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1001. Prove that:

$$\frac{\tan 36^\circ + \tan 72^\circ}{4 \cos 54^\circ} = \phi$$

Proposed by Alpaslan Ceran-Turkey

Solution 1 by Avishek Mitra-West Bengal-India

$$\begin{aligned}
 \text{Let } x = \frac{\pi}{10} \Rightarrow 5x = \frac{\pi}{2} \Rightarrow 2x = \frac{\pi}{2} - 3x \Rightarrow \sin 2x = \sin\left(\frac{\pi}{2} - 3x\right) \\
 \Rightarrow \sin 2x = \cos 3x \Rightarrow 2 \sin x \cos x = 4 \cos^3 x - 3 \cos x \\
 \Rightarrow 2 \sin x - 4 \cos^2 x + 3 = 0 \Rightarrow 4 \sin^2 x + 2 \sin x - 1 = 0 \\
 \Rightarrow \sin x = \frac{-2 \pm \sqrt{20}}{8} = \frac{-1 \pm \sqrt{5}}{4}, \quad \sin 18 = \frac{\sqrt{5}-1}{4} = \cos 72 \\
 \cos 18 = \sqrt{1 - \sin^2 18} = \frac{1}{4} \sqrt{20 + 2\sqrt{5}} = \sin 72, \quad \tan 72 = \frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5-1}} \\
 \text{Now, } \cos 36 = 1 - 2 \sin^2 18 = \frac{\sqrt{5}+1}{4}, \quad \sin 36 = \sqrt{1 - \cos^2 36} = \frac{\sqrt{10-2\sqrt{5}}}{4} \\
 \tan 36 = \frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}, \quad \cos 54 = \sin 36 = \frac{1}{4} \sqrt{10 - 2\sqrt{5}} \\
 \text{Hence } \frac{\tan 36 + \tan 72}{4 \cos 54} = \frac{\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1} + \frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5-1}}}{\sqrt{10-2\sqrt{5}}} = \frac{1}{\sqrt{5}+1} + \frac{1}{\sqrt{5}-1} \sqrt{\frac{10+2\sqrt{5}}{10-2\sqrt{5}}} \\
 = \frac{1}{\sqrt{5}+1} + \frac{1}{\sqrt{5}-1} \cdot \frac{10+2\sqrt{5}}{4\sqrt{5}} = \frac{4\sqrt{5}(\sqrt{5}-1) + (10+2\sqrt{5})(\sqrt{5}+1)}{4\sqrt{5} \cdot 4} \\
 = \frac{40+8\sqrt{5}}{16\sqrt{5}} = \frac{5+\sqrt{5}}{2\sqrt{5}} = \frac{5\sqrt{5}+5}{10} = \frac{\sqrt{5}+1}{2} = \phi \quad (\text{proved})
 \end{aligned}$$

Solution 2 by Nelson Javier Villaherrera Lopez-El Salvador

$$\begin{aligned}
 \frac{\tan(36^\circ) + \tan(72^\circ)}{4 \cos(54^\circ)} &= \frac{\sin(36^\circ) \cos(72^\circ) + \cos(36^\circ) \sin(72^\circ)}{4 \sin(36^\circ) \cos(36^\circ) \cos(72^\circ)} = \\
 &= \frac{\sin(108^\circ)}{2 \sin(72^\circ) \cos(72^\circ)} = \frac{\sin(108^\circ)}{\sin(144^\circ)} = \frac{\cos(18^\circ)}{\cos(54^\circ)} = \frac{\cos(18^\circ)}{\sin(36^\circ)} = \frac{1}{2 \sin(18^\circ)} \\
 &= \frac{1}{\sqrt{2 - 2 \cos(36^\circ)}} = \frac{1}{\sqrt{2 - \frac{(\sqrt{5}+1)}{2}}} = \sqrt{\frac{2}{3-\sqrt{5}}} = \sqrt{\frac{3+\sqrt{5}}{2}} = \sqrt{\frac{6+2\sqrt{5}}{4}} =
 \end{aligned}$$



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$$= \sqrt{\left(\frac{\sqrt{5} + 1}{2}\right)^2} = \frac{\sqrt{5} + 1}{2} = \varphi$$

1002. In ΔABC , M – Mittenpunkt, I_a, I_b, I_c – excenters. Prove that:

$$\frac{[I_b MI_c]}{\cos^2 \frac{A}{2}} = \frac{[I_c MI_a]}{\cos^2 \frac{B}{2}} = \frac{[I_a MI_b]}{\cos^2 \frac{C}{2}}$$

Proposed by Mustafa Tarek-Cairo-Egypt

Solution by Thanasis Gakopoulos-Athens-Greece

PLAGIAGONAL system: $BC \rightarrow Bx, BA \rightarrow By$

$$\text{Let } k = 2ab + 2bc + 2ac - a^2 - b^2 - c^2, m_1 = \frac{ac(a+b-c)}{k}, m_2 = \frac{ac(-a+b+c)}{k}$$

$$I_a(i_{a_1}, i_{a_2}), i_{a_1} = \frac{ac}{-a + b + c}, i_{a_2} = -\frac{ac}{-a + b + c}$$

$$I_b(i_b, i_b), i_b = \frac{ac}{a - b + c}$$

$$I_c(i_{c_1}, i_{c_2}), i_{c_1} = -\frac{ac}{a + b - c}, i_{c_2} = \frac{ac}{a + b - c}$$

$$\left\{ \begin{array}{l} [IaMIc] = \frac{\sin B}{2} \begin{vmatrix} 1 & 1 & 1 \\ i_{c_1} & i_{a_1} & m_1 \\ i_{c_2} & i_{a_2} & m_2 \end{vmatrix} = \frac{\sin B}{2} \frac{4a^2b^2c^2}{k(-a + b + c)(a + b - c)} \\ \cos^2 \frac{B}{2} = \frac{(a + b + c)(a - b + c)}{4ac}, so \frac{[IaMIc]}{\cos^2 \frac{B}{2}} = \frac{a^3b^2c^3}{k \cdot S^2} \cdot \frac{\sin B}{2} \end{array} \right\}$$

$$\left\{ \begin{array}{l} [IaMib] = \frac{\sin B}{2} \begin{vmatrix} 1 & 1 & 1 \\ i_{a_1} & m_1 & i_b \\ i_{a_2} & m_2 & i_b \end{vmatrix} = \frac{\sin B}{2} \cdot \frac{4a^2bc^3}{k(a - b + c)(-a + b + c)} \\ \cos^2 \frac{C}{2} = \frac{(a + b + c)(a + b - c)}{4ab}, so \frac{[IaMIc]}{\cos^2 \frac{C}{2}} = \frac{a^3b^2c^3}{kS^2} \cdot \frac{\sin B}{2} \end{array} \right\}$$

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$$\left\{ \begin{array}{l} [IbMIC] = \frac{\sin B}{2} \begin{vmatrix} 1 & 1 & 1 \\ Ic_1 & m_1 & ib \\ Ic_2 & m_2 & i_b \end{vmatrix} = \frac{\sin B}{2} \cdot \frac{4a^3bc^2}{k(a-b+c)(a+b-c)} \\ \cos^2 \frac{A}{2} = \frac{(a+b+c)(-a+b+c)}{4bc}, \text{ so, } \frac{[IbMIC]}{\cos^2 \frac{A}{2}} = \frac{a^3b^2c^3}{kS^2} \cdot \frac{\sin B}{2} \end{array} \right\}$$

$$\text{Finally: } \frac{[IbMIC]}{\cos^2 \frac{A}{2}} = \frac{[IcMIA]}{\cos^2 \frac{B}{2}} = \frac{[IaMIB]}{\cos^2 \frac{C}{2}}$$

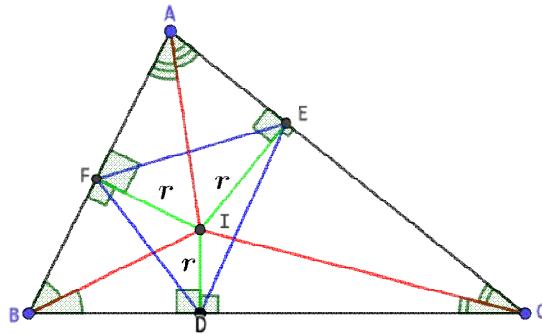
1003. ΔDEF pedal triangle of I incentre in ΔABC , R_a, R_b, R_c – circumradii of $\Delta AEF, \Delta BFD, \Delta CDE$, $\varphi_a, \varphi_b, \varphi_c$ – circumradii in $\Delta BIC, \Delta CIA, \Delta AIB$.

Prove that:

$$\frac{R_a \cdot R_b \cdot R_c}{\varphi_a \cdot \varphi_b \cdot \varphi_c} = \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution by Soumava Chakraborty-Kolkata-India



$$\text{From } \Delta BIC, \frac{1}{2} BI \cdot CI \cdot \sin \angle BIC = \frac{BI \cdot CI \cdot BC}{4\varphi_a} \Rightarrow \frac{r^2 \sin(90^\circ + \frac{A}{2})}{2 \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{r^2 a}{4\varphi_a \sin \frac{B}{2} \sin \frac{C}{2}} \Rightarrow$$

$$\Rightarrow \varphi_a \stackrel{(1)}{=} 2R \sin \frac{A}{2}. \text{ Similarly, } \varphi_b \stackrel{(2)}{=} 2R \sin \frac{B}{2} \text{ & } \varphi_c \stackrel{(3)}{=} 2R \sin \frac{C}{2}$$

$$\text{Now, from } \Delta AIF, \cos \frac{A}{2} = \frac{AF}{r} \Rightarrow AF \stackrel{(a)}{=} r \cot \frac{A}{2}$$

$$\text{Also, from } \Delta AIE, \cos \frac{A}{2} = \frac{AE}{r} \Rightarrow AE \stackrel{(b)}{=} r \cot \frac{A}{2}$$



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From ΔFIE , $FE^2 = 2r^2 - 2r^2 \cos(180^\circ - A) = 4r^2 \cos^2 \frac{A}{2} \Rightarrow FE \stackrel{(c)}{=} 2r \cos \frac{A}{2}$

Using (a), (b), (c), from ΔAFE , we get, $\frac{1}{2} \left(r \cot \frac{A}{2}\right)^2 \sin A = \frac{\left(r \cot \frac{A}{2}\right)^2 \cdot 2r \cos \frac{A}{2}}{4R_a}$

$\Rightarrow 2 \sin \frac{A}{2} \cos \frac{A}{2} = \frac{r \cos \frac{A}{2}}{R_a} \stackrel{(i)}{\Rightarrow} R_a = \frac{r}{2 \sin \frac{A}{2}}$. Similarly, $R_b \stackrel{(ii)}{=} \frac{r}{2 \sin \frac{B}{2}}$ & $R_c \stackrel{(iii)}{=} \frac{r}{2 \sin \frac{C}{2}}$

(1),(2),(3),(i),(ii),(iii) \Rightarrow

$$\prod \left(\frac{R_a}{\varphi_a} \right) = \prod \left(\frac{r}{4R} \cdot \frac{1}{\sin^2 \frac{A}{2}} \right) = \left\{ \left(\frac{r}{4R} \right) \right\}^3 \frac{1}{\left(\prod \sin \frac{A}{2} \right)^2} = \frac{\left(\prod \sin \frac{A}{2} \right)^3}{\left(\prod \sin \frac{A}{2} \right)^2} = \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

1004. In ΔABC the following relationship holds:

$$\sqrt{\frac{h_b h_c}{rr_a}} + \sqrt{\frac{h_c h_a}{rr_b}} + \sqrt{\frac{h_a h_b}{rr_c}} = \frac{w_b + w_c}{a} + \frac{w_c + w_a}{b} + \frac{w_a + w_b}{c}$$

Proposed by Bogdan Fustei-Romania

Solution by Daniel Sitaru-Romania

$$\begin{aligned} \sum_{cyc} \sqrt{\frac{h_b h_c}{rr_a}} &= \sum_{cyc} \sqrt{\frac{\frac{2S}{b} \cdot \frac{2S}{c}}{\frac{S}{s} \cdot \frac{S}{s-a}}} = 2 \sum_{cyc} \sqrt{\frac{s(s-a)}{bc}} = \\ &= 2 \sum_{cyc} \cos \frac{A}{2} = 2 \sum_{cyc} \left(\frac{2bc}{b+c} \cos \frac{A}{2} \cdot \frac{b+c}{2bc} \right) = \\ &= \sum_{cyc} \left(w_a \cdot \frac{b+c}{bc} \right) = \sum_{cyc} \left(w_a \left(\frac{1}{b} + \frac{1}{c} \right) \right) = \sum_{cyc} \left(\frac{w_a}{b} + \frac{w_a}{c} \right) = \sum_{cyc} \frac{w_b + w_c}{a} \end{aligned}$$

1005. In ΔABC the following relationship holds:

$$r_a r_b (r_a + r_b) + r_b r_c (r_b + r_c) + r_c r_a (r_c + r_a) = 2s^2 (2R - r)$$

Proposed by Bogdan Fustei-Romania

Solution by Daniel Sitaru-Romania

$$\sum_{cyc} r_a r_b (r_a + r_b) = \sum_{cyc} \frac{S}{s-b} \cdot \frac{S}{s-c} \left(\frac{S}{s-b} + \frac{S}{s-c} \right) =$$

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$$\begin{aligned}
 &= S^3 \sum_{cyc} \frac{1}{(s-b)(s-c)} \cdot \frac{s-c+s-b}{(s-b)(s-c)} = S^3 \sum_{cyc} \frac{a}{(s-b)^2(s-c)^2} = \\
 &= S^3 \sum_{cyc} \frac{as^2(s-a)^2}{S^4} = \frac{s^2}{S} \sum_{cyc} a(s-a)^2 = \frac{s^2}{rs} \sum_{cyc} (as^2 - 2sa^2 + a^3) = \\
 &= \frac{s}{r} \left(2s^3 - 2s \sum_{cyc} a^2 + \sum_{cyc} a^3 \right) = \frac{s}{r} \left(2s^3 - 4s(s^2 - r^2 - 4Rr) + \sum_{cyc} a^3 \right) = \\
 &= \frac{s}{r} (4sr^2 + 16Rrs - 2s^3 + 2s(s^2 - 3r^2 - 6Rr)) = \frac{s}{r} (4Rrs - 2sr^2) = 2s^2(2R - r)
 \end{aligned}$$

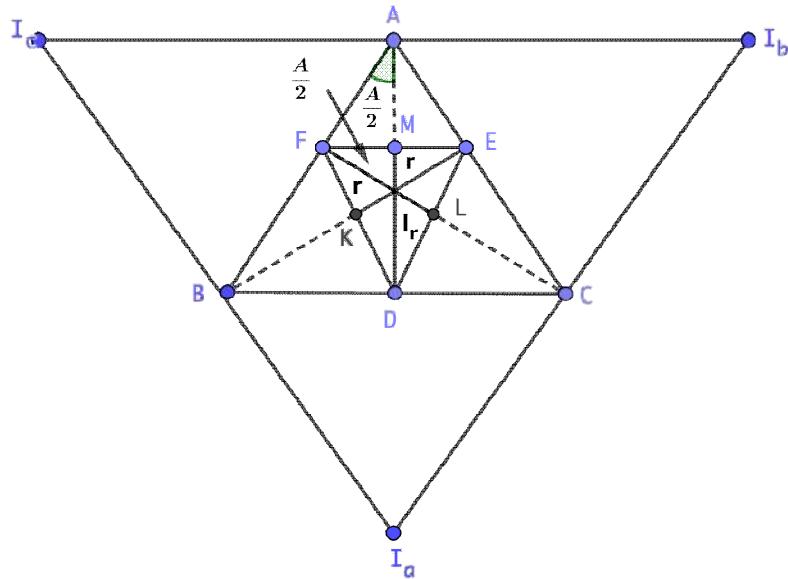
1006. In ΔABC , ΔDEF – is pedal triangle of I – incenter, I_a, I_b, I_c – excenters

If $IM \perp FE, IK \perp DF, IL \perp DE, M \in (EF), K \in (DF), L \in (DE)$ then:

$$\frac{1}{IM^2 \cdot h_a^2} + \frac{1}{IK^2 \cdot h_b^2} + \frac{1}{IL^2 \cdot h_c^2} = \frac{[I_a I_b I_c]}{[ABC]^3} \cdot \frac{r_a + r_b + r_c}{r}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution 1 by Lahiru-Samarakoon-Sri Lanka



$$\Delta = ABC, \text{ From } IM \perp FA, IM = r \sin \frac{A}{2}. \text{ Similarly, } IK = r \sin \frac{B}{2}, IL = r \sin \frac{C}{2}$$

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$$\therefore \frac{1}{IM^2 h_a^2} = \frac{1}{r^2 \sin^2 \frac{A}{2}} \times \frac{a^2}{4\Delta^2} = \frac{4R^2 \times 4 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}}{r^2 \sin^2 \frac{A}{2} \times 4\Delta^2} = \frac{4R^2}{r^2 \Delta^2} \cos^2 \frac{A}{2}$$

$$LHS = \sum \frac{4R^2}{r^2 \Delta^2} \cos^2 \frac{A}{2} = \frac{4R^2}{r^2 \Delta^2} \times \frac{1}{2} (3 + \sum \cos A)$$

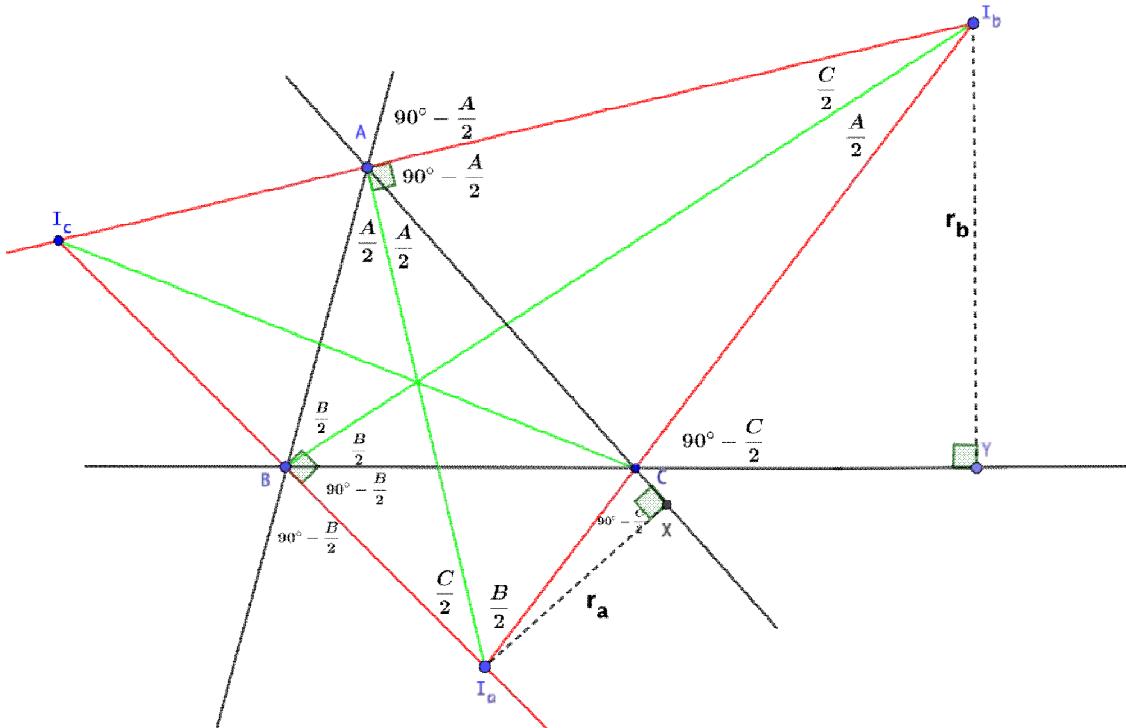
$$= \frac{2R^2}{r^2 \Delta^2} \times \left(3 + 1 + \frac{r}{R} \right) \quad (\because \sum \cos A = 1 + \frac{r}{R}) = \frac{2R}{r} \cdot \frac{1}{\Delta^2} \cdot \frac{(4R + r)}{r}$$

$$But, know that, I_a I_b I_c = 2SR. So, LHS = \frac{I_a I_b I_c}{S} \cdot \frac{1}{r} \cdot \frac{1}{\Delta^2} \cdot \frac{\sum r_a}{r}$$

$$(\because \sum r_a = 4R + r)$$

$$= \frac{I_a I_b I_c}{[ABC]^3} \times \frac{(r_a + r_b + r_c)}{r} \quad (\because sr = \Delta)$$

Solution 2 by Soumava Chakraborty-Kolkata-India



A, B, I_a, I_b are concyclic as $\angle I_a A I_b = \angle I_a B I_b = 90^\circ$

$$\therefore \angle B I_b I_a = \angle B A I_a = \frac{A}{2} \text{ & } \angle I_b I_a A = \angle I_b B A = \frac{B}{2}$$

$$\text{using } \Delta A B I_b, \angle A I_b B = \frac{C}{2} \text{ & using } \Delta A B I_a, \angle A I_a B = \frac{C}{2}$$

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$$\text{From } \Delta CI_b Y, \sin\left(90^\circ - \frac{C}{2}\right) = \frac{I_b Y}{CI_b} \Rightarrow \cos \frac{C}{2} = \frac{r_b}{c I_b} \Rightarrow CI_b \stackrel{(1)}{=} \frac{r_b}{\cos \frac{C}{2}}$$

$$\text{From } \Delta CI_a X, \sin\left(90^\circ - \frac{C}{2}\right) = \frac{I_a X}{CI_a} \Rightarrow \cos \frac{C}{2} = \frac{r_a}{c I_a} \Rightarrow CI_a \stackrel{(2)}{=} \frac{r_a}{\cos \frac{C}{2}}$$

$$(1)+(2) \Rightarrow I_a I_b = \frac{r_a + r_b}{\cos \frac{C}{2}} = \frac{s \left(\frac{\sin \frac{A}{2} + \sin \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} \right)}{\cos \frac{C}{2}} = \frac{s \sin \left(\frac{A+B}{2} \right)}{\left(\prod \cos \frac{A}{2} \right)} = \frac{s \cos \frac{C}{2}}{\frac{s}{4R}} \Rightarrow I_a I_b \stackrel{(a)}{=} 4R \cos \frac{C}{2}$$

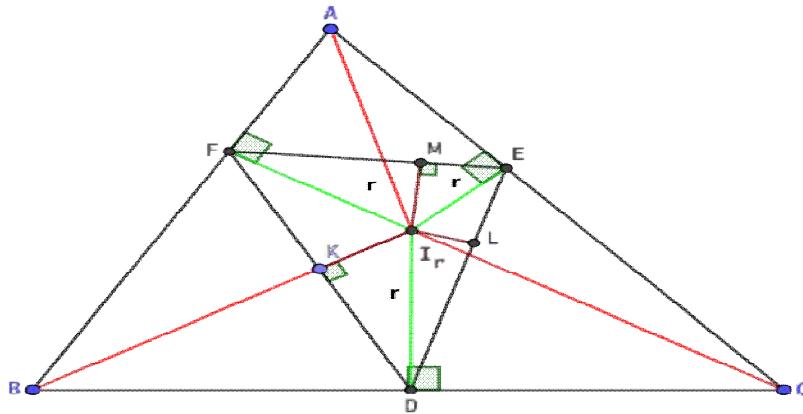
$$\text{Similarly, } I_b I_c \stackrel{(b)}{=} 4R \cos \frac{A}{2} \text{ & } I_c I_a \stackrel{(c)}{=} 4R \cos \frac{B}{2}$$

$$\therefore [I_a I_b I_c] = \frac{1}{2} (I_a I_b) (I_b I_c) (\sin \angle I_c I_b I_a)$$

$$= 8R^2 \cos \frac{C}{2} \cos \frac{A}{2} \sin \left(\frac{A+C}{2} \right) \quad (\text{using (a), (b)}) = 8R^2 \left(\prod \cos \frac{A}{2} \right) = 8R^2 \left(\frac{s}{4R} \right) = 2Rs$$

$$\Rightarrow [I_a I_b I_c] \stackrel{(i)}{=} 2Rs$$

$$ID = IE = IF = r$$



(\because in radius \perp each side of ΔABC) $\therefore r$ is the circumradius of ΔDEF $\therefore \angle FDE = \frac{1}{2} \angle FIE$

($\because \angle$ at circumference $= \frac{1}{2} \angle$ at center)

$$= \frac{1}{2} (180^\circ - A) = 90^\circ - \frac{A}{2} \therefore FE = 2r \sin \left(90^\circ - \frac{A}{2} \right) \Rightarrow FE \stackrel{(x)}{=} 2r \cos \frac{A}{2}$$

$$\text{Now, } \frac{1}{2} IF \cdot IE \cdot \sin(180^\circ - A) = \frac{1}{2} FE \cdot IM (= \text{arc}(\Delta FIE))$$

$$\Rightarrow \frac{r^2 2 \sin \frac{A}{2} \cos \frac{A}{2}}{2} = \frac{1}{2} 2r \cos \frac{A}{2} IM \quad (\text{using (x)})$$

$$\Rightarrow IM \stackrel{(d)}{=} r \sin \frac{A}{2}. \text{ Similarly, } IK \stackrel{(e)}{=} r \sin \frac{B}{2} \text{ & } IL \stackrel{(f)}{=} r \sin \frac{C}{2}$$



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$$\begin{aligned}
 (d), (e), (f) \Rightarrow \sum \frac{1}{IM^2 h_a^2} &= \sum \left(\frac{1}{r^2 \sin^2 \frac{A}{2}} \cdot \frac{16R^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}}{4r^2 s^2} \right) \\
 &= \frac{4R^2}{r^4 s^2} \sum \cos^2 \frac{A}{2} = \frac{2R^2}{r^4 s^2} \sum (1 + \cos A) = \frac{2R^2}{r^4 s^2} \left(3 + 1 + \frac{r}{R} \right) = \frac{2R(4R + r)}{r^4 s^2} \\
 \therefore LHS &\stackrel{(ii)}{=} \frac{2R(4R + r)}{r^4 s^2}
 \end{aligned}$$

Using (i), RHS $= \frac{2RS}{r^3 s^3} \cdot \frac{(4R+r)}{r} = \frac{2R(4R+r)}{r^4 s^2}$ by (ii) **LHS (Hence proved)**

Solution 3 by Thanasis Gakopoulos-Athens-Greece

$$\begin{aligned}
 \underbrace{\frac{1}{IM^2 \cdot h_a^2} + \frac{1}{IK^2 \cdot h_b^2} + \frac{1}{IL^2 \cdot h_c^2}}_A &= \underbrace{\frac{(I_a I_b I_c)}{S^3} \cdot \frac{r_a + r_b + r_c}{r}}_B \\
 KI = r \cdot \sin \frac{B}{2} \rightarrow 2K_1^2 &= r^2(1 - \cos B) \rightarrow \frac{1}{IK^2} = \frac{4ac}{r^2(-a+b+c)(a+b-c)} \\
 \frac{1}{h_b^2} &= \frac{b^2}{4S^2} \\
 \rightarrow \frac{1}{IK^2 h_b^2} &= \frac{abc \cdot b}{r^2 S^2 (-a+b+c)(a+b-c)}
 \end{aligned}
 \right\} \rightarrow$$

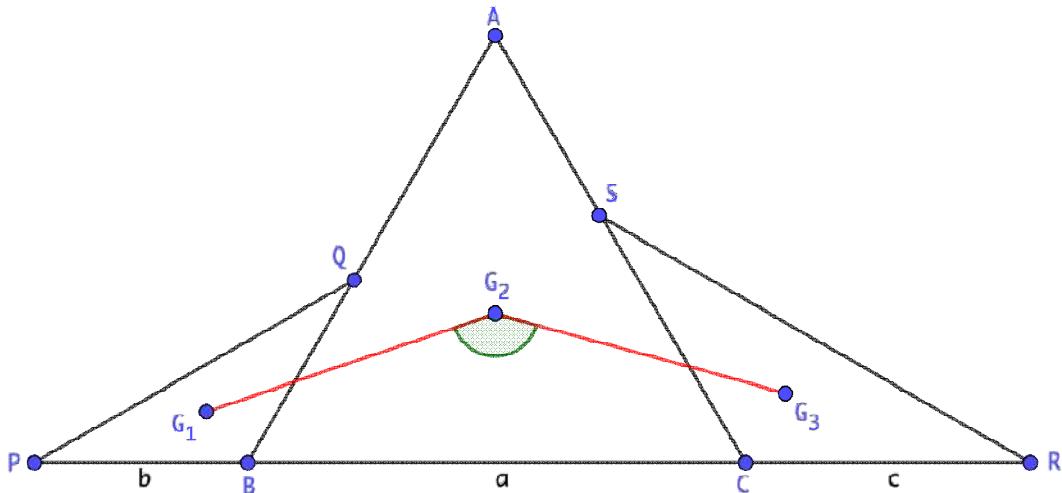
$$\text{Cyclically } \frac{1}{IM^2 \cdot h_a^2} = \frac{abc}{r^2 S^2} \cdot \frac{a}{(a-b+c)(a+b-c)} \cdot \frac{1}{IL^2 h_c^2} = \frac{abc \cdot c}{(-a+b+c)(a-b+c)}$$

$$\begin{aligned}
 A &= \frac{abc}{r^2 S^2} \left[\frac{a}{(a-b+c)(a+b-c)} + \frac{b}{(-a+b+c)(a+b-c)} + \frac{c}{(-a+b+c)(a-b+c)} \right] \rightarrow \\
 A &= \frac{abc}{r^2 S^2} \cdot \frac{a^2 + b^2 + c^2 - 2ab - 2bc - 2ca}{(a-b-c)(a+b-c)(a-b+c)} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 (I_a I_b I_c) &= \frac{(a+b+c)abc}{4S} \quad r_a = \frac{2S}{-a+b+c}, r_b = \frac{2S}{a-b+c}, r_c = \frac{2S}{a+b-c} \\
 B &= \frac{abc(a+b+c)}{4SS^2} \frac{1}{r} 2S \left(\frac{1}{-a+b+c} + \frac{1}{a-b+c} + \frac{1}{a+b-c} \right) = \\
 &= \frac{abc \cdot 2S}{S^3 2S \cdot r} \cdot \frac{a+b+c}{2} \frac{a^2 + b^2 + c^2 - 2ab - 2bc - 2ca}{(a-b-c)(a+b-c)(a-b+c)} = \\
 &= \frac{abc}{S^3 r} \cdot \frac{s}{r} \cdot \frac{a^2 + b^2 + c^2 - 2ab - 2bc - 2ca}{(a-b-c)(a+b-c)(a-b+c)} \quad (2)
 \end{aligned}$$

$$(1), (2) \rightarrow A = B \rightarrow \frac{1}{IM^2 h_a^2} + \frac{1}{IK^2 h_b^2} = \frac{[I_a I_b I_c]}{[ABC]^3} \cdot \frac{r_a + r_b + r_c}{r}$$

1007.



