{?}{\ge}0\Leftrightarrow(R-2r)(2R+r)\stackrel{?}{\ge}0\to$ *true* $\because R\stackrel{Euler}{\ge}2r$ *(Done)*

1091. If in $\Delta ABC, r_a=2, r_b=3, r_c=4$ then:

$$2r^2s<\frac{4a}{3}+\frac{8b}{9}+\frac{2c}{3}<rsR$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\sqrt{\sum r_a r_b} = s = \sqrt{2\cdot 3 + 3\cdot 4 + 2\cdot 4} = \sqrt{26}$$

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{13}{12} \Rightarrow r = \frac{12}{13} \Rightarrow R = \frac{\sum r_a - r}{4} = \frac{105}{52}$$

Hence, we must show that: $\frac{288}{169}\sqrt{26} < \frac{4a}{3} + \frac{8b}{9} + \frac{2c}{3} < \frac{315\sqrt{26}}{169}$.

Now: $r_1 r_a = r_2 r_b = r_3 r_c = \Delta = \frac{12}{13}\sqrt{26} \Rightarrow r_1 = \frac{6\sqrt{26}}{13}; r_2 = \frac{4\sqrt{26}}{13}; r_3 = \frac{3\sqrt{26}}{13}$

$$\Rightarrow a = r_2 + r_3 = \frac{7\sqrt{26}}{13}; b = r_1 + r_3 = \frac{9\sqrt{26}}{13}; c = r_2 + r_1 = \frac{10\sqrt{26}}{13}$$

$$\Rightarrow \Omega = \frac{4a}{3} + \frac{8b}{9} + \frac{2c}{3} = \left(\frac{28}{39} + \frac{72}{117} + \frac{20}{39}\right)\sqrt{26} = \frac{24}{13}\sqrt{26}$$

$$\Rightarrow \frac{288}{169}\sqrt{26} < \frac{24}{13}\sqrt{26} < \frac{315\sqrt{26}}{169}.$$ ***Proved.***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$r_a = s\tan\frac{A}{2}, \textit{etc} \therefore \frac{4a}{3} + \frac{8b}{9} + \frac{2c}{3} = \frac{4}{3s}\left(4R\cos^2\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right) +$$

$$+\frac{8}{3s}\left(4R\cos^2\frac{B}{2}\right)\left(s\tan\frac{B}{2}\right) + \frac{2}{3s}\left(4R\cos^2\frac{C}{2}\right)\left(s\tan\frac{C}{2}\right)$$

$$= \frac{4}{3s}\left(\frac{4Rcos^2 A}{2}\right)(2) + \frac{8}{3s}\left(4R\cos^2\frac{B}{2}\right)3 + \frac{2}{3s}\left(4R\cos^2\frac{C}{2}\right)(4)$$

$$= \frac{16R}{3s}\sum\left(2\cos^2\frac{A}{2}\right) = \frac{16R}{3s}\sum(1+\cos A) = \frac{16R(4R+r)}{3sR} = \frac{16(4R+r)}{3s}$$

$$\therefore \frac{4a}{3} + \frac{8b}{9} + \frac{2c}{3} \overset{(1)}{=} \frac{16(4R+r)}{3s} < rsR \Leftrightarrow 3R(rs^2) > 64R + 16r$$

$$\Leftrightarrow 3R(2\cdot 3\cdot 4) > 64R + 16r \Leftrightarrow 8R > 16r \to \textbf{\textit{true (Euler)}}$$

$$(\because \Delta ABC \textit{ is non-equilateral}, \therefore R \textit{ strictly} > 2r) \Rightarrow \frac{4a}{3} + \frac{8b}{9} + \frac{2c}{3} < rsR$$

Again, $\frac{4a}{3} + \frac{8b}{9} + \frac{2c}{3} > 2r^2 s \overset{by\ (1)}{\Leftrightarrow} \frac{16(4R+r)}{3s} > 2r^2 s \Leftrightarrow 16(4R+r) > 6r(rs^2)$

$$\Leftrightarrow 16(4R+r) > 6r(2\cdot 3\cdot 4) \Leftrightarrow 64R > 128r \to \textbf{\textit{true (Euler)}}$$

$$(\because \Delta ABC \textit{ is non-equilateral}, \therefore R \textit{ strictly} > 2r) \Rightarrow 2r^2 s < \frac{4a}{3} + \frac{8b}{9} + \frac{2c}{3}$$

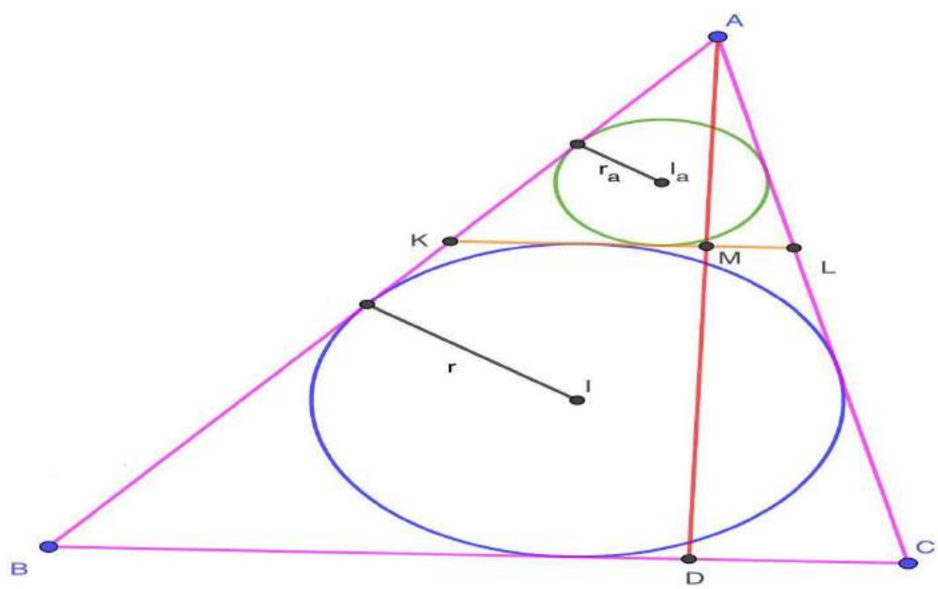
1092. **Let triangle ABC circumscribed to circle $C(I,r)$; let three tangent line at this circle which are parallel with the sides of triangle. In this way are forming other three triangles inside of triangle ABC; if r_1, r_2, r_3 are the rays of the inscribed circles of these three triangles, and**

$m \in R_+$ then prove that:

$$\frac{1}{r_1^m}+\frac{1}{r_2^m}+\frac{1}{r_3^m}\ge\frac{3^{m+1}}{r^m}$$

Proposed by D. M. Batinetu Giurgiu, Neculai Stanciu-Romania

Solution 1 by Omran Kouba-Damascus-Syria



Triangles ABC and AKL are similar.

If $h_a = AD$ is the hight form A in ABC, then $h_a - 2r = AM$ is the hight from A in AKL.

Thus $\frac{r}{r_a}=\frac{h_a}{h_a-2r}=\frac{ah_a}{ah_a-2ra}=\frac{2sr}{2sr-2ar}=\frac{s}{s-a}$ where s is the semiperimer of ABC.

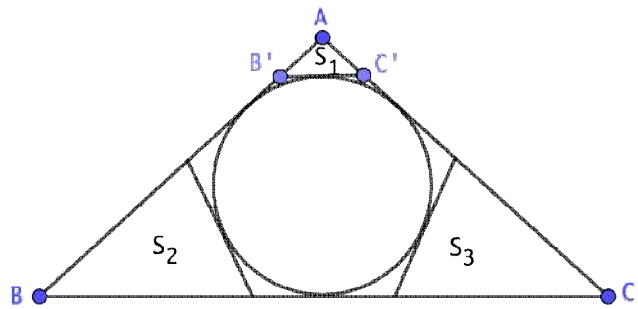
Multiplying similar relations for r_a, r_b and r_c we get:

$$\frac{r}{r_a}\cdot\frac{r}{r_b}\cdot\frac{r}{r_c}=\frac{s^4}{s(s-a)(s-b)(s-c)}=\frac{s^4}{s^2r^2}=\frac{s^2}{r^2}\ge 27$$

Finally, the AM-GM inequality shows that:

$$\frac{r^m}{r_a^m}+\frac{r^m}{r_b^m}+\frac{r^m}{r_c^m}\ge 3\left(\sqrt[3]{\frac{r^3}{r_ar_br_c}}\right)^m = 3^{m+1}$$

Solution 2 by Marian Ursărescu-Romania



$AB'C' \sim ABC \Rightarrow \frac{S_{AB'C'}}{S_{ABC}}=\frac{S_1}{S}=\left(\frac{h_a-2r}{h_a}\right)^2$. ***Similarly,*** $\frac{S_2}{S}=\left(\frac{h_b-2r}{h_b}\right)^2, \frac{S_4}{S}=\left(\frac{h_c-2r}{h_c}\right)^2$ ***(1)***

Let s = ***semiperimeter of*** ABC, s_1 = ***semiperimeter of*** $AB'C'$; s_2 ***of*** S_2, s_3 ***of*** $S_3 \Rightarrow$

$$\frac{s_1}{s}+\frac{s_2}{s}+\frac{s_3}{s}=\frac{h_a-2r}{h_a}+\frac{h_b-2r}{h_b}+\frac{h_c-2r}{h_c}=3-2\left(\frac{r}{h_a}+\frac{r}{h_b}+\frac{r}{h_c}\right)=3-2r\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right) \quad (2)$$