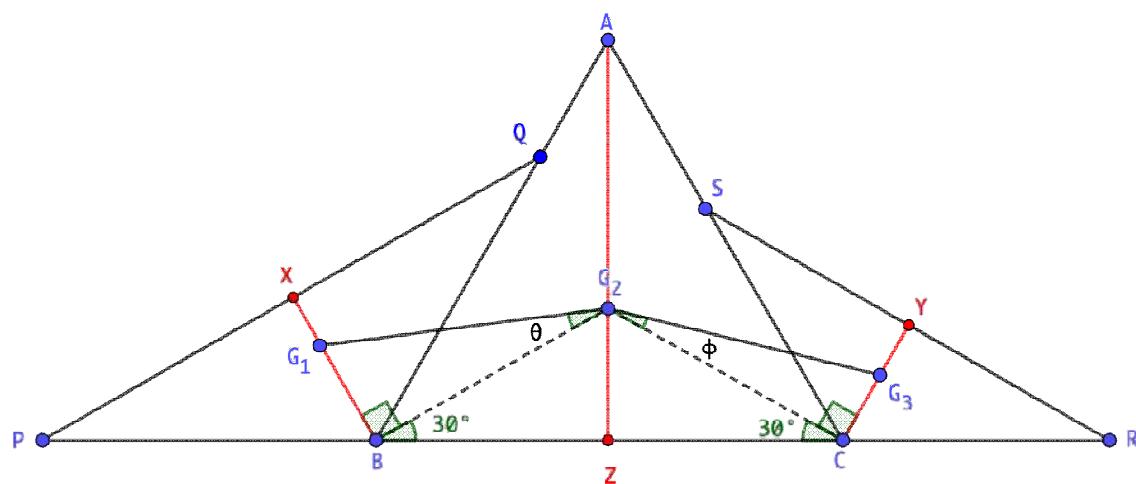
ΔABC equilateral, ΔBQP isosceles, ΔCRS isosceles

$BC = a$, $BP = b$, $CR = c$, G_1, G_2, G_3 centroids

$3a(a - b - c) = bc$. Calculate $m(\angle F)$

Proposed by Thanasis Gakopoulos-Athens-Greece

Solution by Soumava Chakraborty-Kolkata-India



$$\angle PBQ = 120^\circ \Rightarrow \angle P = \angle Q = 30^\circ; \frac{BX}{BP} = \sin 30^\circ \Rightarrow BX = \frac{b}{2} \Rightarrow BG_1 = \frac{2}{3} BX \stackrel{(1)}{=} \frac{b}{3}$$

$$\text{Also, } \angle SCR = 120^\circ \Rightarrow \angle S = \angle R = 30^\circ; \frac{CY}{CR} = \sin 30^\circ \Rightarrow CY = \frac{c}{2} \Rightarrow CG_3 = \frac{2}{3} CY \stackrel{(2)}{=} \frac{c}{3}$$



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$$\frac{AZ}{AB} = \sin 60^\circ \Rightarrow AZ = \frac{\sqrt{3}}{2}a \Rightarrow BG_2 = CG_2 = AG_2 = \frac{2}{3}AZ \stackrel{(3)}{=} \frac{a}{\sqrt{3}}$$

$\because G_2$ is also incenter of $\Delta ABC \therefore \angle ABG_2 = 30^\circ$ (i)

Again $\angle QBX = \frac{1}{2}\angle PBQ \stackrel{(ii)}{=} 60^\circ \therefore \angle G_1BG_2 = 90^\circ$ (by (i)+(ii))

$$\tan \theta = \frac{BG_1}{BG_2} = \frac{b}{3} \times \frac{\sqrt{3}}{a} \quad (\text{from (1), (3)}) \stackrel{(a)}{=} \frac{b}{\sqrt{3}a}$$

Similarly, $\angle G_2CG_3 = 90^\circ$ & so, $\tan \phi = \frac{CG_3}{CG_2} = \frac{c}{3} \times \frac{\sqrt{3}}{a} \quad (\text{from (2), (3)}) \stackrel{(b)}{=} \frac{c}{\sqrt{3}a}$

Now, $\angle BG_2C = 180^\circ - (30^\circ + 30^\circ) = 120^\circ$

$$\tan \angle G_1G_2G_3 = \tan(\theta + \phi + 120^\circ) = \frac{-\sqrt{3} + \tan \theta + \tan \phi + \sqrt{3} \tan \theta \tan \phi}{1 + \sqrt{3} \tan \theta + \sqrt{3} \tan \phi - \tan \theta \tan \phi}$$

$$= \frac{-\sqrt{3} + \frac{b}{\sqrt{3}a} + \frac{c}{\sqrt{3}a} + \sqrt{3} \cdot \frac{b}{\sqrt{3}a} \cdot \frac{c}{\sqrt{3}a}}{1 + \sqrt{3} \cdot \frac{b}{\sqrt{3}a} + \sqrt{3} \cdot \frac{c}{\sqrt{3}a} - \frac{bc}{3a^2}} \quad (\text{using (a), (b)}) = -\sqrt{3} \frac{(3a^2 - ab - ac - bc)}{(3a^2 + 3ab + 3ac - bc)}$$

$$= -\sqrt{3} \frac{(3a^2 - 3ab - 3ac - bc) + 2(ab + ac)}{(3a^2 - 3ab - 3ac - bc) + 6(ab + ac)} = -\frac{1}{\sqrt{3}} \quad (\because 3a^2 - 3ab - 3ac - bc = 0)$$

$\Rightarrow \angle G_1G_2G_3 = 150^\circ$ (Answer)

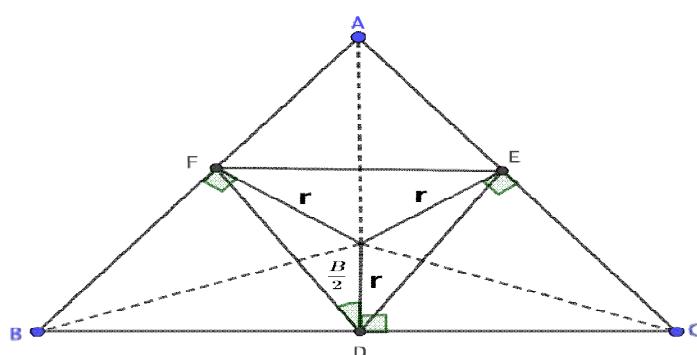
1008. If ΔDEF is pedal triangle of I – incenter in ΔABC and

R_a, R_b, R_c are the circumradii of $\Delta AFE, \Delta BDF, \Delta CDE$ then:

$$R_a^2 + R_b^2 + R_c^2 = \frac{s^2 + r^2 - 8Rr}{4}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution by Lahiru Samarakoon-Sri Lanka





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$$DF = 2r \cos \frac{B}{2}, EF = 2r \cos \frac{A}{2}, DE = 2r \cos \frac{C}{2}$$

$$\therefore \Delta AFE \Rightarrow 2R_a = \frac{EF}{\sin A} \Rightarrow 2R_a = \frac{2r \cos \frac{A}{2}}{\sin A}; R_a = \frac{r}{2 \sin \frac{A}{2}}$$

$$\text{Similarly, } R_b = \frac{r}{2 \sin \frac{B}{2}}, R_c = \frac{r}{2 \sin \frac{C}{2}}$$

$$\begin{aligned} \sum R_a^2 &= \frac{r^2}{4} \left[\sum \frac{1}{\sin^2 \frac{A}{2}} \right] = \frac{r^2}{4} \left[\sum \frac{bc}{(s-a)(s-b)} \right] = \frac{R^2}{4} \frac{[s \Sigma(bc) - 3abc]}{sr^2} \\ &= \frac{r^2}{4sr^2} [s \times (s^2 + r^2 + 4Rr) - 3 \times 4srR] = \frac{(s^2 + r^2 - 8Rr)}{4} \end{aligned}$$

1009. In ΔABC , R_k is the circumradii of pedal triangle of Lemoine's point.

Prove that:

$$R_k = \frac{m_a m_b m_c}{m_a^2 + m_b^2 + m_c^2}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution 1 by Shafiqur Rahman-Bangladesh

$$\begin{aligned} \frac{LL_A}{a} &= \frac{LL_B}{b} = \frac{aLL_A + bLL_B + cLL_C}{a^2 + b^2 + c^2} = \frac{2S}{a^2 + b^2 + c^2} = \frac{3S}{2(m_a^2 + m_b^2 + m_c^2)} = K \Rightarrow \\ \Rightarrow LL_A &= \frac{3aS}{2(m_a^2 + m_b^2 + m_c^2)}, LL_B = \frac{3bS}{2(m_a^2 + m_b^2 + m_c^2)} \text{ and } LL_C = \frac{3cS}{2(m_a^2 + m_b^2 + m_c^2)} \end{aligned}$$

Where L_A, L_B, L_C are the feet of perpendiculars from the Lemoine point L to the sides

BC, CA and AB

$$\begin{aligned} \text{Now, } L_B L_C &= \sqrt{(LL_B)^2 + (LL_C)^2 - 2LL_B \cdot LL_C \cdot \cos(\pi - A)} = \\ &= K \sqrt{b^2 + c^2 + 2bc \frac{b^2 + c^2 - a^2}{2bc}} = K \sqrt{2b^2 + 2c^2 - a^2} = 2Km_a. \text{ Similarly,} \end{aligned}$$

$$L_C L_A = 2Km_b \text{ and } L_A L_B = 2Km_c$$

\therefore Area of $\Delta L_A L_B L_C = (2K)^2 \times$ Area of triangle with sides of length m_a, m_b and

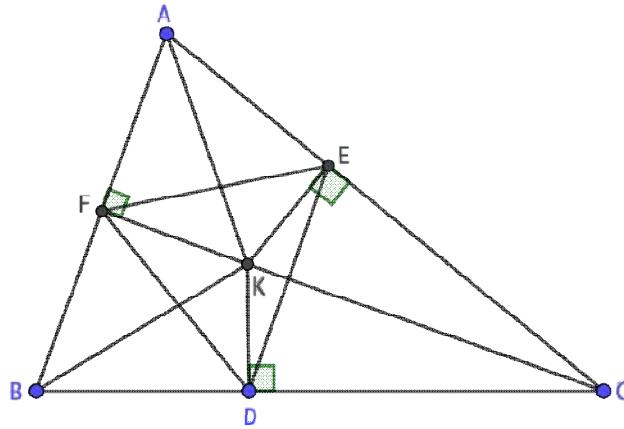
$$m_c = 4K^2 \times \left(\frac{3}{2}\right)^2 \times \text{Area of } \Delta BCG = 9K^2 \times \frac{S}{3} = \frac{27S^3}{4(m_a^2 + m_b^2 + m_c^2)^2}$$

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$$\text{Thus } R_k = \frac{L_A L_B \times L_B L_C \times L_C L_A}{4 \times \text{Area of } \Delta L_A L_B L_C} = \frac{8 \left(\frac{3S}{2(m_a^2 + m_b^2 + m_c^2)} \right)^3 m_a m_b m_c}{4 \times \frac{27S^3}{4(m_a^2 + m_b^2 + m_c^2)^2}} \therefore R_k = \frac{m_a m_b m_c}{m_a^2 + m_b^2 + m_c^2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India



Let K be the Lemoine's point & DEF be the pedal Δ of K.

We have: $KD \stackrel{(1)}{=} \frac{2aS}{\sum a^2}$, $KE \stackrel{(2)}{=} \frac{2bS}{\sum a^2}$, $KF \stackrel{(3)}{=} \frac{2cS}{\sum a^2}$; $FE \stackrel{(4)}{=} \frac{4m_a S}{\sum a^2}$, $DF \stackrel{(5)}{=} \frac{4m_b S}{\sum a^2}$ & $DE \stackrel{(6)}{=} \frac{4m_a S}{\sum a^2}$

From quad. $BDKF$, $\angle DKF \stackrel{(7)}{=} 180^\circ - B$

Similarly, $\angle DKE \stackrel{(8)}{=} 180^\circ - C$ & $\angle FKE \stackrel{(9)}{=} 180^\circ - A$

$\therefore \text{Area}(\Delta DEF) = \text{Area}(\Delta FKE) + \text{Area}(\Delta DKF) + \text{Area}(\Delta DKE)$

$$= \frac{1}{2} (FK \cdot KE \sin A + DK \cdot FK \sin B + DK \cdot KE \sin C)$$

$$(\text{using (7), (8), (9)}): = \frac{1}{2} \left[\frac{4bcS^2}{(\sum a^2)^2} \cdot \frac{a}{2R} + \frac{4caS^2}{(\sum a^2)^2} \cdot \frac{b}{2R} + \frac{4abS^2}{(\sum a^2)^2} \cdot \frac{c}{2R} \right]$$

(using (1), (2), (3))

$$= \frac{abcS^2}{R} \cdot \frac{3}{(\sum a^2)^2} = \frac{4RS^3 \cdot 3}{R(\sum a^2)^2} = \frac{12S^3}{(\sum a^2)^2} \therefore [\Delta DEF] \stackrel{(10)}{=} \frac{12S^3}{(\sum a^2)^2}$$

$$\text{Now, } R_k = \frac{FE \cdot DF \cdot DE}{4[\Delta DEF]} = \frac{64m_a m_b m_c S^3}{(\sum a^2)^3 4 \cdot \frac{12S^3}{(\sum a^2)^2}} \quad (\text{using (4), (5), (6), (10)})$$

$$= \frac{4}{3} \cdot \frac{m_a m_b m_c}{\sum a^2} = \frac{m_a m_b m_c}{\sum m_a^2} \quad \left(\because \frac{\sum m_a^2}{4} = \sum a^2 \right)$$

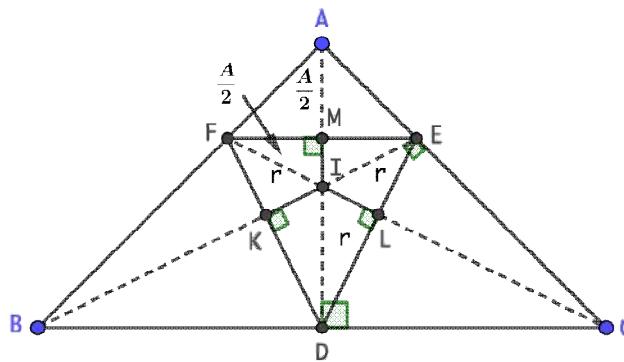
1010. In ΔABC , ΔDEF – is pedal triangle of I – incenter

If $IM \perp FE, IK \perp DF, IL \perp DE, M \in (EF), K \in (DF), L \in (DE)$ then:

$$IM^2 + IK^2 + IL^2 = r^2 \left(1 - \frac{r}{2R}\right)$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution 1 by Lahiru Samarakoon-Sri Lanka

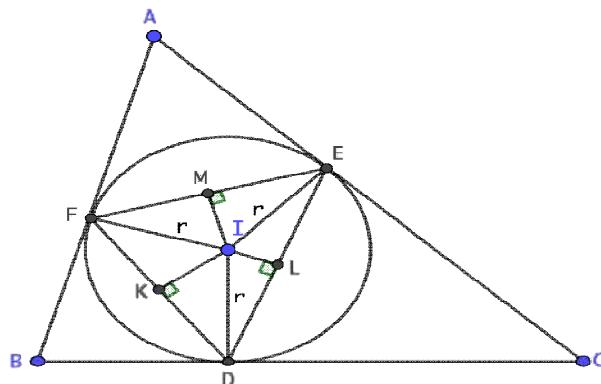


From ΔFMI , $IM = r \sin \frac{A}{2}$. Similarly, $IK = r \sin \frac{B}{2}$, $IL = r \sin \frac{C}{2}$

$$\therefore LHS = IM^2 + IK^2 + IL^2 = r^2 \left(\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} \right) = r^2 \left(1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)$$

$$\text{But, } r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = r^2 \left(1 - 2 \times \frac{r}{4R} \right) = r^2 \left(1 - \frac{r}{2R} \right)$$

Solution 2 by Thanasis Gakopoulos-Athens-Greece



$$\Delta KID: \cos \left(\frac{\overline{FID}}{2} \right) = \frac{KI}{r} \rightarrow KI^2 = r^2 \cos^2 \left(90^\circ - \frac{\widehat{B}}{2} \right) \rightarrow$$



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$$2KI^2 = r^2 \cdot 2 \sin^2 \frac{\widehat{B}}{2} = r^2(1 - \cos B) = r^2 \left(1 - \frac{a^2 + c^2 - b^2}{2ac}\right) \rightarrow$$

$$IK^2 = \frac{r^2}{4ac}(-a + b + c)(a + b - c) \quad (1)$$

$$\text{Cyclically: } IM^2 = \frac{r^2}{4bc}(a - b + c)(a + b - c) \quad (2)$$

$$IL^2 = \frac{r^2}{4ab}(-a + b + c)(a - b + c) \quad (3)$$

$$IM^2 + IK^2 + IL^2 \stackrel{(1)}{=} r^2 \left[1 - \frac{(-a+b+c)(a-b+c)(a+b-c)}{4abc}\right] \quad (4)$$

$$r^2 \left(1 - \frac{r}{2R}\right) = r^2 \left(1 - \frac{\frac{s}{abc}}{\frac{2s}{2S}}\right) = r^2 \left(1 - \frac{2s^2}{s \cdot abc}\right) \rightarrow$$

$$\rightarrow r^2 \left(1 - \frac{r}{2R}\right) = r^2 \left[1 - \frac{(-a+b+c)(a-b+c)(a+b-c)}{4abc}\right] \quad (5)$$

$$(4), (5) \rightarrow IM^2 + IK^2 + IL^2 = r^2 \left(1 - \frac{r}{2R}\right)$$

1011. In ΔABC the following relationship holds:

$$\frac{bc}{h_a^2} + \frac{ca}{h_b^2} + \frac{ab}{h_c^2} = \frac{r + r_a}{h_a} + \frac{r + r_b}{h_b} + \frac{r + r_c}{h_c}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution 1 by Marian Ursărescu-Romania

$$\frac{bc}{h_a^2} + \frac{ac}{h_b^2} + \frac{ab}{h_c^2} = \frac{a^2bc}{4s^2} + \frac{ab^2c}{4s^2} + \frac{abc^2}{4s^2} = \frac{abc}{4s^2}(a + b + c) = \frac{abc}{4s^2} \cdot s = \frac{sabc}{2s^2} \quad (1)$$

$$\text{But } abc = 4RS \text{ and } s = sr \quad (2) \text{ From (1)+(2)} \Rightarrow \frac{bc}{h_a^2} + \frac{ac}{h_b^2} + \frac{ab}{h_c^2} = \frac{2R}{r} \quad (3)$$

$$\frac{r+r_a}{h_a} + \frac{r+r_b}{h_b} + \frac{r+r_c}{h_c} = r \left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \right) + \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} \quad (4)$$

$$\text{But } \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r} \quad (5)$$

$$\begin{aligned} \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} &= \frac{s}{2S} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = \frac{s}{2S} \left(\frac{a+b+c}{abc} \right) = \frac{s}{2S} \left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \right) = \\ &= \frac{1}{2} \sum \frac{1}{s-a} = \frac{1}{2} \sum \frac{a}{s-a} = \frac{1}{2} \sum \frac{a}{2s-a} = \end{aligned}$$

$$= \frac{1}{2} \cdot \frac{2(2R-r)}{r} = \frac{2R-r}{r} = \frac{2R}{r} - 1 \quad (6)$$



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$$\text{Form (4)+(5)+(6)} \Rightarrow \frac{r+r_a}{h_a} + \frac{r+r_b}{h_b} + \frac{r+r_c}{h_c} = 1 + \frac{2R}{r} - 1 = \frac{2R}{r} \quad (7)$$

From (3)+(7) the relationship is true.

Solution 2 by Thanasis Gakopoulos-Thessaloniki-Greece

$$\begin{aligned}
 \frac{bc}{h_a^2} &= \frac{a^2 bc}{a^2 h_a^2} = \frac{a \cdot abc}{4S^2} \cdot \frac{ca}{h_b^2} = \frac{b \cdot abc}{4S^2} \cdot \frac{ab}{h_c^2} = \frac{c \cdot abc}{4S^2} \\
 \frac{bc}{h_a^2} + \frac{ca}{h_b^2} + \frac{ab}{h_c^2} &= \frac{abc}{4S^2} (a + b + c) \quad (1) \\
 \frac{r+r_a}{h_a} &= \frac{a(r+r_a)}{ah_a} = \frac{2S \cdot a(r+r_a)}{2S \cdot 2S} = \frac{1}{4S^2} \left[2Sa \left(\frac{S}{s} + \frac{S}{s-a} \right) \right] = \\
 &= \frac{1}{4S^2} \left[2S^2 a \frac{s-a+s}{s(s-a)} \right] = \frac{1}{4S^2} \left[8S^2 a \frac{b+c}{(a+b+c)(-a+b+c)} \right] \Rightarrow \\
 &\left. \begin{array}{l} \frac{r+r_a}{h_a} = \frac{1}{4S^2} \left[\frac{a}{2} (b+c)(a-b+c)(a+b-c) \right] \\ \text{Similarly, } \frac{r+r_b}{h_b} = \frac{1}{4S^2} \frac{b}{2} (c+a)(-a+b+c)(a+b-c) \\ \frac{r+r_c}{h_c} = \frac{1}{4S^2} \frac{c}{2} (a+b)(-a+b+c)(a-b+c) \end{array} \right\} \rightarrow \\
 \frac{r+r_a}{h_a} + \frac{r+r_b}{h_b} + \frac{r+r_c}{h_c} &= \frac{1}{4S^2} \frac{1}{2} [2abc(a+b+c)] = \\
 \frac{r+r_a}{h_a} + \frac{r+r_b}{h_b} + \frac{r+r_c}{h_c} &= \frac{abc}{4S^2} (a+b+c) \quad (2) \\
 (1), (2) \rightarrow \frac{bc}{h_a^2} + \frac{ca}{h_b^2} + \frac{ab}{h_c^2} &= \frac{r+r_a}{h_a} + \frac{r+r_b}{h_b} + \frac{r+r_c}{h_c}
 \end{aligned}$$

Solution 3 by Bogdan Fustei-Romania

$$\begin{aligned}
 \frac{bc}{h_a^2} + \frac{ca}{h_b^2} + \frac{ab}{h_c^2} &= \frac{2Rh_a}{h_a^2} + \frac{2Rh_c}{h_c^2} + \frac{2Rh_b}{h_b^2} \\
 bc = 2Rh_a \quad (\text{and analogs}) &= \frac{2R}{h_a} + \frac{2R}{h_b} + \frac{2R}{h_c} = \frac{2R}{r}; \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r} \\
 \text{So, finally we have the identity: } \frac{bc}{h_a^2} + \frac{ca}{h_b^2} + \frac{ab}{h_c^2} &= \frac{2R}{r} \\
 \frac{r+r_a}{h_a} + \frac{r+r_b}{h_b} + \frac{r+r_c}{h_c} &= \frac{r}{h_a} + \frac{r}{h_b} + \frac{r}{h_c} + \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} = r \cdot \frac{1}{r} + \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} = \\
 &= 1 + \sum \frac{r_a}{h_a}; \sin^2 \frac{A}{2} = \frac{r}{2R} \cdot \frac{r_a}{h_a} \quad (\text{and analogs})
 \end{aligned}$$



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$$\sin^2 \frac{A}{2} = \frac{r_a - r}{4R} \quad (\text{and analogs}) \Rightarrow \sum \sin^2 \frac{A}{2} = \frac{r_a + r_b + r_c - 3r}{4R}$$

$$r_a + r_b + r_c = 4R + r; \sum \sin^2 \frac{A}{2} = \frac{4R + r - 3r}{4R} = \frac{4R - 2r}{4R} = \frac{2R - r}{2R}$$

$$\sum \sin^2 \frac{A}{2} = \frac{r}{2R} \cdot \sum \frac{r_a}{h_a} \cdot \sum \sin^2 \frac{A}{2} = \frac{2R - r}{2R}$$

$$\frac{r}{2R} \sum \frac{r_a}{h_a} = \frac{2R - r}{2R} \Rightarrow \sum \frac{r_a}{h_a} = \frac{2R - r}{2R} \cdot \frac{2R}{r} = \frac{2R - r}{r}$$

$$\sum \frac{r + r_a}{h_a} = 1 + \sum \frac{r_a}{h_a} = 1 + \frac{2R - r}{r} = 1 + \frac{2R}{r} - 1 = \frac{2R}{r}$$

So, finally we have the following identity: $\frac{bc}{h_a^2} + \frac{ca}{h_b^2} + \frac{ab}{h_c^2} = \frac{r+r_a}{h_a} + \frac{r+r_b}{h_b} + \frac{r+r_c}{h_c} = \frac{2R}{r}$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\frac{bc}{h_a^2} + \frac{ca}{h_b^2} + \frac{ab}{h_c^2} = \frac{r + r_a}{h_a} + \frac{r + r_b}{h_b} + \frac{r + r_c}{h_c}$$

$$bc = 2Rh_a, \text{ etc, } \therefore LHS = \sum \frac{2Rh_a}{h_a^2} = 2R \sum \frac{1}{h_a} \stackrel{(1)}{=} \frac{2R}{r}$$

$$\begin{aligned} RHS &= r \sum \frac{1}{h_a} + \sum \frac{r_a}{h_a} = \frac{r}{r} + \sum \frac{s \tan \frac{A}{2} a}{2rs} \\ &= 1 + \frac{1}{2r} \sum 4R \sin \frac{A}{2} \cos \frac{A}{2} \tan \frac{A}{2} = 1 + \frac{2R}{r} \sum \sin^2 \frac{A}{2} \\ &= 1 + \frac{R}{r} \sum (1 - \cos A) = 1 + \frac{R}{r} \left(3 - 1 - \frac{r}{R} \right) = 1 + \frac{R}{r} \left(2 - \frac{r}{R} \right) \stackrel{(2)}{=} \frac{2R}{r} \end{aligned}$$

$$(1), (2) \Rightarrow LHS = RHS$$

1012. ΔDEF – pedal triangle of I – incenter in ΔABC ,

φ – inradii of ΔDEF , I_a, I_b, I_c – excenters

$\varphi_a, \varphi_b, \varphi_c$ – inradii of $\Delta I_a BC, \Delta I_b CA, \Delta I_c AB$. Prove that:

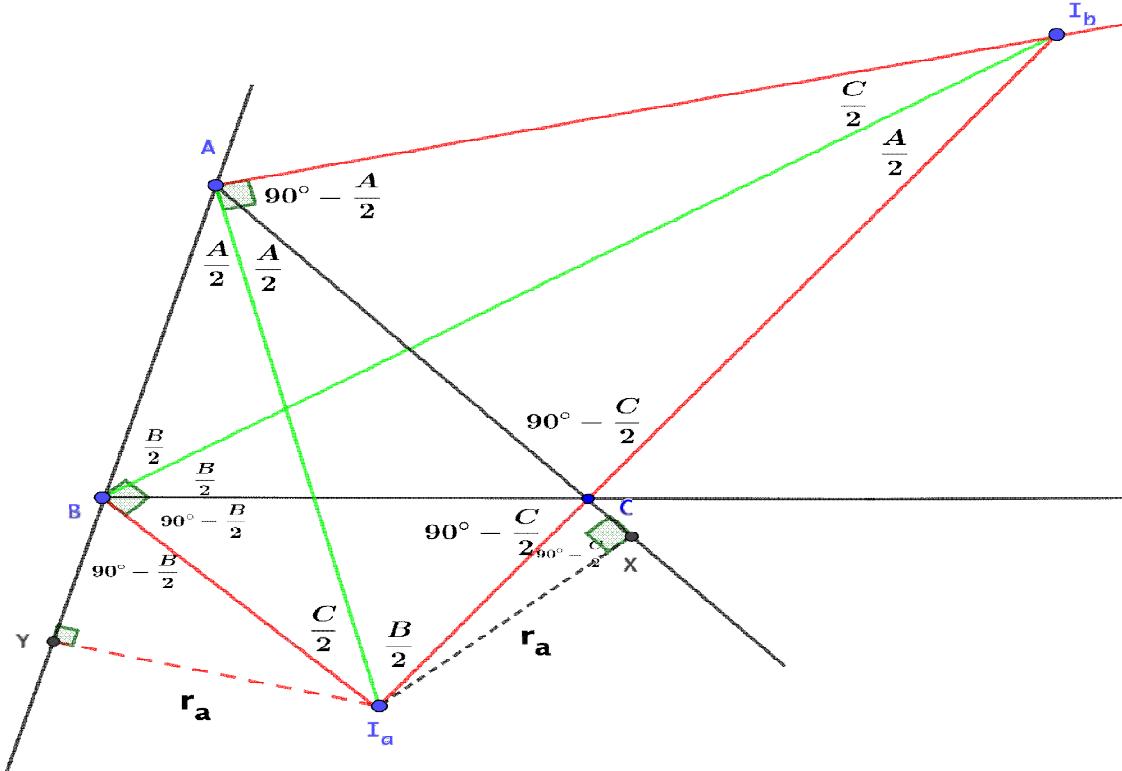
$$\frac{\varphi^2}{\varphi_a^2} + \frac{\varphi^2}{\varphi_b^2} + \frac{\varphi^2}{\varphi_c^2} = \frac{s^2 + r^2 - 8Rr}{4R^2}$$

Proposed by Mehmet Sahin-Ankara-Turkey

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Solution by Soumava Chakraborty-Kolkata-India



A, B, I_a, I_b are concyclic, as $\angle I_a I_b = \angle I_a B I_b = 90^\circ$

$$\therefore \angle B I_b I_a = \angle B A I_a = \frac{A}{2} \text{ & } \angle I_b I_a A = \angle I_b B A = \frac{B}{2}$$

Using $\Delta A B I_b$, $\angle A I_b B = \frac{C}{2}$ & using $\Delta A B I_a$, $\angle A I_a B = \frac{C}{2}$. Using $\Delta I_a X C$, $I_a C \stackrel{(1)}{=} \frac{r_a}{\cos \frac{C}{2}}$ & using

$$\Delta I_a Y B, I_a B \stackrel{(2)}{=} \frac{r_a}{\cos \frac{B}{2}} \therefore [I_a B C] = \frac{1}{2} IB \cdot IC \cdot \sin \left(\frac{B+C}{2} \right)$$

$$\text{by (1),(2)} \frac{1}{2} \cdot \frac{r_a^2 \cos^2 \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2} \cos \frac{A}{2}} = \frac{r_a^2 \cos^2 \frac{A}{2}}{2 \left(\frac{s}{4R} \right)} \stackrel{(3)}{=} \frac{2R}{s} r_a^2 \cos^2 \frac{A}{2}$$

Let circumradius of $\Delta I_a B C = R_0$

$$= \frac{I_a B \cdot I_a C \cdot BC}{4[I_a B C]} = \frac{\frac{a r_a^2}{\cos \frac{B}{2} \cos \frac{C}{2}}}{4 \frac{2R}{s} r_a^2 \cos^2 \frac{A}{2}} \quad (\text{using (1), (2), (3)}) = \frac{as}{8R \cos^2 \frac{A}{2} \left(\frac{s}{4R} \right)} \stackrel{(4)}{=} \frac{a}{2 \cos^2 \frac{A}{2}}$$

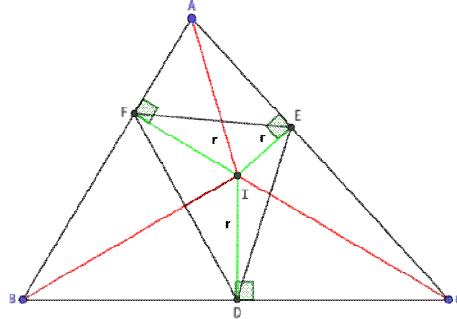
$$\text{Now, } \frac{\varphi_a}{4R_0} = \pi \cos \frac{A}{2} = \frac{s}{4R} \Rightarrow \varphi_a = R_0 \left(\frac{s}{R} \right) \stackrel{\text{by (4)}}{=} \frac{a}{2 \cos^2 \frac{A}{2}} \left(\frac{s}{R} \right) = \frac{4R \sin^2 \frac{A}{2} \cos^2 \frac{A}{2} s}{2R \cos^2 \frac{A}{2}} \stackrel{(a)}{=} 2s \sin \frac{A}{2}$$