But $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$ ***(3)***

From (2)+(3)$\Rightarrow \frac{s_1}{s}+\frac{s_2}{s}+\frac{s_3}{s}=1$ ***(4)***

From (1)+(4)$\Rightarrow \frac{r_1}{r}+\frac{r_2}{r}+\frac{r_3}{r}=\frac{S_1}{s_1}\cdot\frac{s}{S}+\frac{S_2}{s_2}\cdot\frac{s}{S}+\frac{S_3}{s_3}\cdot\frac{s}{S}=\frac{S_1}{s}\cdot\frac{s}{s_1}+\frac{S_2}{S}\cdot\frac{s}{s_2}+\frac{S_3}{S}\cdot\frac{s}{S_3}=$

$$=\frac{s_1}{s}+\frac{s_2}{s}+\frac{s_3}{s}=1 \Rightarrow r_1+r_2+r_3=r \quad (5)$$

$$\frac{1}{r_1^m}+\frac{1}{r_2^m}+\frac{1}{r_3^m}\ge 3\sqrt[3]{\frac{1}{(r_1r_2r_3)^m}}$$

We must show this: $3\sqrt[3]{\frac{1}{(r_1r_2r_3)^m}}\ge\frac{3^{m+1}}{r^m}\Leftrightarrow\sqrt[3]{\frac{1}{(r_1r_2r_3)^m}}\ge\frac{3^m}{r^m}\Leftrightarrow\frac{1}{\sqrt[3]{r_1r_2r_3}}\ge\frac{3}{r}\Leftrightarrow$

$\Leftrightarrow\sqrt[3]{r_1r_2r_3}\le\frac{r}{3}\Leftrightarrow\sqrt[3]{r_1r_2r_3}\le\frac{r_1+r_2+r_3}{3}$ ***(from 5) it is true.***

1093. In ΔABC the following relationship holds:

$$\frac{2Rs^2}{(R+r)^2}\le\frac{a^2}{h_b}+\frac{b^2}{h_c}+\frac{c^2}{h_a}\le\frac{3R^2}{2S}\sqrt{91R^2-256r^2}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution 1 by Soumava Chakraborty-Kolkata-India

$$a^3+a^3+b^3\overset{A-G}{\geq}3a^2b,\ b^3+b^3+c^3\overset{A-G}{\geq}3b^2c,\ c^3+c^3+a^3\overset{A-G}{\geq}3c^2a$$

Adding the last three, $3\sum a^3\geq 3\sum a^2b\Rightarrow \sum a^2b\overset{(1)}{\leq}\sum a^3$

$$\therefore \sum\frac{a^2}{h_b}=\sum\frac{a^2b}{2S}=\frac{\sum a^2b}{2S}\overset{by\,(1)}{\leq}\frac{\sum a^3}{2S}=\frac{2s(s^2-6Rr-3r^2)}{2S}\overset{Mitrinovic}{\leq}$$

$$\leq\frac{3\sqrt{3}R(s^2-6Rr-3r^2)}{2S}\overset{?}{\leq}\frac{3R^2}{2S}\sqrt{91R^2-256r^2}\Leftrightarrow$$

$$\Leftrightarrow 3(s^2-6Rr-3r^2)^2\overset{?}{\leq}R^2(91R^2-256r^2)$$

$$\Leftrightarrow 3s^4-6s^2(6Rr+3r^2)+3r^2(6R+3r)^2\underset{(a)}{\overset{?}{\leq}}R^2(91R^2-256r^2)$$

Now, LHS of (a) $\overset{Gerrtesen}{\leq}3s^2(4R^2+4Rr+3r^2)-6s^2(6Rr+3r^2)+3r^2(6R+3r)^2$

$$=s^2(12R^2-24Rr-9r^2)+3r^2(6R+3r)^2\overset{?}{\leq}R^2(91R^2-256r^2)$$

$$\Leftrightarrow s^2(12R^2-24Rr)+3r^2(6R+3r)^2\underset{(b)}{\overset{?}{\leq}}R^2(91R^2-256r^2)+9r^2s^2$$

Now, LHS of (b) $\underset{(i)}{\overset{Gerretsen}{\leq}}(4R^2+4Rr+3r^2)(12R^2-24Rr)+3r^2(6R+3r)^2$ ***&***

RHS of (b) $\underset{(ii)}{\overset{?}{\geq}}R^2(91R^2-256r^2)+9r^2(16Rr-5r^2)$

(i),(ii)⇒ in order to prove (b), it suffices to prove:

$$R^2(91R^2-256r^2)+9r^2(16Rr-5r^2)\geq(4R^2+4Rr+3r^2)(12R^2-24Rr)+$$

$$+3r^2(6R+3r)^2\Leftrightarrow 43t^4+48t^3-304t^2+108t-72\geq 0\ \left(t=\frac{R}{r}\right)$$

$\Leftrightarrow (t-2)(43t^3+116t^2+18t(t-2)+36)\geq 0\rightarrow$ ***true*** $\because t\overset{Euler}{\geq}2\Rightarrow$ ***(b)⇒(a) is true***

$$\Rightarrow\sum\frac{a^2}{h_b}\leq\frac{3R^2}{2S}\sqrt{91R^2-256r^2}$$

Again, $\sum\frac{a^2}{h_b}\overset{Bergstrom}{\geq}\frac{4s^2}{\sum h_a}=\frac{8Rs^2}{\sum ab}\overset{?}{\geq}\frac{2Rs^2}{(R+r)^2}\Leftrightarrow s^2+4Rr+r^2\overset{?}{\leq}4(R+r)^2$