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$$\text{Similarly, } \varphi_b \stackrel{(b)}{=} 2s \sin \frac{B}{2} \text{ & } \varphi_c \stackrel{(c)}{=} 2s \sin \frac{C}{2}$$



r is the circumcenter of ΔDEF , $\angle D = \frac{1}{2}(180^\circ - A) = 90^\circ - \frac{A}{2}$

$$\angle E = 90^\circ - \frac{B}{2} \text{ & } \angle F = 90^\circ - \frac{C}{2}$$

$$\text{Now, } \frac{\varphi}{4r} = \sin D \sin E \sin F = \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \frac{s}{4R} \Rightarrow \varphi \stackrel{(d)}{=} \frac{sr}{R}$$

$$(a), (b), (c), (d) \Rightarrow \varphi^2 \left(\sum \frac{1}{\varphi_a^2} \right) = \frac{s^2 r^2}{R^2} \cdot \frac{1}{4s^2} \sum \csc^2 \frac{A}{2}$$

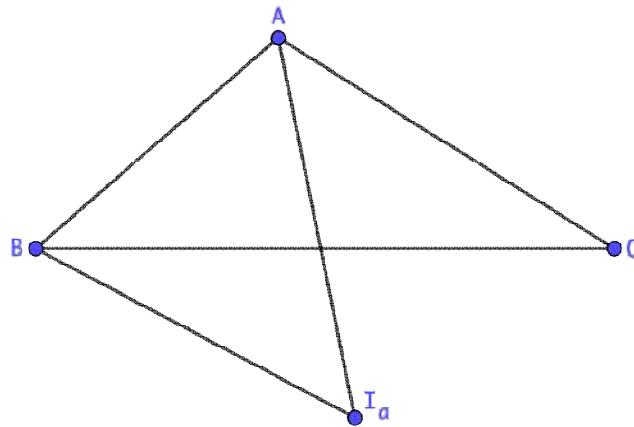
$$= \frac{r^2}{4R^2} \sum \frac{bc(s-a)}{r^2 s} = \frac{1}{4R^2 s} \{s(s^2 + 4Rr + r^2) - 12Rrs\} = \frac{s^2 - 8Rr + r^2}{4R^2} \quad (\text{Proved})$$

1013. If in ΔABC , I_a, I_b, I_c – excenters then:

$$AI_c^2 + BI_a^2 + CI_b^2 = 16R^2 - s^2 - r^2$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution by Marian Ursărescu-Romania



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$$AI_a = \frac{s}{\cos \frac{A}{2}}. \text{ In } \Delta ABI_a, \text{ by the cosine law} \Rightarrow BI_a^2 = AI_a^2 + c^2 - 2AI_a \cdot c \cos \frac{A}{2} \Rightarrow$$

$$BI_a^2 = \frac{s^2}{\cos^2 \frac{A}{2}} + c^2 - \frac{2s}{\cos \frac{A}{2}} \cdot c \cos \frac{A}{2} \Rightarrow BI_a^2 = \frac{s^2}{\cos^2 \frac{A}{2}} + c^2 - 2sc \Rightarrow$$

$$AI_c^2 + BI_a^2 + CI_c^2 = s^2 \sum \frac{1}{\cos^2 \frac{A}{2}} + \sum a^2 - 2s \sum a \quad (1)$$

$$\text{But } \sum \frac{1}{\cos^2 \frac{A}{2}} = 1 + \frac{(4R+r)^2}{s^2} \quad (2)$$

$$\sum a^2 = 2(s^2 - r^2 - 4Rr) \quad (3)$$

From (1)+(2)+(3) \Rightarrow

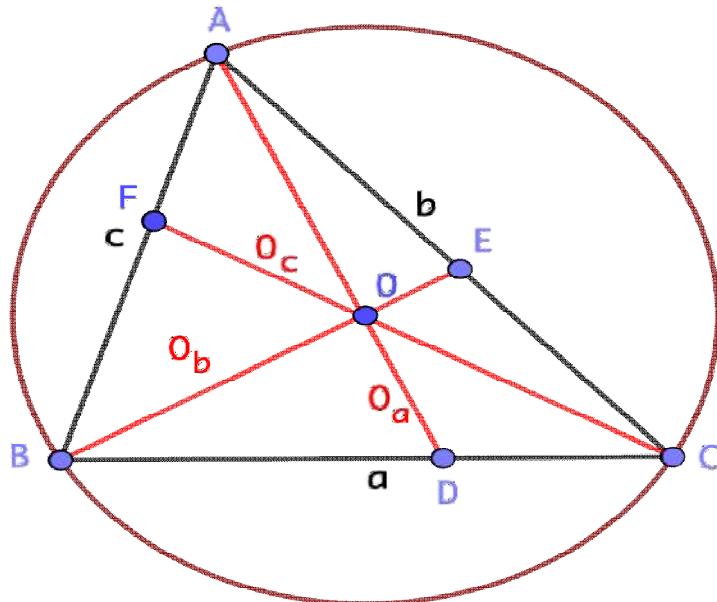
$$AI_c^2 + BI_a^2 + CI_c^2 = s^2 + 16R^2 + 8Rr + r^2 + 2s^2 - 2r^2 - 8Rr - 4s^2 = 16R^2 - s^2 - r^2$$

1014. In acute ΔABC , o_a , o_b , o_c – circumcevians. Prove that:

$$\frac{1}{o_a} + \frac{1}{o_b} + \frac{1}{o_c} = \frac{2}{R}$$

Proposed by Mustafa Tarek-Cairo-Egypt

Solution 1 by Thanasis Gakopoulos-Athens-Greece





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Let $S = \text{area of } \Delta ABC$, $\cos B = \frac{a^2+c^2-b^2}{2ac}$ (1) PLAGIAGONAL system:

$$BC \equiv Bx, BA \equiv By$$

$$B(0,0), C(a,0), A(0,c), O(o_1, o_2), o_1 = \frac{ac^2(a^2+b^2-c^2)}{16S^2}, o_2 = \frac{a^2c(-a^2+b^2+c^2)}{16S^2}$$

$$\frac{2}{R} = \frac{2}{\frac{abc}{4S}} = \frac{8S}{abc} \quad (*)$$

$$\left\{ BO: \frac{x}{o_1} = \frac{y}{o_2}, AC: \frac{x}{a} + \frac{y}{c} = 1 \right\} \rightarrow E(e_1, e_2) \quad e_1 = \frac{o_1 \cdot a \cdot c}{o_1 c + o_2 a} \quad (2)$$

$$e_2 = \frac{o_2 \cdot a \cdot c}{o_1 c + o_2 a} \quad (3)$$

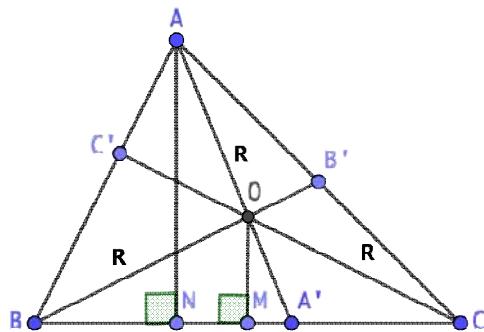
$$BE^2 = e_1^2 + e_2^2 + 2e_1 e_2 \cdot \cos B \rightarrow \\ \rightarrow o_b^2 = \frac{a^2 b^2 c^2 S^2}{(a^2 b^2 + b^2 c^2 - a^4 - c^4)^2} \rightarrow \frac{1}{o_b} = \frac{2a^2 c^2 + a^2 b^2 + b^2 c^2 - a^4 - c^4}{4abc \cdot S} \quad (4)$$

$$\text{Cyclically: } \frac{1}{o_a} = \frac{2b^2 c^2 + c^2 a^2 + a^2 b^2 - b^4 - c^4}{4abcS} \quad (5), \quad \frac{1}{o_c} = \frac{2a^2 b^2 + b^2 c^2 + c^2 a^2 - a^4 - b^4}{4abcS} \quad (6)$$

$$(4), (5), (6) \rightarrow \frac{1}{o_a} + \frac{1}{o_b} + \frac{1}{o_c} = \frac{2(2a^2 b^2 + 2b^2 c^2 + 2c^2 a^2 - a^4 - b^4 - c^4)}{4abcS} = \\ = \frac{2(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4abcS} = \frac{2 \cdot 16S^2}{4abcS} = \frac{8S}{abc} \quad (**)$$

$$(*), (**) \rightarrow \frac{1}{o_a} + \frac{1}{o_b} + \frac{1}{o_c} = \frac{2}{R}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam



$$\Delta OMA' \sim \Delta AHA' \Rightarrow \frac{OA'}{AA'} = \frac{OM}{AH}; \quad OM = \sqrt{R^2 - \frac{BC^2}{4}} = \sqrt{R^2 - \frac{a^2}{4}}$$

$$AH = \frac{2S}{a}; \quad OA' = AA' - R = o_a - R \Rightarrow \frac{o_a - R}{o_a} = \frac{\sqrt{R^2 - \frac{a^2}{4}}}{\frac{2S}{a}} = \frac{a\sqrt{R^2 - \frac{a^2}{4}}}{2S}$$

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$$\Rightarrow 1 - \frac{R}{O_a} = \frac{a\sqrt{R^2 - \frac{a^2}{4}}}{2S} \Rightarrow \frac{R}{O_a} = 1 - \frac{a\sqrt{R^2 - \frac{a^2}{4}}}{2S} \Rightarrow O_a = \frac{R}{1 - \frac{a\sqrt{R^2 - \frac{a^2}{4}}}{2S}} = \frac{R \cdot 2S}{2S \cdot a\sqrt{R^2 - \frac{a^2}{4}}} \Rightarrow$$

$$\Rightarrow \frac{1}{O_a} = \frac{2S - a\sqrt{R^2 - \frac{a^2}{4}}}{R \cdot 2S} \quad (\text{etc}) \Rightarrow \sum \frac{1}{O_a} = \sum \frac{2S - a\sqrt{R^2 - \frac{a^2}{4}}}{R \cdot 2S}$$

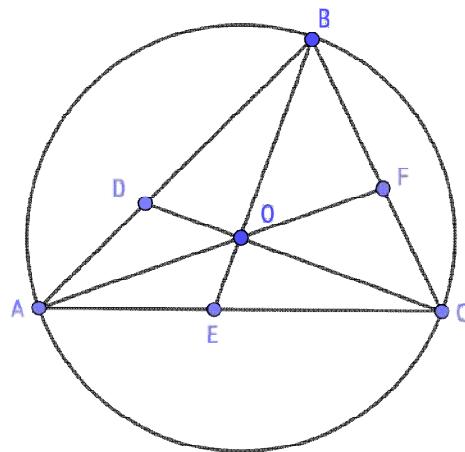
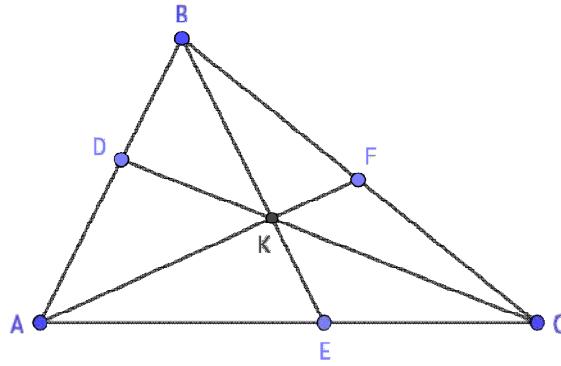
We must show that: $\sum \left(2S - a\sqrt{R^2 - \frac{a^2}{4}} \right) = 4S \Leftrightarrow 6S - \sum a\sqrt{R^2 - \frac{a^2}{4}} = 4S \Leftrightarrow 2S = \sum a\sqrt{R^2 - \frac{a^2}{4}} \quad (*)$

$$\because a = 2R \sin A; \quad (\text{etc}) \Rightarrow \sum a\sqrt{R^2 - \frac{a^2}{4}}$$

$$= \sum (2R \sin A) \sqrt{R^2 - R^2 \sin^2 A} = \sum (2R^2 \sin A) \cos A$$

$$= R^2 \sum \sin 2A = 4R^2 \sin A \sin B \sin C = 4R^2 \cdot \frac{sr}{2R^2} = 2sr = 2S \Rightarrow (*) \text{ true.}$$

Solution 3 by Adil Abdullayev-Baku-Azerbaijan





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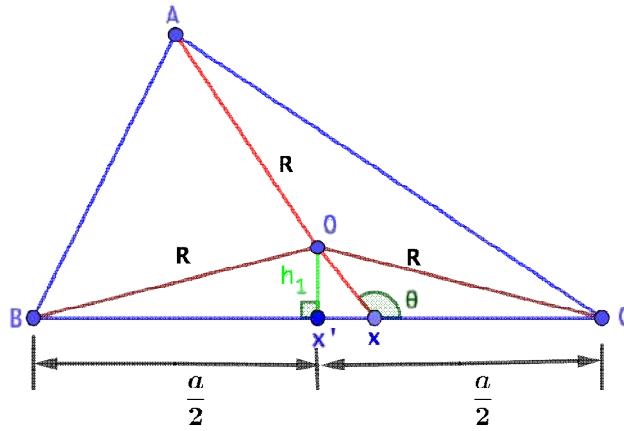
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Von-Aubel theorem: (1) $\frac{BK}{KE} = \frac{BD}{DA} + \frac{BF}{FC}$; (2) $\frac{KE}{BE} = \frac{KF}{AF} = \frac{KD}{CD} = 1$; (3) $\frac{BK}{BE} + \frac{AK}{AF} + \frac{CK}{CD} = 2$

$$AF = o_a, BE = o_b, CD = o_c; AO = OB = OC = R$$

$$(3) \Rightarrow \frac{AO}{o_a} + \frac{BO}{o_b} + \frac{CO}{o_c} = 2 \Rightarrow \frac{R}{o_a} + \frac{R}{o_b} + \frac{R}{o_c} = 2 \Rightarrow \frac{1}{o_a} + \frac{1}{o_b} + \frac{1}{o_c} = \frac{2}{R}$$

Solution 4 by Soumava Chakraborty-Kolkata-India



$$\begin{aligned} \text{Firstly, } \sum a \cos A &= R \sum 2 \sin A \cos A = R(\sin 2A + \sin 2B + \sin 2C) \\ &= R\{2 \sin(A+B) \cos(A-B) + 2 \sin C \cos C\} = 2R \sin C \{\cos(A-B) - \cos(A+B)\} \end{aligned}$$

$$= 4R \sin C \sin A \sin B = 4R \left(\frac{abc}{8R^3} \right) \stackrel{(1)}{=} \frac{abc}{2R^2}$$

$$\text{From } \Delta ACX, \frac{AX}{\sin C} = \frac{b}{\sin \theta} \Rightarrow \frac{o_a}{\sin C} = \frac{b}{\sin \theta} \Rightarrow \sin \theta \stackrel{(2)}{=} \frac{b \sin C}{o_a}$$

$$\text{From } \Delta BOX', h_1^2 = R^2 - \frac{a^4}{4} = R^2 - \frac{4R^2 \sin^2 A}{4} = R^2 \cos^2 A \Rightarrow h_1 \stackrel{(3)}{=} R \cos A$$

$$(\because \Delta ABC \text{ is acute}). \text{ Again, from } \Delta XOX', \frac{h_1}{Ox} = \sin(180^\circ - \theta) = \sin \theta$$

$$\begin{aligned} \Rightarrow \frac{R \cos A}{o_a - R} &= \sin \theta \stackrel{(2)}{=} \frac{b \sin C}{o_a} \Rightarrow \frac{o_a - R}{o_a} = \frac{R \cos A}{b \sin C} \Rightarrow \frac{R}{o_a} = 1 - \frac{R \cos A}{b \sin C} \\ &= 1 - \frac{R \cos A}{bc} \stackrel{(a)}{=} 1 - 2R^2 \left(\frac{a \cos A}{abc} \right) \end{aligned}$$

$$\text{Similarly, } \frac{R}{o_b} \stackrel{(b)}{=} 1 - \frac{2R^2}{abc} (b \cos B) \text{ & } \frac{R}{o_c} \stackrel{(c)}{=} 1 - \frac{2R^2}{abc} (c \cos C)$$

$$(a) + (b) + (c) \Rightarrow \frac{R}{o_a} + \frac{R}{o_b} + \frac{R}{o_c} = 3 - \frac{2R^2}{abc} (\sum a \cos A) \stackrel{(b)}{=} 3 - \frac{2R^2}{abc} \left(\frac{abc}{2R^2} \right)$$

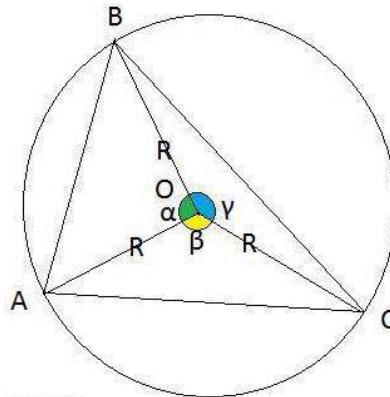
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$$= 3 - 1 = 2 \Rightarrow \frac{1}{o_a} + \frac{1}{o_b} + \frac{1}{o_c} = \frac{2}{R} \quad (\text{Done})$$

Solution 5 by Mansur Mansurov-Azerbaijan



$$\begin{cases} o_a \cdot R \cdot (\sin \alpha + \sin \beta) = 2S \\ o_b \cdot R \cdot (\sin \alpha + \sin \gamma) = 2S \\ o_c \cdot R \cdot (\sin \gamma + \sin \beta) = 2S \end{cases} \rightarrow \frac{R}{s} \cdot (\sin \alpha + \sin \beta + \sin \gamma) = \frac{1}{o_a} + \frac{1}{o_b} + \frac{1}{o_c}$$

$$S = \frac{R^2 \cdot (\sin \alpha + \sin \beta + \sin \gamma)}{2} \rightarrow \frac{1}{o_a} + \frac{1}{o_b} + \frac{1}{o_c} = \frac{2}{R}$$

1015. Prove that $x_1 = \cos A$, $x_2 = \cos B$, $x_3 = \cos C$ are the roots of equation:

$$4R^2x^3 - 4R(R+r)x^2 + (s^2 + r^2 - 4R^2)x + (2R+r)^2 - s^2 = 0$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

We have: $s = a + (s - a) = 2R \sin A + r \cot \frac{A}{2}$ (*)

(Because: $a = 2R \sin A$; and $r = (s - a) \tan \frac{A}{2} \Leftrightarrow s - a = \frac{r}{\tan \frac{A}{2}} = r \cot \frac{A}{2}$)

More, in any triangle ABC: $\sin A > 0$, $\cot \frac{A}{2} > 0$.

Then (*) $\Leftrightarrow s = 2R\sqrt{\sin^2 A} + r\sqrt{\cot^2 \frac{A}{2}}$



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$$\begin{aligned}
 & \Leftrightarrow s = 2R\sqrt{(1 - \cos^2 A)} + r \sqrt{\frac{\cos^2 \frac{A}{2}}{\sin^2 \frac{A}{2}}} \\
 & \Leftrightarrow s = 2R\sqrt{(1 - \cos A)(1 + \cos A)} + r \sqrt{\frac{1 + \cos A}{1 - \cos A}} \\
 & \Leftrightarrow s\sqrt{1 - \cos A} = (\sqrt{1 + \cos A})(2R(1 - \cos A) + r) \\
 & \Leftrightarrow s^2(1 - \cos A) = (1 + \cos A)[2R(1 - \cos A) + r]^2 \\
 & \Leftrightarrow s^2 - s^2 \cos A = (1 + \cos A)(4R^2 \cos^2 A - 8R^2 \cos A + 4R^2 + r^2 + 4Rr - 4Rr \cos A) \\
 & \Leftrightarrow 4R^3 \cos^3 A - 4R(R + r) \cos^2 A + (s^2 + r^2 - 4R^2) \cos A + (2R + r)^2 - s^2 = 0 \quad (2) \\
 & \stackrel{(2)}{\Rightarrow} x_1 = \cos A \text{ is root of equation:} \\
 & 4R^2 x^3 - 4R(R + r)x^2 + (s^2 + r^2 - 4R^2)x + (2R + r)^2 - s^2 = 0 \quad (*) \\
 & \text{Similarly: } x_2 = \cos B, x_3 = \cos C \text{ are roots of (*) Proved.}
 \end{aligned}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 & 4R^2 x^3 - 4R(R + r)x^2 + (s^2 + r^2 - 4R^2)x + (2R + r)^2 - s^2 \stackrel{(1)}{=} 0 \\
 & \text{Let } \alpha, \beta, \gamma \text{ be the roots of (1). Then, } \sum \alpha = \frac{4R(R+r)}{4R^2} = 1 + \frac{r}{R} = \sum \cos A \Rightarrow \sum \alpha \stackrel{(a)}{=} \sum \cos A \\
 & \text{Again, } 2 \sum \cos A \cos B = (\sum \cos A)^2 - \sum \cos^2 A = \left(\frac{R+r}{R}\right)^2 - (3 - \sum \sin^2 A) \\
 & = \left(\frac{R+r}{R}\right)^2 - 3 + \frac{\sum a^2}{4R^2} = \left(\frac{R+r}{R}\right)^2 - 3 + \frac{s^2 - 4Rr - r^2}{2R^2} = \frac{2(R+r)^2 - 6R^2 + s^2 - 4Rr - r^2}{2R^2} \\
 & = \frac{s^2 + r^2 - 4R^2}{2R^2} \Rightarrow \sum \cos A \cos B \stackrel{(i)}{=} \frac{s^2 + r^2 - 4R^2}{4R^2}. \text{ Also, } \sum \alpha \beta \stackrel{(ii)}{=} \frac{s^2 + r^2 - 4R^2}{4R^2} \\
 & (i), (ii) \Rightarrow \sum \alpha \beta \stackrel{(b)}{=} \sum \cos A \cos B \\
 & \text{Again, } \cos A \cos B \cos C \stackrel{(iii)}{=} \frac{s^2 - (2R+r)^2}{4R^2}. \text{ Also, } \alpha \beta \gamma \stackrel{(iv)}{=} \frac{s^2 - (2R+r)^2}{4R^2} \\
 & (iii), (iv) \Rightarrow \alpha \beta \gamma \stackrel{(c)}{=} \cos A \cos B \cos C \\
 & (a), (b), (c) \Rightarrow \cos A, \cos B, \cos C \text{ are roots of (1) (Proved)}
 \end{aligned}$$

1016. In ΔABC , I – incentre, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$.

Prove that:



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$$\left(\frac{ar_a}{R_a}\right)^2 + \left(\frac{br_b}{R_b}\right)^2 + \left(\frac{cr_c}{R_c}\right)^2 = 2s^2\left(2 - \frac{r}{R}\right)$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$R_a = 2R \sin \frac{A}{2} \text{ (etc)}$$

$$\Rightarrow \left(\frac{ar_a}{R_a}\right)^2 = \left(\frac{2R \sin A \cdot s \cdot \tan \frac{A}{2}}{2R \sin \frac{A}{2}}\right)^2 = s^2 \left(\frac{2 \sin \frac{A}{2} \cos \frac{A}{2} \cdot \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}}}{\sin \frac{A}{2}} \right)^2$$

$$= s^2 \left(2 \sin \frac{A}{2}\right)^2 = 4s^2 \cdot \frac{1 - \cos A}{2} = 2s^2(1 - \cos A)$$

$$\Rightarrow LHS = 2s^2 \sum (1 - \cos A) = 2s^2 \left(3 - \sum \cos A\right) = 2s^2 \left(3 - \left\{1 + \frac{r}{R}\right\}\right) = 2s^2 \left(2 - \frac{r}{R}\right)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} LHS &= \sum \frac{a^2 r_a^2}{4R^2 \sin^2 \frac{A}{2}} = \sum \frac{a^2 r^2 s^2 bc}{4R^2 (s-a)^2 (s-b)(s-c)} = \sum \frac{4Rrs \cdot r^2 s^2 a}{4R^2 \cdot r^2 s (s-a)} \\ &= \frac{rs^2}{R} \sum \frac{a}{s-a} = \frac{rs^2}{R} \sum \frac{a-s+s}{s-a} \\ &= -\frac{3rs^2}{R} + \frac{rs^3 \sum (s-b)(s-c)}{R \cdot r^2 s} = -\frac{3rs^2}{R} + \frac{s^2}{Rr} \sum (s^2 - s(b+c) + bc) \\ &= -\frac{3rs^2}{R} + \frac{s^2}{Rr} (3s^2 - 4s^2 + s^2 + 4Rr + r^2) = -\frac{3rs^2}{R} + \frac{s^2(4R+r)}{R} = \frac{(4R-2r)s^2}{R} \\ &= 2s^2 \left(2 - \frac{r}{R}\right) \end{aligned}$$



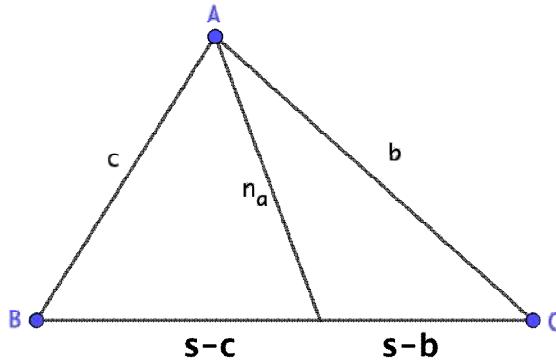
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1017. Find $k \in \mathbb{R}$ such that in any scalene ΔABC the following relationship holds:

$$\sum_{cyc} \frac{n_a^2 - m_a^2}{(b-c)^2} = k + \frac{1}{2} \sum_{cyc} \frac{b+c}{a} n_a, n_b, n_c - \text{Nagel's cevians}$$

Proposed by Bogdan Fustei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India



By Stewart's theorem, $4c^2(s-b) + 4b^2(s-c) = 4an_a^2 + 4a(s-b)(s-c)$

$$\begin{aligned} \Rightarrow 4an_a^2 &= 2c^2(c+a-b) + 2b^2(a+b-c) - a(c+a-b)(a+b-c) \\ &= 2(b^3 + c^3 - b^2c - bc^2) + a(3b^2 + 3c^2 - 2bc - a^2) \\ &\stackrel{(1)}{=} 2(b+c)(b-c)^2 + a(3b^2 + 3c^2 - 2bc - a^2) \\ &\stackrel{(2)}{=} 4am_a^2 = a(2b^2 + 2c^2 - a^2) \end{aligned}$$

$$(1)-(2) \Rightarrow 4an_a^2 - 4am_a^2 = 2(b+c)(b-c)^2 + a(b^2 + c^2 - 2bc)$$

$$= (b-c)^2(2b+2c+a) \Rightarrow \frac{4an_a^2 - 4am_a^2}{(b-c)^2} = 2b+2c+a$$

$$\Rightarrow \frac{n_a^2 - m_a^2}{(b-c)^2} = \frac{2b+2c+a}{4a} \stackrel{(i)}{=} \frac{1}{4} + \frac{1}{2} \left(\frac{b+c}{a} \right)$$

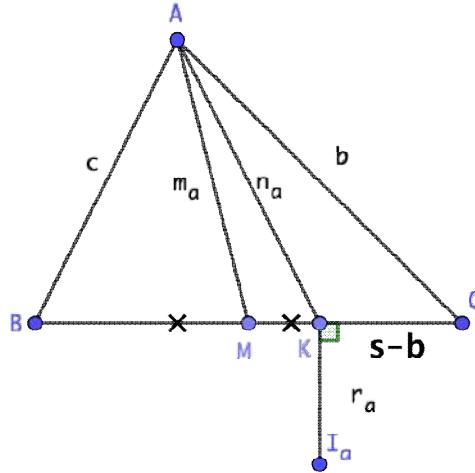
Similarly, $\frac{n_b^2 - m_b^2}{(c-a)^2} \stackrel{(ii)}{=} \frac{1}{4} + \frac{1}{2} \left(\frac{c+a}{b} \right)$ & $\frac{n_c^2 - m_c^2}{(a-b)^2} \stackrel{(iii)}{=} \frac{1}{4} + \left(\frac{a+b}{c} \right) \frac{1}{2}$

$$(i) + (ii) + (iii) \Rightarrow \sum \frac{n_a^2 - m_a^2}{(b-c)^2} = \frac{3}{4} + \frac{1}{2} \sum \left(\frac{b+c}{a} \right) \therefore k = \frac{3}{4} \quad (\text{Answer})$$

R M M

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Solution 2 by Tran Hong-Dong Thap-Vietnam



In ΔABK we have: $AB = c$; $AK = n_a$; $BM = \frac{a}{2}$; $MK = BK - BM = s - c - \frac{a}{2} = \frac{b-c}{2}$

By Stewart's theorem: $c^2 \cdot MK + n_a^2 \cdot BM = BK \cdot (m_a^2 + BM \cdot MK)$

$$\begin{aligned}
&\Leftrightarrow c^2 \left(\frac{b-c}{2} \right) + n_a^2 \cdot \frac{a}{2} = (s-c) \left(m_a^2 + \frac{a}{2} \cdot \frac{b-c}{2} \right) \\
&\Leftrightarrow n_a^2 \cdot \frac{a}{2} = (s-c)m_a^2 + (s-c) \cdot \frac{a(b-c)}{4} - \frac{c^2(b-c)}{2} \\
&\Leftrightarrow n_a^2 = \frac{2(s-c)m_a^2}{a} + \frac{(s-c)(b-c)}{2} - \frac{c^2(b-c)}{a} \\
&\Rightarrow n_a^2 - m_a^2 = \left(\frac{2(s-c)}{a} - 1 \right) m_a^2 + \frac{(s-c)(b-c)}{2} - \frac{c^2(b-c)}{a} \\
&= \frac{(b-c)}{a} \cdot m_a^2 + \frac{(s-c)(b-c)}{2} - \frac{c^2(b-c)}{a} = (b-c) \left[\frac{m_a^2}{a} - \frac{c^2}{a} + \frac{(a+b-c)}{4} \right] \\
&= (b-c) \left[\frac{2(b^2+c^2)-a^2}{4a} - \frac{c^2}{a} + \frac{(a+b-c)}{4} \right] = (b-c) \left[\frac{2(b^2-c^2)-a^2}{4a} + \frac{(a+b-c)}{4} \right] \\
&= \frac{(c-b)}{4a} \cdot [2(b^2-c^2) - a^2 + a(a+b-c)] \\
&= \frac{(b-c)}{4a} [2(b^2-c^2) + ab - ac] = \frac{(b-c)^2}{4a} [a + 2(b+c)] \text{ (etc)} \\
&\Rightarrow \sum \frac{n_a^2 - m_a^2}{(b-c)^2} = \sum \frac{a + 2(b+c)}{4a} = \sum \left(\frac{1}{4} + \frac{(b+c)}{2a} \right) = \frac{3}{4} + \frac{1}{2} \sum \frac{b+c}{a} \Rightarrow k = \frac{3}{4}
\end{aligned}$$



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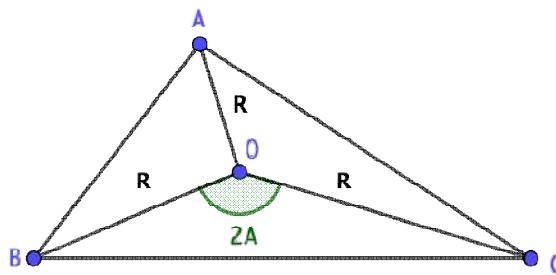
1018. In ΔABC , O – circumcenter, r_1, r_2, r_3 – inradii of $\Delta BOC, \Delta COA, \Delta AOB$.