$\Leftrightarrow s^2\overset{?}{\leq}4R^2+4Rr+3r^2\rightarrow$ ***true (Gerretsen)*** $\Rightarrow\frac{2Rs^2}{(R+r)^2}\leq\sum\frac{a^2}{h_b}$ ***(proof completed)***

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\sum\frac{a^2}{h_b}\overset{(Schwarz)}{\geq}\frac{(a+b+c)^2}{\sum h_b}=\frac{4s^2}{\frac{s^2+r^2+4Rr}{2R}}=\frac{8s^2R}{s^2+r^2+4Rr}$$

Must show that: $\frac{8s^2R}{s^2+r^2+4Rr}\geq\frac{2Rs^2}{(R+r)^2}\Leftrightarrow 4(R+r)^2\geq s^2+r^2+4Rr$

$$\Leftrightarrow s^2\leq 4R^2+3r^2+4Rr\ \textit{(true)}$$

$$\sum\frac{a^2}{h_b}=\sum\frac{a^2b}{bh_b}=\frac{\sum a^2b}{2S}\leq\frac{\sum a^3}{2S}=\frac{2s(s^2-6Rr-3r^2)}{2S}$$

$\overset{(Lebniz)}{\leq}\frac{3\sqrt{3}R(s^2-6Rr-3r^2)}{2S}$. **Must show that:** $\frac{3\sqrt{3}R(s^2-6Rr-3r^2)}{2S}\leq\frac{3R^2}{2S}\sqrt{91R^2-256r^2}$

$$\Leftrightarrow 3(s^2-6Rr-3r^2)^2\leq R^2(91R^2-256r^2)\ \therefore\ s^2\leq 4R^2+4Rr+3r^2$$

Must show that: $3(4R^2-2Rr)^2\leq R^2(91R^2-256r^2)$

$$\Leftrightarrow 12(2R-r)^2\leq 91R^2-256r^2\Leftrightarrow 48R^2-48Rr+12r^2\leq 91R^2-256r^2$$

$$\Leftrightarrow 268r^2\overset{(1)}{\leq}43R^2+48Rr$$

$\because$ **(1) true because:** $R\geq 2r\Rightarrow 43R^2+48Rr\geq 43\cdot 4r^2+48\cdot 2r^2=268r^2$ **Proved.**

1094. In ΔABC the following relationship holds:

$$\max(\Omega_1,\Omega_2)\leq(s+3R)^2$$

$$\Omega_1=(a+w_a)^2+(b+w_b)^2+(c+w_c)^2$$

$$\Omega_2=(a+h_a)^2+(b+h_b)^2+(c+h_c)^2$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution 1 by Marian Ursărescu-Romania

Because $h_a\leq w_a\Rightarrow\max(\Omega_1,\Omega_2)=\Omega_1\Rightarrow$

$$(a+w_a)^2+(b+w_b)^2+(c+w_c)^2\leq(s+3R)^2$$

But $w_a\leq\sqrt{s(s-a)}\Rightarrow$ **we must show:** $\sum\left(a+\sqrt{s(s-a)}\right)^2\leq(s+3R)^2\Leftrightarrow$

$$\sum a^2+2\sum a\sqrt{s(s-a)}+s^2\leq s^2+6sR+9R^2\Leftrightarrow$$

$$\sum a^2+2\sum a\sqrt{s(s-a)}\leq 6sR+9R^2$$

But $\sum a^2 \le 9R^2 \Rightarrow$ **we must show:** $\sum a\sqrt{s(s-a)} \le 3sR \Leftrightarrow \sum a\sqrt{(s-a)} \le 3\sqrt{s}R$ **(1)**

From Cauchy $\Rightarrow \left(\sum a\sqrt{s-a}\right)^2 \le 3\sum a^2(s-a)$ **(2)**

From (1)+(2) we must show: $3\sum a^2(s-a) \le 9sR \Leftrightarrow \sum a^2\,(s-a) = 3sR^2$ **(3)**

But $\sum a^2(s-a) = 4rs(R+r)$ **(4)**

From (3)+(4) we must show: $4rs(R+r) \le 3sR^2 \Leftrightarrow 4Rr + 4r^2 \le 3R^2$

$$\left.\begin{matrix} But\ R^2 \ge 4r^2 \\ 2R^2 \ge 4Rr \end{matrix}\right\} \Rightarrow 3R^2 \ge 4r^2 + 4Rr$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\max(\Omega_1, \Omega_2) \overset{(1)}{\le} (s+3R)^2$$

$\because w_a \ge h_a$, **etc** $\therefore \sum(a+w_a)^2 \ge \sum(a+h_a)^2 \Rightarrow \max(\Omega_1,\Omega_2) = \Omega_1$

$$\therefore (1) \Leftrightarrow \sum a^2 + 2\sum aw_a + \sum w_a^2 \overset{(2)}{\le} (s+3R)^2$$

WLOG, we may assume $a \ge b \ge c$**. Then** $w_a \le w_b \le w_c$

$$\therefore 2\sum aw_a \overset{Chebyshev}{\le} \frac{2}{3}\left(\sum a\right)\left(\sum w_a\right) \le \frac{2}{3}(2s)\left(\sum m_a\right)$$

$$\overset{(i)}{\le} \frac{2}{3}(2s)(4R+r) = \frac{4s(4R+r)}{3}$$

Also, $\sum w_a^2 \overset{(ii)}{\le} \sum s(s-a) = s^2$ & $\sum a^2 \underset{(iii)}{\overset{Leibnitz}{\le}} 9R^2$

(i) + (ii) + (iii) $\Rightarrow$ **LHS of (2)** $\le 9R^2 + s^2 + \frac{4s(4R+r)}{3}$

$$\overset{?}{\le} (s+3R)^2 = s^2 + 9R^2 + 6sR \Leftrightarrow 18sR \overset{?}{\ge} 4s(4R+r) \Leftrightarrow 2sR \overset{?}{\ge} 4sr$$

$\Leftrightarrow R \overset{?}{\ge} 2r \rightarrow$ **true (Euler) (Proved)**

1095. In acute ΔABC **the following relationship holds:**

$$\frac{m_a^2}{r_b^2 + r_c^2} + \frac{m_b^2}{r_c^2 + r_a^2} + \frac{m_c^2}{r_a^2 + r_b^2} \le \frac{3}{2}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution by Soumava Chakraborty-Kolkata-India

For acute-angled ΔABC, $m_a \leq R(1+\cos A) \Rightarrow m_a \leq 2R\cos^2\frac{A}{2} \Rightarrow m_a^2 \overset{(1)}{\leq} 4R^2\cos^4\frac{A}{2}$

Also, $r_b^2 + r_c^2 \geq \frac{1}{2}(r_b + r_c)^2 = \frac{1}{2}s^2\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)^2$

$$= \frac{s^2}{2}\left(\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)^2 = \frac{s^2}{2}\left(\frac{\cos^2\frac{A}{2}}{\frac{s}{4R}}\right)^2 = \frac{s^2}{2}\cdot\frac{16R^2\cos^4\left(\frac{A}{2}\right)}{s^2} = 8R^2\cos^4\frac{A}{2}$$

$$\Rightarrow \frac{1}{r_b^2 + r_c^2} \overset{(2)}{\leq} \frac{1}{8R^2\cos^4\frac{A}{2}}$$

(1).(2)$\Rightarrow \frac{m_a^2}{r_b^2+r_c^2} \overset{(a)}{\leq} \frac{1}{2}$. Similarly, $\frac{m_b^2}{r_c^2+r_a^2} \overset{(b)}{\leq} \frac{1}{2}$ & $\frac{m_c^2}{r_a^2+r_b^2} \overset{(c)}{\leq} \frac{1}{2}$

(a)+(b)+(c)$\Rightarrow LHS \leq \frac{3}{2}$ (Proved)

1096. In ΔABC the following relationship holds:

$$s^3 \geq \frac{3\sqrt{3}r^2(4R+r)^3}{(2R-r)(2R+5r)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$\because s \geq 3\sqrt{3}r$, $\therefore$ it suffices to prove: $s^2 \overset{(1)}{\geq} \frac{r(4R+r)^3}{(2R-r)(2R+5r)}$

Now, LHS of (1) $\overset{Gerretsen}{\geq} 16Rr - 5r^2 \overset{?}{\geq} \frac{r(4R+r)^3}{(2R-r)(2R+5r)}$

$$\Leftrightarrow (16R-5r)(2R-r)(2R+5r) - (4R+r)^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow 5R^2 - 11Rr + 2r^2 \overset{?}{\geq} 0 \Leftrightarrow (5R-r)(R-2r) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$\because R \overset{Euler}{\geq} 2r \Rightarrow$ (1) is true (Proved)

Solution 2 by Tran Hong-Dong Thap-Vietnam

$s^3 \geq \frac{3\sqrt{3}r^2(4R+r)^3}{(2R-r)(2R+5r)}$ $\because s \geq 3\sqrt{3}r$ and $s^2 \geq 16Rr - 5r^2$

$$\Rightarrow s^3 \ge 3\sqrt{3}r^2(16R-5r) \overset{(1)}{\ge} \frac{3\sqrt{3}r^2(4R+r)^3}{(2R-r)(2R+5r)}$$

$$(1) \Leftrightarrow (16R-5r)(2R-r)(2R+5r) \ge (4R+r)^3$$

$$\Leftrightarrow 5R^2-11R+2r^2 \ge 0 \Leftrightarrow 5\left(R-\frac{r}{5}\right)(R-2r) \ge 0\ (\because R \ge 2r).$$ ***True. Proved.***

1097. If $x,y,z \ge 0$ then in ΔABC the following relationship holds:

$$\frac{x}{2}\csc\frac{A}{2}+\frac{y}{2}\csc\frac{B}{2}+\frac{z}{2}\csc\frac{C}{2} \ge \sqrt{xy}+\sqrt{yz}+\sqrt{zx}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

We must show: $\frac{x}{\sin\frac{A}{2}}+\frac{y}{\sin\frac{B}{2}}+\frac{z}{\sin\frac{C}{2}} \ge 2\left(\sqrt{xy}+\sqrt{yz}+\sqrt{xz}\right)$ *(1)*