Prove that:

$$2 \sum_{cyc(A,B,C)} (\sec A + \tan A) = \frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution 1 by Marian Ursărescu-Romania



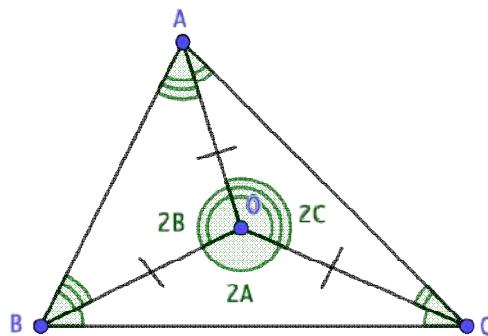
$$\begin{aligned} s &= \frac{a+b+c}{2}, r_1 = \frac{s_{BOC}}{s_{BOC}}, S_{BOC} = \frac{OB \cdot OC \cdot \sin 2A}{2} = \frac{R^2 \sin 2A}{2} \\ p_{BOC} &= \frac{R+R+a}{2} = \frac{2R+a}{2} \end{aligned} \Rightarrow r_1 = \frac{R^2 \sin 2A}{2R+a} \Rightarrow$$

$$\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} = \sum \frac{a(2R+a)}{R^2 \sin 2A} = \sum \frac{2aR + a^2}{R^2 \sin 2A} =$$

$$= 2 \sum \frac{a}{R \sin 2A} + \sum \frac{a^2}{R^2 \sin 2A} = 2 \sum \frac{2R \sin A}{R \cdot 2 \sin A \cos A} + \sum \frac{4R^2 \sin^2 A}{R^2 - 2 \sin A \cos A} =$$

$$= 2 \sum \frac{1}{\cos A} + 2 \sum \tan A = 2 \left(\sum \sec A + \tan A \right)$$

Solution 2 by Soumava Chakraborty-Kolkata-India



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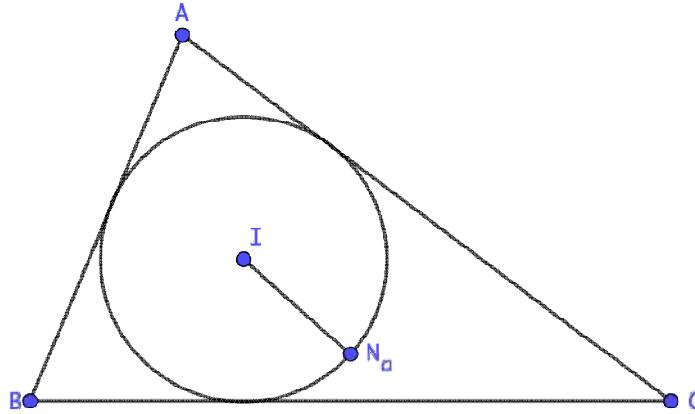
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Let R_1 be the circumradius of ΔBOC . $R_1 = \frac{OB \cdot OC \cdot BC}{4 \cdot \frac{1}{2} OB \cdot OC \sin 2A} \stackrel{(1)}{=} \frac{a}{2 \sin 2A}$

$$\begin{aligned} \text{Again, } \frac{r_1}{4R_1} &= \sin \angle \frac{BOC}{2} \sin \angle \frac{OBC}{2} \sin \angle \frac{OCB}{2} = \sin A \sin^2 \left(45^\circ - \frac{A}{2} \right) \\ &= \sin A \left(\frac{1}{\sqrt{2}} \cos \frac{A}{2} - \frac{1}{\sqrt{2}} \sin \frac{A}{2} \right)^2 = \frac{\sin A (1 - \sin A)}{2} \Rightarrow r_1 = 2R_1 \sin A (1 - \sin A) \\ &\stackrel{\text{by (1)}}{=} \frac{a \sin A (1 - \sin A)}{\sin 2A} \Rightarrow \frac{a}{r_1} = \frac{\sin 2A}{\sin A (1 - \sin A)} \\ \Rightarrow \frac{a}{r_1} &= \frac{2 \sin A \cos A (1 + \sin A)}{\sin A (1 - \sin^2 A)} = \frac{2 \cos A (1 + \sin A)}{\cos^2 A} \stackrel{(a)}{=} 2(\sec A + \tan A) \\ \text{Similarly, } \frac{b}{r_2} &\stackrel{(2)}{=} 2(\sec B + \tan B) \quad \& \frac{c}{r_3} \stackrel{(c)}{=} 2(\sec C + \tan C) \\ (a) + (b) + (c) \Rightarrow 2 \sum (\sec A + \tan A) &= \frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} \quad (\text{proved}) \end{aligned}$$

1019.



$AB = k, BC = k + 3, AC = k + 2, k > 0, I$ – incenter of ΔABC

N_a – Nagel's point of ΔABC , r – inradius of ΔABC , $N_a \in (I, r)$

Find: $S(abc)$ (area)

Proposed by Thanasis Gakopoulos-Greece

Solution by Soumava Chakraborty-Kolkata-India

$a = k + 3, b = k + 2, c = k, N_a \in (I, r), S(ABC) = ?$



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$$\begin{aligned}
 IN_a^2 = r^2 &\Rightarrow 9IG^2 = r^2 \Rightarrow \frac{9}{18s} \left(2 \sum a^2b + 2 \sum ab^2 - \sum a^3 - 9abc \right) = r^2 \\
 &\Rightarrow \frac{1}{2s} \left(2 \sum ab(2s - c) - 2s(s^2 - 6Rr - 3r^2) - 36Rrs \right) = r^2 \\
 &\Rightarrow \frac{1}{2s} (4s(s^2 + 4Rr + r)^2 - 2s(s^2 - 6Rr - 3r^2) - 60Rrs) = r^2 \\
 &\Rightarrow \frac{1}{2s} (2s^3 + 10sr^2 - 32Rrs) = r^2 \Rightarrow s^2 - 16Rr + 5r^2 = r^2 \Rightarrow s^2 \stackrel{(1)}{=} 16Rr - 4r^2
 \end{aligned}$$

$$\text{Now, } abc = 4Rrs \Rightarrow k(k+2)(k+3) = 2Rr(k+k+2+k+3) \Rightarrow 2Rr \stackrel{(2)}{=} \frac{k(k+2)(k+3)}{3k+5}$$

$$\begin{aligned}
 \text{Again, } \sum ab &= s^2 + 4Rr + r^2 \Rightarrow (k+3)(k+2) + k(k+2) + k(k+3) \\
 &= \left(\frac{k+k+2+k+3}{2} \right)^2 + 2 \left(\frac{k(k+2)(k+3)}{3k+5} \right) + r^2 \\
 &\Rightarrow 4(3k+5)r^2 = 4(3k+5)(3k^2 + 10k + 6) - (3k+5)^3 - 8k(k+2)(k+3) \\
 &\Rightarrow 4(3k+5)r^2 = (k^2 - 1)(k+5) \Rightarrow r \stackrel{(3)}{=} \frac{(k^2 - 1)(k+5)}{4(3k+5)} \\
 \text{Plugging (2), (3) in (1): } &\frac{(3k+5)^2}{4} = \frac{8k(k+2)(k+3)}{3k+5} - \frac{4(k^2 - 1)(k+5)}{4(3k+5)} \\
 &\Rightarrow \frac{(3k+5)^2}{4} = \frac{7k^3 + 35k^2 + 49k + 5}{3k+5} \Rightarrow k^3 + 5k^2 - 29k - 105 = 0 \\
 &\Rightarrow (k+3)(k+7)(k-5) = 0 \Rightarrow k = 5 \therefore a = 8, b = 7, c = 5 \Rightarrow s = 10 \\
 &\Rightarrow S(ABC) = \sqrt{(10)(10-8)(10-7)(10-5)} = 10\sqrt{3} \text{ (Answer)}
 \end{aligned}$$

1020. URSARESCU's REFINEMENT OF EULER'S INEQUALITY

In ΔABC the following relationship holds:

$$R \geq \frac{1}{6} \left(\frac{a(b+c-a)}{h_a} + \frac{b(c+a-b)}{h_b} + \frac{c(a+b-c)}{h_c} \right) \geq 2r$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum \frac{2a(s-a)}{h_a} = \sum \frac{2a(s-a)a}{2rs} = \frac{1}{rs} \sum a^2(s-a) =$$



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$$\frac{s \cdot 2(s^2 - 4Rr - r^2) - 2s(s^2 - 6Rr - 3r^2)}{rs} = \frac{2s(2Rr + 2r^2)}{rs} = 4(R + r)$$

$$\therefore \frac{1}{6} \sum \frac{2a(s-a)}{h_a} = \frac{2}{3}(R+r)$$

$$\therefore R \geq \frac{1}{6} \sum \frac{2a(s-a)}{h_a} \Leftrightarrow R \geq \frac{2}{3}(R+r) \Leftrightarrow R \geq 2r \rightarrow \text{true}$$

$$\& \frac{1}{6} \sum \frac{2a(s-a)}{h_a} \geq 2r \Leftrightarrow \frac{R+r}{3} \geq r \Leftrightarrow R \geq 2r \rightarrow \text{true (proved)}$$

Solution 2 by Tran Hong-Vietnam

$$\begin{aligned} \text{We have: } & \frac{1}{6} \sum \frac{a(b+c-a)}{h_a} = \frac{1}{6} \sum \frac{a(b+c-a)}{\frac{2sr}{a}} = \frac{1}{6sr} \sum \frac{a^2}{2} (b+c-a) = \frac{abc}{6sr} (\cos A + \cos B + \cos C) \\ & = \frac{4Rsr}{6sr} \left(1 + \frac{r}{R}\right) = \frac{2}{3}(R+r) \end{aligned}$$

$$R \geq \frac{1}{6} \sum \frac{a(b+c-a)}{h_a} = \frac{2}{3}(R+r) \Leftrightarrow R \geq 2r \text{ (true)}$$

$$2r \leq \frac{1}{6} \sum \frac{a(b+c-a)}{h_a} = \frac{2}{3}(R+r) \Leftrightarrow R \geq 2r \text{ (true)}$$

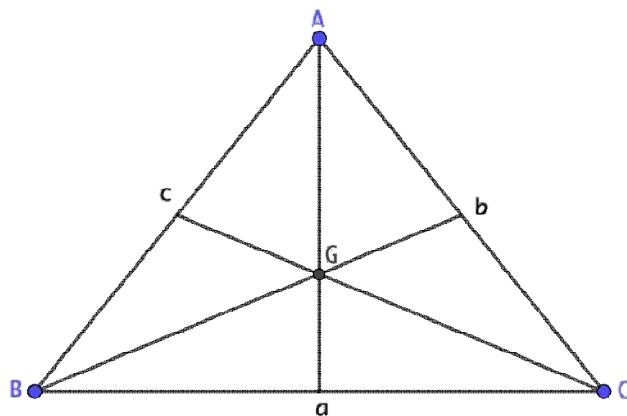
1021. In ΔABC the following relationship holds:

$$\left(\frac{2m_a + 2m_b}{m_c}\right)^7 + \left(\frac{2m_b + 2m_c}{m_a}\right)^7 + \left(\frac{2m_c + 2m_a}{m_b}\right)^7 > \left(\frac{3a}{m_a}\right)^7 + \left(\frac{3b}{m_b}\right)^7 + \left(\frac{3c}{m_c}\right)^7$$

Proposed by Daniel Sitaru – Romania

Solution by Lahiru Samarakoon-Sri Lanka

$$\sum \left(\frac{2m_a + 2m_b}{m_c}\right)^7 > \sum \left(\frac{3a}{m_a}\right)^7$$





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$\Delta ACG, AG = \frac{2}{3}m_a$ and $CG = \frac{2}{3}m_a$. So, to have: $AG + GC > AC$. So, $\frac{2m_a}{3} + \frac{2}{3}m_c > b$

$$(2m_a + 2m_c) > 3b; \left(\frac{2m_a + 2m_c}{m_b}\right)^7 > \left(\frac{3b}{m_b}\right)^7 (\because m_b > 0)$$

So, similarly, from ΔAGB and ΔBGC , and by summation: $\sum \left(\frac{2m_a + 2m_c}{m_b}\right)^7 > \sum \left(\frac{3b}{m_b}\right)^7$

1022. In ΔABC the following relationship holds:

$$\sqrt{3 \left(\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2} \right)} \leq \frac{m_a m_b m_c}{s^2}$$

Proposed by Bogdan Fustei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$(1) \Leftrightarrow 3 \left(\frac{\sum a^2}{4s^2} \right) \leq \frac{m_a^2 m_b^2 m_c^2}{s^4} \Leftrightarrow 4m_a^2 m_b^2 m_c^2 \stackrel{(2)}{\geq} 6r^2 s^2 (s^2 - 4Rr - r^2)$$

$$\begin{aligned} \text{Now, } m_a^2 m_b^2 m_c^2 &= \frac{(2b^2 + 2c^2 - a^2)(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2)}{64} = \\ &\stackrel{(a)}{=} \frac{-4 \sum a^6 + 6(\sum a^4 b^2 + \sum a^2 b^4) + 3a^2 b^2 c^2}{64} \end{aligned}$$

$$\text{Now, } \sum a^6 = (\sum a^2)^3 - 3(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) =$$

$$= (\sum a^2)^3 - 3(\sum a^2 - c^2)(\sum a^2 - a^2)(\sum a^2 - b^2)$$

$$\begin{aligned} &= (\sum a^2)^3 - 3 \left\{ (\sum a^2)^3 - (\sum a^2)^3 + (\sum a^2)(\sum a^2 b^2) - a^2 b^2 c^2 \right\} = \\ &\stackrel{(b)}{=} (\sum a^2)^3 - 3(\sum a^2)(\sum a^2 b^2) + 3a^2 b^2 c^2 \end{aligned}$$

$$\text{Also, } \sum a^4 b^2 + \sum a^2 b^4 = \sum a^2 b^2 (\sum a^2 - c^2) \stackrel{(c)}{=} (\sum a^2)(\sum a^2 b^2) - 3a^2 b^2 c^2$$

$$\begin{aligned} (a), (b), (c) \Rightarrow m_a^2 m_b^2 m_c^2 &= \frac{1}{64} \left\{ -4(\sum a^2)^3 + 18(\sum a^2)(\sum a^2 b^2) - 27a^2 b^2 c^2 \right\} \\ &= \frac{1}{64} \left[-32(s^2 - 4Rr - r^2)^3 + 36(s^2 - 4Rr - r^2) \{ (s^2 + 4Rr + r^2)^2 - 2abc(2s) \} \right. \\ &\quad \left. - 432R^2 r^2 s^2 \right] \\ &\stackrel{(d)}{=} \frac{1}{16} \left\{ s^6 - s^4(12Rr - 33r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4) - 64R^3 r^3 - 48R^2 r^4 \right\} \\ &\quad - 12Rr^5 - r^6 \end{aligned}$$



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$$(d) \Rightarrow (2) \Leftrightarrow s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - 64R^3r^3 - 48R^2r^4 - 12Rr^5 - r^6 \geq 24r^2s^2(s^2 - 4Rr - r^2)$$

$$\Leftrightarrow s^6 - s^4(12Rr - 9r^2) - s^2(60R^2r^2 + 24Rr^3 + 9r^4) - 64R^3r^3 - 48R^2r^4 - 12Rr^5 - r^6 \stackrel{(3)}{\geq} 0$$

$$\begin{aligned} \text{Now, LHS of (3)} &\stackrel{\text{Gerretsen}}{\geq} s^4(4Rr + 4r^2) - s^2(60R^2r^2 + 24Rr^3 + 9r^4) - 64R^3r^3 - \\ &\quad - 48R^2r^4 - 12Rr^5 - r^6 \stackrel{\text{Gerretsen}}{\stackrel{(4)}{\geq}} 0 \end{aligned}$$

$$\begin{aligned} \text{Now, LHS of (4)} &\stackrel{\text{Gerretsen}}{\geq} s^2\{(16Rr - 5r^2)(4Rr + 4r^2) - (60R^2r^2 + 24Rr^3 + 9r^4)\} - \\ &\quad - 64R^3r^3 - 48R^2r^4 - 12Rr^5 - r^6 \stackrel{?}{\geq} 0 \Leftrightarrow \\ &\Leftrightarrow s^2(4R^2 + 20Rr - 29r^2) - 64R^3r - 48R^2r^2 - 12Rr^3 - r^4 \stackrel{?}{\geq} 0 \end{aligned}$$

$$\begin{aligned} \text{Now, } 4R^2 + 20Rr - 29r^2 &\stackrel{\text{Euler}}{\geq} 4R^2 + 40r^2 - 29r^2 > 0 \therefore \text{LHS of (5)} \\ &\stackrel{\text{Gerretsen}}{\geq} (16Rr - 5r^2)(4R^2 + 20Rr - 29r^2) - 64R^3r - 48R^2r^2 - 12Rr^3 - r^4 \stackrel{?}{\geq} 0 \\ &\Leftrightarrow 7R^2 - 16Rr + 4r^2 \stackrel{?}{\geq} 0 \Leftrightarrow (R - 2r)(7R - 2r) \stackrel{?}{\geq} 0 \rightarrow \text{true (Euler) (Proved)} \end{aligned}$$

1023. In ΔABC the following relationship holds:

$$2 \left(\sqrt{\cos \frac{A}{2}} + \sqrt{\cos \frac{B}{2}} + \sqrt{\cos \frac{C}{2}} \right) - (\sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C}) \geq \frac{\frac{5}{1}}{2^{\frac{3}{2}}}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Daniel Sitaru-Romania

$$f: (0, \pi) \rightarrow \mathbb{R}, f(x) = (\sin x)^{\frac{1}{2}}, f''(x) = -\frac{1}{2} \sin x (\sin x)^{-\frac{1}{2}} - \frac{1}{4} \cos^2 x (\sin x)^{-\frac{3}{2}} < 0,$$

f – concave

$$\frac{1}{3} \sum_{cyc(A,B,C)} f(A) + f\left(\frac{A+B+C}{3}\right) \leq \frac{2}{3} \sum_{cyc(A,B,C)} f\left(\frac{B+C}{2}\right)$$

$$\frac{1}{3} \sum_{cyc(A,B,C)} \sqrt{\sin A} + \sin\left(\frac{\pi}{3}\right) \leq \frac{2}{3} \sum_{cyc(A,B,C)} \sqrt{\sin\left(\frac{B+C}{2}\right)}$$



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$$\sum_{cyc(A,B,C)} \sqrt{\sin A} + 3 \sqrt{\frac{\sqrt{3}}{2}} \leq 2 \sum_{cyc(A,B,C)} \sqrt{\sin\left(\frac{\pi - A}{2}\right)}$$

$$2 \sum_{cyc(A,B,C)} \sqrt{\cos \frac{A}{2} - \sum_{cyc(A,B,C)} \sqrt{\sin A}} \geq \frac{3^{\frac{5}{4}}}{2^{\frac{1}{2}}}$$

Solution 2 by Lahiru Samarakoon-Sri Lanka

We have to prove, $\sum \sqrt{\sin A} + \frac{3^{\frac{5}{4}}}{2^{\frac{1}{2}}} \leq 2 \left(\sum \sqrt{\cos \frac{A}{2}} \right)$. Let's consider, $f(x) = \sqrt{\sin x}$

$$f'(x) = \frac{\cos x}{2\sqrt{\sin x}}; f''(x) = \frac{\sqrt{\sin x}(-\sin x) - \cos x \frac{\cos x}{2\sqrt{\cos x}}}{2\sin x} = \frac{-(2\sin^2 x + \cos^2 x)}{4\sin x \sqrt{\cos x}}$$

Then, $f''(x) < 0$

$$\frac{1}{3} \sum \sqrt{\sin A} + \sqrt{\sin\left(\frac{A+B+C}{2}\right)} \leq \frac{2}{3} \sum \sin\left(\frac{B+C}{2}\right)$$

$$\frac{1}{3} \sum \sqrt{\sin A} + \left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{2}} \leq \frac{2}{3} \sum \cos \frac{A}{2}; \sum \sqrt{\sin A} + \frac{3^{\frac{5}{4}}}{2^{\frac{1}{2}}} \leq 2 \sum \cos \frac{A}{2}; (\text{it's true})$$

1024. In acute ΔABC , I – incenter the following relationship holds:

$$\frac{m_a}{AI^2} + \frac{m_b}{BI^2} + \frac{m_c}{CI^2} \leq \frac{4R+r}{4r^2}$$

Proposed by Bogdan Fustei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum \frac{m_a}{AI^2} &\leq \frac{4R+r}{4r^2} \because m_a \leq R(1 + \cos A), \text{ etc}, \therefore \sum \frac{m_a}{AI^2} \leq \sum \frac{R \cdot 2 \cos^2 \frac{A}{2} \sin^2 \frac{A}{2}}{r^2} = \sum \frac{R \sin^2 A}{2r^2} = \sum \frac{R \cdot a^2}{2r^2 \cdot 4R^2} = \\ &= \frac{1}{8Rr^2} \sum a^2 \stackrel{?}{\leq} \frac{4R+r}{4r^2} \Leftrightarrow \sum a^2 \stackrel{?}{\leq} 8R^2 + 2Rr \Leftrightarrow \\ &\Leftrightarrow s^2 - 4Rr - r^2 \stackrel{?}{\leq} 4R^2 + Rr \Leftrightarrow s^2 \stackrel{?}{\leq}_{(1)} 4R^2 + 5Rr + r^2 \end{aligned}$$

$$\begin{aligned} \text{Now, LHS of (1)} &\stackrel{\text{Gerretsen}}{\leq} 4R^2 + 4Rr + 3r^2 \stackrel{?}{\leq} 4R^2 + 5Rr + r^2 \Leftrightarrow Rr \stackrel{?}{\geq} 2r^2 \Leftrightarrow \\ &\Leftrightarrow R \stackrel{?}{\geq} 2r \rightarrow \text{true (Euler) (Proved)} \end{aligned}$$



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1025. In ΔABC the following relationship holds:

$$\sum \sqrt{r_a(r_b + r_c)} \leq (m_a + m_b + m_c) \sqrt{\frac{R}{r}}$$

Proposed by Bogdan Fustei – Romania

Solution 1 by Marian Ursărescu-Romania

From Cauchy's inequality $\Rightarrow (\sum \sqrt{r_a(r_b + r_c)})^2 \leq 3 \sum r_a(r_b + r_c)$

$$\Rightarrow \sum \sqrt{r_a(r_b + r_c)} \leq \sqrt{6 \sum r_a r_b} \quad (1)$$

$$\text{But } \sum r_a r_b = s^2 \quad (2)$$

$$\text{From (1)+(2)} \Rightarrow \sum \sqrt{r_a(r_b + r_c)} \leq \sqrt{6s} \quad (3)$$

$$\left. \begin{aligned} m_a + m_b + m_c &\geq 3\sqrt[3]{m_a m_b m_c} \\ m_a &\geq \sqrt{s(s-a)} \end{aligned} \right\} \Rightarrow m_a + m_b + m_c \geq 3\sqrt[3]{sS} \Rightarrow$$

$$m_a + m_b + m_c \geq 3\sqrt[3]{s^2 r} \quad (4)$$

From (3)+(4) we must show:

$$3\sqrt[3]{s^2 r} \cdot \sqrt{\frac{R}{r}} \geq \sqrt{6s} \Leftrightarrow 3^6 s^4 r^2 \cdot \frac{R^3}{r^3} \geq 6^3 s^6 \Leftrightarrow 3^6 \frac{R^3}{r} \geq 3^3 \cdot 2^3 \cdot s^2 \Leftrightarrow 27R^3 \geq 8s^2 r \quad (5)$$

From Mitrinovic's inequality: $27R^2 \geq 4s^2 \Rightarrow 27R^3 \geq 4Rs^2 \quad (6)$

From (5)+(6) we must show: $4Rs^2 \geq 8s^2 r \Leftrightarrow R \geq 2r$, true (Euler)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum \sqrt{r_a(r_b + r_c)} &\leq \left(\sum m_a \right) \sqrt{\frac{R}{r}} \\ \sum \sqrt{r_a(r_b + r_c)} &= \sum \sqrt{s^2 \tan \frac{A}{2} \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right)} \\ &= \sum \sqrt{s^2 \tan \frac{A}{2} \left(\frac{\cos \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} \right)} = \sum \sqrt{s^2 \tan \frac{A}{2} \left(\frac{\cos^2 \frac{A}{2}}{\frac{s}{4R}} \right)} \end{aligned}$$



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$$= \sum \sqrt{4Rs \cos \frac{A}{2} \sin \frac{A}{2}} = \sum \sqrt{sa} \stackrel{CBS}{\leq} \sqrt{s} \sqrt{3} \sqrt{2s} = \sqrt{6}s \therefore LHS \stackrel{(1)}{\leq} \sqrt{6}s$$

$$\text{Again, RHS} \stackrel{\text{Tereshin}}{\geq} \sqrt{\frac{R}{r} \sum \left(\frac{b^2+c^2}{4R} \right)} = \sqrt{\frac{R}{r} \left(\frac{\sum a^2}{2R} \right)} \therefore RHS \stackrel{(2)}{\geq} \sqrt{\frac{R}{r} \left(\frac{\sum a^2}{2R} \right)}$$

$$(1), (2) \Rightarrow \text{it suffices to prove: } \sqrt{\frac{R}{r} \left(\frac{\sum a^2}{2R} \right)} \geq \sqrt{6}s \Leftrightarrow (\sum a^2)^2 \geq 24Rrs^2$$

$$\Leftrightarrow s^4 + r^2(4R+r)^2 - 2s^2(4Rr+r^2) \geq 6Rrs^2$$

$$\Leftrightarrow s^4 + r^2(4R+r)^2 \stackrel{(3)}{\geq} 2(7Rr+r^2)s^2$$

$$\text{Now, LHS of (3)} \stackrel{\text{Gerretsen}}{\geq} s^2(16Rr-5r^2) + r^2(4R+r)^2 \stackrel{?}{\geq} 2(7Rr+r^2)s^2 \Leftrightarrow$$

$$\Leftrightarrow s^2(2R-4r) + r(4R+r)^2 \stackrel{(4)}{\geq} 3rs^2$$

$$\text{Now, LHS of (4)} \stackrel{\text{Gerretsen}}{\geq} (2R-4r)(16Rr-5r^2) + r(4R+r)^2$$

$$\& \text{RHS of (4)} \stackrel{\text{Gerretsen}}{\leq} (b) 3r(4R^2+4Rr+3r^2)$$

(a), (b) \Rightarrow in order to prove (4), it suffices to prove:

$$(2R-4r)(16R-5r) + (4R+r)^2 \geq 3(4R^2+4Rr+3r^2) \geq 0$$

$$\Leftrightarrow (R-2r)(6R-r) \geq 0 \rightarrow \text{true} \Rightarrow (4) \text{ is true} \Rightarrow (3) \text{ is true (Proved)}$$

1026. If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{x}{y+z} \cdot r_a^2 + \frac{y}{z+x} \cdot r_b^2 + \frac{z}{x+y} \cdot r_c^2 \geq \frac{91r^2 - 16R^2}{2}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution by Soumitra Mandal-Chandar Nagore-India

We know, $\sum_{cyc} r_a = 4R + r$ and $\sum_{cyc} r_a r_b = s^2$

$$\sum_{cyc} \frac{x}{y+z} r_a^2 = (x+y+z) \sum_{cyc} \frac{r_a^2}{y+z} - \sum_{cyc} r_a^2$$



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$$\begin{array}{l} \text{BERGSTROM'S} \\ \text{INEQUALITY} \end{array} \geq \frac{(r_a + r_b + r_c)^2}{2} - \left(\sum_{\text{cyc}} r_a \right)^2 + 2 \sum_{\text{cyc}} r_a r_b = 2s^2 - \frac{(4R + r)^2}{2}$$

$$\text{We need to prove, } 2s^2 - \frac{(4R+r)^2}{2} \geq \frac{91r^2 - 16R^2}{2} \Leftrightarrow s^2 \geq 23r^2 + 2Rr$$

We know, $s^2 \geq 16Rr - 5r^2$ we need to prove, $16Rr - 5r^2 \geq 23r^2 + 2Rr$
 $\Leftrightarrow 14R(R - 2r) \geq 0$, which is true. Hence proved

1027. In ΔABC , I – incentre, $AI = x, BI = y, CI = z$

the following relationship holds:

$$\frac{2r^3}{27} (x + y + z)^3 + r^2(x^4 + y^4 + z^4) \geq x^2y^2z^2$$

Proposed by Mustafa Tarek-Cairo-Egypt

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} r^2 \sum AI^4 &= r^6 \sum \frac{1}{\sin^4 \frac{A}{2}} = r^6 \sum \frac{b^2 c^2 (s-a)^2}{(s-b)^2 (s-c)^2 (s-a)^2} = \\ &= \left(\frac{r^6}{r^4 s^2} \right) \left(\sum b^2 c^2 (s-a)^2 \right) \stackrel{(1)}{=} \frac{r^2 \sum b^2 c^2 (s^2 - 2as + a^2)}{s^2} \end{aligned}$$

$$\begin{aligned} \text{Now, } \sum b^2 c^2 (s^2 - 2as + a^2) &= s^2 [(\sum ab)^2 - 2abc(2s)] - 2sabc(\sum ab) + 3(16R^2 r^2 s^2) \\ &= s^2 (s^2 + 4Rr + r^2)^2 - 8Rrs^2(s^2 + 4Rr + r^2) - 16Rrs^4 + 48R^2 r^2 s^2 \\ &= s^2 (s^2 + 4Rr + r^2)(s^2 - 4Rr + r^2) - 16Rrs^4 + 48R^2 r^2 s^2 \\ &= s^2 [(s^2 + r^2)^2 - 16R^2 r^2 - 16Rrs^2 + 48R^2 r^2] \\ &\stackrel{(2)}{=} s^2 (s^4 + r^4 + 2s^2 r^2 - 16Rrs^2 + 32R^2 r^2) \end{aligned}$$

$$(1), (2) \Rightarrow r^2 \sum AI^4 \stackrel{(3)}{=} r^2 (s^4 + r^4 + 2s^2 r^2 - 16Rrs^2 + 32R^2 r^2)$$

$$\text{Now, } \frac{2r^3}{27} (\sum AI)^3 \stackrel{A-G}{\geq} 2r^3 (\prod AI) = \frac{2r^6}{\frac{r}{4R}} = 8Rr^5 \Rightarrow \frac{2r^3}{27} (\sum AI)^3 \stackrel{(4)}{\geq} 8Rr^5$$

$$(3) + (4) \Rightarrow LHS \geq 8Rr^5 + r^2 (s^4 + r^4 + 2s^2 r^2 - 16Rrs^2 + 32R^2 r^2) \stackrel{?}{\geq} (\prod AI)^2$$



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$$\begin{aligned}
 & \Leftrightarrow 8Rr^5 + r^2(s^4 + r^4 + 2s^2r^2 - 16Rrs^2 + 32R^2r^2) \stackrel{?}{\geq} \left(\frac{r^3}{4R}\right)^2 = 16R^2r^4 \\
 & \Leftrightarrow r^2(s^4 - s^2(16Rr - 2r^2) + 16R^2r^2 + 8Rr^3 + r^4) \stackrel{?}{\geq} 0 \\
 & \Leftrightarrow s^4 - s^2(16Rr - 2r^2) + 16R^2r^2 + 8Rr^3 + r^4 \stackrel{?}{\geq} 0 \\
 & \Leftrightarrow s^4 - s^2(16Rr - 2r^2) + r^2(4R + r)^2 \stackrel{?}{\geq} 0 \quad (5)
 \end{aligned}$$

Now, RHS of (5) $\stackrel{\text{Trucht}}{\geq} s^4 - s^2(16Rr - 2r^2) + 3r^2s^2 = s^2(s^2 - 16Rr + 2r^2 + 3r^2)$
 $= s^2(s^2 - (16Rr - 5r^2)) \stackrel{\text{Gerretsen}}{\geq} 0 \Rightarrow (5) \text{ is true (proved)}$

Solution 2 by Marian Ursărescu-Romania

We must show: $\frac{2r^3}{27}(AI + BI + CI)^3 + r^2(AI^4 + BI^4 + CI^4) \geq (AI \cdot BI \cdot CI)^2 \quad (1)$

$$\text{But } AI = \frac{r}{\sin \frac{A}{2}} \text{ and } AI \cdot BI \cdot CI = 4Rr^2 \quad (2)$$

From (1)+(2) we must show: $\frac{2r^3}{27}(AI + BI + CI)^3 + r^2(AI^4 + BI^4 + CI^4) \geq 16R^2r^4 \Leftrightarrow$

$$\frac{2r}{27}(AI + BI + CI)^3 + (AI^4 + BI^4 + CI^4) \geq 16R^2r^2 \quad (3)$$

$$AI + BI + CI \geq \sqrt[3]{AI \cdot BI \cdot CI} \quad (4)$$

From (3)+(4) we must show: $2r \cdot AI \cdot BI \cdot CI + (AI^4 + BI^4 + CI^4) \geq 16R^2r^2 \stackrel{(2)}{\Leftrightarrow}$

$$AI^4 + BI^4 + CI^4 \geq 8Rr^2(2R - r) \quad (5)$$

From Cauchy's inequality: $AI^4 + BI^4 + CI^4 \geq \frac{(AI^2 + BI^2 + CI^2)^2}{3}$ and

$$AI^2 + BI^2 + CI^2 = s^2 + r^2 - 8Rr \Rightarrow$$

$$AI^4 + BI^4 + CI^4 \geq \frac{(s^2 + r^2 - 8Rr)^2}{3} \quad (6)$$

From (5)+(6) we must show: $(s^2 + r^2 - 8Rr)^2 \geq 24Rr^2(2R - r) \quad (7)$

From Gerretsen's inequality we have: $s^2 \geq 16Rr - 5r^2 \quad (8)$

From (7)+(8): $(8Rr - 4r^2)^2 \geq 24Rr^2(2R - r) \Leftrightarrow R \geq 2r \text{ true.}$



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Solution 3 by Tran Hong-Vietnam

$$x = AI = \frac{r}{\sin \frac{A}{2}}; y = \frac{r}{\sin \frac{B}{2}}; z = \frac{r}{\sin \frac{C}{2}}; \text{Hence, inequality} \Leftrightarrow$$

$$\begin{aligned} \frac{2r^3}{27} \left(r \sum \frac{1}{\sin \frac{A}{2}} \right)^3 + r^2 r^4 \sum \left(\frac{1}{\sin \frac{A}{2}} \right)^4 &\geq r^6 \left(\frac{1}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} \right)^2 \\ \Leftrightarrow \frac{2}{27} \left(\sum \frac{1}{\sin \frac{A}{2}} \right)^3 + \sum \left(\frac{1}{\sin \frac{A}{2}} \right)^4 &\geq \left(\frac{1}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} \right)^2 \quad (*) \end{aligned}$$

$$\text{Let } a = \frac{1}{\sin \frac{A}{2}}, b = \frac{1}{\sin \frac{B}{2}}, c = \frac{1}{\sin \frac{C}{2}} \text{ then } abc = 4 \cdot \frac{R}{r}; a^2 + b^2 + c^2 = \frac{s^2 + r^2 - 8Rr}{r^2}$$

$$\text{Hence } (*) \text{ becomes: } \frac{2}{27} (a + b + c)^3 + (a^4 + b^4 + c^4) \geq (abc)^2 = 16 \frac{R^2}{r^2}$$

$$\frac{2}{27} (a + b + c)^3 \stackrel{\text{Cauchy}}{\geq} 2 \cdot 4 \cdot \frac{R}{r} = 8 \frac{R}{r}$$

$$a^4 + b^4 + c^4 \geq \frac{(a^2 + b^2 + c^2)^2}{3} = \frac{(s^2 + r^2 - 8Rr)^2}{3r^4}$$

$$\text{We must show that: } 8 \frac{R}{r} + \frac{(s^2 + r^2 - 8Rr)^2}{3r^4} \geq 16 \frac{R^2}{r^2} \quad (**)$$

$$\text{But } s^2 \geq 16Rr - 5r^2$$

$$LHS_{(**)} \geq 8 \cdot \frac{R}{r} + \frac{(8Rr - 4r^2)^2}{3r^4} = 8 \cdot \frac{R}{r} + \frac{16}{3} \left(\frac{2R}{r} - 1 \right)^2$$

$$\text{Must show: } 8t + \frac{16}{3} (2t - 1)^2 \geq 16r^2 \left(t = \frac{R}{r} \geq 2 \right)$$

$$\Leftrightarrow \frac{2}{3}t^2 - \frac{5}{3}t + \frac{2}{3} \geq 0 \Leftrightarrow (t - 2) \left(t - \frac{1}{2} \right) \geq 0$$

(true because $t \geq 2$). Proved.

1028. In ΔABC the following relationship holds:

$$\frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \left(\frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \right)^2$$

Proposed by Adil Abdullayev-Baku-Azerbaijan



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Solution 1 by Bogdan Fustei-Romania

In ΔABC the following relationship: $\frac{a^2}{R_a^2} + \frac{b^2}{R_b^2} + \frac{c^2}{R_c^2} \leq 8 + \left(\frac{ab+bc+ac}{a^2+b^2+c^2}\right)^2$

(I – incenter in ΔABC); R_a, R_b, R_c – circumradii $\Delta BIC, \Delta CIA, \Delta AIB$)

Using two additional inequalities:

$$1) \frac{R}{r} \geq \frac{abc+a^2+b^2+c^2}{2abc}$$

$$2) x, y, z > 0: \frac{x^3+y^3+z^3}{4xyz} + \frac{1}{4} \geq \left(\frac{x^2+y^2+z^2}{xy+yz+zx}\right)^2$$

From the two inequalities from above we can write the following:

$$\frac{R}{2r} \stackrel{(1)}{\geq} \frac{a^3+b^3+c^3}{4abc} + \frac{1}{4} \stackrel{(2)}{\geq} \left(\frac{a^2+b^2+c^2}{ab+bc+ac}\right)^2. \text{ So, finally: } \frac{R}{2r} \geq \left(\frac{a^2+b^2+c^2}{ab+bc+ac}\right)^2$$

$$R_a = 2R \sin \frac{A}{2} \text{ (and the analogs); } \sin \frac{A}{2} = \sqrt{\frac{r_a - r}{4R}} \text{ (and the analogs)}$$

$$a^2 = (r_b + r_c)(r_a - r) \text{ (and the analogs)}$$

$$\Rightarrow R_a = 2R \cdot \sqrt{\frac{r_a - r}{R}} = \sqrt{4R^2 \frac{(r_a - r)}{4R}} = \sqrt{R(r_a - r)} \text{ (and the analogs)}$$

$$R_a^2 = R(r_a - r) \text{ (and the analogs)} \Rightarrow \frac{a^2}{R_a^2} = \frac{(r_b + r_c)(r_a - r)}{R(r_a - r)} = \frac{r_b + r_c}{R}$$

$$\text{So, } \frac{a^2}{R_a^2} = \frac{r_b + r_c}{R} \text{ (and the analogs)}$$

$$\frac{a^2}{R_a^2} + \frac{b^2}{R_b^2} + \frac{c^2}{R_c^2} = \frac{r_b + r_c}{R} + \frac{r_a + r_c}{R} + \frac{r_a + r_b}{R} = \frac{2(r_a + r_b + r_c)}{R} = \frac{2(4R + r)}{R}$$

$$(r_a + r_b + r_c = 4R + r) \Rightarrow \frac{a^2}{R_a^2} + \frac{b^2}{R_b^2} + \frac{c^2}{R_c^2} = \frac{8R + 2r}{R} = 8 + \frac{2r}{R}$$

The inequality from enunciation becomes: $8 + \frac{2r}{R} \leq 8 + \left(\frac{ab+bc+ac}{a^2+b^2+c^2}\right)^2 \Rightarrow$

$$\Rightarrow \frac{R}{2r} \geq \left(\frac{a^2 + b^2 + c^2}{ab + bc + ac}\right)$$

From the above, the inequality from enunciation is proved.



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Solution 2 by Soumava Chakraborty-Kolkata-India

$$R \geq 2r \left(\frac{\sum a^2}{\sum ab} \right)^2 \Leftrightarrow R(S^2 + 4Rr + r^2)^2 \geq 8r(S^2 - 4Rr - r^2)^2$$

$$\Leftrightarrow R\{S^4 + r^2(4R+r)^2 + 2S^2(4Rr+r^2)\} \geq 8r\{S^4 + r^2(4R+r)^2 - 2S^2(4Rr+r^2)\}$$

$$\Leftrightarrow (R-2r)S^4 + 2S^2(4Rr+r^2)(R+8r) + r^2(4R+r)^2(R-8r) \stackrel{(1)}{\geq} 6rs^4$$

$$\text{Now, LHS of (1)} \stackrel{\text{Gerretsen}}{\geq_{(a)}} S^2(R-2r)(16Rr-5r^2) +$$

$$+ 2S^2(4Rr+r^2)(R+8r) + r^2(4R+r)^2(R-8r)$$

$$= S^2r(24R^2 + 29Rr + 26r^2) + r^2(4R+r)^2(R-8r)$$

$$\text{& RHS of (1)} \stackrel{\text{Gerretsen}}{\leq_{(b)}} 6rs^2(4R^2 + 4Rr + 3r^2)$$

(a), (b) \Rightarrow in order to prove (1), it suffices to prove:

$$S^2(5Rr + 8r^2) + r(4R+r)^2(R-8r) \stackrel{(2)}{\geq} 0$$

$$\text{Now, LHS of (2)} \stackrel{\text{Gerretsen}}{\geq} (16Rr-5r^2)(5Rr+8r^2) + r(4R+r)^2(R-8r) \stackrel{?}{\geq} 0$$

$$\Leftrightarrow 2t^3 - 5t^2 + 5t - 6 \stackrel{?}{\geq} 0 \quad \left(t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t-2)\{2t(t-2) + 3t + 3\} \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2$$

$$\Rightarrow R \geq 2r \left(\frac{\sum a^2}{\sum ab} \right)^2 \Rightarrow R \cdot 4rs^2 \geq 8r^2S^2 \left(\frac{\sum a^2}{\sum ab} \right)^2 \Leftrightarrow \frac{Sabc}{8S^2} \stackrel{(3)}{\geq} \left(\frac{\sum a^2}{\sum ab} \right)^2$$

Applying (3) on a triangle with sides $\frac{2}{3}m_a, \frac{2}{3}m_b, \frac{2}{3}m_c$ (whose area will be $= \frac{S}{3}$), we get,

$$\frac{\left(\frac{1}{3}\sum m_a\right)\frac{8}{27}m_am_bm_c}{8\left(\frac{S^2}{9}\right)} \geq \frac{\left(\frac{4}{9}\right)^2(\sum m_a^2)^2}{\left(\frac{4}{9}\right)^2(\sum m_am_b)^2} \Rightarrow \frac{m_am_bm_c(\sum m_a)}{9S^2} \geq \left(\frac{\sum m_a^2}{\sum m_am_b}\right)^2$$

1029. If $M \in \text{Int}(\Delta ABC)$ then:

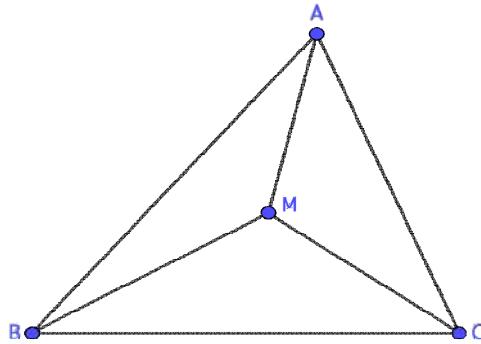
$$27 \cdot [MAB] \cdot [MBC] \cdot [MCA] \leq [ABC]^3$$

Proposed by Daniel Sitaru – Romania

R M M

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Solution 1 by Mehmet Sahin-Ankara-Turkey



Let (x, y, z) be the barycentric coordinates of M .

$$x + y + z = 1 \text{ and } [MBC] = x \cdot [ABC]; [MCA] = y \cdot [ABC]; [MAB] = z \cdot [ABC]$$

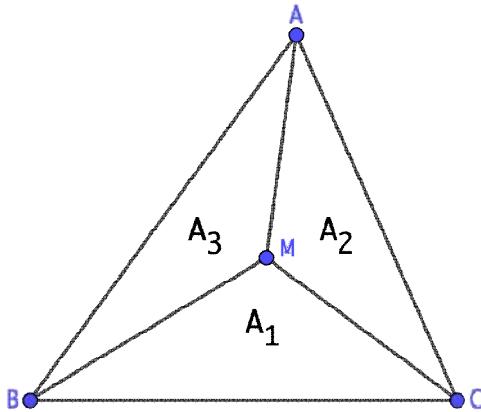
$$[MAB] \cdot [MBC] \cdot [MCA] = xyz[ABC]^3 \quad (1)$$

Using Arithmetic and Geometric Mean inequality:

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz} \Rightarrow \sqrt[3]{xyz} \leq \frac{1}{3} \Rightarrow xyz \leq \frac{1}{27} \quad (2)$$

$$\text{From (1) and (2): } 27[MAB] \cdot [MBC] \cdot [MCA] \leq [ABC]^3$$

Solution 2 by Ravi Prakash-New Delhi-India

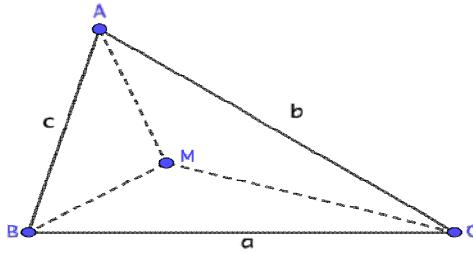


$$27[MBC][MCA][MAB] = 27A_1A_2A_3 = \left[3A_1^{\frac{1}{3}}A_2^{\frac{1}{3}}A_3^{\frac{1}{3}}\right]^3 \leq (A_1 + A_2 + A_3)^3 = [ABC]^3$$

R M M

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Solution 3 by Thanasis Gakopoulos-Athens-Greece



PLAGIAGONAL system: $BC \equiv Bx, BA = By$

$$B(0, 0), C(a, 0), A(0, c), M(m_1, m_2)$$

$$\left\{ \begin{array}{l} (MAB) = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ 0 & m_1 & 0 \\ 0 & m_2 & c \end{vmatrix} \cdot \sin B = \frac{m_1 c \cdot \sin B}{2} \\ (MBC) = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ 0 & m_1 & a \\ 0 & m_2 & 0 \end{vmatrix} \cdot \sin B = \frac{m_2 \cdot a \cdot \sin B}{2} \\ (ABC) = \frac{1}{2} ac \cdot \sin B, (MCA) = (ABC) - (MAB) - (MBC) = \frac{(ac - m_1 c - m_2 a) \sin B}{2} \end{array} \right.$$

AM-GM: $(MAB)(MBC)(MCA) \leq \left[\frac{(MAB)+(MBC)+(MCA)}{3} \right]^3 =$

$$= \frac{1}{27} \left[\frac{m_1 c \cdot \sin B}{2} + \frac{m_2 a \cdot \sin B}{2} + \frac{(ac - m_1 c - m_2 a) \sin B}{2} \right]^3 =$$

$$= \frac{1}{27} \left(\frac{1}{2} ac \cdot \sin B \right)^3 = \frac{1}{27} (ABC)^3 \rightarrow 27(MAB) \cdot (MBC) \cdot (MCA) \leq (ABC)^3$$

1030. In ΔABC the following relationship holds:

$$4(m_a + m_b + m_c) \leq \frac{r_a}{\sin^2 \frac{A}{2}} + \frac{r_b}{\sin^2 \frac{B}{2}} + \frac{r_c}{\sin^2 \frac{C}{2}}$$

Proposed by Bogdan Fustei – Romania

Solution 1 by Lahiru Samarakoon-Sri Lanka

$$4(m_a + m_b + m_c) \leq \sum \frac{r_a}{\sin^2 \frac{A}{2}}$$

$$RHS = \sum \frac{r_a}{\sin^2 \frac{A}{2}} = \sum \frac{s \tan \frac{A}{2}}{\sin^2 \frac{A}{2}} = 2S \sum \frac{1}{\sin A} = 2S \times 2R \frac{(\Sigma ab)}{abc} = \frac{4SR \times (S^2 + r^2 + 4Rr)}{4R \times Sr} = \frac{(S^2 + r^2 + 4Rr)}{r}$$



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$$\text{But } S^2 \geq 16Rr - sr^2$$

$$\geq \frac{(16Rr - sr^2 + r^2 + 4Rr)}{r} = 4(5R - r) \geq 4(4R + r) \quad (\because R \geq 2r) \text{ Euler}$$

We have to prove: $4 \sum m_a \leq 4(4R + r)$

$\sum m_a \leq (4R + r)$ (it's true) ($\because \sum m_a \leq \sum r_a = 4R + r$)

Solution 2 by Marian Ursărescu-Romania

In any ΔABC we have: $\sum \frac{r_a}{\sin^2 \frac{A}{2}} = \frac{s^2 + r^2 + 4Rr}{r} \Rightarrow$ we must show:

$$4(m_a + m_b + m_c) \leq \frac{s^2 + r^2 + 4Rr}{r} \quad (1)$$

But in any ΔABC we have: $m_a + m_b + m_c \leq 4R + r \quad (2)$

From (1)+(2) we must show:

$$16R + 4r \leq \frac{s^2 + r^2 + 4Rr}{r} \Leftrightarrow 16Rr + 4r^2 \leq s^2 + r^2 + 4Rr \Leftrightarrow s^2 \geq 12Rr + 3r^2 \quad (3)$$

Form Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2 \quad (4)$

From (3)+(4) we must show: $16Rr - 5r^2 \geq 12Rr + 3r^2 \Leftrightarrow 4Rr \geq 8r^2 \Leftrightarrow R \geq 2r,$

true (Euler)

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum r_a \csc^2 \frac{A}{2} &= \sum r_a \left(1 + \cot^2 \frac{A}{2}\right) = \sum r_a + \sum s \tan \frac{A}{2} \cot^2 \frac{A}{2} = \sum r_a + \sum s \cot \frac{A}{2} \\ &= \sum r_a + \sum \sqrt[s]{\frac{s(s-a)}{(s-b)(s-c)}} = \sum r_a + \sum \frac{s^2(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}} \\ &= \sum r_a + \sum s^2 \left(\frac{s-a}{\Delta}\right) = \sum r_a = \sum s^2 \left(\frac{1}{r_a}\right) \\ &= \sum r_a + s^2 \left(\frac{\sum r_a r_b}{r_a r_b r_c}\right) = \sum r_a + \frac{S^4}{rs^2} = \sum r_a + \frac{S^2}{r} \\ &\stackrel{\text{Gerretsen}}{\geq} 4R + r + 16R - 5r = 20R - 4r \\ &= 16R + 4(R - r) \stackrel{\text{Euler}}{\geq} 16R + 4r = 4(4R + r) \geq 4 \sum m_a \end{aligned}$$

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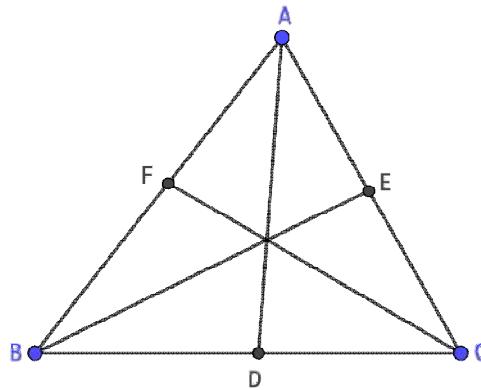
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1031. If in ΔABC , AD, BE, CF – internal bisectors then:

$$AF \cdot BC + BD \cdot AC + CE \cdot AB \geq 18r^2$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Lahiru Samarakoon-Sri Lanka



Because CF, AD bisectors: $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b} \Rightarrow BD = \frac{ac}{b+c}$. So, $BD \cdot AC = \frac{ac}{b+c} b = \frac{abc}{b+c}$

\therefore similarly, for AF, BC and CE, AB set summatting $LHS = \sum BD \cdot AC = abc \sum \left[\frac{1}{b+c} \right]$

$$\begin{aligned} &= abc \left[\frac{12}{b+c} + \frac{12}{a+c} + \frac{2}{b+c} \right] \geq abc \times \frac{(1+1+1)^2}{2(a+b+c)} \\ &= 4RSr \times \frac{9}{4s} \quad (\because \sum a = 2s) = 9Rr, \text{ but } R \geq 2r. \text{ So, } \geq 18r^2 \quad (\text{proved}) \end{aligned}$$

$$\sum BD \cdot AC \geq 18r^2$$

Solution 2 by Soumava Chakraborty-Kolkata-India

By angle-bisector theorem, $\frac{CD}{BD} = \frac{b}{c} \Rightarrow \frac{CD+BD}{BD} = \frac{b+c}{c}$

$$\Rightarrow BD \stackrel{(1)}{=} \frac{ac}{b+c}. \text{ Similarly, } AF \stackrel{(2)}{=} \frac{bc}{a+b} \text{ & } CE \stackrel{(3)}{=} \frac{ab}{c+a}$$

$$(1), (2), (3) \Rightarrow LHS = abc \left(\sum \frac{1}{a+b} \right) = abc \left(\frac{\sum (b+c)(c+a)}{\prod (a+b)} \right)$$

$$= abc \left(\frac{(\sum a^2 + 2 \sum ab) + \sum ab}{2abc + \sum ab(2s - c)} \right) = abc \left(\frac{4s^2 + s^2 + 4Rr + r^2}{2s(s^2 + 4Rr + r^2) - 4Rrs} \right)$$

$$= \frac{4Rrs(5s^2 + 4Rr + r^2)}{2s(s^2 + 2Rr + r^2)} = \frac{2Rr(5s^2 + 4Rr + r^2)}{s^2 + 2Rr + r^2} 18r^2$$



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$$\Leftrightarrow R(5s^2 + 4Rr + r^2) \geq 9r(s^2 + 2Rr + r^2)$$

$$\Leftrightarrow (5R - 9r)s^2 + Rr(4R + r) - 9r^2(2R + r) \stackrel{(4)}{\geq} 0$$

$$\text{Now, LHS of (4)} \stackrel{\text{Gerretsen}}{\geq} (5R - 9r)(16Rr - 5r^2) + Rr(4R + r) - 9r^2(2R + r) \stackrel{?}{\geq} 0$$

$$\Leftrightarrow 42R^2 - 93Rr + 18r^2 \stackrel{?}{\geq} (R - 2r)(42R - 9r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \geq 2r$$

(Euler) \Rightarrow (4) is true (proved)

1032. In acute ΔABC the following relationship holds:

$$\frac{2\sqrt{3}}{R} \leq \frac{1}{a \cos A} + \frac{1}{b \cos B} + \frac{1}{c \cos C} \leq \frac{\sqrt{3}}{4R \cos A \cos B \cos C}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Lahiru Samarakoon-Sri Lanka

$$\text{Consider, } \frac{2\sqrt{3}}{R} \leq \sum \frac{1}{a \cos A} = \sum \frac{1}{2R \sin A \cos A} \therefore \text{we have to prove, } \sum \frac{1}{\sin 2A} \geq 2\sqrt{3}$$

$$\text{Consider, } f(n) = \frac{1}{\sin n} = \csc n \quad (n \in (0, \pi)) \because \text{ABC acute triangle, so,}$$

$$0 < 2A, 2B, 2C < \pi$$

$$f'(n) = -\csc n \cdot \cot n \Leftrightarrow f''(n) = \csc^2 n (\csc n + \cot n) \text{ then } f''(n) \geq 0$$

$$\therefore \sum \frac{1}{\sin 2A} \geq 3 \frac{1}{\sin(\frac{2A+2B+2C}{3})} = \frac{3}{\sin(\frac{2\pi}{3})} = \frac{3 \times 2}{\sqrt{3}} = 2\sqrt{3} \text{ it's true.}$$

$$\text{Consider, } \sum \frac{1}{a \cos A} \leq \frac{\sqrt{3}}{4R \cos A \cos B \cos C}. \text{ We have to prove,}$$

$$\frac{\sum bc \cdot \cos B \cdot \cos C}{4R \Delta \cdot \cos A \cos B \cos C} \leq \frac{\sqrt{3}}{4R \cos A \cos B \cos C}$$

$$\text{So, we have to prove, } \sum \frac{\cos B \cdot \cos C}{\sin A} \leq \frac{\sqrt{3}}{2} \left(\because \frac{1}{2}bc \sin A = \Delta \right)$$

$$LHS = \frac{1}{4 \prod \sin A} \sum \sin 2B \cdot \sin 2C$$

$$\leq \frac{1}{4 \prod \cos A} \frac{(\sin 2A + \sin 2B + \sin 2C)^2}{3} \left(\because \sin 2A, \sin 2B, \sin 2C \geq 0 \atop \text{acute } \Delta \right)$$

$$= \frac{1}{4 \prod \sin A} \times \frac{1}{3} \times (4 \prod \sin A)^2 = \frac{4}{3} (\sin A \cdot \sin B \cdot \sin C) \leq \frac{4}{3} \times \frac{3\sqrt{3}}{8} = \frac{\sqrt{3}}{2} \text{ it's true (Proved)}$$



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Solution 2 by Marian Ursărescu-Romania

$$\text{From AM-GM} \Rightarrow \frac{1}{a \cos A} + \frac{1}{b \cos B} + \frac{1}{c \cos C} \geq 3 \sqrt[3]{\frac{1}{abc \cos A \cos B \cos C}} \Rightarrow$$

$$\begin{aligned} \text{We must show this: } & \frac{3}{\sqrt[3]{abc \cos A \cos B \cos C}} \geq \frac{2\sqrt{3}}{R} \Leftrightarrow \\ & \Leftrightarrow \frac{ab \cos A \cos B \cos C}{27} \leq \frac{R^3}{8 \cdot 3\sqrt{3}} \Leftrightarrow ab \cos A \cos B \cos C \leq \frac{3\sqrt{3}}{7} R^3 \quad (1) \end{aligned}$$

$$\left. \begin{aligned} \text{But } abc &\leq 3\sqrt{3}R^3 \\ \text{and } \cos A \cos B \cos C &\leq \frac{1}{8} \end{aligned} \right\} \Rightarrow (1) \text{ it's true.}$$

Let $a \leq b \leq c \Rightarrow \cos A \geq \cos B \geq \cos C$. From Chebyshev's inequality \Rightarrow

$$\frac{1}{a \cos A} + \frac{1}{b \cos B} + \frac{1}{c \cos C} \leq \frac{1}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \left(\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C} \right) \Rightarrow$$

$$\text{We must show this: } \frac{1}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \left(\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C} \right) \leq \frac{\sqrt{3}}{4R \cos A \cos B \cos C}$$

$$\Leftrightarrow \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) (\cos A \cos B + \cos A \cos C + \cos C \cos A) \leq \frac{3\sqrt{3}}{4R} \quad (2)$$

$$\text{But } \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{\sqrt{3}}{2r} \quad (3). \text{ From (2)+(3) we must show:}$$

$$\sum \cos A \cos B \leq \frac{3r}{2R} \quad (4)$$

$$\text{But } \sum \cos A \cos B = \frac{s^2 + r^2 - 4R^2}{4R^2} \quad (5)$$

$$\text{From (4)+(5) we must show: } \frac{s^2 + r^2 - 4R^2}{4R^2} \leq \frac{3r}{2R} \Leftrightarrow s^2 + r^2 - 4R^2 \leq 6Rr \quad (6)$$

From Gerretsen's inequality:

$$s^2 \leq 4R^2 + 4Rr + 3r^2 \Rightarrow s^2 + r^2 - 4R^2 \leq 4Rr + 4r^2 \quad (7)$$

From (6)+(7) we must show: $4Rr + 4r^2 \leq 6Rr \Leftrightarrow 4r^2 \leq 2Rr \Leftrightarrow 2r \leq R$ (true Euler)

1033. In acute ΔABC the following relationship holds:

$$\cos A \sin(\sin A) + \cos B \sin(\sin B) + \cos C \sin(\sin C) \leq \frac{3}{2} \sin\left(\frac{\sqrt{3}R}{4r}\right)$$

Proposed by Marian Ursărescu – Romania



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Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

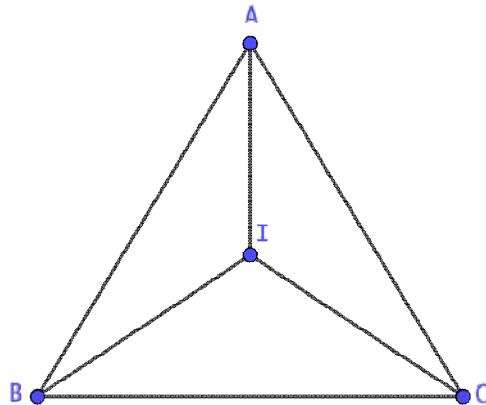
$$\begin{aligned}
 A, B, C &\in \left(0; \frac{\pi}{2}\right) \\
 f(x) &= \cos x \cdot \sin(\sin x) \\
 f'(x) &= -\sin x \cdot \sin(\sin x) + \cos^2 x \cdot \cos(\sin x) \\
 f''(x) &= -\cos x \cdot \sin(\sin x) - \sin^2 x \cdot \cos(\sin x) - 2 \cdot \cos x \\
 &\quad \cdot \sin x \cdot \cos(\sin x) - \cos^3 x \cdot \sin(\sin x) = \\
 &= -\left((\cos x + \cos^3 x) \cdot \sin(\sin x) + (\sin^2 x + 2 \cos x \cdot \sin x) \cdot \cos(\sin x)\right) < 0 \\
 f''(x) &< 0 \\
 \sum \cos A \cdot \sin(\sin A) &\leq 3 \cdot \cos \frac{A+B+C}{3} \cdot \sin \left(\sin \frac{A+B+C}{3}\right) = \\
 &= \frac{3}{2} \cdot \sin \left(\sin \frac{\pi}{3}\right) = \frac{3}{2} \sin \left(\frac{\sqrt{3}}{2}\right) \stackrel{\text{Acute Euler}}{\leq} \frac{3}{2} \cdot \sin \left(\frac{\sqrt{3}R}{4r}\right)
 \end{aligned}$$

1034. If in $\Delta ABC, I$ – incenter then:

$$\left(\frac{AI+BI}{CI}\right)^5 + \left(\frac{BI+CI}{AI}\right)^5 + \left(\frac{CI+AI}{BI}\right)^5 > \left(\frac{BC}{AI}\right)^5 + \left(\frac{CA}{BI}\right)^5 + \left(\frac{AB}{CI}\right)^5$$

Proposed by Daniel Sitaru – Romania

Solution by Lahiru Samarakoon-Sri Lanka



For ABI triangle, $AI + BI > AB \cdot \left(\frac{AI+BI}{CI}\right) > \left(\frac{AB}{CI}\right)$ ($\because CI > 0$). So, $\left(\frac{AI+BI}{CI}\right)^5 > \left(\frac{AB}{CI}\right)^5$



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∴ similarly, from ΔBIC and ΔAIC , and get summation,

$$\sum \left(\frac{AI + BI}{CI} \right)^5 > \sum \left(\frac{BC}{AI} \right)^5$$

1035. In ΔABC the following relationship holds:

$$\sum_{cyc} \sqrt{\frac{r_a}{m_a}} + \sum_{cyc} \frac{h_b + h_c}{w_a} \geq 6 \sum_{cyc} \sin \frac{A}{2}$$

Proposed by Bogdan Fustei – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\because m_a \leq \frac{R}{2r} h_a \text{ etc., } \therefore \sqrt{\frac{r_a}{m_a}} \geq \sqrt{\frac{2r}{R} \cdot \frac{r_a}{h_a}} \text{ etc.} \Rightarrow \sum \sqrt{\frac{r_a}{m_a}} \stackrel{(1)}{\geq} \sum \sqrt{\frac{2r}{R} \cdot \frac{\Delta}{s-a} \cdot \frac{a}{2\Delta}}$$

$$= \sum \sqrt{\frac{r}{R}} \sqrt{\frac{abcs}{s(s-a)bc}} = \sum \sqrt{\frac{r}{R}} \sqrt{\frac{a^2s}{4Rrs}} \cdot \frac{1}{\cos \frac{A}{2}}$$

$$= \sum \sqrt{\frac{r}{R}} \sqrt{\frac{1}{4Rr}} \cdot \frac{4R \sin \frac{A}{2} \cos \frac{A}{2}}{\cos \frac{A}{2}} = 2 \sum \sin \frac{A}{2}$$

$$\text{Now, } \frac{h_b + h_c}{w_a} \geq 4 \sin \frac{A}{2} \Leftrightarrow \frac{ca+ab}{2R} \cdot \frac{(b+c)}{2bc \cos \frac{A}{2}} \geq 4 \sin \frac{A}{2} \Leftrightarrow a(b+c)^2 \geq (4R \sin \frac{A}{2} \cos \frac{A}{2})(4bc)$$

$$\Leftrightarrow a(b+c)^2 \geq 4abc \Leftrightarrow (b+c)^2 \geq 4bc \rightarrow \text{true} \Rightarrow \frac{h_b + h_c}{w_a} \stackrel{(a)}{\geq} 4 \sin \frac{A}{2}$$

$$\text{Similarly, } \frac{h_c + h_a}{w_b} \stackrel{(b)}{\geq} 4 \sin \frac{B}{2} \text{ & } \frac{h_a + h_b}{h_c} \stackrel{(c)}{\geq} 4 \sin \frac{C}{2}$$

$$(a) + (b) + (c) \Rightarrow \sum \frac{h_b + h_c}{w_a} \stackrel{(2)}{\geq} 4 \sum \sin \frac{A}{2}$$

$$(1) + (2) \Rightarrow \sum \sqrt{\frac{r_a}{m_a}} + \sum \frac{h_b + h_c}{w_a} \geq 6 \sum \sin \frac{A}{2}$$

Solution 2 by Tran Hong-Vietnam

$$\sum \sqrt{\frac{r_a}{m_a}} + \sum \frac{h_b + h_c}{w_a} \geq 6 \sum \sin \frac{A}{2}$$



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We have: $\frac{1}{m_a} \geq \frac{2r}{R} \cdot \frac{1}{h_a} = \frac{2r \cdot a}{R \cdot 2rs} = \frac{a}{Rs} \Rightarrow \sum \sqrt{\frac{r_a}{m_a}} \geq \sum \sqrt{\frac{rs \cdot a}{s-a} \cdot \frac{a}{Rs}} = \sqrt{\frac{ra}{R(s-a)}} =$

$$\sum \sqrt{\tan \frac{A}{2} \cdot 4 \sin \frac{A}{2} \cdot \cos \frac{A}{2}} = 2 \sum \sin \frac{A}{2} \quad (1)$$

$$\sum \frac{h_b + h_c}{w_a} = \sum \frac{2S \left(\frac{1}{b} + \frac{1}{c} \right)}{\frac{2bc}{b+c} \cos \frac{A}{2}} = \sum S \cdot \frac{(b+c)^2}{(bc)^2} \cdot \frac{1}{\cos \frac{A}{2}}$$

$$= \sum bc \sin \frac{A}{2} \cdot \cos \frac{A}{2} \cdot \frac{(b+c)^2}{(bc)^2} \cdot \frac{1}{\cos^2 \frac{A}{2}} = \sum \sin \frac{A}{2} \cdot \frac{(b+c)^2}{bc} \geq 4 \sum \sin \frac{A}{2} \quad (2)$$

$\{ \because (b+c)^2 \geq 4bc \}$. From (1)+(2) \Rightarrow Proved.

1036. If a, b and c are the lengths of the sides of a triangle, then:

$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} - 2 \left[\left(\frac{a-b}{a+b} \right)^2 + \left(\frac{b-c}{b+c} \right)^2 + \left(\frac{c-a}{c+a} \right)^2 \right] \geq 3$$

Proposed by Titu Zvonaru, Neculai Stanciu-Romania

Solution 1 by Marian Ursărescu-Romania

$$\frac{a}{b+c-a} = \frac{2R \sin A}{2R(\sin B + \sin C - \sin A)} = \frac{\sin A}{\sin B + \sin C - \sin A} \quad (1)$$

$$\text{But if } A + B + C = \pi \text{ then: } \sin B + \sin C - \sin A = 4 \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{A}{2} \quad (2)$$

$$\text{From (1)+(2)} \Rightarrow \frac{a}{b+c-a} = \frac{2 \sin^2 \frac{A}{2} \cos \frac{A}{2}}{4 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} \cos \frac{A}{2}} = \frac{\sin^2 \frac{A}{2}}{2 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2}} \quad (3)$$

$$\text{From (3)} \Rightarrow \sum \frac{a}{b+c-a} = \sum \frac{\sin^2 \frac{A}{2}}{2 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2}} = \frac{1}{2} \sum \frac{\sum \sin^2 \frac{A}{2}}{\sin^2 \frac{B}{2} \sin^2 \frac{C}{2}} \quad (4)$$

$$\text{But } \sum \sin^2 \frac{A}{2} = \frac{2R-r}{2R} \text{ and } \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{r}{4R} \quad (5)$$

$$\text{From (4)+(5)} \Rightarrow \sum \frac{a}{b+c-a} = \frac{2R-r}{r} = 2 \frac{R}{r} - 1 \quad (6)$$

From (6) inequality becomes: $2 \frac{R}{r} - 1 - 2 \sum \left(\frac{a-b}{a+b} \right)^2 \geq 3 \Leftrightarrow$

$$\frac{R}{r} - 2 - \sum \left(\frac{a-b}{a+b} \right)^2 \geq 0 \quad (7). \text{ But } (a+b)^2 \geq 4ab \Rightarrow (7) \text{ becomes:}$$

$$\frac{R}{r} - 2 - \sum \frac{(a-b)^2}{4ab} \geq 0 \Leftrightarrow \frac{R}{r} - 2 - \sum \frac{a^2 - 2ab + b^2}{ab} \geq 0 \Leftrightarrow$$



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$$\Leftrightarrow \frac{R}{r} - 2 - \frac{1}{4} \sum \frac{a^2 + b^2}{ab} + \frac{3}{2} \geq 0 \Leftrightarrow \frac{R}{r} - \frac{1}{2} - \frac{1}{4} \sum \frac{a^2 + b^2}{ab} \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{4R}{r} - 2 - \sum \frac{a^2 + b^2}{ab} \geq 0 \quad (8)$$

$$\text{But } \sum \frac{a^2 + b^2}{ab} = \frac{s^2 + r^2 - 2Rr}{2Rr} \quad (9)$$

$$\text{From (8)+(9) we must show this: } 4 \frac{R}{r} - 2 - \frac{s^2 + r^2 - 2Rr}{2Rr} \geq 0 \quad (10)$$

$$\text{But from Gerretsen } s^2 \leq 4R^2 + 4Rr + 3r^2 \quad (11)$$

$$\text{From (10)+(11) we must show: } 4 \frac{R}{r} - 2 - \frac{4R^2 + 2Rr + 4r^2}{2Rr} \geq 0. \text{ Let } \frac{R}{r} = x, x \geq 2$$

$$\Rightarrow 4x - 2 - \frac{4x^2 + 2x + 4}{2x} \geq 0 \Leftrightarrow 2x^2 - 3x - 2 \geq 0 \Leftrightarrow (2x + 1)(x - 2) \geq 0 \text{ true.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\Delta ABC, \sum \frac{a}{b+c-a} - 2 \sum \left(\frac{a-b}{a+b} \right)^2 \stackrel{(1)}{\geq} 3$$

$$(1) \Leftrightarrow 3 + \sum \frac{a}{b+c-a} \geq 6 + 2 \sum \left(\frac{a-b}{a+b} \right)^2$$

$$\Leftrightarrow \sum \left(\frac{a}{b+c-a} + 1 \right) \geq 2 \sum \left(1 + \frac{(a-b)^2}{(a+b)^2} \right) = \sum \frac{4(a^2 + b^2)}{(a+b)^2}$$

$$\Leftrightarrow \frac{1}{2} \sum \left(\frac{b+c}{s-a} \right) \stackrel{(1)}{\geq} \sum \frac{4(a^2 + b^2)}{(a+b)^2}$$

$$\because (a+b)^2 \geq 4ab, \text{ etc., RHS of (1)} \stackrel{(2)}{\leq} \sum \frac{a^2 + b^2}{ab} = \frac{\sum c(a^2 + b^2)}{abc} = \frac{\sum ab(2s-c)}{4Rrs}$$

$$= \frac{2s(s^2 + 4Rr + r^2) - 12Rrs}{4Rrs} = \frac{s^2 - 2Rr + r^2}{2Rr}$$

$$\text{LHS of (1)} = \frac{1}{2} \sum \left(\frac{2s-a}{s-a} \right) = \frac{1}{2} \sum \left(1 + \frac{s}{s-a} \right) = \frac{3}{2} + \frac{s}{2} \left(\frac{\sum (s-b)(s-c)}{(s-a)(s-b)(s-c)} \right)$$

$$= \frac{3}{2} + \frac{s}{2sr^2} (3s^2 - 4s^2 + s^2 + 4Rr + r^2) = \frac{3}{2} + \frac{4R+r}{2r} \stackrel{(3)}{=} \frac{2(R+r)}{r}$$

$$(2), (3) \Rightarrow \text{it suffices to prove: } \frac{2(R+r)}{r} \stackrel{(4)}{\geq} \frac{s^2 - 2Rr + r^2}{2Rr}$$

$$\Leftrightarrow 4R(R+r) + 2Rr - r^2 \geq s^2 \Leftrightarrow s^2 \stackrel{(4)}{\leq} 4R^2 + 6Rr - r^2$$

$$\text{Now, Gerretsen} \Rightarrow s^2 \leq 4R^2 + 4Rr + 3r^2 \stackrel{?}{\leq} 4R^2 + 6Rr - r^2 \Leftrightarrow R \stackrel{?}{\geq} 2r$$

→ true (Euler) ⇒ (4) is true (Proved)



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1037. In ΔABC the following relationship holds:

$$\frac{r_a}{r_b + r_c} + \frac{r_b}{r_c + r_a} + \frac{r_c}{r_a + r_b} + \frac{3}{2} \leq \frac{12}{6 - \frac{R}{r}}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{r_a}{r_b + r_c} + \frac{r_b}{r_c + r_a} + \frac{r_c}{r_a + r_b} + \frac{3}{2} \leq \frac{12}{6 - \frac{R}{r}}$$

$$\sum \left(\frac{r_a}{r_b + r_c} + 1 \right) + \frac{3}{2} - 3 \leq \frac{12r}{6r - R}; \quad \sum r_a \cdot \sum \frac{1}{r_a + r_b} \leq \frac{12r}{6r - R} + \frac{3}{2} = \frac{42r - 3R}{2(6r - R)}$$

$$\sum r_a \cdot \frac{\sum (r_a + r_b)(r_b + r_c)}{\prod (r_a + r_b)} \leq \frac{42r - 3R}{2(6r - R)}$$

$$\sum r_a \cdot \frac{(\sum r_a)^2 + \sum r_a r_b}{\sum r_a \cdot \sum r_a \cdot r_b - r_a r_b r_c} \leq \frac{42r - 3R}{2(6r - R)}$$

$$a) \sum r_a = 4R + r$$

$$b) \sum r_a r_b = s^2$$

$$c) r_a r_b r_c = r \cdot s^2$$

$$(4R + r) \left[\frac{(4R + r)^2 + s^2}{(4R + r)s^2 - rs^2} \right] = (4R + r) \left[\frac{(4R + r)^2 + s^2}{4Rs^2} \right] \leq \frac{42r - 3R}{2(6r - R)}$$

$$(6r - R)(4R + r)^3 + (4R + r)(6r - R) \cdot s^2 \leq 2R(42r - 3r)s$$

$$(6r - R)(4R + r)^3 \leq (61Rr - 2R^2 - 6r^2)s^2$$

$$6(Rr - 2r^2 - 6r^2) > 0$$

$$(6r - R)(4R + r)^3 \leq (61Rr - 2R^2 - 6r^2)(16Rr - 5r^2); \quad \frac{R}{2} = t$$

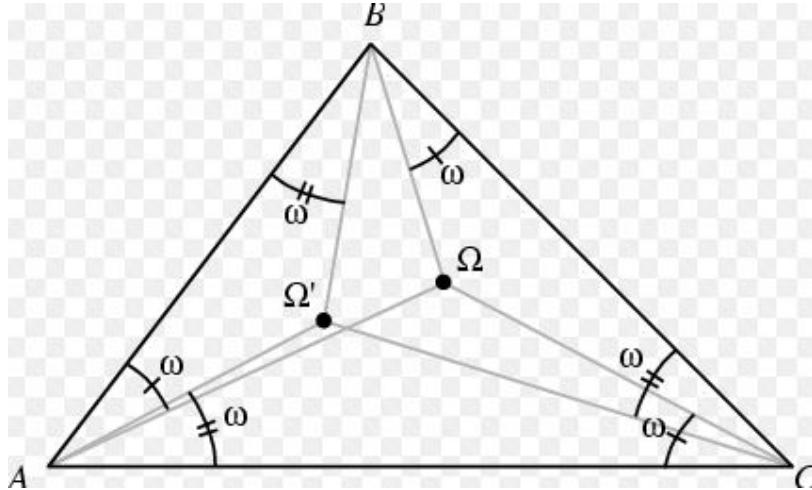
$$(6 - t)(4t + 1)^3 \leq (61t - 2t^2 - 6)(16t - 5)$$

$$-64t^4 + 336t^3 + 276t^2 + 71t + 6 \leq -32t^3 + 986t^2 - 410t + 30$$

$$32t^4 + 184t^3 + 355t^2 - 236t + 12 \geq 0$$

$$\underbrace{(t - 2)^2}_{\geq 0} \underbrace{(32t^2 - 56 + 2)^2}_{\geq 0} \geq 0$$

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If in ΔABC , ω – Brocard angle then:

$$\sin \omega \leq \frac{(a+b)^2 + (b+c)^2 + (c+a)^2}{16(a^2 + b^2 + c^2)} + \frac{s}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\because \sin \omega = \frac{2s}{\sqrt{\sum a^2b^2}}, \therefore \text{given inequality} \Leftrightarrow \frac{s}{\sqrt{\sum a^2b^2}} \leq \frac{\sum(a+b)^2}{16\sum a^2} \Leftrightarrow$$

$$\Leftrightarrow \sqrt{\sum a^2b^2} \sum (a+b)^2 \stackrel{(1)}{\geq} rs \cdot 16 \sum a^2$$

$$\begin{aligned} \text{Now, LHS of (1)} &\geq \sqrt{abc(2S)}\{2(\sum a^2 + \sum ab)\} \\ &= 4S\sqrt{2Rr}(3S^2 - 4Rr - r^2) \stackrel{?}{\geq} 32rs(S^2 - 4Rr - r^2) \\ &\Leftrightarrow 2Rr(3S^2 - 4Rr - r^2)^2 \stackrel{?}{\geq} 64r^2(S^2 - 4Rr - r^2)^2 \\ &\Leftrightarrow R(3S^2 - 4Rr - r^2)^2 \stackrel{?}{\geq} 32r(S^2 - 4Rr - r^2)^2 \\ \Leftrightarrow R\{9S^4 + (4Rr + r^2)^2 - 6S^2(4Rr + r^2)\} &\stackrel{?}{\geq} 32r\{S^4 + (4Rr + r^2)^2 - 2S^2(4Rr + r^2)\} \\ \Leftrightarrow (9R - 18r)S^4 - 6R(4Rr + r^2)S^2 + 64r^2(4R + r)S^2 + & \\ + Rr^2(4R + r)^2 - 32r^3(4R + r)^2 &\stackrel{(2)}{\geq} 14rs^4 \end{aligned}$$



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$$\text{Now, LHS of (2)} \stackrel{\text{Gerretsen}}{\geq_{(a)}} (16Rr - 5r^2)(9R - 18r)S^2 -$$

$-6R(4Rr + r^2)S^2 + 64r^2(4R + r)S^2 + Rr^2(4Rr + r)^2 - 32r^3(4R + r)^2$ & also,

$$\text{RHS of (2)} \stackrel{\text{Gerretsen}}{\leq_{(b)}} 14rS^2(4R^2 + 4Rr + 3r^2)$$

(a), (b) \Rightarrow in order to prove (2), it suffices to prove:

$$s^2\{(16Rr - 5r^2)(9R - 18r) - 6R(4Rr + r^2) + 64r^2(4R + r) - 14r(4R^2 + 4Rr + 3r^2)\} \\ + Rr^2(4R + r)^2 - 32r^3(4R + r)^2 \geq 0$$

$$\Leftrightarrow S^2(64R^2 - 139Rr + 112r^2) + Rr(4R + r)^2 - 32r^2(4R + r)^2 \stackrel{(3)}{\geq} 0$$

$$\because 64R^2 - 139Rr + 112r^2$$

$$= (R - 2r)(64R - 11r) + 90r^2 > 0 \quad (\because R \stackrel{\text{Euler}}{\geq} 2r)$$

$$\therefore \text{LHS of (3)} \stackrel{\text{Gerretsen}}{\geq} (16Rr - 5r^2)(64R^2 - 139Rr + 112r^2) +$$

$$+ Rr(4R + r)^2 - 32r^2(4R + r)^2 \stackrel{?}{\geq} 0 \Leftrightarrow 130t^3 - 381t^2 + 279t - 74 \stackrel{?}{\geq} 0 \quad (t = \frac{R}{r})$$

$$\Leftrightarrow (t - 2)\{130t(t - 2) + 139t + 37\} \stackrel{?}{\geq} 0 \rightarrow \text{true} \quad (\because t \stackrel{\text{Euler}}{\geq} 2 \text{ (Proved)})$$

Solution 2 by Tran Hong-Vietnam

$$\sin \omega = \frac{2S}{\sqrt{\sum a^2 b^2}}. \text{ Inequality} \Leftrightarrow \frac{S}{\sqrt{\sum a^2 b^2}} \leq \frac{\sum a^2 + \sum ab}{8 \sum a^2}$$

$$\Leftrightarrow \frac{2 \sum a^2 b^2 - \sum a^4}{\sum a^2 b^2} \leq \left(\frac{\sum a^2 + \sum ab}{2 \sum a^2} \right)^2 \quad (1)$$

Let $p = \sum a$, $q = \sum ab$, $r = abc$, suppose $c \leq b \leq a$

$$(1) \Leftrightarrow \{8(q^2 - 2pr) - 4(p^4 - 4p^2q + 2q^2 + 4pr)\}(p^2 - 2q)^2 \leq (q^2 - 2pr)(p^2 - q)^2;$$

$$\Leftrightarrow \{-2p(p^2 - q)^2 + 32p(p^2 - 2q)^2\}r + g(p, q) \geq 0$$

$$\Leftrightarrow 2p\{16(p^2 - 2q)^2 - (p^2 - q)^2\}r + g(p, q) \geq 0$$

$$\Leftrightarrow 2p\{15p^4 - 62p^2q + 63q^2\}r + g(p, q) \geq 0$$

$$\text{Let } f(r) = 2p\{15p^4 - 62p^2q + 63q^2\}r + g(p, q)$$

$$15p^4 - 62p^2q + 63q^2 = (3p^2 - 7q)(5p^2 - 9q) > 0 \text{ (because } p^2 \geq 3q)$$

\Rightarrow The function f increasing of $r = abc$, by ABC Theorem we just check:

$$\because c = 0, 0 < a \leq b:$$



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$$(1) \Leftrightarrow \frac{2a^2b^2 - (a^4 + b^4)}{2} \leq \left(\frac{a^2 + b^2 + ab}{2a^2 + 2b^2} \right)^2$$

$$\Leftrightarrow 4(a^4 - b^4)^2 + a^2b^2(a^2 + ab + b^2)^2 \geq 0 \text{ (true)}$$

$\because a = b, c \leq a:$

$$(1) \Leftrightarrow \frac{4a^2c^2 - c^4}{a^4 + 2a^2c^2} \leq \left(\frac{3a^2 + c^2 + 2ac}{4a^2 + 2c^2} \right)^2$$

$$\Leftrightarrow (a - c)^2(9a^6 + 30a^5c + 15a^4c^2 + 28a^3c^3 + 14a^2c^4 + 8ac^5 + 4c^6) \geq 0$$

It is true. Proved. Equality $\Leftrightarrow a = b = c$.

1039. In ΔABC the following relationship holds:

$$16 \left(\sum ab \sin^2 A \right) \left(\sum ab \cos^2 A \right) \leq 729R^4$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} AM &\geq GM \Rightarrow \sqrt{\sum ab \sin^2 A} \sqrt{\sum ab \cos^2 A} \leq \frac{\sum ab \sin^2 A + \sum ab \cos^2 A}{2} \\ &= \frac{ab(\sin^2 A + \cos^2 A) + bc(\sin^2 B + \cos^2 B) + ca(\sin^2 C + \cos^2 C)}{2} \\ &= \frac{\sum ab}{2} (\because \sin^2 A + \cos^2 A = 1, \text{etc.}) \therefore (\sum ab \sin^2 A)(\sum ab \cos^2 A) \leq \frac{(\sum ab)^2}{4} \\ &\Rightarrow 16 \left(\sum ab \sin^2 A \right) \left(\sum ab \cos^2 A \right) \leq 4 \left(\sum ab \right)^2 \stackrel{?}{\leq} 324R^4 \\ &\Leftrightarrow \sum ab \leq 9R^2 \rightarrow \text{true } \because \sum ab \leq \sum a^2 \stackrel{\text{Leibnitz}}{\leq} 9R^2 \\ &\quad (\text{Proved}) \end{aligned}$$

1040. If in ΔABC , I – incentre, $\Delta A'B'C'$ - pedal triangle of incentre then:

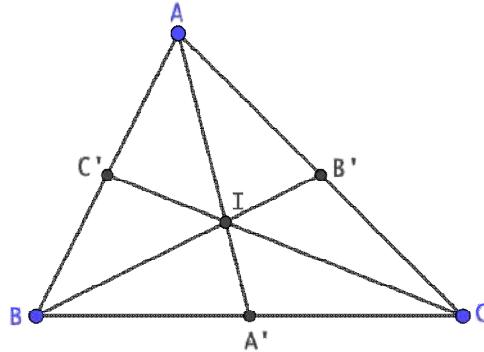
$$\frac{IA \cdot IA'}{w_a} + \frac{IB \cdot IB'}{w_b} + \frac{IC \cdot IC'}{w_c} \leq \frac{3\sqrt{3}}{4S} \cdot IA \cdot IB \cdot IC$$

Proposed by Daniel Sitaru – Romania

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Solution by Soumava Chakraborty-Kolkata-India



$$\text{Angle - bisector theorem} \Rightarrow \frac{A'C}{A'B} = \frac{b}{c} \Rightarrow \frac{a}{A'B} = \frac{b+c}{c} \Rightarrow A'B \stackrel{(1)}{=} \frac{ac}{b+c}$$

$$\text{Angle - bisector on } \Delta ABA' \Rightarrow \frac{IA'}{IA} \stackrel{(b)}{=} \frac{\frac{ac}{b+c}}{c} = \frac{a}{b+c} \Rightarrow IA' = \frac{a}{b+c} IA \Rightarrow IA' \cdot IA = \frac{a}{b+c} IA^2$$

$$= \frac{a}{b+c} \cdot \frac{r^2}{\sin^2 \frac{A}{2}} \Rightarrow \frac{IA \cdot IA'}{w_a} = \frac{ar^2 bc(b+c)}{(b+c)(s-b)(s-c)2bc \cos \frac{A}{2}}$$

$$= \frac{4R \sin \frac{A}{2} \cos \frac{A}{2} r^2}{2(s-b)(s-c) \cos \frac{A}{2}} = \frac{2Rr^2}{(s-b)(s-c)} \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$= \frac{2Rr^2}{\sqrt{bc(s-b)(s-c)}} = \frac{2Rr^2 \sqrt{a(s-a)}}{\sqrt{4Rrs \cdot r^2 S}} = \sqrt{\frac{4R^2 r^4}{4Rr^3 s^2}} \sqrt{a(s-a)} \stackrel{(a)}{=} \sqrt{\frac{Rr}{s^2}} \sqrt{a(s-a)}$$

$$\text{Similarly, } \frac{IB \cdot IB'}{w_b} \stackrel{(b)}{=} \sqrt{\frac{Rr}{s^2}} \sqrt{b(s-b)} \text{ & } \frac{IC \cdot IC'}{w_c} \stackrel{(c)}{=} \sqrt{\frac{Rr}{s^2}} \sqrt{c(s-c)}$$

$$(a)+(b)+(c) \Rightarrow LHS = \sqrt{\frac{Rr}{s^2}} \sum \sqrt{a(s-a)}$$

$$\leq \frac{CBS}{s} \frac{\sqrt{Rr}}{s} \sqrt{3} \sqrt{\sum a(s-a)} = \frac{\sqrt{3Rr}}{s} \sqrt{s(2s) - 2(s^2 - 4Rr - r^2)}$$

$$\stackrel{(i)}{=} \frac{\sqrt{3Rr}}{s} \sqrt{2(4Rr + r^2)} = \frac{r}{s} \sqrt{6R(4R + r)}$$

$$\text{Now, RHS} = \frac{3\sqrt{3}}{4rs} \cdot \frac{r^3}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{3\sqrt{3}r^2}{s(\frac{r}{R})} \stackrel{(ii)}{=} \frac{3\sqrt{3}Rr}{s}$$

(i), (ii) \Rightarrow it suffices to prove: $6R(4R + r) \leq 27R^2 \Leftrightarrow 3R^2 \geq 6Rr \Leftrightarrow R \geq 2r \rightarrow \text{true}$

(Euler) (Proved)



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1041. In ΔABC the following relationship holds:

$$\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \geq \frac{1}{2} \left(\frac{h_b + h_c}{h_a} + \frac{h_c + h_a}{h_b} + \frac{h_a + h_b}{h_c} \right)$$

Proposed by Bogdan Fustei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{In any } \Delta ABC, \sum \frac{m_a}{h_a} \geq \frac{1}{2} \sum \left(\frac{h_b + h_c}{h_a} \right)$$

$$RHS = \frac{1}{2} \sum \left(\frac{\frac{ca + ab}{2R}}{\frac{bc}{2R}} \right) = \frac{1}{2} \sum \left(\frac{ca + ab}{bc} \right) \stackrel{(1)}{=} \frac{\sum a^2 b + \sum ab^2}{2ab}$$

$$LHS \stackrel{\text{Tereshin}}{\geq} \sum \left(\frac{\frac{b^2 + c^2}{4R}}{\frac{bc}{2R}} \right) = \frac{1}{2} \sum \left(\frac{b^2 + c^2}{bc} \right) = \frac{\sum a^2 b + \sum ab^2}{2ab} \stackrel{\text{by (1)}}{=} RHS$$

1042. In ΔABC the following relationship holds:

$$\frac{1}{m_a} \sin \frac{A}{2} + \frac{1}{m_b} \sin \frac{B}{2} + \frac{1}{m_c} \sin \frac{C}{2} \leq \frac{m_a^2 + m_b^2 + m_c^2}{2m_a m_b m_c}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum \frac{\sin \frac{A}{2}}{m_a} = \frac{\sum m_a^2}{2m_a m_b m_c} \Leftrightarrow \sum m_b m_c \sin \frac{A}{2} \stackrel{(1)}{\leq} \frac{\sum m_a^2}{2} = \frac{3 \sum a^2}{8}$$

$$\sum m_b m_c \sin \frac{A}{2} \stackrel{\text{CBS}}{\leq} \sqrt{\sum m_b^2 m_c^2} \sqrt{\sum \sin^2 \frac{A}{2}} = \sqrt{\frac{9 \sum a^2 b^2}{16}} \sqrt{\frac{\sum (1 - \cos A)}{2}}$$

$$= \sqrt{\frac{9 \sum a^2 b^2}{16}} \sqrt{\frac{2R - r}{2R}} \stackrel{?}{\leq} \frac{3 \sum a^2}{8} \Leftrightarrow \frac{9 \sum a^2 b^2}{16} \left(\frac{2R - r}{2R} \right) \stackrel{?}{\leq} \frac{9}{64} \left(\sum a^2 \right)^2$$

$$\Leftrightarrow 2(2R - r) \left(\left(\sum ab \right)^2 - 2abc(2s) \right) \stackrel{?}{\leq} 4R(s^2 - 4Rr - r^2)^2$$

$$\Leftrightarrow 2(2R - r)(s^2 + 4Rr + r^2)^2 - 4R(s^2 - 4Rr - r^2)^2 \stackrel{?}{\leq} 32(2R - r)Rrs^2$$



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$$\begin{aligned}
 &\Leftrightarrow 2R((s^2 + 4Rr + r^2)^2 - (s^2 - 4Rr - r^2)^2) \stackrel{?}{\leq} 16(2R - r)Rrs^2 + r(s^2 + 4Rr + r^2)^2 \\
 &\Leftrightarrow 2R(2s^2)(8Rr + 2r^2) \stackrel{?}{\leq} 16(2R - r)Rrs^2 + r(s^2 + 4Rr + r^2)^2 \\
 &\Leftrightarrow s^4 + r^2(4R + r)^2 + 2s^2(4Rr + r^2) \stackrel{?}{\geq} \frac{24Rrs^2}{(2)}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, LHS of (2)} &\stackrel{\text{Gerretsen}}{\leq} s^2(16Rr - 5r^2) + r^2(4R + r)^2 + 2s^2(4Rr + r^2) \stackrel{?}{\geq} 24Rrs^2 \\
 &\Leftrightarrow r^2(4R + r)^2 \stackrel{?}{\geq} 3r^2s^2 \Leftrightarrow 4R + r \stackrel{?}{\geq} \sqrt{3}s \rightarrow \text{true (Trucht)} \Rightarrow (1) \text{ is true (Done).}
 \end{aligned}$$

1043. In ΔABC : $P = e^{(\sin A+2 \sin B)(\sin B+2 \sin C)(\sin C+2 \sin A)}$

Find: $\max P$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Sagar Kumar-Patna Bihar-India

$$\begin{aligned}
 P &= e^{(\sin A+2 \sin B)(\sin B+2 \sin C)(\sin C+2 \sin A)} \Rightarrow \cos 0 < A, B, C < \pi \Rightarrow \\
 \Rightarrow \sin A, \sin B, \sin C > 0 &\Rightarrow (\sin A + 2 \sin B)(\sin B + 2 \sin C)(\sin C + 2 \sin A) \\
 &\leq \left(\frac{3(\sin A + \sin B + \sin C)}{3} \right)^3
 \end{aligned}$$

$$AM \geq GM \Rightarrow LHS \leq (\sin A + \sin B + \sin C)^3$$

$$\text{and we know that in a } \Delta ABC: \sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2} \Rightarrow LHS \leq \left(\frac{3\sqrt{3}}{2} \right)^3 = \frac{81\sqrt{3}}{8}$$

$$\text{Hence } P_{\max} \leq e^{\left(\frac{81\sqrt{3}}{8} \right)}. \text{ Equality holds when } A = B = C = \frac{\pi}{3}$$

1044. In ΔABC , I – incentre, R_a, R_b, R_c – circumradii in $\Delta BIC, \Delta CIA, \Delta AIB$.

Prove that:

$$2R^2 - 2Rr - r^2 \leq \frac{1}{4R^2} (R_a^4 + R_b^4 + R_c^4) \leq 4R^2 - 8Rr + 3r^2$$

Proposed by Marian Ursărescu – Romania



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Solution 1 by Bogdan Fustei-Romania

$$\begin{aligned} R_a &= 2R \sin \frac{A}{2} \quad (\text{and analogous}) \\ \sin \frac{A}{2} &= \sqrt{\frac{r_a - r}{4R}} \quad (\text{and analogous}) \end{aligned} \quad \left. \begin{aligned} R_a &= \sqrt{R(r_a - r)} \quad (\text{and analogous}) \\ R_a^4 = R^2(r_a - r)^2 & \quad (\text{and analogous}) \Rightarrow R_a^4 + R_b^4 + R_c^4 = R^2 \cdot \sum(r_a - r)^2 \end{aligned} \right\}$$

$$R_a^4 + R_b^4 + R_c^4 = R^2 \left[\sum r_a^2 + 3r^2 - 2r(r_a + r_b + r_c) \right]$$

$$r_a r_b + r_b r_c + r_a r_c = s^2 \Rightarrow \sum r_a^2 = (r_a + r_b + r_c)^2 - 2 \sum r_a r_b$$

$$\sum r_a^2 = (r_a + r_b + r_c)^2 - 2s^2$$

$$R_a^4 + R_b^4 + R_c^4 = R^2[(r_a + r_b + r_c)^2 - 2s^2 - 2r(r_a + r_b + r_c) + 3r^2]$$

$$R_a^4 + R_b^4 + R_c^4 = R^2[(R_a + R_b + R_c)(R_a + R_b + R_c - 2r) - 2s^2 + 3r^2]$$

$$R_a^4 + R_b^4 + R_c^4 = R^2[(4R + r)(4R - r) - s^2 + 3r^2]$$

$$R_a^4 + R_b^4 + R_c^4 = R^2(16R^2 - r^2 - 2s^2 + 3r^2) = 2R^2(8R^2 - s^2 + r^2)$$

$$\frac{R_a^4 + R_b^4 + R_c^4}{4R^2} = \frac{2R^2(8R^2 - s^2 + r^2)}{4R^2} = \frac{8R^2 - s^2 + r^2}{2}. \quad \text{The inequality from enunciation becomes:}$$

$$2R^2 - 2Rr - r^2 \leq \frac{8R^2 - s^2 + r^2}{2} \leq 4R^2 - 8Rr + 3r^2$$

$$\begin{aligned} 4R^2 - 4Rr - 2r^2 &\leq 8R^2 - s^2 + r^2 \Rightarrow s^2 \leq 8R^2 + r^2 - 4R^2 + 4Rr + 2r^2 = \\ &= 4R^2 + 4Rr + 3r^2 \quad (\text{Gerretsen's inequality}) \end{aligned}$$

$$8r^2 - s^2 + r^2 \leq 8R^2 - 16Rr + 6r^2 \Rightarrow 16Rr - 5r^2 \leq s^2 \quad (\text{Gerretsen's inequality})$$

From the above the inequality from enunciation is proved.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$2R^2 - 2Rr - r^2 \stackrel{(a)}{\leq} \frac{\sum R_a^4}{4R^2} \stackrel{(b)}{\leq} 4R^2 - 8Rr + 3r^2$$

$$\text{From } \Delta BIC, \frac{BI \cdot CI \cdot BC}{4R_a} = \frac{1}{2} BC \cdot r \Rightarrow R_a = \frac{r^2 \sin \frac{A}{2}}{2r \sin \frac{B}{2} \sin \frac{C}{2} \sin \frac{A}{2}} = \frac{r \sin \frac{A}{2}}{2 \left(\frac{r}{4R} \right)} = 2R \sin \frac{A}{2}$$

$$\text{Similarly, } R_b = 2R \sin \frac{B}{2} \text{ & } R_c = 2R \sin \frac{C}{2}$$

$$\therefore \frac{\sum R_a^4}{4R^2} = \frac{16R^4 \sum \sin^4 \frac{A}{2}}{4R^2} \quad (\text{using above 3 relations})$$



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$$\begin{aligned}
 &= \sum R^2 \left(2 \sin^2 \frac{A}{2} \right)^2 = R^2 \sum (1 - \cos A)^2 = R^2 \sum (1 + 1 - \sin^2 A - 2 \cos A) \\
 &= 6R^2 - \sum \frac{a^2 \cdot R^2}{4R^2} - 2R^2 \left(1 + \frac{r}{R} \right) = 6R^2 - \frac{s^2 - 4Rr - r^2}{2} - 2R(R + r) \\
 &= \frac{12R^2 - s^2 + 4Rr + r^2 - 4R^2 - 4Rr}{2} \stackrel{(1)}{=} \frac{8R^2 + r^2 - s^2}{2} \\
 (1) \Rightarrow (a) &\Leftrightarrow 8R^2 + r^2 - s^2 \geq 4R^2 - 4Rr - 2r^2 \\
 \Leftrightarrow s^2 &\leq 4R^2 + 4Rr + 3r^2 \rightarrow \text{true by Gerretsen} \Rightarrow (a) \text{ is true} \\
 \text{Also, } (1) \Rightarrow (b) &\Leftrightarrow 8R^2 + r^2 - s^2 \leq 8R^2 - 16Rr + 6r^2 \\
 \Leftrightarrow s^2 &\geq 16Rr - 5r^2 \rightarrow \text{true by Gerretsen} \Rightarrow (b) \text{ is true (Done)}
 \end{aligned}$$

1045. In ΔABC the following relationship holds:

$$a^3 \cos B \cos C + b^3 \cos C \cos A + c^3 \cos A \cos B \geq \frac{27abc}{\left(\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C} \right)^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \frac{abc}{2R^2} \sum a^2 - \sum a^3 \cos A &= \frac{4Rrs}{R^2} (s^2 - 4Rr - r^2) - \sum a^3 \left(\frac{b^2 + c^2 - a^2}{2bc} \right) \\
 &= \frac{4rs}{R} (s^2 - 4Rr - r^2) - \sum \frac{a^4(b^2 + c^2 - a^2)}{2abc} \\
 &= \frac{4rs(s^2 - 4Rr - r^2)}{R} - \frac{\sum a^2 b^2 (\sum a^2 - c^2) - \sum a^6}{8Rrs} \\
 &\stackrel{(1)}{=} \frac{32r^2 s^2 (s^2 - 4Rr - r^2) - (\sum a^2 b^2)(\sum a^2) + 3a^2 b^2 c^2 + \sum a^6}{8Rrs}
 \end{aligned}$$

$$\begin{aligned}
 \text{Numerator} &= 32r^2 s^2 (s^2 - 4Rr - r^2) - (\sum a^2 b^2)(\sum a^2) + 3a^2 b^2 c^2 + 3a^2 b^2 c^2 + \\
 &\quad + \sum a^2 \left(\sum a^4 - \sum a^2 b^2 \right) = \\
 &= 32r^2 s^2 (s^2 - 4Rr - r^2) - 2 \left(\sum a^2 b^2 \right) \left(\sum a^2 \right) + 96R^2 r^2 s^2 + \\
 &\quad + \left(\sum a^2 \right) \left\{ \left(\sum a^2 \right)^2 - 2 \sum a^2 b^2 \right\}
 \end{aligned}$$



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$$\begin{aligned}
&= 32r^2s^2(s^2 - 4Rr - r^2) - 8 \left(\sum a^2b^2 \right) (s^2 - 4Rr - r^2) + \\
&\quad + 96R^2r^2S^2 + 8(s^2 - 4Rr - r^2)^2 \\
&= 8(s^2 - 4Rr - r^2) \left\{ (s^2 - 4Rr - r^2)^2 - \left(\sum ab \right)^2 + 16Rrs^2 + 4r^2s^2 \right\} + \\
&\quad + 96R^2r^2S^2 \\
&= 8(s^2 - 4Rr - r^2) \{(2s^2)(-8Rr - 2r^2) + 16Rrs^2 + 4r^2s^2 + 96R^2r^2S^2\} \\
&\stackrel{(2)}{=} 96R^2r^2S^2 \\
(1), (2) &\Rightarrow \frac{abc}{2R^2} \sum a^2 - \sum a^3 \cos A = \frac{96R^2r^2S^2}{8Rrs} \stackrel{(3)}{=} 12Rrs \\
\text{Now, } \sum a^3 \cos B \cos C &= \frac{1}{2} \sum a^3 (2 \cos B \cos C) = \\
&= \frac{1}{2} \sum a^3 \{\cos(B+C) + \cos(B-C)\} = \\
&= -\frac{1}{2} \sum a^3 \cos A + \frac{1}{2} \sum a^2 \cdot 2R \sin(B+C) \cos(B-C) \\
&= -\frac{1}{2} \sum a^3 \cos A + \frac{R}{2} \sum a^2 (\sin 2B + 2 \sin 2C) \\
&= -\frac{1}{2} \sum a^3 \cos A + \frac{R}{2} \sum a^2 \left(\sum \sin 2A - \sin 2A \right) \\
&= -\frac{1}{2} \sum a^3 \cos A + \frac{R}{2} \left(\sum a^2 \right) \left(4 \frac{abc}{8R^3} \right) - \frac{R}{2} \sum a^2 \cdot 2 \sin A \cos A \\
&= -\frac{1}{2} \sum a^3 \cos A + \frac{R}{2} \left(\sum a^2 \right) \left(\frac{abc}{2R^3} \right) - \frac{1}{2} \sum a^2 \cdot a \cos A \\
&= -\sum a^3 \cos A + \frac{abc}{4R^2} \left(\sum a^2 \right) \\
&= \left(\frac{abc}{2R^2} \left(\sum a^2 \right) - \sum a^3 \cos A \right) - \frac{abc}{4R^2} \left(\sum a^2 \right) \\
&\stackrel{by (3)}{=} (12Rrs) - \frac{4Rrs}{4R^2} \cdot 2(s^2 - 4Rr - r^2) = 12Rrs - \frac{2rs(s^2 - 4Rr - r^2)}{R} \\
&= \frac{12R^2rs - 2rs(s^2 - 4Rr - r^2)}{R} \stackrel{(4)}{=} \frac{2rs(6R^2 - s^2 + 4Rr + r^2)}{R} \\
\text{Now, } 6R^2 - s^2 + 4Rr + r^2 &> 0 \Leftrightarrow s^2 < 6R^2 + 4Rr + r^2 \\
\text{But, } s^2 &\stackrel{Gerretsen}{\leq} 4R^2 + 4Rr + 3r^2 \stackrel{?}{<} 6R^2 + 4Rr + r^2
\end{aligned}$$



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$$\Leftrightarrow R > r \rightarrow \text{true} \therefore 6R^2 - s^2 + 4Rr + r^2 > 0$$

$$(4) \Rightarrow \text{given inequality} \Leftrightarrow \frac{2rs(6R^2 - s^2 + 4Rr + r^2)}{R} \left(\sum \frac{1}{\cos A} \right)^2 \stackrel{(5)}{\geq} 27abc$$

$$\therefore \left(\sum \frac{1}{\cos A} \right)^2 \stackrel{(6)}{\geq} 3 \sum \frac{1}{\cos A \cos B} = \frac{3 \sum \cos A}{\prod \cos A}$$

$$= \frac{3 \left(\frac{R+r}{R} \right)}{\frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2}} = \frac{12R(R+r)}{s^2 - 4R^2 - 4Rr - r^2},$$

$$\therefore (6) \Rightarrow (5) \Leftrightarrow \frac{12(R+r) \cdot 2rs(6R^2 - s^2 + 4Rr + r^2)}{R(s^2 - 4R^2 - 4Rr - r^2)} \stackrel{(7)}{\geq} 108Rrs$$

$$\Leftrightarrow 2(R+r)(6R^2 - s^2 + 4Rr + r^2) \geq 9R(s^2 - 4R^2 - 4Rr - r^2)$$

$$\Leftrightarrow 2(R+r)6R^2 - 2(R+r)s^2 + 2(R+r)(4Rr + r^2) \geq$$

$$\geq 9Rs^2 - 36R^3 - 9R(4Rr + r^2)$$

$$\Leftrightarrow 48R^3 + 12R^2r + (11R + 2r)(4Rr + r^2) \stackrel{(7)}{\geq} (11R + 2r)s^2$$

$$\text{Now, RHS of (7)} \stackrel{\text{Gerretsen}}{\leq} (11R + 2r)(4R^2 + 4Rr + 3r^2)$$

$$\stackrel{?}{\leq} 48R^3 + 12R^2r + (4Rr + r^2)(11R + 2r) \Leftrightarrow 2t^3 + 2t^2 - 11t - 2 \stackrel{?}{\geq} 0 \quad (\text{where } t = \frac{R}{r})$$

$$\Leftrightarrow (t-2)(2t^2 + 6t + 1) \stackrel{?}{\geq} 0 \rightarrow \text{true because } t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (7) \text{ is true} \Rightarrow (5) \text{ is true}$$

(Proved)

1046. In ΔABC the following relationship holds:

$$\left(\frac{h_b h_c}{h_a} \right)^2 + \left(\frac{h_c h_a}{h_b} \right)^2 + \left(\frac{h_a h_b}{h_c} \right)^2 \geq \left(\frac{2S}{R} \right)^2$$

Proposed by Bogdan Fustei – Romania

Solution 1 by Daniel Sitaru – Romania

$$\sum_{cyc} \left(\frac{h_b h_c}{h_a} \right)^2 = \sum_{cyc} \left(\frac{\frac{2S}{b} \cdot \frac{2S}{c}}{\frac{2S}{a}} \right)^2 = 4S^2 \sum_{cyc} \frac{a^2}{b^2 c^2} =$$



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$$= 4S^2 \cdot \frac{1}{a^2 b^2 c^2} \sum_{cyc} a^4 \stackrel{\text{GOLDNER}(1949)}{\geq} \frac{4S^2}{a^2 b^2 c^2} \cdot 16S^2 = \frac{4S^2}{16R^2 S^2} \cdot 16S^2 = \left(\frac{2S}{R}\right)^2$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum \left(\frac{h_b h_c}{h_a} \right)^2 &\stackrel{\text{Chebyshev}}{\geq} \frac{1}{3} \left(\sum \frac{h_b h_c}{h_a} \right)^2 = \frac{1}{3} \left(\sum \frac{4S^2}{2S} \cdot \frac{a^2}{4Rrs} \right)^2 = \\ &= \frac{1}{3} \left(\frac{\sum a^2}{2R} \right)^2 \geq \left(\frac{2S}{R} \right)^2 \Leftrightarrow \frac{1}{\sqrt{3}} \cdot \frac{\sum a^2}{2R} \geq \frac{2S}{R} \Leftrightarrow \sum a^2 \geq 4\sqrt{3}S \end{aligned}$$

→ true (Ionescu – Weitzenbock) (Proved)

1047. In acute ΔABC the following relationship holds:

$$\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C} > A^2 + B^2 + C^2 + \cos A + \cos B + \cos C$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum \frac{1}{\cos A} > \sum A^2 + \sum \cos A \Leftrightarrow \sum \left(\frac{1}{\cos A} - \cos A \right) > \sum A^2 \Leftrightarrow \sum \frac{\sin^2 A}{\cos A} \stackrel{(1)}{>} \sum A^2$$

$$\text{Let } f(x) = \sin^2 x - x^2 \cos x, \forall x \in \left[0, \frac{\pi}{2}\right]$$

$$f'(x) = 2 \sin x \cos x + x^2 \sin x - 2x \cos x \stackrel{(2)}{\geq} 2 \sin x \cos x + x^2 \sin x - 2 \sin x$$

$$\left(\because x \cos x \leq \sin x \text{ as } x \leq \tan x ; \forall x \in \left[0, \frac{\pi}{2}\right] \right)$$

$$= \sin x (2 \cos x + x^2 - 2). \text{ Let } g(x) = 2 \cos x + x^2 - 2 \quad \forall x \in \left[0, \frac{\pi}{2}\right]$$

$$g'(x) = -2 \sin x + 2x \geq 0 \text{ as } \forall x \in \left[0, \frac{\pi}{2}\right], x \geq \sin x \therefore g(x) \stackrel{(3)}{>} g(0) = 0$$

$$(2), (3) \Rightarrow f'(x) \geq 0 \therefore f(x) \geq f(0) = 0$$

$$\Rightarrow \forall x \in \left[0, \frac{\pi}{2}\right], \sin^2 x \geq x^2 \cos x, \text{ with equality at } x = 0$$

$$\therefore \forall x \in \left(0, \frac{\pi}{2}\right), \sin^2 x > x^2 \cos x \Rightarrow \frac{\sin^2 x}{\cos x} \stackrel{(a)}{>} x^2 \because A, B, C \in \left(0, \frac{\pi}{2}\right) \therefore (a) \Rightarrow \frac{\sin^2 A}{\cos A} > A^2 \text{ etc}$$

$$\Rightarrow \sum \frac{\sin^2 A}{\cos A} > \sum A^2 \Rightarrow (1) \text{ is true (Proved)}$$



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1048. In ΔABC the following relationship holds:

$$12R \leq \frac{b^2 + c^2}{h_a} + \frac{c^2 + a^2}{h_b} + \frac{a^2 + b^2}{h_c} \leq \frac{9\sqrt{3}R^2}{S}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution 1 by Marian Ursărescu-Romania

$$b^2 + c^2 \geq 2bc \text{ and } h_a = \frac{2S}{a} \Rightarrow \frac{b^2 + c^2}{h_a} \geq \frac{abc}{S} \Rightarrow \sum \frac{b^2 + c^2}{h_a} \geq \frac{3abc}{S} \quad (1)$$

$$\text{But } abc = 4sRr \text{ and } S = sr \quad (2). \text{ From } (1) + (2) \Rightarrow \sum \frac{b^2 + c^2}{h_a} \geq \frac{12sRr}{sr} = 12R$$

$$\begin{aligned} \text{Now: } \sum \frac{b^2 + c^2}{h_a} &\geq \frac{9\sqrt{3}R^3}{S} \Leftrightarrow \sum \frac{b^2 + c^2}{\frac{2S}{a}} \geq \frac{9\sqrt{3}R^2}{S} \Leftrightarrow \sum a(b^2 + c^2) \geq 18\sqrt{3}R^3 \Leftrightarrow \\ &\Leftrightarrow \sum a^2(b + c) \geq 18\sqrt{3}R^3 \quad (3) \end{aligned}$$

$$\text{But } \sum a^2(b + c) = 2s(s^2 + r^2 - 2Rr) \quad (4)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum \frac{b^2 + c^2}{h_a} \stackrel{\text{Bergrstrom}}{\geq} \frac{(2 \sum a)^2}{2 \sum h_a} = \frac{4(\sum a)^2 R}{\sum ab}$$

$$\sum a^2 \geq 3 \sum ab \stackrel{(12 \sum ab)R}{\geq} \frac{(12 \sum ab)R}{\sum ab} = 12R \Rightarrow \sum \frac{b^2 + c^2}{h_a} \geq 12R. \text{ Now, Tereshin} \Rightarrow b^2 + c^2 \leq 4Rm_a, \text{ etc}$$

$$\therefore \sum \frac{b^2 + c^2}{h_a} \stackrel{(1)}{\leq} 4R \sum \frac{m_a}{h_a}. \text{ WLOG, we may assume } a \geq b \geq c$$

$$\therefore m_a \leq m_b \leq m_c \text{ & } \frac{1}{h_a} \geq \frac{1}{h_b} \geq \frac{1}{h_c} \therefore (1) \Rightarrow \sum \frac{b^2 + c^2}{h_a} \stackrel{\text{Chebyshev}}{\leq} \frac{4R}{3} (\sum m_a) \left(\sum \frac{1}{h_a} \right)$$

$$\sum m_a \leq 4R + r \stackrel{4R(4R + r)}{\leq} \frac{4R(4R + r)}{3r} = \frac{4Rs(4R + r)}{3S} \stackrel{\text{Mitrinovic}}{\leq} \frac{4R \frac{3\sqrt{3}R}{2} (4R + r)}{3S}$$

$$\stackrel{\text{Euler}}{\leq} \frac{4R \left(\frac{\sqrt{3}R}{2} \right) \left(\frac{9R}{2} \right)}{S} = \frac{9\sqrt{3}R^3}{S} \Rightarrow \sum \frac{b^2 + c^2}{h_a} \leq \frac{9\sqrt{3}R^3}{S} \quad (\text{Done})$$

From (3)+(4) we must show: $s(s^2 + r^2 - 2Rr) \geq 9\sqrt{3}R^3 \quad (5)$

$$\text{But } s \leq \frac{3\sqrt{3}}{2} R \quad (6)$$

From (5)+(6) we must show $s^2 + r^2 - 2Rr \leq 6R^2 \quad (7)$

From Gerretsen's inequality $s^2 \leq 4R^2 + 4Rr + 3r^2 \Rightarrow$



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$$\Rightarrow s^2 + r^2 - 2Rr \leq 4R^2 + 2Rr + 4r^2 \quad (8)$$

From (7)+(8) we must show: $4R^2 + 2Rr + 4r^2 \leq 6R^2 \Leftrightarrow Rr + 2r^2 \leq R^2 \quad (9)$

$$\text{But from Euler } r \leq \frac{R}{r} \wedge r^2 \leq \frac{R^2}{4} \Rightarrow Rr + 2r^2 \leq R^2.$$

1049. In ΔABC the following relationship holds:

$$\frac{a \cdot m_a}{\sin \frac{A}{2}} + \frac{b \cdot m_b}{\sin \frac{B}{2}} + \frac{c \cdot m_c}{\sin \frac{C}{2}} \geq 6sR$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\frac{am_a}{\sin \frac{A}{2}} = \frac{2R \sin A m_a}{\sin \frac{A}{2}} = \frac{4R \sin \frac{A}{2} \cos \frac{A}{2} m_a}{\sin \frac{A}{2}} = 4R \cos \frac{A}{2} m_a \Rightarrow$$

$$\text{We must show this: } m_a \cos \frac{A}{2} + m_b \cos \frac{B}{2} + m_c \cos \frac{C}{2} \geq \frac{3}{2}s \quad (1)$$

$$\text{But } m_a \geq \frac{b+c}{2} \cos \frac{A}{2} \quad (2).$$

$$\text{From (1)+(2) we must show: } \sum (b+c) \cos^2 \frac{A}{2} \geq 3s \quad (3)$$

$$\text{But } \cos^2 \frac{A}{2} = \frac{s(s-a)}{bc} \quad (4)$$

$$\text{From (3)+(4) we must show: } \sum \frac{(b+c)(s-a)}{bc} \geq 3 \Leftrightarrow \sum \frac{(b+c)(b+c-a)}{bc} \geq 6 \quad (5)$$

$$\begin{aligned} \text{But } \sum \frac{(b+c)(b+c-a)}{bc} &= \sum \frac{a(b+c)(b+c-a)}{abc} = \\ &= \frac{\sum a(b+c)^2 - \sum a^2(b+c)}{abc} = \frac{\sum (ab^2 + ac^2 + 2abc) - \sum a^2b - \sum a^2c}{abc} \\ &= \frac{\sum ab^2 + \sum ac^2 + 6abc - \sum a^2b - \sum a^2c}{abc} = \frac{6abc}{abc} = 6 \quad (6). \text{ From (6) } \Rightarrow \text{ it's true.} \end{aligned}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{LHS} = \sum \frac{4R \sin \frac{A}{2} \cos \frac{A}{2}}{\sin \frac{A}{2}} m_a \stackrel{m_a \geq \frac{b+c}{2} \cos \frac{A}{2}}{\geq} \sum 4R \cos \frac{A}{2} \cdot \frac{b+c}{2} \cos \frac{A}{2}$$

$$= 2R \sum (b+c) \cdot \frac{s(s-a)}{bc} = \frac{2Rs}{4Rrs} \sum a(b+c)(s-a)$$



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$$\begin{aligned}
 &= \frac{1}{2r} \sum \left(\sum ab - bc \right) (s-a) = \frac{1}{2r} \left\{ \sum ab \sum (s-a) - \sum bc(s-a) \right\} \\
 &= \frac{1}{2r} (s \sum ab - s \sum ab + 12Rrs) = 6sR \text{ (Proved)}
 \end{aligned}$$

1050. In ΔABC the following relationship holds:

$$\frac{\sqrt{b+c}}{r_a} + \frac{\sqrt{c+a}}{r_b} + \frac{\sqrt{a+b}}{r_c} \leq \frac{4R-2r}{r \cdot \sqrt[4]{27r^2}}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 &\text{WLOG, we may assume } a \geq b \geq c. \text{ Then, } \sqrt{b+c} \leq \sqrt{c+a} \leq \sqrt{a+b} \text{ & } \frac{1}{r_a} \leq \frac{1}{r_b} \leq \frac{1}{r_c} \\
 \therefore LHS &\stackrel{\text{Chebyshev}}{\leq} \frac{1}{3} \left(\sum \sqrt{b+c} \right) \left(\sum \frac{1}{r_a} \right) \stackrel{\text{CBS}}{\leq} \frac{\sqrt{3}}{3} \sqrt{4s} \left(\frac{1}{r} \right) = \frac{1}{r} \sqrt{\frac{4s}{3}} \stackrel{?}{\leq} \frac{4R-2r}{r \sqrt[4]{27r^2}} \\
 &\Leftrightarrow \frac{4s}{3} \stackrel{?}{\leq} \frac{4(2R-r)^2}{3\sqrt{3}r} \Leftrightarrow sr\sqrt{3} \stackrel{?}{\leq} (2R-r)^2. \text{ Now, LHS of (1)} \stackrel{\text{Mitrinovic}}{\leq} \frac{3\sqrt{3}R}{2} \cdot r\sqrt{3} = \frac{9Rr}{2} \\
 &\stackrel{?}{\leq} (2R-r)^2 \Leftrightarrow 8R^2 - 17Rr + 2r^2 \stackrel{?}{\geq} 0 \Leftrightarrow (8R-r)(R-2r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2
 \end{aligned}$$

1051. In ΔABC the following relationship holds:

$$(m_a + m_b + m_c) \left(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \right) + \frac{9S^2}{m_a m_b m_c (m_a + m_b + m_c)} \geq 10$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 &\text{We shall first prove: } (\sum a) \left(\sum \frac{1}{a} \right) + \frac{16S^2}{abc(\sum a)} \geq 10 \\
 &\Leftrightarrow \left(\frac{2s}{4Rrs} \right) (s^2 + 4Rr + r^2) + \frac{16r^2s^2}{8Rrs^2} \geq 10 \Leftrightarrow \frac{s^2 + 4Rr + 5r^2}{2Rr} \geq 10 \\
 &\Leftrightarrow s^2 \geq 16Rr - 5r^2 \rightarrow \text{true (Gerretsen)} \therefore (\sum a) \left(\sum \frac{1}{a} \right) + \frac{16S^2}{abc(\sum a)} \stackrel{(1)}{\geq} 10
 \end{aligned}$$

Applying (1) on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ and whose area



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$$\begin{aligned}
 & \text{of course, will be } \frac{s}{3}, \text{ we get: } \left(\frac{2}{3} \sum m_a\right) \left(\frac{3}{2} \sum \frac{1}{m_a}\right) + \frac{16 \left(\frac{s^2}{9}\right)}{\left(\frac{8}{27} \prod m_a\right) \left(\frac{2}{3} \sum m_a\right)} \geq 10 \\
 & \Leftrightarrow (\sum m_a) \left(\sum \frac{1}{m_a}\right) + \frac{9s^2}{(\prod m_a)(\sum m_a)} \geq 10 \text{ (proved)}
 \end{aligned}$$

1052. In ΔABC the following relationship holds:

$$\frac{a^2}{R_a^2} + \frac{b^2}{R_b^2} + \frac{c^2}{R_c^2} \leq 8 + \left(\frac{ab + bc + ca}{a^2 + b^2 + c^2}\right)^2$$

(I – incentre, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$)

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Bogdan Fustei-Romania

$$\begin{aligned}
 & \text{Using two additional inequalities: 1) } \frac{R}{r} \geq \frac{abc + a^2 + b^2 + c^2}{2abc} \\
 & 2) x, y, z > 0: \frac{x^3 + y^3 + z^3}{4xyz} + \frac{1}{4} \geq \left(\frac{x^2 + y^2 + z^2}{xy + yz + zx}\right)^2
 \end{aligned}$$

From the two inequalities from above we can write the following:

$$\frac{R}{2r} \stackrel{(1)}{\geq} \frac{a^3 + b^3 + c^3}{4abc} + \frac{1}{4} \stackrel{(2)}{\geq} \left(\frac{a^2 + b^2 + c^2}{ab + bc + ac}\right)^2. \text{ So, finally: } \frac{R}{2r} \geq \left(\frac{a^2 + b^2 + c^2}{ab + bc + ac}\right)^2$$

$$R_a = 2R \sin \frac{A}{2} \text{ (and the analogs); } \sin \frac{A}{2} = \sqrt{\frac{r_a - r}{4R}} \text{ (and the analogs)}$$

$$a^2 = (r_b + r_c)(r_a - r) \text{ (and the analogs)}$$

$$\Rightarrow R_a = 2R \cdot \sqrt{\frac{r_a - r}{R}} = \sqrt{4R^2 \frac{(r_a - r)}{4R}} = \sqrt{R(r_a - r)} \text{ (and the analogs)}$$

$$R_a^2 = R(r_a - r) \text{ (and the analogs)} \Rightarrow \frac{a^2}{R_a^2} = \frac{(r_b + r_c)(r_a - r)}{R(r_a - r)} = \frac{r_b + r_c}{R}. \text{ So, } \frac{a^2}{R_a^2} = \frac{r_b + r_c}{R} \text{ (and the}$$

$$\text{analog)} \frac{a^2}{R_a^2} + \frac{b^2}{R_b^2} + \frac{c^2}{R_c^2} = \frac{r_b + r_c}{R} + \frac{r_a + r_c}{R} + \frac{r_a + r_b}{R} = \frac{2(r_a + r_b + r_c)}{R} = \frac{2(4R + r)}{R}$$

$$(r_a + r_b + r_c = 4R + r) \Rightarrow \frac{a^2}{R_a^2} + \frac{b^2}{R_b^2} + \frac{c^2}{R_c^2} = \frac{8R + 2r}{R} = 8 + \frac{2r}{R}$$

$$\text{The inequality from enunciation becomes: } 8 + \frac{2r}{R} \leq 8 + \left(\frac{ab + bc + ac}{a^2 + b^2 + c^2}\right)^2 \Rightarrow$$



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$$\Rightarrow \frac{R}{2r} \geq \left(\frac{a^2 + b^2 + c^2}{ab + bc + ac} \right)$$

From the above, the inequality from enunciation is proved.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Area of } \Delta BIC = \frac{1}{2} ar \text{ & also } = \frac{BI \cdot CI \cdot a}{4R_a} \Rightarrow \frac{ar}{2} = \frac{r^2 a}{\sin \frac{B}{2} \sin \frac{C}{2} \cdot 4R_a} \Rightarrow R_a = \frac{r}{2 \sin \frac{B}{2} \sin \frac{C}{2}}$$

$$= \frac{2r \sin \frac{A}{2}}{4\pi \sin \frac{A}{2}} = \frac{2r \sin \frac{A}{2}}{\frac{r}{R}} = 2R \sin \frac{A}{2}$$

$$\Rightarrow R_a = 2R \sin \frac{A}{2} \Rightarrow \frac{a^2}{R_a^2} = \frac{\left(4R \sin \frac{A}{2} \cos \frac{A}{2}\right)^2}{4R^2 \sin^2 \frac{A}{2}} \stackrel{(1)}{=} 4 \cos^2 \frac{A}{2}$$

$$\text{Similarly, } \frac{b^2}{R_b^2} \stackrel{(2)}{=} 4 \cos^2 \frac{B}{2} \text{ & } \frac{c^2}{R_c^2} \stackrel{(3)}{=} 4 \cos^2 \frac{C}{2}$$

$$(1) + (2) + (3) \Rightarrow LHS = 2 \left(\sum 2 \cos^2 \frac{A}{2} \right) = 2 \sum (1 + \cos A)$$

$$= 2 \left(3 + 1 + \frac{r}{R} \right) = \frac{2(4R + r)}{R} \leq 8 + \left(\frac{\sum ab}{\sum a^2} \right)^2 \Leftrightarrow R \geq 2r \left(\frac{\sum a^2}{\sum ab} \right)^2$$

$$\Leftrightarrow R(s^2 + 4Rr + r^2)^2 \geq 8r(s^2 - 4Rr - r^2)^2$$

$$\Leftrightarrow (R - 2r)s^4 + 2s^2(4R + r^2)(R + 8r) + r^2(4R + r)^2(R - 8r) \stackrel{(4)}{\geq} 6rs^4$$

$$\text{Now, } \because (R - 2r)s^4 \stackrel{\text{Gerretsen}}{\geq} s^2(R - 2r)(16Rr - 5r^2)$$

$$\therefore \text{LHS of (4)} \stackrel{(a)}{\geq} s^2r(24R^2 + 29Rr + 26r^2) + r^2(4R + r)^2(R - 8r)$$

$$\text{& RHS of (4)} \stackrel{\substack{\text{Gerretsen} \\ (b)}}{\leq} 6rs^2(4R^2 + 4Rr + 3r^2)$$

(a), (b) \Rightarrow in order to prove (4), it suffices to prove:

$$s^2(5Rr + 8r^2) + r(4R + r)^2(R - 8r) \stackrel{(5)}{\geq} 0$$

$$\text{Now, LHS of (5)} \stackrel{\text{Gerretsen}}{\geq} (16Rr - 5r^2)(5Rr + 8r^2) + r(4R + r)^2(R - 8r) \stackrel{?}{\geq} 0 \Leftrightarrow$$

$$\Leftrightarrow (t - 2)\{2t(t - 2) + 3t + 3\} \stackrel{?}{\geq} 0 \quad (t = \frac{R}{r}) \rightarrow \text{true } b \because t \stackrel{\text{Euler}}{\geq} 2 \text{ (proved).}$$



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1053. If in ΔABC : $a \leq b \leq c$ then:

$$\frac{bm_c}{cm_b} + \frac{am_b}{bm_a} + \frac{cm_a}{am_c} \geq \frac{cm_b}{bm_c} + \frac{bm_a}{am_b} + \frac{am_c}{cm_a}$$

Proposed by Daniel Sitaru – Romania

Solution by Serban George Florin-Romania

$$\begin{aligned} & \left(\frac{bm_c}{cm_b} - \frac{cm_b}{bm_c} \right) + \left(\frac{am_b}{bm_a} - \frac{bm_a}{am_b} \right) + \left(\frac{cm_a}{am_c} - \frac{am_c}{cm_a} \right) \geq 0 \\ & \frac{b^2 m_c^2 - c^2 m_b^2}{bcm_b m_c} + \frac{a^2 m_b^2 - b^2 m_a^2}{abm_a m_b} + \frac{c^2 m_a^2 - a^2 m_c^2}{acm_a m_c} \geq 0 \\ & \Rightarrow \frac{\left(\frac{m_c}{c}\right)^2 - \left(\frac{m_b}{b}\right)^2}{\frac{m_b}{b} \cdot \frac{m_c}{c}} + \frac{\left(\frac{m_b}{b}\right)^2 - \left(\frac{m_a}{a}\right)^2}{\frac{m_a}{a} \cdot \frac{m_b}{b}} + \frac{\left(\frac{m_a}{a}\right)^2 - \left(\frac{m_c}{c}\right)^2}{\frac{m_a}{a} \cdot \frac{m_c}{c}} \geq 0 \end{aligned}$$

$$\text{If } a \leq b \text{ then } \frac{m_a}{a} \geq \frac{m_b}{b} \Leftrightarrow \frac{m_a^2}{a^2} \geq \frac{m_b^2}{b^2}$$

$$\frac{b^2(2b^2 + 2c^2 - a^2)}{4} \geq \frac{a^2(2a^2 + 2c^2 - b^2)}{4}, 2b^4 + 2b^2c^2 - a^2b^2 \geq 2a^4$$

$$+ 2a^2c^2 - a^2b^2, (b^4 - a^4) + c^2(b^2 - a^2) \geq 0, (b^2 - a^2)(b^2 + a^2) + c^2(b^2 - a^2) \geq 0$$

$$(b^2 - a^2)(b^2 + a^2 + c^2) \geq 0 \text{ (true)} b^2 \geq a^2, b^2 - a^2 \geq 0$$

$$\text{Note } \frac{m_a}{a} = x, \frac{m_b}{b} = y, \frac{m_c}{c} = z, a \leq b \leq c \Rightarrow x \geq y \geq z \Rightarrow \frac{z^2 - y^2}{yz} + \frac{y^2 - x^2}{xy} + \frac{x^2 - z^2}{xz} \geq 0$$

$$\Rightarrow \frac{x^2 - z^2}{xz} \geq \frac{y^2 - z^2}{yz} + \frac{x^2 - y^2}{xy}, \frac{(x^2 - y^2) + (y^2 - z^2)}{xz} \geq \frac{y^2 - z^2}{yz} + \frac{x^2 - b^2}{xy}$$

$$\frac{x^2 - y^2}{xz} + \frac{y^2 - z^2}{xz} \geq \frac{y^2 - z^2}{yz} + \frac{x^2 - y^2}{xy}; (x^2 - y^2) \left(\frac{1}{xz} - \frac{1}{xy} \right) \geq (y^2 - z^2) \left(\frac{1}{yz} - \frac{1}{xz} \right)$$

$$\frac{(x - y)(x + y)(y - z)}{xyz} \geq \frac{(y - z)(y + z)(x - y)}{xyz}$$

$$\Rightarrow (x - y)(x + y)(y - z) \geq (y - z)(y + z)(x - y)$$

$$\Rightarrow (x - y)(x + y)(y - z) - (y - z)(y + z)(x - y) \geq 0$$

$$\Rightarrow (x - y)(y - z)(x + y - y - z) \geq 0, (x - y)(y - z)(x - z) \geq 0$$

True

$$x \geq y \Rightarrow x - y \geq 0; y \geq z \Rightarrow y - z \geq 0; x \geq z \Rightarrow x - z \geq 0$$



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1054. In acute ΔABC the following relationship holds:

$$\prod \left(\frac{a}{c} \cos A + \frac{b}{c} \cos B - \cos C \right) \leq \cos A \cos B \cos C$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} & a \cos A, b \cos B, c \cos C > 0; a \cos A + b \cos B - c \cos C \\ & = R(\sin 2A + \sin 2B) - 2R \sin C \cos C = R \cdot 2 \sin C \cos(A - B) - 2R \sin C \cos C \\ & = 2R \sin C \{\cos(A - B) + \cos(A + B)\} = 2R \sin C \cdot 2 \cos A \cos B \\ & = 4R \sin C \cos A \cos B > 0 \quad (\because \cos A, \cos B > 0) \end{aligned}$$

Similarly, $b \cos B + c \cos C - a \cos A > 0$ & $c \cos C + a \cos A - b \cos B > 0$

$\therefore a \cos A, b \cos B, c \cos C$ are sides of a triangle.

Let $a \cos A = x, b \cos B = y, c \cos C = z$. Then, given inequality becomes:

$xyz \geq (x + y - z)(y + z - x)(z + x - y)$, which, of course holds true when x, y, z are 3 sides of a triangle (proved).

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

Since A, B, C are acute – angles. Hence: $\cos A, \cos B, \cos C > 0$. Hence:

$$\begin{aligned} & \left(\frac{a}{c} \cos A + \frac{b}{c} \cos B - \cos C \right) \left(\frac{a}{b} \cos A + \frac{c}{b} \cos B - \cos B \right) \left(\frac{b}{a} \cos B + \frac{c}{a} \cos C - \cos A \right) \\ & \leq \cos A \cos B \cos C \end{aligned}$$

$$\begin{aligned} & \text{If } (a \cos A + b \cos B - \cos C)(b \cos B + c \cos C - a \cos A)(a \cos A + c \cos C - b \cos B) \\ & \leq (a \cos A)(b \cos B)(c \cos C) \end{aligned}$$

Let $a \cos A = x, b \cos B = y, c \cos C = z$. If $(x + y - z)(y + z - x)(z + xy) \leq xyz$

$$\text{Let } x + y - z = m, y + z - x = n, z + x - y = p. \frac{m+n}{2} = y, \frac{n+p}{2} = z, \frac{m+p}{2} = x$$

If $mnp \leq \left(\frac{m+p}{2} \right) \left(\frac{m+n}{2} \right) \left(\frac{n+p}{2} \right)$ and it's true. Therefore,

$$\begin{aligned} & \left(\frac{a}{c} \cos A + \frac{b}{c} \cos B - \cos C \right) \left(\frac{a}{b} \cos A + \frac{c}{b} \cos B - \cos B \right) \left(\frac{b}{a} \cos B + \frac{c}{a} \cos C - \cos A \right) \\ & \leq \cos A \cos B \cos C. \end{aligned}$$

It's true.



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1055. In acute ΔABC the following relationship holds:

$$(am_a + bm_b + cm_c)(s_a m_a + s_b m_b + s_c m_c) \leq \frac{243\sqrt{3}R^4}{8}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mehmet Sahin-Ankara-Turkey

$$(am_a + bm_b + cm_c)^2 \leq (a^2 + b^2 + c^2)(m_a^2 + m_b^2 + m_c^2)$$

$$am_a + bm_b + cm_c \leq \sqrt{9R^2 \cdot \frac{3}{4} \cdot 9R^2} = \frac{9\sqrt{3}R^3}{2} \quad (1)$$

$$s_a \leq m_a, s_b \leq m_b, s_c \leq m_c$$

$$s_a m_a + s_b m_b + s_c m_c \leq m_a^2 + m_b^2 + m_c^2 = \frac{3}{4} \cdot 9R^2 \quad (2)$$

From (1) and (2): $(am_a + bm_b + cm_c)(s_a m_a + s_b m_b + s_c m_c) \leq \frac{9\sqrt{3}}{2} \cdot \frac{27}{4} R^4 \leq \frac{243\sqrt{3}}{8} R^4$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{By Tsintsifas, } m_a \leq \frac{b^2+c^2}{2bc} w_a \Rightarrow \frac{2bc}{b^2+c^2} m_a \leq w_a \Rightarrow s_a \stackrel{(1)}{\leq} w_a$$

$$\text{Similarly, } s_b \stackrel{(2)}{\leq} w_b, s_c \stackrel{(3)}{\leq} w_c$$

$$(1), (2), (3) \Rightarrow \sum s_a m_a \leq \sum w_a m_a \stackrel{CBS}{\leq} \sqrt{\sum w_a^2} \sqrt{\sum m_a^2}$$

$$\stackrel{w_a^2 \leq s(s-a), etc}{\leq} \sqrt{s \sum (s-a)} \sqrt{\sum m_a^2} = s \sqrt{\sum m_a^2}$$

$$\stackrel{\text{Mitrinovic}}{\leq} \frac{3\sqrt{3}R}{2} \sqrt{\sum m_a^2} \therefore \sum s_a m_a \stackrel{(a)}{\leq} \frac{3\sqrt{3}R}{2} \sqrt{\sum m_a^2}$$

$$\text{Also, } \sum am_a \stackrel{CBS}{\leq} \sqrt{\sum a^2} \sqrt{\sum m_a^2} \stackrel{\text{Leibnitz}}{\stackrel{(b)}{\leq}} 3R \sqrt{\sum m_a^2}$$

$$(a), (b) \Rightarrow LHS \leq \frac{9\sqrt{3}R^2}{2} (\sum m_a^2) = \frac{27\sqrt{3}R^2}{8} \sum a^2 \stackrel{\text{Leibnitz}}{\leq} \frac{243\sqrt{3}r^4}{8} \quad (\text{Proved})$$

1056. In ΔABC the following relationship holds:

$$\sum \sqrt{2s - 2\sqrt{a(2s-a)}} \geq (\sqrt{2}-1)(\sqrt{a} + \sqrt{b} + \sqrt{c})$$

Proposed by Daniel Sitaru – Romania



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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \sum \sqrt{2s - 2\sqrt{a(2s-a)}} &= \sum \sqrt{(\sqrt{2s-a})^2 + (\sqrt{a})^2 - 2\sqrt{a(2s-a)}} \\
 &= \sum \sqrt{(\sqrt{2s-a} - \sqrt{a})^2} \stackrel{(1)}{=} \sum (\sqrt{2s-a} - \sqrt{a}) \\
 (\because \sqrt{2s-a} > \sqrt{a} \text{ as } 2s = a+b+c > 2a \because b+c > a) \\
 (1) \Rightarrow \text{it suffices to prove: } \sum \sqrt{b+c} &\geq \sqrt{2} \sum \sqrt{a} \\
 \Leftrightarrow \sum (b+c) + 2 \sum \sqrt{(b+c)(c+a)} &\geq 2 \sum a + 4 \sum \sqrt{ab} \\
 \Leftrightarrow \sum \sqrt{(b+c)(c+a)} &\geq 2 \sum \sqrt{ab} \\
 \Leftrightarrow \sum (b+c)(c+a) + 2\sqrt{(a+b)(b+c)(c+a)} \left(\sum \sqrt{a+b} \right) &\\
 &\geq 4 \sum ab + 8\sqrt{abc} \left(\sum \sqrt{a} \right) \\
 \Leftrightarrow \sum a^2 + 3 \sum ab + 2\sqrt{(a+b)(b+c)(c+a)} \left(\sum \sqrt{a+b} \right) &\\
 &\geq 4 \sum ab + 8\sqrt{abc} \left(\sum \sqrt{a} \right) \\
 \Leftrightarrow \sum a^2 + 2\sqrt{(a+b)(b+c)(c+a)} \left(\sum \sqrt{a+b} \right) &\stackrel{(2)}{\geq} \sum ab + 8\sqrt{abc} \left(\sum \sqrt{a} \right)
 \end{aligned}$$

Let $a+b = x, b+c = y, c+a = z$

Then, $x+y > z, y+z > x, z+x > y \Rightarrow x, y, z \rightarrow \text{sides of a } \Delta$

we have $\sqrt{x} + \sqrt{y} + \sqrt{z} \stackrel{(a)}{\geq} \sqrt{y+z-x} + \sqrt{z+x-y} + \sqrt{x+y-z}$

When, x, y, z are sides of a triangle, Re-substituting the values of x, y, z , (a) \Rightarrow

$$\sum \sqrt{a+b} \geq \sum \sqrt{(b+c) + (c+a) - (a+b)} = \sum \sqrt{2c} \Rightarrow \sum \sqrt{a+b} \stackrel{(i)}{\geq} \sqrt{2} \sum \sqrt{a}$$

$$\text{Also, } 2\sqrt{(a+b)(b+c)(c+a)} \stackrel{(ii)}{\geq} \stackrel{(A-G)}{2\sqrt{8abc}}$$

$$(i). (ii) \Rightarrow 2\sqrt{(a+b)(b+c)(c+a)} \left(\sum \sqrt{a+b} \right) \stackrel{(iii)}{\geq} 2\sqrt{8abc} \cdot \sqrt{2} \sum \sqrt{a} = 8\sqrt{abc} \left(\sum \sqrt{a} \right)$$

$$\text{Moreover, } \sum a^2 \stackrel{(iv)}{\geq} \sum ab$$

(iii) + (iv) \Rightarrow (2) is true (Hence proved)



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1057. In ΔABC the following relationship holds:

$$\sqrt{h_a + h_b} + \sqrt{h_b + h_c} + \sqrt{h_c + h_a} \leq \frac{a + b + c}{\sqrt{R}}$$

Proposed by Bogdan Fustei – Romania

Solution 1 by Ertan Yildirim-Turkey

$$\sqrt{h_a + h_b} + \sqrt{h_a + h_c} + \sqrt{h_b + h_c} \leq \frac{a + b + c}{\sqrt{R}}$$

$$\sqrt{\frac{bc + ac}{2R}} + \sqrt{\frac{bc + ab}{2R}} + \sqrt{\frac{ac + ab}{2R}} \leq \frac{a + b + c}{\sqrt{R}}$$

$$\sqrt{c(a+b)} + \sqrt{b(a+c)} + \sqrt{a(b+c)} \stackrel{?}{\leq} \sqrt{2}(a+b+c)$$

$$CSE: (\sqrt{c} \cdot \sqrt{a+b} + \sqrt{b} \cdot \sqrt{a+c} + \sqrt{a} \cdot \sqrt{b+c})^2 \leq$$

$$(c + b + a) \cdot 2(a + b + c) = 2(a + b + c)^2$$

$$\Rightarrow \sqrt{c(a+b)} + \sqrt{b(a+c)} + \sqrt{a(b+c)} \leq \sqrt{2}(a+b+c) \text{ (true)}$$

Solution 2 by Marian Ursărescu-Romania

$$\text{Inequality} \Leftrightarrow (\sqrt{h_a + h_b} + \sqrt{h_b + h_c} + \sqrt{h_c + h_a})^2 \leq \frac{4s^2}{R} \quad (1)$$

$$\text{From Cauchy's Inequality} \Rightarrow (\sqrt{h_a + h_b} + \sqrt{h_b + h_c} + \sqrt{h_c + h_a})^2 \leq 6(h_a + h_b + h_c) \quad (2)$$

$$\text{From (1)+(2) we must show: } 3(h_a + h_b + h_c) \leq \frac{2s^2}{R} \quad (3)$$

$$\text{But } h_a + h_b + h_c = \frac{s^2 + r^2 + 4Rr}{2R} \quad (4) \text{ From (3)+(4) we must show:}$$

$$\frac{3(s^2 + r^2 + 4Rr)}{2R} \leq \frac{2s^2}{R} \Leftrightarrow 3(s^2 + r^2 + 4Rr) \leq 4s^2 \Leftrightarrow$$

$$s^2 \geq 3r^2 + 12Rr \quad (5)$$

$$\text{From Gerretsen's inequality we have: } s^2 \geq 16Rr - 5r^2 \quad (6)$$

$$\text{From (5)+(6) we must show: } 16Rr - 5r^2 \geq 3r^2 + 12Rr \Leftrightarrow 4Rr \geq 8r^2 \Leftrightarrow$$

R ≥ 2r, true because it's Euler's inequality.



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Solution 3 by Vo Thanh Long-Vietnam

$$\begin{aligned}
 \text{We have } \sum \sqrt{h_a + h_b} &= \sum \sqrt{\frac{2S}{a} + \frac{2S}{b}} = \sum \sqrt{\frac{2S(a+b)}{ab}} \stackrel{\text{Bunhiakovsky}}{\leq} \sqrt{6S \sum \frac{a+b}{ab}} \\
 &= \sqrt{\frac{12S(ab + bc + ca)}{abc}} \leq \sqrt{\frac{(a+b+c)^2}{R}} = \frac{a+b+c}{\sqrt{R}} \\
 &\quad " = " \text{ when } \Delta ABC \text{ is equilateral triangle.}
 \end{aligned}$$

1058. In ΔABC the following relationship holds:

$$a(2s - a) \cos \frac{A}{2} + b(2s - b) \cos \frac{B}{2} + c(2s - c) \cos \frac{C}{2} \geq 36\sqrt{3}r^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\text{We must show: } a(b+c) \cos \frac{A}{2} + b(a+c) \cos \frac{B}{2} + c(a+b) \cos \frac{C}{2} \geq 36\sqrt{3}r^2 \quad (1)$$

$$\text{But } a(b+c) \cos \frac{A}{2} + b(a+c) \cos \frac{B}{2} + c(a+b) \cos \frac{C}{2} \geq 3 \sqrt[3]{abc(a+b)(b+c)(a+c) \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} \quad (2)$$

From (1)+(2) we must show:

$$\sqrt[3]{abc(a+b)(b+c)(a+c) \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} \geq 12\sqrt{3}r^2 \quad (3)$$

$$\text{But } abc = 4sRr \quad (4), \quad (a+b)(b+c)(a+c) = 2s(s^2 + r^2 + 2Rr) \quad (5)$$

$$\text{and } \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \frac{s}{4R} \quad (6). \text{ From (4)+(5)+(6) we must show:}$$

$$\sqrt[3]{4sRr \cdot 2s(s^2 + r^2 + 2Rr) \cdot \frac{s}{4R}} \geq 12\sqrt{3}r^2 \Leftrightarrow$$

$$\left. \begin{aligned} s \sqrt[3]{2r(s^2 + r^2 + 2Rr)} &\geq 12\sqrt{3}r^2 \quad (7) \\ \text{From Mitrinovic } s \geq 3\sqrt{3}r \quad (8) \end{aligned} \right\} \text{we must show}$$

$$\sqrt[3]{2r(s^2 + r^2 + 2Rr)} \geq 4r \Leftrightarrow 2r(s^2 + r^2 + 2Rr) \geq 64r^3 \Leftrightarrow$$

$$s^2 + r^2 + 2Rr \geq 32r^2 \quad (9)$$

$$\text{From Gerretsen we have } s^2 \geq 16Rr - 5r^2 \quad (10)$$

$$\text{From (9)+(10) we must show: } 18Rr - 4r^2 \geq 32r^2 \Leftrightarrow$$

$$\Leftrightarrow 18Rr \geq 36r^2 \Leftrightarrow R \geq 2r \text{ true (Euler)}$$



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Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 & \text{By Bogdan Fustei, } \frac{b+c}{2} \stackrel{(1)}{\geq} \sqrt{2r(r_b + r_c)} = \sqrt{2rs \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right)} = \sqrt{2rs \left(\frac{\cos \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} \right)} \\
 & = \sqrt{\frac{2rs \cos^2 \frac{A}{2}}{\frac{s}{4R}}} = \sqrt{8Rr \cos^2 \frac{A}{2}} = 2\sqrt{2Rr} \cos \frac{A}{2}. \text{ Now, } \sum a(2s - a) \cos \frac{A}{2} = \sum a(b + c) \cos \frac{A}{2} \\
 & \stackrel{\text{by (1)}}{\geq} \sum a \left(4\sqrt{2Rr} \cos^2 \frac{A}{2} \right) = 4\sqrt{2Rr} \sum a \frac{s(s-a)}{bc} \\
 & = \frac{4s\sqrt{2Rr}}{4Rrs} \sum a^2(s-a) = \frac{\sqrt{2Rr}}{Rr} \left(s \sum a^2 - \sum a^3 \right) \\
 & = \frac{\sqrt{2Rr}}{Rr} \left\{ s \sum a^2 - 3abc - 2s \left(\sum a^2 - \sum ab \right) \right\} \\
 & = \frac{\sqrt{2Rr}}{Rr} \left\{ 2s \left(\sum ab \right) - 2s(s^2 - 4Rr - r^2) - 12Rrs \right\} \\
 & = \frac{2s\sqrt{2Rr}}{Rr} (s^2 + 4Rr + r^2 - s^2 + 4Rr + r^2 - 6Rr) \\
 & = \frac{2s\sqrt{2Rr}}{Rr} (2Rr + 2r^2) = \frac{4s\sqrt{2Rr}(R+r)}{R} \stackrel{?}{\geq} 36\sqrt{3}r^2 \Leftrightarrow \frac{s^2 \cdot 2Rr(R+r)^2}{R^2} \stackrel{?}{\geq} 243r^4 \\
 & \text{Now, LHS of (2)} \stackrel{s \geq 3\sqrt{3}r}{\geq} \frac{27 \cdot 2r^3(R+r)^2}{R} \stackrel{?}{\geq} 243r^4 \\
 & \Leftrightarrow 2(R+r)^2 \stackrel{?}{\geq} 9Rr \Leftrightarrow 2R^2 - 5Rr + 2r^2 \stackrel{?}{\geq} 0 \Leftrightarrow (R-2r)(2R-r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \\
 & \because R \stackrel{\text{Euler}}{\geq} 2r \Rightarrow (2) \text{ is true (Proved)}
 \end{aligned}$$

1059. In ΔABC the following relationship holds:

$$\left(1 + \frac{1}{\sin A} + \frac{1}{\sin B + \sin C}\right) \left(1 + \frac{1}{\sin B} + \frac{1}{\sin A + \sin C}\right) \left(1 + \frac{1}{\sin C} + \frac{1}{\sin A + \sin B}\right) \geq (1 + \sqrt{3})^3$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Soumitra Mandal-Chandar Nagore-India

We know, $\frac{3\sqrt{3}}{2} \geq \sum_{cyc} \sin A$ and $\frac{3\sqrt{3}}{8} \geq \prod_{cyc} \sin A$



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$$\begin{aligned}
 \prod_{cyc} \left(1 + \frac{1}{\sin A} + \frac{1}{\sin B + \sin C} \right) &\stackrel{\text{HOLDER'S INEQUALITY}}{\geq} \left(1 + \frac{1}{\sqrt[3]{\sin A \sin B \sin C}} + \frac{1}{\sqrt[3]{\prod_{cyc} (\sin A + \sin B)}} \right)^3 \\
 &\stackrel{\text{REVERSE AM} \geq \text{GM}}{\geq} \left(1 + \frac{2}{\sqrt{3}} + \frac{3}{2 \sum_{cyc} \sin A} \right)^3 \geq \left(1 + \frac{2}{\sqrt{3}} + \frac{3}{3\sqrt{3}} \right)^3 = (1 + \sqrt{3})^3
 \end{aligned}$$

(proved)

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\begin{aligned}
 \text{In a triangle we have: } \sin A + \sin B + \sin C &\leq \frac{3\sqrt{3}}{2} \text{ and we get } \frac{1}{\sqrt[3]{\sin A \sin B \sin C}} \geq \frac{2}{\sqrt{3}} \\
 \text{and } \frac{1}{\sqrt[3]{(\sin A + \sin B)(\sin B + \sin C)(\sin C + \sin A)}} &\geq \frac{1}{\sqrt{3}} \\
 \text{Hence } \left(1 + \frac{1}{\sqrt[3]{\sin A \sin B \sin C}} + \frac{1}{\sqrt[3]{(\sin A + \sin B)(\sin B + \sin C)(\sin C + \sin A)}} \right) &\geq 1 + \sqrt{3} \\
 \Rightarrow \left(1 + \frac{1}{\sqrt[3]{\sin A \sin B \sin C}} + \frac{1}{\sqrt[3]{(\sin A + \sin B)(\sin B + \sin C)(\sin C + \sin A)}} \right)^3 &\geq (1 + \sqrt{3})^3 \\
 \Rightarrow \left(1 + \frac{1}{\sin A} + \frac{1}{\sin B + \sin C} \right) \left(1 + \frac{1}{\sin B} + \frac{1}{\sin C + \sin A} \right) \left(1 + \frac{1}{\sin C} + \frac{1}{\sin A + \sin B} \right) &\geq (1 + \sqrt{3})^3
 \end{aligned}$$

Therefore, it's true.

1060. If in ΔABC : $ab = 12R^2 \sin^2 \frac{C}{2}$ then:

$$r \leq \frac{c\sqrt{3}}{6}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Lahiru Samarakoon-Sri Lanka

$$\begin{aligned}
 ab &= 12 \left(\frac{abc}{4\Delta} \right)^2 \times \frac{(s-a)(s-b)}{ab}; \quad 16\Delta^2 = 12c^2(s-a)(s-b) \\
 \text{So, } 2\Delta &= \sqrt{3}c\sqrt{(s-a)(s-b)}; \quad \frac{\sqrt{3}c}{6} = \frac{2\Delta}{3\sqrt{(s-a)(s-b)}} \\
 \text{but, } r &= \frac{\Delta}{s} \text{ so, consider } \frac{\sqrt{3}c}{6} - r = \frac{2\Delta}{3\sqrt{(s-a)(s-b)}} - \frac{\Delta}{s} = \frac{\Delta}{3s\sqrt{(s-a)(s-b)}} [2s - 3\sqrt{(s-a)(s-b)}] \\
 \text{but, } AM \geq GM: \frac{(s-a)+(s-b)}{2} &\geq \sqrt{(s-a)(s-b)}; \quad \frac{c}{2} \geq \sqrt{(s-a)(s-b)}
 \end{aligned}$$



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$$\text{So, } \underbrace{\frac{\Delta}{3s\sqrt{(s-a)(s-b)}} \left[2s - \frac{3c}{2} \right]}_{(+)} \cdot \underbrace{\frac{\Delta}{3s\sqrt{(s-a)(s-b)}} \left[a + b - \frac{c}{2} \right]}_{(+)} \cdot \text{So, } \frac{\sqrt{3}c}{6} - r \geq c; \quad r \leq \frac{c\sqrt{3}}{6}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} ab &= 12R^2 \sin^2 \frac{C}{2} \Rightarrow ab = 12 \left(\frac{abc}{4\Delta} \right)^2 \frac{(s-a)(s-b)}{ab} \\ &\Rightarrow a^2b^2 = \frac{3}{4} \cdot \frac{a^2b^2c^2(s-a)(s-b)}{s(s-a)(s-b)(s-c)} \\ &\Rightarrow 4s(s-c) = 3c^2 \Rightarrow (a+b+c)(a+b-c) = 3c^2 \\ &\Rightarrow (a+b)^2 - c^2 = 3c^2 \Rightarrow a+b = 2c \Rightarrow a+b+c = 3c \\ &\Rightarrow s = \frac{3c}{2} \stackrel{s \geq 3\sqrt{3}r}{\geq} 3\sqrt{3}r \Rightarrow c \geq 2\sqrt{3}r \Rightarrow \frac{c\sqrt{3}}{6} \geq r \Rightarrow r \leq \frac{c\sqrt{3}}{6} \text{ (proved)} \end{aligned}$$

1061. In ΔABC the following relationship holds:

$$27a^2b^2c^2 \leq (8R - 10r)^6$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\text{We must show: } 3\sqrt[3]{a^2b^2c^2} \leq (8R - 10r)^2 \quad (1)$$

$$\text{But } \sqrt[3]{a^2b^2c^2} \leq \frac{a^2+b^2+c^2}{3} \quad (2)$$

$$\text{From (1)+(2) we must show: } a^2 + b^2 + c^2 \leq (8R - 10r)^2 \quad (3)$$

$$\text{But } a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr) \quad (4)$$

$$\text{From (3)+(4) we must show: } s^2 - r^2 - 4Rr \leq 2(4R - 5r)^2 \quad (5)$$

$$\text{From Gerretsen's inequality: } s^2 \leq 4R^2 + 4Rr + 3r^2 \quad (6)$$

$$\text{From (5)+(6) we must show: } 4R^2 + 2r^2 \leq 2(4R - 5r)^2 \Leftrightarrow$$

$$\Leftrightarrow 2R^2 + r^2 \leq 16R^2 - 40Rr + 25r^2 \Leftrightarrow$$

$$\Leftrightarrow 14R^2 - 40Rr + 24r^2 \geq 0 \Leftrightarrow 7R^2 - 20Rr + 12r^2 \geq 0$$

Which is true because $R \geq 2r$ ⇒

$$7R^2 - 20Rr + 12r^2 \geq 28r^2 - 40r^2 + 12r^2 = 0$$