From Cauchy's inequality $\Rightarrow$

$$\left(\frac{x}{\sin\frac{A}{2}}+\frac{y}{\sin\frac{B}{2}}+\frac{z}{\sin\frac{C}{2}}\right)\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right) \ge \left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2 \Rightarrow$$

$$\frac{x}{\sin\frac{A}{2}}+\frac{y}{\sin\frac{B}{2}}+\frac{z}{\sin\frac{C}{2}} \ge \frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}} \quad (2)$$

From (1)+(2) we must show: $\frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}} \ge 2\left(\sqrt{xy}+\sqrt{yz}+\sqrt{xz}\right)$ *(3)*

But in any ΔABC we have: $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2} \le \frac{3}{2}$ *(4)*

$$\frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}} \ge \frac{2}{3}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2 \quad (4)$$

From (3)+(4) we must show: $\frac{2}{3}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2 \ge 2\left(\sqrt{xy}+\sqrt{yz}+\sqrt{xz}\right) \Leftrightarrow$

$$\Leftrightarrow \left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2 \ge 3\left(\sqrt{xy}+\sqrt{yz}+\sqrt{xz}\right)$$

$$\Leftrightarrow x+y+z \ge \sqrt{xy}+\sqrt{yz}+\sqrt{xz} \quad \textit{(true)}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

Suppose: $x=\max\{x;y;z\}$. *We have:*

$$\Rightarrow \sin\frac{A}{2} \le \sin\frac{B}{2} \le \sin\frac{C}{2} \Rightarrow \frac{1}{\sin\frac{A}{2}} \ge \frac{1}{\sin\frac{B}{2}} \ge \frac{1}{\sin\frac{C}{2}}$$

By Chebyshev's inequality, we have:

$$\frac{1}{2}\left(x\cdot\frac{1}{\sin\frac{A}{2}} + y\cdot\frac{1}{\sin\frac{B}{2}} + z\cdot\frac{1}{\sin\frac{C}{2}}\right) \ge \frac{1}{2}\cdot\frac{1}{3}(x+y+z)\left(\sum\frac{1}{\sin\frac{A}{2}}\right)$$

$$\overset{(Jensen)}{\ge} \frac{1}{2}\cdot\frac{1}{3}\cdot(x+y+z)\cdot\frac{3}{\sin\left(\frac{A+B+C}{6}\right)} = \frac{1}{2}\cdot\frac{1}{3}\cdot(x+y+z)\cdot\frac{3}{\sin\left(\frac{\pi}{6}\right)} = x+y+z$$

But: $x+y+z \overset{(BCS)}{\ge} \sqrt{xy}+\sqrt{yz}+\sqrt{zx} \Rightarrow LHS \ge RHS$

Case: $x \ge z \ge y$. *Then we suppose:* $A \le C \le B$

$$\Rightarrow \sin\frac{A}{2} \le \sin\frac{C}{2} \le \sin\frac{B}{2} \Rightarrow \frac{1}{\sin\frac{A}{2}} \ge \frac{1}{\sin\frac{C}{2}} \ge \frac{1}{\sin\frac{B}{2}}$$

By Chebyshev's inequality, we have:

$$\frac{1}{2}\left(x\cdot\frac{1}{\sin\frac{A}{2}} + z\cdot\frac{1}{\sin\frac{C}{2}} + y\cdot\frac{1}{\sin\frac{B}{2}}\right) \ge \frac{1}{2}\cdot\frac{1}{3}\cdot(x+z+y)\left(\sum\frac{1}{\sin\frac{A}{2}}\right)$$

$$\overset{(Jensen)}{\ge} \frac{1}{2}\cdot\frac{1}{3}(x+y+z)\cdot\frac{3}{\sin\left(\frac{A+B+C}{6}\right)} = x+y+z$$

But: $x+y+z \overset{BCS}{\ge} \sqrt{xy}+\sqrt{xz}+\sqrt{yz} \Rightarrow LHS \ge RHS$

1098. MARIAN URSĂRESCU's REFINEMENT OF EULER'S INEQUALITY

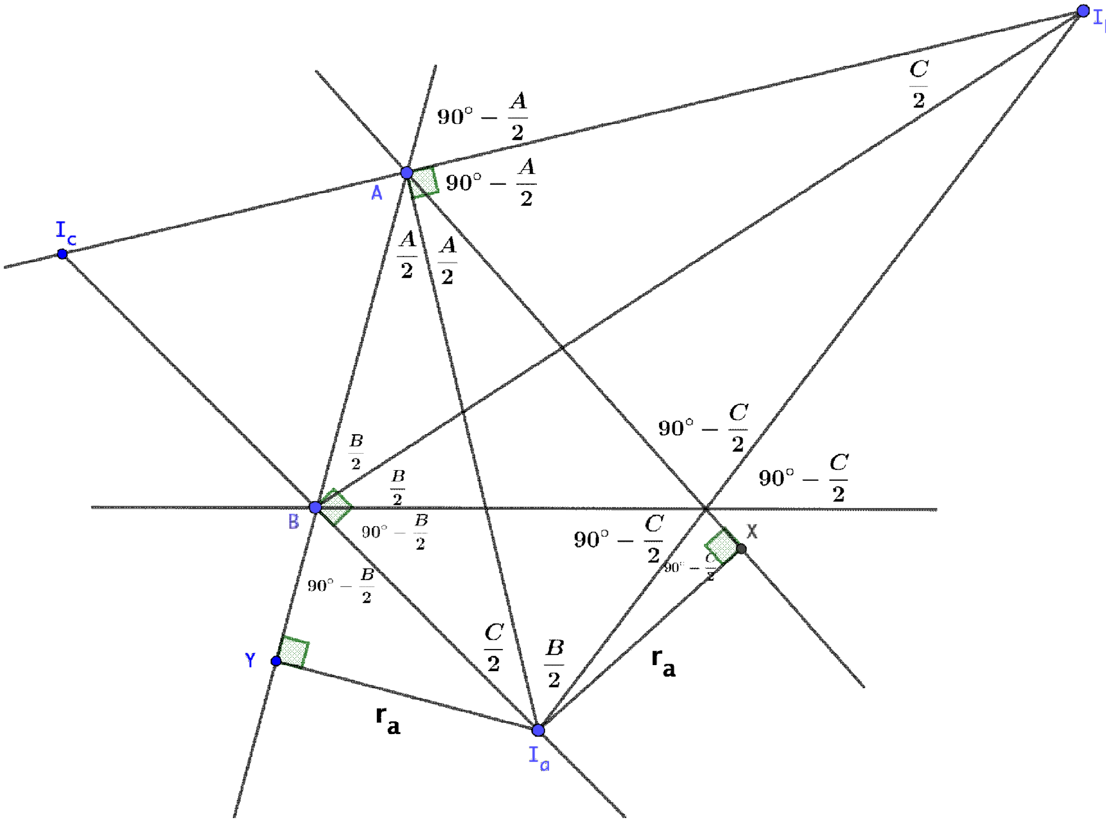
In $\Delta ABC, I_a, I_b, I_c$ – excenters. Prove that:

$$R \ge \frac{4}{9}\left(\frac{[I_aBC]}{a} + \frac{[I_bCA]}{b} + \frac{[I_cAB]}{c}\right) \ge 2r$$

Proposed by Marian Ursărescu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$R \overset{(i)}{\ge} \frac{4}{9}\left(\frac{[I_aBC]}{a} + \frac{[I_bCA]}{b} + \frac{[I_cAB]}{c}\right) \overset{(ii)}{\ge} 2r$$



From ΔI_aCX, $\sin\left(90^\circ-\frac{C}{2}\right)=\frac{r_a}{I_aC}\Rightarrow I_aC\overset{(1)}{=}\frac{r_a}{\cos\frac{C}{2}}$

From ΔI_aBY, $\sin\left(90^\circ-\frac{B}{2}\right)=\frac{r_a}{I_aB}\Rightarrow I_aB\overset{(2)}{=}\frac{r_a}{\cos\frac{B}{2}}$

Using (1), (2), $[I_aBC]=\frac{1}{2}\cdot\frac{r_a^2}{\cos\frac{B}{2}\cos\frac{C}{2}}\sin\left(\frac{B+C}{2}\right)=\frac{r_a^2\cos^2\frac{A}{2}}{2\left(\frac{s}{4R}\right)}=\frac{2R}{s}s^2\tan^2\frac{A}{2}\cos^2\frac{A}{2}$

$$=2Rs\left(\sin^2\frac{A}{2}\right)=\frac{2Rs(s-b)(s-c)}{bc}$$

$$\therefore\frac{[I_aBC]}{a}=\frac{2Rs(s-a)(s-c)}{4Rrs}\overset{(a)}{=}\frac{(s-b)(s-c)}{2r}$$

Similarly, $\frac{[I_bCA]}{b}\overset{(b)}{=}\frac{(s-c)(s-a)}{2r}$ & $\frac{[I_cAB]}{c}\overset{(c)}{=}\frac{(s-a)(s-b)}{2r}$

(a)+(b)+(c)$\Rightarrow\frac{4}{9}\left(\frac{[I_aBC]}{a}+\frac{[I_bCA]}{b}+\frac{[I_cAB]}{c}\right)=\frac{4}{9\cdot 2r}\{\sum(s-b)(s-c)\}$

$$=\frac{2}{9r}(3s^2-4s^2+s^2+4Rr+r^2)=\frac{2}{9r}(4Rr+r^2)\overset{(d)}{=}\frac{2(4R+r)}{9}\overset{Euler}{\geq}\frac{2\cdot 2r}{9}=2r$$

$\therefore$ *(ii) is true.*

Also, (d)$\Rightarrow \frac{4}{9}\sum\frac{[I_aBC]}{a}\overset{Euler}{\leq}\frac{8R+r}{9}=R\Rightarrow$ *(i) is true (Proved)*

1099. In ΔABC the following relationship holds:

$$\frac{a}{b+h_c}+\frac{b}{c+h_a}+\frac{c}{a+h_b}\geq\frac{64r-5R}{9R+3s}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\frac{a}{b+h_c}\geq\frac{64r-5R}{9R+3s}$$

$$\frac{a}{b+h_c}+\frac{b}{c+h_a}+\frac{c}{a+h_b}\overset{Ma\geq Mg}{\geq}$$

$$\geq 3\sqrt[3]{\frac{abc}{(a+h_b)(b+h_c)(c+h_a)}}=3\cdot\frac{1}{\sqrt[3]{\left(\frac{a+h_b}{a}\right)\cdot\left(\frac{b+h_c}{b}\right)\cdot\left(\frac{c+h_a}{c}\right)}}=$$

$$=3\cdot\frac{1}{\sqrt[3]{\left(1+\frac{h_b}{a}\right)\left(1+\frac{h_c}{b}\right)\left(1+\frac{h_a}{c}\right)}}\overset{Ma\geq Mg}{\geq}\frac{9}{3+\sum\frac{h_a}{a}}=$$

$$=\frac{9}{3+\sum\frac{bc}{2R\cdot c}}=\frac{9}{3+\frac{a+b+c}{2R}}=\frac{9}{3+\frac{s}{R}}=$$

$$=\frac{9R}{3R+r}=\frac{27R}{9R+3s}=\frac{32R-5r}{9R+3s}\overset{R\geq 2r}{\geq}\frac{64r-5R}{9R+3s}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS=\frac{a^2}{ab+ah_c}+\frac{b^2}{bc+bh_a}+\frac{c^2}{ac+ch_b}\overset{Bergstrom}{\geq}\frac{4s^2}{\sum ab+\frac{\sum a^2b}{2R}}$$

$$\overset{CBS}{\geq}\frac{4s^2}{\sum ab+\frac{\sqrt{\sum a^2}\sqrt{\sum a^2b^2}}{2R}}\overset{\substack{Leibnitz\\Goldsone}}{\geq}\frac{4s^2}{\sum ab+\frac{3R\cdot 2Rs}{2R}}\overset{3\sum ab\leq(\sum a)^2}{\geq}\frac{4s^2}{\frac{4s^2}{3}+3Rs}$$

$$=\frac{12s}{9R+4s}\overset{?}{\geq}\frac{64r-5R}{9R+3s}\Leftrightarrow\frac{12s}{9R+4s}+\frac{5R-64r}{9R+3s}\overset{?}{\geq}0$$

$$\Leftrightarrow 12s(9R+3s)+(5R-64r)(9R+4s)\overset{?}{\geq}0$$

$$\Leftrightarrow 128Rs+36s^2+45R^2\underset{(1)}{\overset{?}{\geq}}256rs+576Rr$$

Now, $128Rs\underset{(a)}{\overset{Euler}{\geq}}256rs$

Again, $36s^2+45R^2\overset{Gerretsen}{\geq}36(16Rr-5r^2)+45R^2=576Rr+45R^2-180r^2$

$$=576Rr+45(R+2r)(R-2r)\overset{Euler}{\geq}576Rr\Rightarrow 36s^2+45R^2\overset{(b)}{\geq}576Rr$$

(a)+(b)⇒ (1) is true (Proved)

1100. In ΔABC the following relationship holds:

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\leq\frac{1}{4}\left(\frac{h_b+h_c}{w_a}+\frac{h_c+h_a}{w_b}+\frac{h_a+h_b}{w_c}\right)$$

Proposed by Bogdan Fustei-Romania

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\sin\frac{A}{2}\cdot 1\overset{AM\geq GM}{\leq}\sum\sin\frac{A}{2}\cdot\frac{\left(\frac{b+c}{2}\right)^2}{bc}=\sum\frac{bc\cdot\sin A}{8}\cdot\frac{1}{\cos\frac{A}{2}}\cdot\left(\frac{b+c}{bc}\right)^2=$$

$$=\frac{\Delta}{4}\cdot\sum\left(\frac{b+c}{bc}\right)^2\cdot\cos\frac{A}{2}=\frac{\Delta}{2}\cdot\sum\frac{b+c}{bc}\cdot\frac{1}{\frac{2bc\cdot\cos\frac{A}{2}}{b+c}}$$

$$=\frac{\Delta}{2}\cdot\sum\left(\frac{1}{b}+\frac{1}{c}\right)\cdot\frac{1}{w_a}=\frac{1}{4}\sum\frac{h_b+h_c}{w_a}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{4}\sum\left(\frac{h_b+h_c}{w_a}\right)=\frac{1}{4}\sum\left(\frac{\frac{ca+ab}{2R}}{\frac{2bc}{b+c}\cos\frac{A}{2}}\right)=\sum\left(\frac{a(b+c)^2}{16Rbc\cos\frac{A}{2}}\right)$$

$$=\sum\left(\frac{4R\sin\frac{A}{2}\cos\frac{A}{2}(b+c)^2}{16Rbc\cos\frac{A}{2}}\right)=\sum\left(\frac{\sin\frac{A}{2}(b+c)^2}{4bc}\right)\overset{A-G}{\geq}\sum\sin\frac{A}{2}\quad\textit{(Proved)}$$

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It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru