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**859 INEQUALITY IN TRIANGLE
ROMANIAN MATHEMATICAL MAGAZINE
2018**

MARIN CHIRCIU

1) In ΔABC the following relationship holds:

$$\sum \frac{s^2 - r_a^2}{s^2 + r_a^2} \geq \frac{3r}{R}$$

Proposed by Daniel Sitaru - Romania

Proof.
Lemma.
2) In ΔABC:

$$\sum \frac{s^2 - r_a^2}{s^2 + r_a^2} = 1 + \frac{r}{R}$$

Proof.

Using $r_a = s \tan \frac{A}{2}, \cos A = \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}, \sum \cos A = 1 + \frac{r}{R}$, *we obtain:*

$$\sum \frac{s^2 - r_a^2}{s^2 + r_a^2} = \sum \frac{s^2 - s^2 \tan^2 \frac{A}{2}}{s^2 + s^2 \tan^2 \frac{A}{2}} = \sum \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} = \sum \cos A = 1 + \frac{r}{R}.$$

□

Back to the main problem:

*Using **Lemma** we write the inequality:*

$$1 + \frac{r}{R} \geq \frac{3r}{R} \Leftrightarrow R \geq 2r \text{ (Euler's inequality).}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having an opposite sense:

3) In ΔABC:

$$\sum \frac{s^2 - r_a^2}{s^2 + r_a^2} \leq \frac{3}{2}$$

Proof.

Using ***Lemma*** *we write the inequality:*

$$1+\frac{r}{R}\leq\frac{3}{2}\Leftrightarrow R\geq 2r \text{ (Euler's inequality).}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

4) In ΔABC:

$$\frac{3r}{R}\leq\sum\frac{s^2-r_a^2}{s^2+r_a^2}\leq\frac{3}{2}$$

Proof.

See inequalities 1) and 3).

Equality holds if and only if the triangle is equilateral.

□

Remark.

In the same class of inequalities we can propose:

5) In acute-angled ΔABC:

$$\sum\frac{s^2+r_a^2}{s^2-r_a^2}\geq 6$$

Proposed by Marin Chirciu - Romania

Proof.

Lemma.

6) In acute-angled ΔABC:

$$\sum\frac{s^2+r_a^2}{s^2-r_a^2}=\frac{s^2+r^2-4Rr}{s^2-(2R+r)^2}$$

Proof.

Using $r_a=s\tan\frac{A}{2},\cos A=\frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}},\sum\frac{1}{\sum\cos A}=\frac{s^2+r^2-4Rr}{s^2-(2R+r)^2}$, *we obtain:*

$$\sum\frac{s^2+r_a^2}{s^2-r_a^2}=\sum\frac{s^2+s^2\tan^2\frac{A}{2}}{s^2-s^2\tan^2\frac{A}{2}}=\sum\frac{1+\tan^2\frac{A}{2}}{1-\tan^2\frac{A}{2}}=\sum\frac{1}{\cos A}=\frac{s^2+r^2-4Rr}{s^2-(2Rr+r)^2}.$$

□

Back to the main problem:

Using ***Lemma*** *we write the inequality:*

$$\frac{s^2+r^2-4Rr}{s^2-(2R+r^2)}\geq 6\Leftrightarrow 5s^2\leq 20R^2+24Rr+7r^2, \text{ which follows from}$$

Gerretsen's inequality: $s^2\leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$5(4R^2+4Rr+3r^2)\leq 20R^2+24Rr+7r^2\Leftrightarrow R\geq 2r \text{ (Euler's inequality).}$$

Equality holds if and only if the triangle is equilateral.

□

7) In ΔABC:

$$\sum \frac{s^2-h_a^2}{s^2+h_a^2} \le \frac{3R}{4r}$$

Proposed by Marin Chirciu - Romania

Proof.

Lemma.

8) In ΔABC:

$$\sum \frac{s^2-h_a^2}{s^2+h_a^2} = \frac{s^4+s^2(12R^2-8Rr-6r^2)+r^2(16R^2+40Rr-39r^2)}{s^4+s^2(4R^2-8Rr+10r^2)+r^2(16R^2-24Rr+9r^2)}.$$

Proof.

Using $h_a = \dfrac{2S}{a}$, *we obtain:*

$$\sum \frac{s^2-h_a^2}{s^2+h_a^2} = \sum \frac{s^2-\frac{S^2}{a^2}}{s^2+\frac{S^2}{a^2}} = \sum \frac{a^2s^2-r^2s^2}{a^2s^2+r^2s^2} = \sum \frac{a^2-4r^2}{a^2+4r^2}.$$

$$\sum \frac{a^2-4r^2}{a^2+4r^2} = \frac{\sum(a^2-4r^2)(b^2+4r^2)(c^2+4r^2)}{\prod(a^2+4r^2)} =$$

$$= \frac{s^4+s^2(12R^2-8Rr-6r^2)+r^2(16R^2+40Rr-39r^2)}{s^4+s^2(4R^2-8Rr+10r^2)+r^2(16R^2-24Rr+9r^2)}$$

$$\sum(a^2-4r^2)(b^2+4r^2)(c^2+4r^2) = 4r^2[s^4+s^2(12R^2-8Rr-6r^2)+r^2(16R^2+40Rr-39r^2)]$$

$$\prod(a^2+4r^2) = 4r^2[s^4+s^2(4R^2-8Rr+10r^2)+r^2(16R^2-24Rr+9r^2)]$$

□

Back to the main problem:

Using ***Lemma*** *we write the inequality:*

$$\frac{s^4+s^2(12R^2-8Rr-6r^2)+r^2(16R^2+40Rr-39r^2)}{s^4+s^2(4R^2-8Rr+10r^2)+r^2(16R^2-24Rr+9r^2)} \le \frac{3R}{4r} \Leftrightarrow$$

$$s^2[s^2(3R-4r)+2(6R^3-36R^2r+31Rr^2+12r^3)]+r^2(48R^3-136R^2r-133Rr^2+156r^3) \ge 0$$

which follows from Gerretsen's inequality $s^2 \ge 16Rr-5r^2$.

It remains to prove that:

$$(16Rr-5r^2)[(16Rr-5r^2)(3R-4r)+2(6R^3-36R^2r+31Rr^2+12r^3)]+$$

$$+r^2(48R^3-136R^2r-133Rr^2+156r^3) \ge 0 \Leftrightarrow$$

$$48R^4-99R^3r-72R^2r^2+164Rr^3-16r^4 \ge 0 \Leftrightarrow (R-2r)(48R^3-3R^2r-78Rr^2+8r^3) \ge 0$$

obviously from Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

9) In ΔABC:

$$\sum \frac{s^2+h_a^2}{s^2-h_a^2} = \frac{s^4-s^2(12R^2+8Rr-10r^2)+r^2(16R^2-24Rr-55r^2)}{s^4-s^2(4R^2+8Rr+6r^2)+r^2(16R^2+40Rr+25r^2)}$$

Proposed by Marin Chirciu - Romania

Proof.

$$\text{Using } h_a = \frac{2S}{a}, \text{ we obtain:}$$

$$\sum \frac{s^2 + h_a^2}{s^2 - h_a^2} = \sum \frac{s^2 + \frac{S^2}{a^2}}{s^2 - \frac{S^2}{a^2}} = \sum \frac{a^2 s^2 + r^2 s^2}{a^2 s^2 - r^2 s^2} = \sum \frac{a^2 + 4r^2}{a^2 - 4r^2}.$$

$$\sum \frac{a^2 + 4r^2}{a^2 - 4r^2} = \frac{\sum (a^2 + 4r^2)(b^2 - 4r^2)(c^2 - 4r^2)}{\prod (a^2 - 4r^2)} =$$

$$= \frac{s^4 - s^2(12R^2 + 8Rr - 10r^2) + r^2(16R^2 - 24Rr - 55r^2)}{s^4 - s^2(4R^2 + 8Rr + 6r^2) + r^2(16R^2 + 40Rr + 25r^2)}$$

$$\sum (a^2+4r^2)(b^2-4r^2)(c^2-4r^2) = 4r^2[-s^4+s^2(12R^2+8Rr-10r^2)+r^2(-16R^2+24Rr+55r^2)]$$

$$\prod (a^2 - 4r^2) = 4r^2[-s^4 + s^2(4R^2 + 8Rr + 6r^2) - r^2(16R^2 + 40Rr + 25r^2)]$$

□

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871 INEQUALITY IN TRIANGLE ROMANIAN MATHEMATICAL MAGAZINE 2018

MARIN CHIRCIU

1) In ΔABC:

$$\sum\left(\frac{1}{b}+\frac{1}{c}\right)\frac{h_a}{r_a}\le\frac{9R}{2S}$$

Proposed by Mehmet Şahin - Ankara - Turkey

Proof.

Lemma.

2) In ΔABC:

$$\sum\left(\frac{1}{b}+\frac{1}{c}\right)\frac{h_a}{r_a}=\frac{s^2-r^2-4Rr}{Rrs}$$

Proof.

Using $h_a=\frac{2S}{a}$ *and* $r_a=\frac{S}{s-a}$ *we obtain:*

$$\sum\left(\frac{1}{b}+\frac{1}{c}\right)\frac{h_a}{r_a}=\sum\frac{b+c}{bc}\cdot\frac{\frac{2S}{a}}{\frac{S}{s-a}}=\frac{2}{abc}\sum(b+c)(s-a)=\frac{2}{4Rrs}\cdot 2(s^2-r^2-4Rr)=$$

$$=\frac{s^2-r^2-4Rr}{Rrs}\ \textit{because}\ \sum(b+c)(s-a)=\sum a^2=2(s^2-r^2-4Rr)$$

□

Let's pass to solving the inequality from enunciation.

*Using the **Lemma** the inequality can be written:*

$$\frac{s^2-r^2-4Rr}{Rrs}\le\frac{9R}{2rs}\Leftrightarrow 2s^2\le 9R^2+8Rr+2r^2,$$

which follows from Gerretsen's inequality: $s^2\le 4R^2+4Rr+3r^2$.

It remains to prove that:

$2(4R^2+4Rr+3r^2)\le 9R^2+8Rr+2r^2\Leftrightarrow R^2\ge 4r^2$, *obviously form Euler's inequality* $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having an opposite sense for:

3) In ΔABC:

$$\boldsymbol{\sum\left(\frac{1}{b}+\frac{1}{c}\right)\frac{h_a}{r_a}\geq\frac{18r}{Rs}}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using **Lemma** the inequality can be written:*

$$\frac{s^2-r^2-4Rr}{Rrs}\geq\frac{18r}{Rs}\Leftrightarrow s^2\geq 4Rr+19r^2,$$

which follows from Gerretsen's inequality: $s^2\geq 16Rr-5r^2$.

It remains to prove that:

$16Rr-5r^2\geq 4Rr+19r^2\Leftrightarrow R\geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

4) In ΔABC:

$$\boldsymbol{\frac{18r}{Rs}\leq\sum\left(\frac{1}{b}+\frac{1}{c}\right)\frac{h_a}{r_a}\leq\frac{9R}{2S}}.$$

Proof.

*See inequalities **1** and **3**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Changing between them h_a with r_a we propose:

5) In ΔABC:

$$\boldsymbol{\frac{9R}{2S}\leq\sum\left(\frac{1}{b}+\frac{1}{c}\right)\frac{r_a}{h_a}\leq\frac{9R^3}{8Sr^2}}$$

Proposed by Marin Chirciu - Romania

Proof.

Lemma.

6). In ΔABC:

$$\boldsymbol{\sum\left(\frac{1}{b}+\frac{1}{c}\right)\frac{r_a}{h_a}=\frac{s^2(2R-r)-r^2(4R+r)}{4sRr^2}}$$

Proof.

Using $h_a=\dfrac{2S}{a}$ *and* $r_a=\dfrac{S}{s-a}$ *we obtain:*

$$\sum\left(\frac{1}{b}+\frac{1}{c}\right)\frac{r_a}{h_a}=\sum\frac{b+c}{bc}\cdot\frac{\frac{S}{s-a}}{\frac{2S}{a}}=\frac{1}{2}\sum\frac{a(b+c)}{bc(s-a)}=\frac{1}{2}\cdot\frac{s^2(2R-r)-r^2(4R+r)}{2sRr^2}=$$

$$=\frac{s^2(2R-r)-r^2(4R+r)}{4sRr^2}\ \textit{because}\ \sum\frac{a(b+c)}{bc(s-a)}=\frac{s^2(2R-r)-r^2(4R+r)}{2sRr^2}$$

□

Let's pass to solving the inequality from enunciation.

Using ***Lemma*** *the inequality from the right can be written.*

$$\frac{s^2(2R-r)-r^2(4R+r)}{4sRr^2} \le \frac{9R^3}{8sr^3} \Leftrightarrow 2s^2r(2R-r) \le 9R^4+8Rr^3+2r^4,$$

which follows from Gerretsen's inequality: $s^2 \le 4R^2+4Rr+3r^2$.

It remains to prove that:

$$2(4R^2+4Rr+3r^2)r(2R-r) \le 9R^4+8Rr^3+2r^4 \Leftrightarrow 9R^4-16R^3r-8R^2r^2+4Rr^3+8r^4 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(9R^3+2R^2r-4Rr^2-4r^3) \ge 0,$ *obviously from Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral

Using ***Lemma*** *the inequality from the left can be written:*

$$\frac{s^2(2R-r)-r^2(4R+r)}{4sRr^2} \ge \frac{9R}{2S} \Leftrightarrow s^2(2R-r) \ge r(18R^2+4Rr+r^2),$$

which follows from Gerretsen's inequality: $s^2 \ge 16Rr-5r^2$.

It remains to prove that:

$$(16Rr-5r^2)(2R-r) \ge r(18R^2+4Rr+r^2) \Leftrightarrow 7R^2-15Rr+2r^2 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(7R-r) \ge 0,$ *obviously from Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Linking the sums $\sum\left(\frac{1}{b}+\frac{1}{c}\right)\frac{h_a}{r_a}$ *and* $\sum\left(\frac{1}{b}+\frac{1}{c}\right)\frac{r_a}{h_a}$

we can write the sequence of inequalities:

7) In ΔABC:

$$\boldsymbol{\frac{18r}{Rs} \le \sum\left(\frac{1}{b}+\frac{1}{c}\right)\frac{h_a}{r_a} \le \frac{9R}{2S} \le \sum\left(\frac{1}{b}+\frac{1}{c}\right)\frac{r_a}{h_a} \le \frac{9R^3}{8Sr^2}}.$$

Proposed by Mehmet Şahin - Turkey and Marin Chirciu - Romania

Proof.

See ***4)*** *and* ***5)***.

Equality holds if and only if the triangle is equilateral.

□

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878 INEQUALITY IN TRIANGLE
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MARIN CHIRCIU

1) In ΔABC:

$$\sum \frac{a}{r_b^2 + r_c^2} \leq \frac{2R - r}{S}$$

Proposed by Mehmet Şahin - Ankara - Turkey

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC:

$$\sum \frac{a}{r_b r_c} = \frac{2(2R - r)}{S}$$

Proof.

Using $r_a = \dfrac{S}{s-a}, \sum a r_a = 2s(2R - r), r_a r_b r_c = rs^2$ *we obtain:*

$$\sum \frac{a}{r_b r_c} = \frac{\sum a r_a}{r_a r_b r_c} = \frac{2s(2R - r)}{rs^2} = \frac{2(2R - r)}{S}$$

□

Let's get back to the main problem:

Using $r_b^2 + r_c^2 \geq 2rbr_c$ *and* ***Lemma*** *we obtain:*

$$\sum \frac{a}{r_b^2 + r_c^2} \leq \sum \frac{a}{2r_b r_c} = \frac{1}{2} \cdot \frac{2(2R - r)}{S} = \frac{2R - r}{S}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

If we replace r_a with h_a we obtain:

3) In ΔABC:

$$\sum \frac{a}{h_b^2 + h_c^2} \leq \frac{3R}{2S}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.
4) In ΔABC

$$\boldsymbol{\sum \frac{a}{h_b h_c} = \frac{3R}{S}}.$$

Proof.

Using $h_a = \frac{2S}{a}, \sum a h_a = 6S, h_a h_b h_c = \frac{2S^2}{R}$ *we obtain:*

$$\sum \frac{a}{h_b h_c} = \frac{\sum a h_a}{h_a h_b h_c} = \frac{6S}{\frac{2S^2}{R}} = \frac{3R}{S}.$$

□

Let's get back to the main problem:

Using $h_b^2 + h_c^2 \geq 2h_b h_c$ *and Lemma we obtain:*

$$\sum \frac{a}{h_b^2 + h_c^2} \leq \sum \frac{a}{2h_b h_c} = \frac{1}{2} \cdot \frac{3R}{S} = \frac{3R}{2S}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum \frac{a}{h_b h_c}$ *and* $\sum \frac{a}{r_b r_c}$ *we can write the relationship:*

5) In ΔABC:

$$\boldsymbol{\sum \frac{a}{h_b h_c} \leq \sum \frac{a}{r_b r_c}}$$

Proof.

Using the sums $\sum \frac{a}{h_b h_c} = \frac{3R}{S}$ *and* $\sum \frac{a}{r_b r_c} = \frac{2(2R - r)}{S}$ *we write the inequality:*

$$\frac{3R}{S} \leq \frac{2(2R - r)}{S} \Leftrightarrow R \geq 2r \text{ (Euler's inequality)}.$$

Equality holds if and only if the triangle is equilateral.

□

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897 INEQUALITY IN TRIANGLE ROMANIAN MATHEMATICAL MAGAZINE 2018

MARIN CHIRCIU

1) In ΔABC:

$$\sum \frac{a^2}{h_b + h_c} \geq 6r$$

Proposed by Seyran Ibrahimov - Azerbaijan

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC:

$$\sum \frac{a^2}{h_b + h_c} \geq \frac{4R(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr}$$

Proof.

Using $h_a = \frac{2S}{a}$ *we obtain:*

$$\sum \frac{a^2}{h_b + h_c} = \sum \frac{a^2}{\frac{2S}{b} + \frac{2S}{c}} = \frac{abc}{2S} \sum \frac{a}{b+c} = \frac{4RS}{2S} \cdot \frac{2(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr} = \frac{4R(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr},$$

which follows from $\sum \frac{a}{b+c} = \frac{2(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr}$

□

Let's get back to the main problem:

*Using the **Lemma** the inequality can be written:*

$$\frac{4R(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr} \geq 6r \Leftrightarrow s^2(2R - 3r) \geq r(2R^2 + 8Rr + 3r^2),$$

which follows from Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2$.

It remains to prove that:

$$(16Rr - 5r^2)(2R - 3r) \geq r(2R^2 + 8Rr + 3r^2) \Leftrightarrow 5R^2 - 11Rr + 6r^2 \geq 0 \Leftrightarrow (R - 2r)(5R - r) \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasises an inequality having an opposite sense.

3) In ΔABC:

$$\sum \frac{a^2}{h_b+h_c} \leq \frac{3R^2}{2r}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma** the inequality can be written:*

$$\frac{4R(s^2-r^2-Rr)}{s^2+r^2+2Rr} \leq \frac{3R^2}{2r} \Leftrightarrow s^2(3R-8r)+r(6R^2+11Rr+8r^2) \geq 0$$

We distinguish the following cases:

Case 1). If $(3R-8r) \geq 0$, the inequality is obviously.

Case 2). If $(3R-8r) < 0$, we rewrite the inequality:

$$r(6R^2+11Rr+8r^2) \geq s^2(8r-3r)$$

which follows from Gerretsen's inequality: $s^2 \leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$r(6R^2+11Rr+8r^2) \geq (4R^2+4Rr+3r^2)(8r-3R) \Leftrightarrow 6R^3-7R^2r-6R^2r-6Rr^2-8r^3 \geq 0$$

$\Leftrightarrow (R-2r)(6R^2+5Rr+4r^2) \geq 0$, *obviously, from Euler's inequality $R \geq 2r$.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

4) In ΔABC:

$$6r \leq \sum \frac{a^2}{h_b+h_c} \leq \frac{3R^2}{2r}$$

Proof.

*See inequalities **1)** and **3)**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

If we replace h_a with r_a we propose:

5) In ΔABC

$$6r \leq \sum \frac{a^2}{r_b+r_c} \leq \frac{3R^2}{2r}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.

6) In ΔABC:

$$\sum \frac{a^2}{r_b+r_c} = 2(2R-r).$$

Proof.

$$\text{Using } r_a = \frac{S}{s-a} \text{ we obtain:}$$

$$\sum \frac{a^2}{r_b + r_c} = \sum \frac{a^2}{\frac{S}{s-b} + \frac{S}{s-c}} = \frac{1}{S} \sum a(s-b)(s-c) = \frac{1}{sr} \cdot 2sr(2R-r) = 2(2R-r),$$

$$\text{which follows from } \sum a(s-b)(s-c) = 2sr(2R-r).$$

□

Let's get back to the main problem:

The left side inequality:

Using the ***Lemma*** *the left side inequality can be written:*

$$2(2R-r) \geq 6r \Leftrightarrow R \geq 2r \text{ (Euler's inequality).}$$

Equality holds if and only if the triangle is equilateral.

The right hand inequality:

Using the ***Lemma*** *the right hand inequality:*

$$2(2R-r) \leq \frac{3R^2}{2r} \Leftrightarrow 3R^2 - 8Rr + 4r^2 \geq 0 \Leftrightarrow (R-2r)(3R-2r) \geq 0,$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum \frac{a^2}{h_b + h_c}$ *and* $\sum \frac{a^2}{r_b + r_c}$ *the relationship can be written:*

7) In ΔABC:

$$\sum \frac{a^2}{h_b + h_c} \leq \sum \frac{a^2}{r_b + r_c}$$

Proposed by Marin Chirciu - Romania

Proof.

$$\text{Using the sums } \sum \frac{a^2}{h_b + h_c} = \frac{4R(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr} \text{ and } \sum \frac{a^2}{r_b + r_c} = 2(2R-r)$$

$$\text{the inequality can be written: } \frac{4R(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr} \leq 2(2R-r) \Leftrightarrow s^2 \leq s^2 \leq 6R^2 + 2Rr - r^2$$

which follows from Gerretsen's inequality $s^2 \leq 4R^2 + 4Rr + 3r^2$.

It remains to prove that:

$$4R^2 + 4Rr + 3r^2 \leq 6R^2 + 2Rr - r^2 \Leftrightarrow R^2 - Rr - 2r^2 \geq 0 \Leftrightarrow (R-2r)(R+r) \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

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SECLAMAN INEQUALITY ROMANIAN MATHEMATICAL MAGAZINE 2018

MARIN CHIRCIU

1) In ΔABC:

$$\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}\le 2R-r$$

Proposed by Dan Seclaman - Romania

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC:

$$\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}=\frac{3(s^2-r^2-4Rr)}{2(4Rr+r)}$$

Proof.

$$\textit{Using } \sum m_a^2=\frac{3}{4}\sum a^2, \sum a^2=2(s^2-r^2-4Rr), \sum r_a=4R+r.$$

□

Let's get back to the main problem.

*Using the **Lemma** we write the inequality:*

$$\frac{3(s^2-r^2-4Rr)}{2(4R+r)}\le 2R-r \Leftrightarrow 3s^2\le(4R+r)^2$$

which follows from Gerretsen's inequality: $s^2\le 4R^2+4Rr+3r^2$.

It remains to prove that:

$$3(4R^2+4Rr+3r^2)\le(4R+r)^2 \Leftrightarrow R^2-Rr-2r^2\ge 0 \Leftrightarrow (R-2r)(2R+r)\ge 0,$$

obviously from Euler's inequality $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasises an inequality having an opposite sense.

3) In ΔABC:

$$\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}\ge 3r.$$

Proof.

*Using the **Lemma** the inequality can be written:*

$$\frac{3(s^2-r^2-4Rr)}{2(4R+r)} \geq 3r \Leftrightarrow s^2 \geq 12Rr + 3r^2,$$

which follows from Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2$.

It remains to prove that:

$$16Rr - 5r^2 \geq 12Rr + 3r^2 \Leftrightarrow R \geq 2r \text{ (Euler's inequality).}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

4) In ΔABC:

$$\boldsymbol{3r \leq \frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c} \leq 2R-r.}$$

Proof.

*See inequalities **1)** and **3)**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Replacing the sum $r_a+r_b+r_c$ *with* $h_a+h_b+h_c$ *we propose:*

5) In ΔABC:

$$\boldsymbol{\frac{3R}{2} \leq \frac{m_a^2+m_b^2+m_c^2}{h_a+h_b+h_c} \leq \frac{3R^2}{4r}}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.

6) In ΔABC

$$\boldsymbol{\frac{m_a^2+m_b^2+m_c^2}{h_a+h_b+h_c} = 3R \cdot \frac{s^2-r^2-4Rr}{s^2+r^2+4Rr}}$$

Proof.

Using $\sum m_a^2 = \frac{3}{4}\sum a^2, \sum a^2 = 2(s^2-r^2-4Rr), \sum h_a = \frac{s^2+r^2+4Rr}{2R}$

□

Let's get back to the main problem:

The left side inequality:

*Using the **Lemma** the inequality from the left side can be written:*

$$3R\cdot\frac{s^2-r^2-4Rr}{s^2+r^2+4Rr}\geq\frac{3R}{2}\Leftrightarrow s^2\geq 12Rr+3r^2$$

which follows from Gerretsen's inequality: $s^2\geq 16Rr-5r^2$.

It remains to prove that:

$$16Rr-5r^2\geq 12Rr+3r^2\Leftrightarrow R\geq 2r\ \textit{(Euler's inequality)}$$

Equality holds if and only if the triangle is equilateral.

The inequality from the right:

$$3R\cdot\frac{s^2-r^2-4Rr}{s^2+r^2+4Rr}\leq\frac{3R^2}{4r}\Leftrightarrow s^2(R-4r)+r(4R^2+17Rr+4r^2)\geq 0$$

We distinguish the following cases:

Case 1) If $(R-4r)\geq 0$, *the inequality is obviously.*

Case 2) If $(R-4r)<0$, *the inequality can be rewritten:*

$$r(4R^2+17Rr+4r^2)\geq s^2(4r-R)$$

which follows from Gerretsen's inequality: $s^2\leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$r(4R^2+17Rr+4r^2)\geq(4R^2+4Rr+3r^2)(4r-R)\Leftrightarrow R^3-2R^2r+Rr^2-2r^3\geq 0$$

$\Leftrightarrow (R-2r)(R^2+r^2)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}$ *and* $\frac{m_a^2+m_b^2+m_c^2}{h_a+h_b+h_c}$ *we can write the relationship:*

7) In ΔABC:

$$\boldsymbol{\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}\leq\frac{m_a^2+m_b^2+m_c^2}{h_a+h_b+h_c}}$$

Proof.

The inequality is equivalent with:

$$\frac{1}{r_a+r_b+r_c}\leq\frac{1}{h_a+h_b+h_c}\Leftrightarrow h_a+h_b+h_c\leq r_a+r_b+r_c\Leftrightarrow\frac{s^2+r^2+4Rr}{2R}\leq 4R+r\Leftrightarrow$$

$s^2\leq 8R^2-2Rr-r^2$, *which follows from Gerretsen's inequality:* $s^2\leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$4R^2+4Rr+3r^2\leq 8R^2-2Rr-r^2\Leftrightarrow 2R^2-3Rr-2r^2\geq 0\leq(R-2r)(2R+r)\geq 0,$$

obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

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ROMANIAN MATHEMATICAL MAGAZINE
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MARIN CHIRCIU

1) In ΔABC:

$$\sum \frac{l_b l_c}{l_a} \geq \sum \frac{h_b h_c}{h_a}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC:

$$\sum \frac{l_b l_c}{l_a} = \frac{2s^2(2R+r)+2r^2(4R+r)}{s^2+r^2+2Rr}$$

3) In ΔABC:

$$\sum \frac{h_b h_c}{h_a} = \frac{s^2-r^2-4Rr}{R}$$

Proof.

$$\textit{Using } l_a = \frac{2bc}{b+c}\cos\frac{A}{2} \textit{ and } h_a \frac{2S}{a}$$

□

Let's get back to the main problem.

Using ***Lemma*** *the inequality can be written:*

$$\frac{2s^2(2R+r)+2r^2(4R+r)}{s^2+r^2+2Rr} \geq \frac{s^2-r^2-4Rr}{R} \Leftrightarrow s^2(4R^2+4Rr-s^2)+r^2(4Rr+r)^2 \geq 0$$

We distinguish the following cases:

Case 1). If $(4R^2+4Rr-s^2) \geq 0$, *the inequality is obvious.*

Case 2). If $(4R^2+4Rr-s^2) < 0$, *we rewrite the inequality:*

$r^2(4R+r)^2 \geq s^2(s^2-4R^2-4Rr)$, *which follows from Blundon-Gerretsen:*

$s^2 \leq \frac{R(4R+r)^2}{2(2R-r)} \leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$r^2(4R+r)^2 \geq \frac{R(4R+r)^2}{2(2R-r)}(4R^2+4Rr+3r^2-4R^2-4Rr) \Leftrightarrow 2(2R-r) \geq 3R \Leftrightarrow R \geq 2r.$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

If h_a with r_a we propose:

4) In ΔABC:

$$\sum \frac{l_b l_c}{l_a} \leq \sum \frac{r_b r_c}{r_a}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.
5) In ΔABC:

$$\sum \frac{r_b r_c}{r_a} = \frac{s^2 - 2r^2 - 8Rr}{r}$$

Proof.

$$\text{Using } r_a = \frac{S}{s-a}$$

□

Let's get back to the main problem.

*Using the **Lemma** the inequality can be written:*

$$\frac{2s^2(2R+r) + 2r^2(4R+r)}{s^2+r^2+2Rr} \leq \frac{s^2-2r^2-8Rr}{r} \Leftrightarrow s^2(s^2-3r^2-10Rr) \geq 4r^2(4R^2+5Rr+r^2)$$

which follows from Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2$.

It remains to prove that:

$$(16Rr-5r^2)(16Rr-5r^2-3r^2-10Rr) \geq 4r^2(4R^2+5Rr+r^2) \Leftrightarrow 40R^2-89Rr+18r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(40R-9r) \geq 0,$ *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

6) In ΔABC:

$$\sum \frac{h_b h_c}{h_a} \leq \sum \frac{r_b r_c}{r_a}$$

Proof.

*Using the **Lemma** the inequality can be written:*

$$\frac{s^2-r^2-8Rr}{r} \leq \frac{s^2-r^2-8Rr}{r} \Leftrightarrow s^2(R-r) \geq r(8R^2-2Rr-r^2),$$

which follows from Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2$

It remains to prove that:

$$(16Rr-5r^2)(R-r) \geq r(8R^2-2Rr-r^2) \Leftrightarrow 8R^2-19Rr+6r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(8R-3r) \geq 0,$ *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum \frac{l_b l_c}{l_a}, \sum \frac{h_b h_c}{h_a}$ *and* $\sum \frac{r_b r_c}{r_a}$ *the following inequalities hold:*

7) In ΔABC:

$$\sum \frac{r_b r_c}{r_a} \geq \sum \frac{l_b l_c}{l_a} \geq \sum \frac{h_b h_c}{h_a}$$

Proposed by Marin Chirciu - Romania

Proof.

*See inequalities **1), 4)** and **5).***

Equality holds if and only if the triangle is equilateral.

□

Remark.

For the sums $\sum \frac{l_b l_c}{l_a}, \sum \frac{h_b h_c}{h_a}$ *and* $\sum \frac{r_b r_c}{r_a}$ *we can write the relationships:*

8) In ΔABC:

$$3(R+r) \leq \sum \frac{l_b l_c}{l_a} \leq 8R - 7r$$

Proposed by Marin Chirciu - Romania

Proof.

*Using **Lemma 2)** we write the inequality:*

$$3(R+r) \leq \frac{2s^2(2R+r) + 2r^2(4R+r)}{s^2 + r^2 + 2Rr} \leq 8R - 7r$$

The left hand inequality can be written:

$$s^2(R-r) \geq r(6R^2 + Rr + r^2)$$

which follows from Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2$

It remains to prove that:

$$(16Rr - 5r^2)(R-r) \geq r(6R^2 + Rr + r^2) \Leftrightarrow 5R^2 - 11Rr + 2r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(5R-r) \geq 0,$ *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

The right hand inequality can be written:

$$s^2(4Rr - 9r) + r(16R^2 - 14Rr - 9r^2) \geq 0$$

We distinguish the following cases:

Case 1). If $(4R - 9r) \geq 0$ *the inequality is obvious.*

Case 2). If $(4R - 9r) < 0$, *the inequality can be written:*

$r(16R^2 - 14Rr - 9r^2) \geq s^2(9r - 4R)$, *which follows from Gerretsen's inequality:* $s^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$$r(16R^2 - 14Rr - 9r^2) \geq (4R^2 + 4Rr + 3r^2)(9r - 4R) \Leftrightarrow 8R^3 - 2R^2 r - 19Rr^2 - 18r^3 \geq 0$$

$\Leftrightarrow (R-2r)(8R^2 + 14Rr + 9r^2) \geq 0,$ *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

9) In ΔABC:

$$9r \le \sum \frac{h_b h_c}{h_a} \le 4R + r$$

Proposed by Marin Chirciu - Romania

Proof.

*Using **Lemma 3)** the inequality can be written:*

$$9r \le \frac{s^2 - r^2 - 4Rr}{R} \le 4R + r$$

The left hand inequality can be written: $s^2 \ge 13Rr + r^2$, *which follows from Gerretsen's inequality:* $s^2 \ge 16Rr - 5r^2$ *and Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

The right hand inequality can be written: $s^2 \le 4R^2 + 5Rr + r^2$ *which follows from Gerretsen's inequality:* $s^2 \le 4R^2 + 4Rr + 3r^2$ *and Euler's inequality:* $R \ge 2R$.

Equality holds if and only if the triangle is equilateral.

□

10) In ΔABC:

$$8R - 7r \le \sum \frac{r_b r_c}{r_a} \le \frac{(2R - r)^2}{r}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using **Lemma 3)** we write the inequality:* $8R - 7r \le \dfrac{s^2 - 2r^2 - 8Rr}{r} \le \dfrac{(2R - r)^2}{r}$

The left hand inequality can be written: $s^2 \ge 16Rr - 5r^2$ *(Gerretsen's inequality)*

Equality holds if and only if the triangle is equilateral.

The right hand inequality can be written: $s^2 \le 4R^2 + 4Rr + 3r^2$, *(Gerretsen's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the sequence of inequalities:

11) In ΔABC:

$$9r \le \sum \frac{h_b h_c}{h_a} \le \sum \frac{l_b l_c}{l_a} \le 8R - 7r \le \sum \frac{r_b r_c}{r_a} \le \frac{(2R - r)^2}{r}$$

Proof.

*See inequalities **9), 1), 8)** and 10).*

Equality holds if and only if the triangle is equilateral.

□

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83 IDENTITY IN TRIANGLE
ROMANIAN MATHEMATICAL MAGAZINE
2018

MARIN CHIRCIU

1) In ΔABC:

$$\sum \frac{h_a + h_b}{r_a + r_b} = 2\left(1 + \frac{r}{R}\right)$$

Proposed by Bogdan Fustei - Romania

Proof.

Using $h_a = \frac{2S}{a}$ *and* $r_a = \frac{S}{s-a}$ *we obtain:*

$$\sum \frac{h_a + h_b}{r_a + r_b} = \sum \frac{\frac{2S}{a} + \frac{2S}{b}}{\frac{S}{s-a} + \frac{S}{s-b}} = \frac{2}{abc} \sum (a+b)(s-a)(s-b) = 2\left(1 + \frac{r}{R}\right)$$

which follows from: $abc = 4srR$ *and* $\sum (a+b)(s-a)(s-b) = 4sr(R+r)$

□

Remark.

Let's emphasises a double inequality with the above sum:

2) In ΔABC

$$\frac{6r}{R} \leq \sum \frac{h_a + h_b}{r_a + r_b} \leq 3$$

Proof.

*Using identity **1)** the inequality can be written:* $\frac{6r}{R} \leq 2\left(1 + \frac{r}{R}\right) \leq 3,$

which follows from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Reversing the fraction from the above sum we propose:

3) In ΔABC:

$$3 \leq \sum \frac{r_b + r_c}{h_b + h_c} \leq \frac{3R}{2r}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.
4) In ΔABC:

$$\sum \frac{r_b + r_c}{h_b + h_c} = \frac{R}{r} \cdot \frac{s^2 + 5r^2 + 8Rr}{s^2 + r^2 + 2Rr}$$

Proof.

Using $h_a = \frac{2S}{a}$ *and* $r_a = \frac{S}{s-a}$ *we obtain:*

$$\sum \frac{r_b + r_c}{h_b + h_c} = \sum \frac{\frac{S}{s-b} + \frac{S}{s-c}}{\frac{2S}{b} + \frac{2S}{c}} = \frac{abc}{2} \sum \frac{1}{(b+c)(s-b)(s-c)} = \frac{R}{r} \cdot \frac{s^2 + 5r^2 + 8Rr}{s^2 + r^2 + 2Rr}$$

which follows from: $abc = 4srR$ *and* $\sum \frac{1}{(b+c)(s-b)(s-c)} = \frac{s^2 + 5r^2 + 8Rr}{2r^2 s(s^2 + r^2 + 2Rr)}$

□

Let's get back to the main problem.

The left side inequality:

Using the ***Lemma****, the left side inequality can be written:*

$$\frac{R}{r} \cdot \frac{s^2 + 5r^2 + 8Rr}{s^2 + r^2 + 2Rr} \geq 3 \Leftrightarrow s^2(R - 3r) + r(8R^2 - Rr - 3r^2) \geq 0$$

We distinguish the following cases:

Case 1). If $(R - 3r) \geq 0$*, the inequality is obvious.*

Case 2). If $(R-3r) < 0$*, we rewrite the inequality:* $r(8R^2 - Rr - 3r^2) \geq s^2(3r - R)$*,*

which follows from Gerretsen's inequality: $s^2 \leq 4R^2 + 4Rr + 3r^2$*.*

It remains to prove that:

$$r(8R^2 - Rr - 3r^2) \geq (4R^2 + 4Rr + 3r^2)(3r - R) \Leftrightarrow 2R^3 - 5Rr^2 - 6r^3 \geq 0$$

$\Leftrightarrow (R - 2r)(2R^2 + 4Rr + 3r^2) \geq 0$*, obviously from Euler's inequality* $R \geq 2r$*.*

Equality holds if and only if the triangle is equilateral.

The right hand inequality:

Using ***Lemma*** *the right hand inequality can be written:*

$$\frac{R}{r} \cdot \frac{s^2 + 5r^2 + 8Rr}{s^2 + r^2 + 2Rr} \leq \frac{3R}{2r} \Leftrightarrow s^2 \leq 12Rr + 3r^2,$$

which follows from Gerretsen's inequality: $s^2 \geq 16Rr - 5r$*.*

It remains to prove that:

$16Rr - 5r^2 \geq 12Rr + 3r^2 \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum \frac{h_b + h_c}{r_b + r_c}$ *and* $\sum \frac{r_b + r_c}{h_b + h_c}$ *the relationship can be written:*

5) In ΔABC:

$$\sum \frac{h_b + h_c}{r_b + r_c} \leq \sum \frac{r_b + r_c}{h_b + h_c}$$

Proof.

Using the sums $\sum \frac{h_b + h_c}{r_b + r_c} = 2\left(1 + \frac{r}{R}\right)$ *and* $\sum \frac{r_b + r_c}{h_b + h_c} = \frac{R}{r} \cdot \frac{s^2 + 5r^2 + 8Rr}{s^2 + r^2 + 2Rr}$

we write the inequality: $2\left(1 + \frac{r}{R}\right) \leq \frac{R}{r} \cdot \frac{s^2 + 5r^2 + 8Rr}{s^2 + r^2 + 2Rr} \Leftrightarrow$

$$\Leftrightarrow s^2(R^2 - 2Rr - 2r^2) + r(8R^3 + R^2r - 6Rr^2 - 2r^3) \geq 0$$

We distinguish the following cases:

Case 1) If $(R^2 - 2Rr - 2r^2) \geq 0$, *the inequality is obviously.*

Case 2). If $(R^2 - 2Rr - 2r^2) < 0$, *inequality can be written:*

$$r(8R^3 + R^2r - 6Rr^2 - 2r^3) \geq s^2(2r^2 + 2Rr - r^2),$$

which follows from Gerretsen's inequality: $s^2 \leq 4R^2 + 4Rr + 3r^2$.

It remains to prove that:

$$r(8R^3 + R^2r - 6Rr^2 - 2r^2) \geq (4R^2 + 4Rr + 3r^2)(2r^2 + 2Rr - R^2) \Leftrightarrow$$

$$\Leftrightarrow R^4 + R^3r - 3R^2r^2 - 5Rr^3 - 2r^4 \geq 0 \Leftrightarrow (R - 2r)(R + r)^3 \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

The sequence of inequalities can be written:

6) In ΔABC:

$$\frac{6r}{R} \leq \sum \frac{h_a + h_b}{r_a + r_b} \leq 3 \leq \sum \frac{r_b + r_c}{h_b + h_c} \leq \frac{3R}{2r}$$

Proposed by Marin Chirciu - Romania

Proof.

*See inequalities **2)** and **3)**.*

Equality holds if and only if the triangle is equilateral.

□

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ABOUT PROBLEM JP.128
ROMANIAN MATHEMATICAL MAGAZINE
AUTUMN EDITION 2018

MARIN CHIRCIU

1. In ΔABC:

$$\sum \frac{1}{m_b + m_c} \leq \frac{1}{2r}$$

Proposed by Marian Ursărescu - Romania

Proof.

Using the inequality $\frac{1}{x+y} \leq \frac{1}{4}\left(\frac{1}{x} + \frac{1}{y}\right), x, y > 0,$ *we obtain:*

$$\sum \frac{1}{m_b + m_c} \leq \sum \frac{1}{4}\left(\frac{1}{m_b} + \frac{1}{m_c}\right) = \frac{1}{2} \sum \frac{1}{m_a} \leq \frac{1}{2} \sum \frac{1}{h_a} = \frac{1}{2} \cdot \frac{1}{r} = \frac{1}{2r}$$

We took into account that $m_a \geq h_a$ *and* $\sum \frac{1}{h_a} = \frac{1}{r}$

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasise an inequality having an opposite sense.

2) In ΔABC:

$$\sum \frac{1}{m_b + m_c} \geq \frac{1}{R}$$

Proof.

Using Bergström's inequality, we obtain:

$\sum \frac{1}{m_a + m_b} \geq \frac{9}{\sum (m_b + m_c)} = \frac{9}{2 \sum m_a} \geq \frac{1}{R}$, *where the last inequality follows from*

$$\sum m_a \leq \frac{9R}{2}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

3) In ΔABC:

$$\frac{1}{R} \leq \sum \frac{1}{m_b + m_c} \leq \frac{1}{2r}.$$

Proof.

*See inequalities **1)** and **2)**.*
Equality holds if and only if the triangle is equilateral.

□

Remark.

In the same class of inequalities we can propose:

4) In ΔABC:

$$\frac{1}{R} \geq \sum \frac{1}{l_b + l_c} \leq \frac{1}{2r}.$$

Proof.

*Analogous **1)** and **2)**. Using* $\sum l_a \leq \frac{9R}{2}$.
Equality holds if and only if the triangle is equilateral.

□

5) In ΔABC:

$$\frac{1}{R} \leq \sum \frac{1}{h_b + h_c} \leq \frac{1}{2r}$$

Proof.

We prove using the following lemma:

Lemma.
6) In ΔABC:

$$\sum \frac{1}{h_b + h_c} = \frac{s^4 + s^2(16Rr + 2r^2) + r^2(4R + r)^2}{4rs^2(s^2 + r^2 + 2Rr)}$$

Proof.

We use $h_a = \frac{2S}{a}$

□

Back to the main problem:
*Using the **Lemma** the inequality from the left side can be written:*

$$\frac{s^4 + s^2(16Rr + 2r^2) + r^2(4R + r)^2}{4rs^2(s^2 + r^2 + 2Rr)} \geq \frac{1}{R} \Leftrightarrow$$

$$s^2[s^2(R - 4r) + r(16R^2 - 6Rr - 4r^2)] + Rr^2(4R + r)^2 \geq 0$$

We distinguish the following cases:
Case 1). If $(R - 4r) \geq 0$*, the inequality is obvious.*
Case 2). If $(R - 4r) < 0$*, the inequality can be written:*
$Rr^2(4R + r)^2 \geq s^2[s^2(4r - R) + r(4r^2 + 6Rr - 16R^2)]$ *which follows from*
Blundon - Gerretsen's inequality $s^2 \leq \frac{R(4R + r)^2}{2(2R - r)} \leq 4R^2 + 4Rr + 3r^2$.
It remains to prove that:

$$Rr^2(4R+r)^2 \ge \frac{R(4R+r)^2}{2(2R-r)}[(4R^2+4Rr+3r^2)(4r-R)+r(4r^2+6Rr-16R^2)] \Leftrightarrow$$

$$\Leftrightarrow 4R^3+4R^2r-15Rr^2-18r^3 \ge 0 \Leftrightarrow (R-2r)(R^2+12Rr+9r^2) \ge 0$$

obviously, from Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

*Inequality from the right side can be proven in the same manner with **1)**.*

Analogous with the left side inequality.

□

7) In ΔABC:

$$\frac{1}{R}\sum \frac{1}{r_b+r_c} \le \frac{1}{2r}$$

Proof.

We prove using the following lemma:

Lemma.
8) In ΔABC:

$$\sum \frac{1}{r_b+r_c} = \frac{1}{4R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$$

Proof.

We use $r_a = \dfrac{S}{s-a}$

□

Back to the main problem:

*Using the **Lemma** the left side inequality can be written:*

$\frac{1}{4R}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \ge \frac{1}{R} \Leftrightarrow (4R+r)^2 \ge 3s^2$, *which follows from Gerretsen's inequality:* $s^2 \le 4R^2+4Rr+3r^2$ *and Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

*The right side inequality can be proven in the same manner as **1)**.*

Analogous, the left side inequality.

□

Remark.

Between the sums $\sum \frac{1}{h_b+h_c}$ *and* $\sum \frac{1}{r_b+r_c}$ *the following relationship can be written:*

9) In ΔABC:

$$\sum \frac{1}{h_b+h_c} \ge \sum \frac{1}{r_b+r_c}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the sums from the above lemmas the inequality can be written:

$$\frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{4rs^2(s^2+r^2+2Rr)} \geq \frac{1}{4R}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \Leftrightarrow$$

$$\Leftrightarrow s^2[s^2(R-r)-2r^2(4R+r)] \geq r^2(4R+r)^2(R+r) \text{ which follows}$$

from Gerretsen's inequality: $s^2 \geq 16Rr-5r^2$. *It remains to prove that:*

$$(16Rr-5r^2)[(16Rr-5r^2)(R-r)-2r^2(4R+r)] \geq r^2(4R+r)^2(R+r) \Leftrightarrow$$

$$30R^3-71R^2r+23Rr^2-2r^3 \geq 0 \Leftrightarrow (R-2r)(30R^2-11Rr+r^2) \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the sequence of inequalities:

10) In ΔABC:

$$\frac{1}{R} \leq \sum \frac{1}{r_b+r_c} \leq \sum \frac{1}{h_b+h_c} \leq \frac{1}{2r}$$

Proof.

*See inequalities **7), 9)** and **5).***

Equality holds if and only if the triangle is equilateral.

□

11) In ΔABC:

$$\frac{1}{R} \leq \sum \frac{1}{m_b+m_c} \leq \sum \frac{1}{l_b+l_c} \leq \sum \frac{1}{h_b+h_c} \leq \frac{1}{2r}.$$

Proof.

*See inequalities **2), 5)** and* $h_a \leq l_a \leq m_a$.

Equality holds if and only if the triangle is equilateral.

□

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ABOUT 881 INEQUALITY IN TRIANGLE ROMANIAN MATHEMATICAL MAGAZINE 2018

MARIN CHIRCIU

1) In ΔABC:

$$\sum r_a(h_b + h_c)^2 \geq 12Ss$$

Proposed by Mehmet Şahin - Ankara - Turkey

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC:

$$\sum r_a h_b h_c = 2s^2 r\left(2 - \frac{r}{R}\right)$$

Proof.

Using $r_a = \frac{S}{s-a}, h_a = \frac{2S}{a}$ *we obtain:*

$$\sum r_a h_b h_c = \sum \frac{S}{s-a} \cdot \frac{2S}{b} \cdot \frac{2S}{c} = 4S^3 \sum \frac{1}{bc(s-a)} = 2s^2 r\left(2 - \frac{r}{R}\right)$$

which follows from $\sum \frac{1}{bc(s-a)} = \frac{2R-r}{2Rr^2 s}$.

□

Back to the main problem:

Using $(h_b + h_c)^2 \geq 4h_b h_c$ *and the* ***Lemma*** *we obtain:*

$\sum r_a(h_b + h_c)^2 \geq 4\sum r_a h_b h_c = 4 \cdot 2s^2 r\left(2 - \frac{r}{R}\right) \geq 12s^2 r$ *where the last inequality is equivalent with* $R \geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

If we interchange r_a *with* h_a *we propose:*

1) In ΔABC:

$$\sum h_a(r_b + r_c)^2 \geq 12Ss$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC:

$$\boldsymbol{\sum h_a r_b r_c = \frac{s^2(s^2+r^2-8Rr)}{2R}}$$

Proof.

Using $r_a = \dfrac{S}{s-a}, h_a = \dfrac{2S}{a}$ *we obtain:*

$$\sum h_a r_b r_c = \sum \frac{2S}{a} \cdot \frac{S}{s-b} \cdot \frac{S}{s-c} = 2S^3 \sum \frac{1}{a(s-b)(s-c)} = \frac{s^2(s^2+r^2-8Rr)}{2R}$$

which follows from $\displaystyle\sum \frac{1}{a(s-b)(s-c)} = \frac{s^2+r^2-8Rr}{4sRr^3}$

□

Back to the main problem:

Using $(r_b+r_c)^2 \geq 4r_b r_c$ *and* ***Lemma*** *we obtain:*

$\displaystyle\sum h_a(r_b+r_c)^2 \geq 4\sum h_a r_b r_c = 4 \cdot \frac{s^2(s^2+r^2-8Rr)}{2R} \geq 12s^2 r$, *where the last inequality is equivalent with* $s^2 \geq 14Rr - r^2$, *which follows from Gerretsen's inequality:* $s^2 \geq 16Rr - 5r^2$. *It remains to prove that:*

$16Rr - 5r^2 g \geq 14Rr - r^2 \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum r_a h_b h_c$ *and* $\sum h_a r_b r_c$ *the following relationship can be written:*

3) In ΔABC:

$$\boldsymbol{\sum r_a h_b h_c \leq \sum h_a r_b r_c.}$$

Proof.

Using the sums $\displaystyle\sum r_a h_b h_c = 2s^2 r\left(2 - \frac{r}{R}\right)$ *and* $\displaystyle\sum h_a r_b r_c = \frac{s^2(s^2+r^2-8Rr)}{2R}$

the inequality can be written: $\displaystyle 2s^2 r\left(2-\frac{r}{R}\right) \leq \frac{s^2(s^2+r^2-8Rr)}{2R} \Leftrightarrow s^2 \geq 16Rr - 5r^2$

(Gerretsen's inequality)

Equality holds if and only if the triangle is equilateral.

□

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INEQUALITY IN TRIANGLE 931
ROMANIAN MATHEMATICAL MAGAZINE
2018

MARIN CHIRCIU

1) In ΔABC

$$\sum \frac{h_a}{h_b h_c} \leq \frac{R}{2r^2}$$

Proposed by Bogdan Fustei - Romania

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC

$$\sum \frac{h_a}{h_b h_c} = \frac{s^4 + s^2(2r^2 - 8Rr) + r^2(4R + r)^2}{8s^2 r^2 R}$$

Proof.

Using $h_a = \frac{2S}{a}$, *we obtain:*

$$\sum \frac{h_a}{h_b h_c} = \sum \frac{\frac{2S}{a}}{\frac{2S}{b} \cdot \frac{2S}{c}} = \frac{1}{2S} \sum \frac{bc}{a} = \frac{s^4 + s^2(2r^2 - 8Rr) + r^2(4R + r)^2}{8s^2 r^2 R},$$

which follows from:

$$\sum \frac{bc}{a} = \frac{s^4 + s^2(2r^2 - 8Rr) + r^2(4R + r)^2}{4srR}$$

□

Back to the main problem:

*Using the **Lemma** the inequality can be written:*

$$\frac{s^4 + s^2(2r^2 - 8Rr) + r^2(4R + r^2)}{8s^2 r^2 R} \leq \frac{R}{2r^2} \Leftrightarrow s^2(4R^2 + 8Rr - 2r^2 - s^2) \geq r^2(4R + r)^2$$

which follows from Gerretsen's inequality $16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$.

It remains to prove that:

$$(16Rr - 5r^2)(4R^2 + 8Rr - 2r^2 - 4R^2 - 4Rr - 3r^2) \geq r^2(4R + r)^2 \Leftrightarrow$$

$$\Leftrightarrow 4R^2 - 9Rr + 2r^2 \geq 0 \Leftrightarrow (R - 2r)(R - r) \geq 0$$

obviously form Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasise an inequality having an opposite sense.

3) In ΔABC:

$$\sum \frac{h_a}{h_b h_c} \geq \frac{1}{r}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma** we can write:*

$$\frac{s^4 + s^2(2r^2 - 8Rr) + r^2(4Rr + r)^2}{8s^2r^2R} \geq \frac{1}{r} \Leftrightarrow s^2(s^2 + 2r^2 - 16Rr) + r^2(4R + r)^2 \geq 0$$

We distinguish the following cases:

Case 1). If $(s^2 + 2r^2 - 16Rr) \geq 0$, the inequality is obvious.

Case 2). If $(s^2 + 2r^2 - 16Rr) < 0$, the inequality can be rewritten:

$r^2(4R + r)^2 \geq s^2(16Rr - 2r^2 - s^2)$, *which follows from Blundon-Gerretsen's inequality:* $16Rr - 5r^2 \leq s^2 \leq \dfrac{R(4R + r)^2}{2(2R - r)}$

It remains to prove that:

$$r^2(4R+r)^2 \geq \frac{R(4R + r)^2}{2(2R - r)}(16Rr - 2r^2 - 16Rr + 5r^2) \Leftrightarrow R \geq 2r \text{ (Euler's inequality).}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

The double inequality can be written:

4) In ΔABC:

$$\frac{1}{r} \leq \sum \frac{h_a}{h_b h_c} \leq \frac{R}{2r^2}$$

Proof.

*See inequalities **1)** and **3)**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

If we replace $\dfrac{h_a}{h_b h_c}$ *with* $\dfrac{r_a}{r_b r_c}$ *we propose:*

5) In ΔABC:

$$\frac{2}{r}\left(1 - \frac{r}{R}\right) \leq \sum \frac{r_a}{r_b r_c} \leq \frac{1}{r}\left(\frac{R}{r} - 1\right)$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.

6) In ΔABC:

$$\boldsymbol{\sum \frac{r_a}{r_b r_c} = \frac{1}{r}\left[\left(\frac{4R+r}{s}\right)^2 - 2\right]}$$

Proof.

Using $r_a = \dfrac{S}{s-a}$, *using:*

$$\sum \frac{r_a}{r_b r_c} = \sum \frac{\frac{S}{s-a}}{\frac{S}{s-b}\cdot\frac{S}{s-c}} = \frac{1}{S}\sum \frac{(s-b)(s-c)}{s-a} = \frac{1}{rs}\cdot\frac{(4R+r)^2-2s^2}{s} = \frac{1}{r}\left[\left(\frac{4R+r}{s}\right)^2-2\right]$$

which follows from:

$$\sum \frac{(s-b)(s-c)}{s-a} = \frac{(4R+r)^2-2s^2}{s}$$

Back to the main problem.

□

*Using the **Lemma** the inequality can be written:*

$$\frac{2}{r}\left(1-\frac{r}{R}\right) \le \frac{1}{r}\left[\left(\frac{4R+r}{s}\right)^2 - 2\right] \le \frac{1}{r}\left(\frac{R}{r}-1\right),$$ *which follows from*

Blundon-Gerretsen's inequality: $\dfrac{r(4R+r)^2}{R+r} \le s^2 \le \dfrac{R(4R+r)^2}{2(2R-r)}$

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum \dfrac{h_a}{h_b h_c}$ *and* $\sum \dfrac{r_a}{r_b r_c}$ *the following relationship holds:*

7) In ΔABC:

$$\boldsymbol{\sum \frac{h_a}{h_b h_c} \ge \left(\frac{2r}{R}\right)^2 \sum \frac{r_a}{r_b r_c}}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the identities **2)** and **6)** the inequality can be written:*

$$\frac{s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2}{8s^2r^2R} \ge \left(\frac{2r}{R}\right)^2 \cdot \frac{(4R+r)^2-2s^2}{rs^2} \Leftrightarrow$$

$$\Leftrightarrow s^2[Rs^2+2r(32r^2+Rr-4R^2)] \ge r^2(4R+r)^2(32r-R),$$

which follows from Gerretsen's inequality $s^2 \ge 16Rr - 5r^2$.

It remains to prove that:

$$(16Rr-5r^2)[R(16Rr-5r^2)+2r(32r^2+Rr-4R^2)] \ge r^2(4R+r)^2(32r-R) \Leftrightarrow$$

$$\Leftrightarrow 9R^3 - 37R^2r + 49Rr^2 - 22r^3 \ge 0 \Leftrightarrow (R-2r)(9R^2-19Rr+11r^2) \ge 0$$

obviously from Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

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MARIN CHIRCIU

1) In ΔABC the following relationship holds:

$$\frac{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}{\tan^{n+2}\frac{A}{2}+\tan^{n+2}\frac{B}{2}}+\frac{\tan^n\frac{B}{2}+\tan^n\frac{C}{2}}{\tan^{n+2}\frac{B}{2}+\tan^{n+2}\frac{C}{2}}+\frac{\tan^n\frac{C}{2}+\tan^n\frac{A}{2}}{\tan^{n+2}\frac{C}{2}+\tan^{n+2}\frac{A}{2}}\leq$$
$$\leq 1+\frac{4R}{r}; n\in\mathbb{N}, n\geq 1.$$

Proposed by Marian Ursărescu - Romania

Proof.

We prove the following lemma:

Lemma.
2) If $x, y > 0$ then:

$$\frac{x^n+y^n}{x^{n+2}+y^{n+2}}\leq\frac{1}{xy}$$

Proof.

Inequality is equivalent with $(x-y)(x^{n+1}-y^{n+1})\geq 0$, true because the factors $(x-y)$ and $(x^{n+1}-y^{n+1})$ have the same sign.

□

Back to the main problem:
*Using the **Lemma** we obtain:*

$$\sum\frac{\tan^n\frac{B}{2}+\tan^n\frac{C}{2}}{\tan^{n+2}\frac{B}{2}+\tan^{n+2}\frac{C}{2}}\leq\sum\frac{1}{\tan\frac{B}{2}\tan\frac{C}{2}}=\sum\cot\frac{B}{2}\cot\frac{C}{2}=1+\frac{4R}{r}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

In the same class of problems we can propose:

3) In ΔABC the following relationship holds:

$$\frac{\cot^n\frac{A}{2}+\cot^n\frac{B}{2}}{\cot^{n+2}\frac{A}{2}+\cot^{n+2}\frac{B}{2}}+\frac{\cot^n\frac{B}{2}+\cot^n\frac{C}{2}}{\cot^{n+2}\frac{B}{2}+\cot^{n+2}\frac{C}{2}}+\frac{\cot^n\frac{C}{2}+\cot^n\frac{A}{2}}{\cot^{n+2}\frac{C}{2}+\cot^{n+2}\frac{A}{2}}\leq 1; n\in\mathbb{N}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma** we obtain:*

$$\sum \frac{\cot^n \frac{B}{2} + \cot^n \frac{C}{2}}{\cot^{n+2} \frac{B}{2} + \cot^{n+2} \frac{C}{2}} \le \sum \frac{1}{\cot \frac{B}{2} \cot \frac{C}{2}} = \sum \tan \frac{B}{2} \tan \frac{C}{2} = 1$$

Equality holds if and only if the triangle is equilateral.

□

4) In ΔABC the following relationship holds:

$$\frac{\sin^n \frac{A}{2} + \sin^n \frac{B}{2}}{\sin^{n+2} \frac{A}{2} + \sin^{n+2} \frac{B}{2}} + \frac{\sin^n \frac{B}{2} + \sin^n \frac{C}{2}}{\sin^{n+2} \frac{B}{2} + \sin^{n+2} \frac{C}{2}} + \frac{\sin^n \frac{C}{2} + \sin^n \frac{A}{2}}{\sin^{n+2} \frac{C}{2} + \sin^{n+2} \frac{A}{2}} \le \frac{6R}{r}; n \in \mathbb{N}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma** we obtain:*

$$\sum \frac{\sin^n \frac{B}{2} + \sin^n \frac{C}{2}}{\sin^{n+2} \frac{B}{2} + \sin^{n+2} \frac{C}{2}} \le \sum \frac{1}{\sin \frac{B}{2} \sin \frac{C}{2}} = \frac{\sum \sin \frac{A}{2}}{\prod \sin \frac{A}{2}} \le \frac{\frac{3}{2}}{\frac{r}{4R}} = \frac{6R}{r},$$

which follows from:

$$\sum \sin \frac{A}{2} \le \frac{3}{2} \text{ and } \prod \sin \frac{A}{2} = \frac{r}{4R}$$

Equality holds if and only if the triangle is equilateral.

□

5) In ΔABC the following relationship holds:

$$\frac{\sin^n \frac{A}{2} + \sin^n \frac{B}{2}}{\sin^{n+2} \frac{A}{2} + \sin^{n+2} \frac{B}{2}} + \frac{\sin^n \frac{B}{2} + \sin^n \frac{C}{2}}{\sin^{n+2} \frac{B}{2} + \sin^{n+2} \frac{C}{2}} + \frac{\sin^n \frac{C}{2} + \sin^n \frac{A}{2}}{\sin^{n+2} \frac{C}{2} + \sin^{n+2} \frac{A}{2}} \le \frac{6R}{r}; n \in \mathbb{N}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma** we obtain:*

$$\sum \frac{\cos^n \frac{B}{2} + \cos^n \frac{C}{2}}{\cos^{n+2} \frac{B}{2} + \cos^{n+2} \frac{C}{2}} \le \sum \frac{1}{\cos \frac{B}{2} \cos \frac{C}{2}} = \frac{\sum \cos \frac{A}{2}}{\prod \cos \frac{A}{2}} \le \frac{\frac{3\sqrt{3}}{2}}{\frac{s}{4R}} = \frac{6\sqrt{3}R}{s}$$

which follows from

$$\sum \cos \frac{A}{2} \le \frac{3\sqrt{3}}{2} \text{ and } \prod \cos \frac{A}{2} = \frac{s}{4R}$$

Equality holds if and only if the triangle is equilateral.

□

6) In ΔABC the following relationship holds:

$$\frac{a^n + b^n}{a^{n+2} + b^{n+2}} + \frac{b^n + c^n}{b^{n+2} + c^{n+2}} + \frac{c^n + a^n}{c^{n+2} + a^{n+2}} \le \frac{1}{2Rr}; n \in \mathbb{N}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the ***Lemma*** *we obtain:*

$$\sum \frac{b^n+c^n}{b^{n+2}+c^{n+2}} \le \sum \frac{1}{bc} = \frac{\sum a}{\prod a} \le \frac{2s}{4Rrs} = \frac{1}{2Rr}$$

Equality holds if and only if the triangle is equilateral.

□

7) In ΔABC the following relationship holds:

$$\frac{h_a^n+h_b^n}{h_a^{n+2}+h_b^{n+2}} + \frac{h_b^n+h_c^n}{h_b^{n+2}+h_c^{n+2}} + \frac{h_c^n+h_a^n}{h_c^{n+2}+h_a^{n+2}} \le \frac{1}{3r^2}; n \in \mathbb{N}.$$

Proposed by Marin Chirciu - Romania

Proof.

Using the ***Lemma*** *we obtain:*

$$\sum \frac{h_b^n+h_c^n}{h_b^{n+2}+h_c^{n+2}} \le \sum \frac{1}{h_bh_c} = \frac{\sum h_a}{\prod h_a} = \frac{\frac{s^2+r^2+4Rr}{2R}}{\frac{2s^2r^2}{R}} = \frac{s^2+r^2+4Rr}{4s^2r^2}$$

which follows from:

$$\sum h_a = \frac{s^2+r^2+4Rr}{2R} \text{ and } \prod h_a = \frac{2s^2r^2}{R}$$

It remains to prove that $\frac{s^2+r^2+4Rr}{4s^2r^2} \le \frac{1}{3r^2} \Leftrightarrow s^2 \ge 12Rr+3r^2$, *true from*

Gerretsen's inequality $s^2 \ge 16Rr-5r^2$ *and Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

8) In ΔABC the following relationship holds:

$$\frac{r_a^n+r_b^n}{r_a^{n+2}+r_b^{n+2}} + \frac{r_b^n+r_c^n}{r_b^{n+2}+r_c^{n+2}} + \frac{r_c^n+r_a^n}{r_c^{n+2}+r_a^{n+2}} \le \frac{1}{3r^2}; n \in \mathbb{N}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the ***Lemma*** *we obtain:*

$$\sum \frac{r_b^n+r_c^n}{r_b^{n+2}+r_c^{n+2}} \le \sum \frac{1}{r_br_c} = \frac{\sum r_a}{\prod r_a} = \frac{4R+r}{s^2r},$$

which follows from

$$\sum h_a = 4R+r \text{ and } \prod r_a = s^2r.$$

It remains to prove that $\frac{4R+r}{s^2r} \le \frac{1}{3r^2} \Leftrightarrow s^2 \ge 12Rr+3r^2$, *true from*

Gerretsen's inequality $s^2 \ge 16Rr-5r^2$ *and Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

9) In ΔABC the following relationship holds:

$$\frac{m_a^n+m_b^n}{m_a^{n+2}+m_b^{n+2}}+\frac{m_b^n+m_c^n}{m_b^{n+2}+m_c^{n+2}}+\frac{m_c^n+m_a^n}{m_c^{n+2}+m_a^{n+2}}\le\frac{1}{3r^2};n\in\mathbb{N}.$$

Proposed by Marin Chirciu - Romania

Proof.

Using the ***Lemma*** *we obtain:*

$$\sum\frac{m_b^n+m_c^n}{m_b^{n+2}+m_c^{n+2}}\le\sum\frac{1}{m_bm_c}=\frac{\sum m_a}{\prod m_a}\le\frac{4R+r}{s^2r},$$

which follows from

$$\sum m_a\le 4R+r \text{ and } \prod r_a=s^2r.$$

It remains to prove that $\frac{4R+r}{s^2r}\le\frac{1}{3r^2}\Leftrightarrow s^2\ge 12Rr+3r^2$, *true from Gerretsen's inequality* $s^2\ge 16Rr-5r^2$ *and Euler's inequality* $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

10) In ΔABC the following relationship holds:

$$\frac{w_a^n+w_b^n}{w_a^{n+2}+w_b^{n+2}}+\frac{w_b^n+w_c^n}{w_b^{n+2}+w_c^{n+2}}+\frac{w_c^n+w_a^n}{w_c^{n+2}+w_a^{n+2}}\le\frac{4R+r}{27r^3};n\in\mathbb{N}.$$

Proposed by Marin Chirciu - Romania

Proof.

Using the ***Lemma*** *we obtain:*

$$\sum\frac{w_b^n+w_c^n}{w_b^{n+2}+w_c^{n+2}}\le\sum\frac{1}{w_bw_c}=\frac{\sum w_a}{\prod w_a}\le\frac{4R+r}{27r^3}$$

which follows from

$$\sum w_a\le\sum m_a\le 4R+r \text{ and } \prod w_a\ge 27r^3.$$

Equality holds if and only if the triangle is equilateral.

□

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MARIN CHIRCIU

1) In ΔABC:

$$(a+b+c)(a^2+b^2+c^2) \geq 2\sum(b+c)h_a^2$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

Proof.

We prove the following lemmas:

Lemma 1.
2) In ΔABC:

$$\sum(b+c)h_a^2 = \frac{s[s^4+s^2(2r^2-10Rr)+r^2(4R+r)(2R+r)]}{2R^2}$$

Proof.

Using $h_a = \dfrac{2S}{a}$, $\sum b^2c^2(b+c) = 2s[s^4+s^2(2r^2-10Rr)+r^2(4R+r)(2R+r)]$ *and*

$$\sum \frac{b+c}{a^2} = \frac{s^4+s^2(2r^2-10Rr)+r^2(4R+r)(2R+r)}{8sr^2R^2}$$

□

We prove the following lemma:

Lemma 2.
3) In ΔABC:

$$s^4-2s^2(2R^2+10Rr-r^2)+r(4R+r)^3 \leq 0$$

Proof.

Using Blundon's inequality:

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr} \leq s^2 \leq 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}$$

□

Back to the main problem:

Using the ***Lemma 1*** *we write the inequality:*

$$2s\cdot 2(s^2-r^2-4Rr)\geq 2\cdot\frac{s[s^4+s^2(2r^2-10Rr)+r^2(4R+r)(2R+r)]}{2R^2}\Leftrightarrow$$

$$\Leftrightarrow s^4-2s^2(2R^2+5Rr-r^2)+r(16R^3+12R^2r+6Rr^2+r^3)\leq 0$$

Using ***Lemma 2*** *it suffices to prove that:*

$$2s^2(2R^2+10Rr-r^2)-r(4R+r)^3\leq 2s^2(2R^2+5Rr-r^2)-r(16R^3+12R^2r+6Rr^2+r^3)\Leftrightarrow$$

$\Leftrightarrow 5s^2\leq 24R^2+18Rr+3r^2$, *which follows from Gerretsen's inequality:*

$s^2\leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$5(4R^2+4Rr+3r^2)\leq 24R^2+18Rr+3r^2\Leftrightarrow 2R^2-Rr-6r^2\geq 0\Leftrightarrow(R-2r)(2R+3r)\geq 0$$

obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

For $\sum(b+c)h_a^2$ *the double inequality can be written:*

4) In ΔABC:

$$\mathbf{36sr^2\leq\sum(b+c)h_a^2\leq 9sR^2}$$

Proof.

See ***1), 2)*** *Gerretsen's inequality:* $16Rr-5r^2\leq s^2\leq 4R^2+4Rr+3r^2$

and Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Replacing h_a *with* r_a *we propose:*

5) In ΔABC:

$$\mathbf{(a+b+c)(a^2+b^2+c^2)\leq 2\sum(b+c)r_a^2}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 3.
6) In ΔABC:

$$\mathbf{\sum(b+c)r_a^2=2s(8R^2+6Rr+r^2-s^2)}$$

Proof.

Using $r_a = \dfrac{S}{s-a}, \sum (b+c)(s-b)^2(s-c)^2 = 2sr^2(8R^2+6Rr+r^2-s^2)$ *and*

$$\sum \frac{b+c}{(s-a)^2} = \frac{2(8R^2+6Rr+r^2-s^2)}{sr^2}$$

□

Back to the main problem.

Using ***Lemma 1*** *the inequality can be written:*

$2s \cdot 2(s^2-r^2-4Rr) \le 2 \cdot 2s(8R^2+6Rr+r^2-s^2) \Leftrightarrow s^2 \le 4R^2+5Rr+r^2,$

which follows from Gerretsen's inequality: $s^2 \le 4R^2+4Rr+3r^2$ *and Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

For $\sum (b+c)r_a^2$ *the double inequality can be written:*

7) In ΔABC:

$$\boldsymbol{4s(2R^2+r^2) \le \sum (b+c)r_a^2 \le 4s(4R^2-7r^2)}$$

Proof.

See ***6)****, Gerretsen's inequality:* $16Rr-5r^2 \le s^2 \le 4R^2+4Rr+3r^2$

and Euler's inequality $R \ge 2r$

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum (b+c)h_a^2$ *and* $\sum (b+c)r_a^2$ *the following relationship holds:*

8) In ΔABC:

$$\boldsymbol{\sum (b+c)h_a^2 \le \sum (b+c)r_a^2.}$$

Proof.

See inequalities ***1)*** *and* ***4)****.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality ***6)*** *can be refined:*

9) In ΔABC:

$$\boldsymbol{\sum (b+c)h_a^2 \le \frac{1}{2}(a+b+c)(a^2+b^2+c^2) \le \sum (b+c)r_a^2}$$

Proof.

*See inequalities **1)** and **4)**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the sequence of inequalities:

10) In ΔABC:

$$36sr^2 \leq \sum(b+c)h_a^2 \leq \frac{1}{2}(a+b+c)(a^2+b^2+c^2) \leq \sum(b+c)r_a^2 \leq 4s(4R^2-7r^2)$$

Proof.

*See inequalities **4)**, **9)**, and **7)**.*

Equality holds if and only if the triangle is equilateral.

□

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PROBLEM JP.152
ROMANIAN MATHEMATICAL MAGAZINE
2018

MARIN CHIRCIU

1) In ΔABC

$$\sum \frac{h_a r_a}{l_a^2} \geq 3$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC

$$\sum \frac{h_a r_a}{l_a^2} = \frac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr}$$

Proof.

Using $h_a = \dfrac{2S}{a}, r_a = \dfrac{S}{s-a}$ *and* $l_a = \dfrac{2bc}{b+c}\cos\dfrac{A}{2}$ *we obtain:*

$$\sum \frac{h_a r_a}{l_a^2} = \sum \frac{\frac{2S}{a} \cdot \frac{S}{s-a}}{\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)^2} = \frac{r}{8R}\sum \frac{(b+c)^2}{(s-a)^2} = \frac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr}$$

which follows from: $\displaystyle\sum \frac{(b+c)^2}{(s-a)^2} = \frac{2(8R^2 + 8Rr + 3r^2 - s^2)}{r^2}$

□

Let's get back to the main problem:

Using the ***Lemma*** *we write the inequality:*

$\dfrac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr} \geq 3 \Leftrightarrow s^2 \leq 8R^2 - 4Rr + 3r^2$ *which follows from Gerretsen's inequality:* $s^2 \leq 4R^2 + 4Rr + 3r^2$.

It remains to prove that: $4R^2 + 4Rr + 3r^2 \leq 8R^2 - 4Rr + 3r^2 \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

The inequality can be strengthened:

3) In ΔABC:

$$\sum \frac{h_a r_a}{l_a^2} \geq 2 + \frac{R}{2r}.$$

Proof.

*Using the **Lemma** we write the inequality:*

$$\frac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr} \geq \frac{R}{2r} + 2 \Leftrightarrow s^2 \leq 6R^2 + 3r^2$$

which follows from Blundon-Gerretsen inequality: $s^2 \leq \dfrac{R(4R+r)^2}{2(2R-r)}$

It remains to prove that: $\dfrac{R(4R+r)^2}{2(2R-r)} \leq 6R^2 + 3r^2 \Leftrightarrow 8R^3 - 20R^2r + 11Rr^2 - 6r^3 \geq 0 \Leftrightarrow$

$\Leftrightarrow (R-2r)(8R^2 - 4r + 3r^2) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality 3) is stronger than inequality 1):

4) In ΔABC

$$\sum \frac{h_a r_a}{l_a^2} \geq 2 + \frac{R}{2r} \geq 3.$$

Proof.

See inequality 3) and Euler's inequality $R \geq 2r$.

□

Remark.

Let's emphasise an inequality having an opposite sense:

5) In ΔABC :

$$\sum \frac{h_a r_a}{l_a^2} \leq 2\left(\frac{R}{r} + \frac{r}{R} - 1\right)$$

Proof.

*Using the **Lemma** the inequality can be written:*

$\dfrac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr} \leq 2\left(\dfrac{R}{r} + \dfrac{r}{R} - 1\right) \Leftrightarrow s^2 \geq 16Rr - 5r^2$ *(Gerretsen's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

6) In ΔABC

$$2+\frac{R}{2r}\le\sum\frac{h_ar_a}{l_a^2}\le 2\left(\frac{R}{r}+\frac{r}{R}-1\right)$$

Proposed by Marin Chirciu - Romania

Proof.

See inequalities 3) and 5).

Equality holds if and only if the triangle is equilateral.

□

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PROBLEM JP.159
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MARIN CHIRCIU

1) In ΔABC

$$\sum a^2 h_b h_c \le 4(R+r)^4$$

Proposed by Marian Ursărescu - Romania

Proof.

We prove the following lemma:

Lemma.
In ΔABC:

$$\sum a^2 h_b h_c = \frac{2r}{R} s^2(s^2 - 3r^2 - 6Rr)$$

Proof.

Using $h_a = \frac{2S}{a}$, *we obtain:*

$$\sum a^2 h_b h_c = \sum a^2 \cdot \frac{2S}{b} \cdot \frac{2S}{c} = 4S^2 \sum \frac{a^2}{bc} = \frac{2r}{R} s^2(s^2 - 3r^2 - 6Rr)$$

which follows from: $\sum \frac{a^2}{bc} = \frac{s^2 - 3r^2 - 6Rr}{2Rr}$.

□

Let's get back to the main problem:

Using the ***Lemma*** *the inequality can be written:*

$$\frac{2r}{R} s^2(s^2 - 3r^2 - 6Rr) \le 4(R+r)^4 \Leftrightarrow s^2(s^2 - 3r^2 - 6Rr) \le \frac{2R}{r}(R+r)^4$$

We have: $s^2(s^2 - 3r^2 - 6Rr) = s^4 - s^2(3r^2 + 6Rr)$ *and we use 1) and 2):*

1): $s^4 \le s^2(4R^2 + 20Rr - 2r^2) - r(4R+r)^3$, *ture from:*

$$2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2 - 2Rr} \le s^2 \le 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr},$$

Bloundon-Rouche's inequality,

2): Blundon-Gerretsen: $s^2 \le \frac{R(4R+r)^2}{2(2R-r)}$

We obtain:

$$s^2(s^2 - 3r^2 - 6Rr) = s^4 - s^2(3r^2 + 6Rr) \le s^2(4R^2 + 20Rr - 2r^2) - r(4R+r)^3 - s^2(3r^2 + 6Rr) =$$

$$= s^2(4R^2 + 14Rr - 5r^2) - r(4R+r)^3 \le \frac{R(4R+r)^2}{2(2R-r)}(4R^2 + 14Rr - 5r^2) - r(4R+r)^3 =$$

$$= (4R+r)^2\frac{4R^3-2R^2r-Rr^2+2r^3}{2(2R-r)}.$$

It remains to prove that:

$$(4R+r)^2\frac{4R^3-2R^2r-Rr^2+2r^3}{2(Rr-r)} \leq \frac{2R}{r}(R+r)^4 \Leftrightarrow$$

$$\Leftrightarrow 8R^6-36R^5r+32R^4r^2+36R^3r^3-30R^2r^4-19Rr^5-2r^6 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(8R^5-20R^4r-8R^3r^2+20R^2r^3+10Rr^4+r^5) \geq 0$, *obviously from Euler* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasise an inequality having an opposite sense:

3) In ΔABC:

$$\boldsymbol{\sum a^2h_bh_c \geq 324r^4}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma** we write the inequality:*

$\frac{2r}{R}s^2(s^2-3r^2-6Rr) \geq 324r^4$, *which follows from Gerretsen's inequality:*

$s^2 \geq 16Rr-5r^2$. *It remains to prove that:*

$$\frac{2r}{R}(16Rr-5r^2)(16Rr-5r^2-3r^2-6Rr) \geq 324r^4 \Leftrightarrow 8R^2-17Rr+2r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(8R-r) \geq 0$, *obviously from Euler's inequality:* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

4) In ΔABC:

$$\boldsymbol{324r^4 \leq \sum a^2h_bh_c \leq 4(R+r)^4.}$$

Proof.

See inequalities 1) and 3).

Equality holds if and only if the triangle is equilateral.

□

Remark.

If we replace h_bh_c with r_br_c we propose:

5) In ΔABC:

$$\boldsymbol{12s^2r^2 \leq \sum a^2r_br_c \leq 6s^2Rr}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.

6) In ΔABC:

$$\sum a^2 r_b r_c = 4s^2 r(R+r)$$

Proof.

Using $r_a = \frac{S}{s-a}$, *we obtain:*

$$\sum a^2 r_b r_c = \sum a^2 \cdot \frac{S}{s-b} \cdot \frac{S}{s-c} = S^2 \sum \frac{a^2}{(s-b)(s-c)} = 4s^2 r(R+r)$$

which follows from: $\sum \frac{a^2}{(s-b)(s-c)} = \frac{4(R+r)}{r}$.

□

Let's get back to the main problem:

*Using the **Lemma** the inequality holds:*

$12s^2r^2 \leq 4s^2 r(R+r) \leq 6s^2 Rr \Leftrightarrow 6r \leq 2(R+r) \leq 3R$, *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum a^2 h_b h_c$ *and* $\sum a^2 r_b r_c$ *the following relationship exists:*

7) In acute-angled ΔABC:

$$\sum a^2 r_b r_c \leq \sum a^2 h_b h_c$$

Proposed by Marin Chirciu - Romania

Proof.

Using the identities 2) and 6) we write the inequality:

$4s^2 r(R+r) \leq \frac{2r}{R} s^2(s^2 - 3r^2 - 6Rr) \Leftrightarrow s^2 \geq 2R^2 + 8Rr + 3r^2$, *(Walker's inequality).*

true only for the acute-angled triangle.

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the sequence of inequalities:

1) In acute-angled ΔABC:

$$324r^4 \leq 12S^2 \leq \sum a^2 r_b r_c \leq \sum a^2 h_b h_c \leq 4(R+r)^4.$$

Proof.

See inequalities 1), 5), 7) and Mitrinovič's inequality $s^2 \geq 27r^2$.

Equality holds if and only if the triangle is equilateral.

□

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INEQUALITY IN TRIANGLE 946 ROMANIAN MATHEMATICAL MAGAZINE 2018

MARIN CHIRCIU

1) In ΔABC:

$$4\sum m_b m_c - 4R\sum \frac{h_b h_c}{h_a} \leq s^2 + r(4R+r)$$

Proposed by Bogdan Fustei - Romania

Proof.

We prove the following lemmas:

Lemma 1.
2) In ΔABC:

$$\sum \frac{h_b h_c}{h_a} = \frac{s^2 - r(4R+r)}{R}$$

Proof.

Using $h_a = \frac{2S}{a}$, *we obtain:*

$$\sum \frac{h_b h_c}{h_a} = \sum \frac{\frac{2S}{b} \cdot \frac{2S}{c}}{\frac{2S}{a}} = 2S\sum \frac{a}{bc} = 2rs \cdot \frac{s^2 - r(4R+r)}{2Rrs} = \frac{s^2 - r(4R+r)}{R}$$

which follows from: $\sum \frac{a}{bc} = \frac{s^2 - r(4R+r)}{2Rrs}$

□

Lemma 2.
3) In ΔABC:

$$\sum m_b m_c \leq \frac{5s^2 - 3r(4R+r)}{4}.$$

Proof.

Using $4m_b m_c \leq 2a^2 + bc$, $\sum a^2 = 2s^2 - 2r(4R+r)$ *and* $\sum bc = s^2 + r(4R+r)$ *we obtain:*

$$4\sum m_b m_c \leq 4\sum(2a^2 + bc) = 8\sum a^2 + 4\sum bc = 5s^2 - 3r(4R+r).$$

□

Back to the main problem.

Using ***Lemma 1*** *and* ***Lemma 2*** *it suffices to prove that:*

$$5s^2-3r(4R+r)-4R\cdot\frac{s^2-r(4R+r)}{R}\le s^2+r(4R+r),\ \textit{obviously with equality.}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

If we replace $\frac{h_bh_c}{h_a}$ *with* $\frac{r_br_c}{r_a}$ *we propose:*

1) In ΔABC:

$$\boldsymbol{4\sum m_bm_c-4r\sum\frac{r_br_c}{r_a}\le s^2+5r(4R+r)}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 3.
2) In ΔABC:

$$\boldsymbol{\sum\frac{r_br_c}{r_a}=\frac{s^2-2r(4R+r)}{R}}$$

Proof.

Using $r_a=\frac{S}{s-a}$, *we obtain:*

$$\sum\frac{r_br_c}{r_a}=\sum\frac{\frac{S}{s-a}\cdot\frac{S}{s-c}}{\frac{S}{s-a}}=S\sum\frac{s-a}{(s-b)(s-c)}=rs\cdot\frac{s^2-2r(4R+r)}{r^2s}=\frac{s^2-2r(4R+r)}{r},$$

which follows from: $\sum\frac{(s-b)(s-c)}{s-a}=\frac{s^2-2r(4R+r)}{r^2s}$

□

Back to the main problem:

□

Using ***Lemma 1*** *and* ***Lemma 3*** *it suffices to prove that:*

$$5s^2-3r(4R+r)-4r\cdot\frac{s^2-2r(4R+r)}{r}\le s^2+5r(4R+r),\ \textit{obviously, with equality.}$$

Equality holds if and only if the triangle is equilateral.

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INEQUALITY IN TRIANGLE 914
ROMANIAN MATHEMATICAL MAGAZINE
2018

MARIN CHIRCIU

1. In ΔABC

$$\sum \frac{a^3}{h_b+h_c} \geq 2Rs$$

Proposed by Seyran Ibrahimov - Azerbaijan

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC:

$$\sum \frac{a^3}{h_b+h_c} = \frac{4Rs(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}$$

Proof.

Using $h_a = \frac{2S}{a}$ *we obtain:*

$$\sum \frac{a^3}{h_b+h_c} = \frac{\sum a^3(h_a+h_b)(h_a+h_c)}{\prod(h_b+h_c)} = \frac{4Rs(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}$$

which follows from:

$$\sum a^3(h_a+h_b)(h_a+h_c) = \frac{4rs^3(s^2-3r^2-4Rr)}{R} \text{ and } \prod(h_b+h_c) = \frac{rs^2(s^2+r^2+2Rr)}{R^2}$$

□

Let's get back to the main problem:

Using the ***Lemma*** *the inequality can be written:*

$\frac{4Rs(s^2-3r^2-4Rr)}{s^2+r^2+2Rr} \geq 2Rs \Leftrightarrow s^2 \geq 10Rr+7r^2$, *which follows from Gerretsen's inequlaity:*

$s^2 \geq 16Rr-5r^2$. *It remains to prove that:*

$16Rr-5r^2 \geq 16Rr-5r^2 \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral. □

Remark.

Let's emphasise an inequality having an opposite sense:

3) In ΔABC:

$$\sum \frac{a^3}{h_b+h_c} \leq \frac{R^2s}{r}$$

Proof.

Using the ***Lemma*** *the inequality can be written:*

$$\frac{4Rs(s^2-3r^2-4Rr)}{s^2+r^2+2Rr} \le \frac{R^2s}{r} \Leftrightarrow s^2(R-4r)+r(2R^2+17Rr+12r^2) \ge 0$$

We distinguish the following cases:

Case 1). If $(R-4r) \ge 0$, the inequality is obvious.

Case 2) If $(R-4r) < 0$, the inequality can be rewritten:

$r(2R^2+17Rr+12r^2) \ge s^2(4r-R)$, *which follows from Gerretsen's inequality:*

$$s^2 \le 4R^2+4Rr+3r^2$$

It remains to prove that:

$$r(2R^2+17Rr+12r^2) \ge (4R^2+4Rr+3r^2)(4r-R) \Leftrightarrow 2R^2-5Rr+2r^2 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(2R-r) \ge 0$, *obviously from Euler's inequality $R \ge 2r$.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

4) In ΔABC:

$$\boldsymbol{2Rs \le \sum \frac{a^3}{h_b+h_c} \le \frac{R^2s}{r}}$$

Proposed by Marin Chirciu - Romania

Proof.

See inequalities 1) and 3).

Equality holds if and only if the triangle is equilateral.

□

Remark.

If we interchange h_a with r_a we propose:

5) In ΔABC:

$$\boldsymbol{2Rs \le \sum \frac{a^3}{r_b+r_c} \le \frac{R^2s}{r}}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.

6) In ΔABC:

$$\boldsymbol{\sum \frac{a^3}{r_b+r_c} = 4s(R-r).}$$

Proof.

Using $r_a = \dfrac{S}{s-a}$ *we obtain:*

$$\sum \frac{a^3}{r_b + r_c} = \frac{\sum a^3(r_a + r_b)(r_a + r_c)}{\prod (r_b + r_c)} = 4s(R-r), \text{ which follows from:}$$

$$\sum a^3(r_a + r_b)(r_a + r_c) = 16s^3R(R-r) \text{ and } \prod (r_b + r_c) = 4s^2R.$$

□

Let's get back to the main problem.

Using the ***Lemma*** *the inequality can be written:*

$$2Rs \le 4s(R-r) \le \frac{R^2 s}{r} \Leftrightarrow 2Rr \le 4r(R-r) \le R^2, \text{ which follows from Euler's inequality: } R \ge 2r.$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum \dfrac{h_a}{r_b + r_c}$ *and* $\sum \dfrac{r_a}{h_b + h_c}$ *we can write the following relationship:*

7) In ΔABC:

$$\sum \frac{a^3}{h_b + h_c} \le \sum \frac{a^3}{r_b + r_c}$$

Proposed by Marin Chirciu - Romania

Proof.

Using ***Lemma 2)*** *and* ***Lemma 3)*** *we write the following inequality:*

$$\frac{4Rs(s^2 - 3r^2 - 4Rr)}{s^2 + r^2 + 2Rr} \le 4s(R-r) \Leftrightarrow s^2 \le 6R^2 + 2Rr - r^2, \text{ which follows from}$$

Gerretsen's inequality $s^2 \le 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$$4R^2 + 4Rr + 3r^2 \le 6R^2 + 2Rr - r^2 \Leftrightarrow R^2 - Rr - 2r^2 \ge 0 \Leftrightarrow (R-2r)(R+r) \ge 0,$$

obviously, from Euler's inequality: $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the sequence of inequalities:

1) In ΔABC:

$$2Rs \le \sum \frac{a^3}{h_b + h_c} \le \sum \frac{a^3}{r_b + r_c} \le \frac{R^2 s}{r}.$$

Proof.

See inequalities ***1), 7)*** *and* ***5).***

Equality holds if and only if the triangle is equilateral.

□

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INEQUALITY IN TRIANGLE 939
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2018

MARIN CHIRCIU

1) In ΔABC:

$$\frac{32}{27R^2r} \leq \prod\left(\frac{1}{r_b}+\frac{1}{r_c}\right) \leq \frac{4R}{27r^4}$$

Proposed by Adil Abdullayev - Baku - Azerbaijan

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC:

$$\prod\left(\frac{1}{r_b}+\frac{1}{r_c}\right) = \frac{4R}{s^2r^2}$$

Proof.

Using $r_a = \frac{S}{s-a}$, *we obtain:*

$$\prod\left(\frac{1}{r_b}+\frac{1}{r_c}\right) = \prod\left(\frac{s-b}{S}+\frac{s-c}{S}\right) = \prod\frac{a}{S} = \frac{abc}{S^3} = \frac{4Rrs}{s^3r^3} = \frac{4R}{s^2r^2}$$

□

Back to the main problem:

*Using the **Lemma** we write the inequality:*

$\frac{32}{27R^2r} \leq \frac{4R}{s^2r^2} \leq \frac{4R}{27r^4}$, *which follows from Mitrinovič's inequality:* $27r^2 \leq s^2 \leq \frac{27R^2}{4}$

and Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

The inequality can be strengthened:

3) In ΔABC:

$$\frac{16}{27Rr^2} \leq \prod\left(\frac{1}{r_b}+\frac{1}{r_c}\right) \leq \frac{8}{27r^3}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma** we write the inequality:*

$$\frac{16}{27Rr^2} \leq \frac{4R}{s^2r^2} \leq \frac{8}{27r^3}, \text{ which follows from Mitrinovič's inequality:}$$

$$27r^2 \leq s^2 \leq \frac{27R^2}{4}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **3)** is stronger than inequality **1)**.*

4) In ΔABC:

$$\frac{32}{27R^2r} \leq \frac{16}{27Rr^2} \leq \prod\left(\frac{1}{r_b} + \frac{1}{r_c}\right) \leq \frac{8}{27r^3} \leq \frac{4R}{27r^4}$$

Proof.

See inequality 3) and Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

If we replace $\frac{1}{r_b} + \frac{1}{r_c}$ with $\frac{1}{h_b} + \frac{1}{h_c}$ we propose:

5) In ΔABC

$$\frac{16}{27Rr^2} \leq \prod\left(\frac{1}{h_b} + \frac{1}{h_c}\right) \leq \frac{8}{27r^3}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.
6) In ΔABC:

$$\prod\left(\frac{1}{h_b} + \frac{1}{h_c}\right) = \frac{s^2 + r^2 + 2Rr}{4s^2r^3}$$

Proposed by Marin Chirciu - Romania

Using $h_a = \frac{2S}{a}$, we obtain:

$$\prod\left(\frac{1}{h_b}+\frac{1}{h_c}\right) = \prod\left(\frac{b}{2S}+\frac{c}{2S}\right) = \prod \frac{b+c}{2S} = \frac{\prod(b+c)}{8S^3} = \frac{2s(s^2+r^2+2Rr)}{8s^3r^3} = \frac{s^2+r^2+2Rr}{4s^2r^3}$$

□

Let's get back to the main problem:

Using the ***Lemma*** *we write the inequality:* $\frac{16}{27Rr^2} \le \frac{s^2+r^2+2Rr}{4s^2r^3} \le \frac{8}{27r^3}$

The left hand inequality:

$$\frac{16}{27Rr^2} \le \frac{s^2+r^2+2Rr}{4s^2r^3} \Leftrightarrow s^2(27R-64r)+27Rr(2R+r) \ge 0$$

We distinguish the following cases:

Case 1). If $(27R-64r) \ge 0$*, the inequality is obvious.*

Case 2). If $(27R-64r) < 0$*, the inequality can be rewritten:*

$27Rr(2R+r) \ge s^2(64r-27R)$*, which follows from Gerretsen's inequality:*

$s^2 \le 4R^2+4Rr+3r^2$*. It remains to prove that:*

$27Rr(2R+r) \ge (4R^2+4Rr+3r^2)(64r-27R) \Leftrightarrow 54R^3-47R^2r-148Rr^2-96r^3 \ge 0 \Leftrightarrow$

$\Leftrightarrow (R-2r)(54R^2+61Rr+48r^2) \ge 0$*, obviously from Euler's inequality* $R \ge 2r$*.*

Equality holds if and only if the triangle is equilateral.

The right hand inequality:

$\frac{s^2+r^2+2Rr}{4s^2r^3} \le \frac{8}{27r^3} \Leftrightarrow 5s^2 \ge 54Rr+27r^2$*, which follows from Gerretsen's inequality:* $s^2 \ge 16Rr-5r^2$*. It remains to prove that:*

$5(16Rr-5r^2) \ge 54Rr+27r^2 \Leftrightarrow R \ge 2r$ *(Eulers's inequality).*

Equality holds if and only if the triangle is equilateral.

Remark.

Between the products $\prod\left(\frac{1}{r_b}+\frac{1}{r_c}\right)$ *and* $\prod\left(\frac{1}{h_b}+\frac{1}{h_c}\right)$ *the following relationship holds:*

7) In ΔABC:

$$\prod\left(\frac{1}{r_b}+\frac{1}{r_c}\right) \le \prod\left(\frac{1}{h_b}+\frac{1}{h_c}\right)$$

Proposed by Marin Chirciu - Romania

Proof.

Using identities 2) and 6) the inequality can be written:

$\frac{4R}{s^2r^2} \le \frac{s^2+r^2+2Rr}{4s^2r^3} \Leftrightarrow s^2 \ge 14Rr-r^2$*, which follows from Gerretsen's inequality* $s^2 \ge 16Rr-5r^2$*. It remains to prove that:*

$16Rr-5r^2 \ge 14Rr-r^2 \Leftrightarrow R \ge 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

The following sequence of inequalities can be written:

8) In ΔABC:

$$\frac{16}{27Rr^2} \le \prod\left(\frac{1}{r_b}+\frac{1}{r_c}\right) \le \prod\left(\frac{1}{h_b}+\frac{1}{h_c}\right) \le \frac{8}{27r^3}$$

Proposed by Marin Chirciu - Romania

Proof.

See inequalities 3), 5) and 7).

Equality holds if and only if the triangle is equilateral.

□

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INEQUALITY IN TRIANGLE 949
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MARIN CHIRCIU

1) In ΔABC:

$$\sum(r_b - r_c)^2 \leq \frac{3s^2(R-2r)}{r}$$

Proposed by Adil Abdullayev - Baku - Azerbaijan

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC:

$$\sum(r_b - r_c)^2 = 2(4R+r)^2 - 6s^2.$$

Proof.

Using $r_a = \frac{S}{s-a}$, *we obtain:*

$$\sum(r_b - r_c)^2 = 2\sum r_a^2 - 2\sum r_b r_c = 2\Big(\sum r_a\Big)^2 - 6\sum r_b r_c = 2(4R+r)^2 - 6s^2$$

which follows from: $\sum r_a = 4R + r$ *and* $\sum r_b r_c = s^2$.

□

Getting back to the main problem:
*Using the **Lemma** we write the inequality:*

$2(4R+r)^2 - 6s^2 \leq \frac{3s^2(R-2r)}{r} \Leftrightarrow 2r(4R+r)^2 \leq 3s^2R$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r}$. *It remains to prove that:*

$$2r(4R+r)^2 \leq 3R \cdot \frac{r(4R+r)^2}{R+r} \Leftrightarrow R \geq 2r \text{ (Euler's inequality).}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasise an inequality having an opposite sense:

3) In ΔABC:

$$\sum(r_b - r_c)^2 \geq \frac{3s^2(R-2r)}{2R-r}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma** we write the inequality:*

$$2(4R+r)^2 - 6s^2 \geq \frac{3s^2(R-2r)}{2R-r} \Leftrightarrow 2(2R-r)(4R+r)^2 \geq 3s^2(5R-4r)$$

which follows from Blundon-Gerretsen's inequality $s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$

It remains to prove that:

$$2(2R-r)(4R+r)^2 \geq 3 \cdot \frac{R(4R+r)^2}{2(2R-r)}(5R-4r) \Leftrightarrow (R-2r)^2 \geq 0, \text{ obvious.}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

4) In ΔABC:

$$\frac{3s^2(R-2r)}{2R-r} \leq \sum(r_b - r_c)^2 \leq \frac{3s^2(R-2r)}{r}$$

Proof.

*See inequalities **1)** and **3)**.*

Equality holds if and only if the triangle is equilateral.

□

5) In ΔABC:

$$s^2(R-2r)\frac{r^2}{2R^2(R-r)} \leq \sum(h_b - h_c)^2 \leq s^2(R-2r)\frac{4R-3r}{2R^2}.$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.

6) In ΔABC:

$$\sum(h_b - h_c)^2 = \frac{s^4 + s^2(2r^2 - 16Rr) + r^2(4R+r)^2}{2R^2}$$

Proof.

Using $h_a = \dfrac{2S}{a}$, *we obtain:*

$$\sum(h_b-h_c)^2 = 2\sum h_a^2 - 2\sum h_b h_c = 2\Big(\sum h_a\Big)^2 - 6\sum h_b h_c = 2\Big(\frac{s^2+r^2+4Rr}{2R}\Big)^2 - 6\frac{2s^2r}{R} =$$

$$= \frac{s^4+s^2(2r^2-16Rr)+r^2(4R+r)^2}{2R^2}, \text{ which follows from:}$$

$$\sum h_a = \frac{s^2+r^2+4Rr}{2R} \text{ and } \sum h_b h_c = \frac{2s^2r}{R}$$

□

Let's get back to the main problem:

*Using the **Lemma** we write the inequality:*

$$\frac{2s^2(R-2r)(R-r)}{R^2} \le \sum(h_b-h_c)^2 \le s^2(R-2r)\frac{4R-3r}{2R^2}.$$

The left side inequality: $\sum(h_b-h_c)^2 \ge s^2(R-2r)\dfrac{r^2}{2R^2(R-r)}$ *it follows from:*

$$\sum(h_b-h_c)^2 = \frac{s^4+s^2(2r^2-16Rr)+r^2(4R+r)^2}{2R^2} = \frac{s^2(s^2+2r^2-16Rr)+r^2(4R+r)^2}{2R^2} \ge$$

$$\ge \frac{s^2(s^2+2r^2-16Rr)+r^2\cdot 3s^2}{2R^2} = \frac{s^2(s^2+5r^2-16Rr)}{2R^2} \ge$$

$$\ge \frac{s^2(16Rr-5r^2+\frac{r^2(R-2r)}{R-r}+5r^2-16Rr)}{2R^2} = s^2(R-2r)\frac{r^2}{2R^2(R-r)}$$

which follows from Yang Xue Zhi's inequality:

$$s^2 \ge 16Rr-5r^2+\frac{r^2(R-2r)}{R-r}$$

□

Note.
Yang Xue Zhi's inequality:

$$16Rr-5r^2+\frac{r^2(R-2r)}{R-r} \le s^2 \le 4R^2+4Rr+3r^2-\frac{r^2(R-2r)}{R-r},$$

strengthen Gerretsen's inequality:

$$16Rr-5r^2 \le s^2 \le 4R^2+4Rr+3r^2.$$

Right hand inequality: $\sum(h_b-h_c)^2 \le s^2(R-2r)\dfrac{4R-3r}{2R^2}$ *is equivalent with:*

$$\frac{s^4+s^2(2r^2-16Rr)+r^2(4R+r)^2}{2R^2} \le s^2(R-2r)\frac{4R-3r}{2R^2} \Leftrightarrow$$

$\Leftrightarrow r^2(4R+r)^2 \le s^2(4R^2+5Rr+4r^2-s^2)$, *which follows from Gerretsen's inequality:*

$$\frac{r(4R+r)^2}{R+r} \le 16Rr-5r^2 \le s^2 \le 4R^2+4Rr+3r^2.$$

It remains to prove that:

$r^2(4R+r)^2 \le \dfrac{r(4R+r)^2}{R+r}(4R^2+5Rr+4r^2-4R^2-4Rr-3r^2)$, *obviously, with equality.*

Equality holds if and only if the triangle is equilateral.

Remark.

Between the sums $\sum(h_b-h_c)^2=\dfrac{s^4+s^2(2r^2-16Rr)+r^2(4R+r)^2}{2R^2}$ *and*

$\sum(r_b-r_c)^2=2(4R+r)^2-6s^2$ *the following relationship exist:*

7) In ΔABC

$$\boldsymbol{\sum(h_b-h_c)^2\le\sum(r_b-r_c)^2}$$

Proof.

*Using the identities **2)** and **6)** the inequality can be written:*

$$\frac{s^4+s^2(2r^2-16Rr)+r^2(4R+r)^2}{2R^2}\le 2(4R+r)^2-6s^2\Leftrightarrow$$

$\Leftrightarrow s^2(s^2+2r^2-16Rr+12R^2)\le(4R+r)^2(4R^2-r^2)$, *which follows from*

Blundon-Gerretsen's inequality: $s^2\le\dfrac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2$.

It remains to prove that:

$$\frac{R(4R+r)^2}{2(2R-r)}(4R^2+4Rr+3r^2+2r^2-16Rr+12R^2)\le(4R+r)^2(4R^2-r^2)\Leftrightarrow$$

$\Leftrightarrow 4R^2-9Rr+2r^2\ge0\Leftrightarrow(R-2r)(4R-r)\ge0$, *obviously, from Euler's inequality* $R\ge2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the following sequence of inequalities:

8) In ΔABC:

$$\boldsymbol{s^2(R-2r)\cdot\frac{r^2}{2R^2(R-r)}\le\sum(h_b-h_c)^2\le\sum(r_b-r_c)^2\le s^2(R-2r)\cdot\frac{3}{r}}$$

Proof.

*See inequalities **1)**, **5)** and **7)**.*

Equality holds if and only if the triangle is equilateral.

□

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PROBLEM 3647
CRUX MATHEMATICORUM 2012
NR.10/2018

MARIN CHIRCIU

1) In ΔABC

$$\sum \frac{(r_a + r_b)(r_a + r_c)}{bc} \geq 9$$

Proposed by Panagiote Ligouras - Italy

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC

$$\sum \frac{(r_a + r_b)(r_a + r_c)}{bc} = \frac{4R + r}{r}$$

Proof.

Using $r_a = \dfrac{S}{s-a}$ *we obtain:*

$$\sum \frac{(r_a + r_b)(r_a + r_c)}{bc} = \sum \frac{(\frac{S}{s-a} + \frac{S}{s-b})(\frac{S}{s-a} + \frac{S}{s-a})}{bc} = S^2 \sum \frac{\frac{c}{(s-a)(s-b)} \cdot \frac{b}{(s-a)(s-b)}}{bc} =$$

$$= S^2 \sum \frac{1}{(s-a)\prod(s-a)} = \frac{r^2 s^2}{\prod(s-a)} \sum \frac{1}{s-a} = \frac{r^2 s^2}{r^2 s} \cdot \frac{4R + r}{rs} = \frac{4R + r}{r}$$

□

Let's get back to the main problem.

Using the Lemma the inequality can be written: $\dfrac{4R + r}{r} \geq 9 \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having an opposite sense.

3) In ΔABC

$$\sum \frac{(r_a + r_b)(r_a + r_c)}{bc} \leq \frac{9R}{2r}.$$

Proof.

Using the ***Lemma*** *the inequality can be written:* $\frac{4R+r}{r} \leq \frac{9R}{2r} \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

The double inequality can be written:

4) In ΔABC

$$\mathbf{9 \leq \sum \frac{(r_a+r_b)(r_a+r_c)}{bc} \leq \frac{9R}{2r}}$$

Proof.

See inequalities ***1)*** *and* ***3)****.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Replacing h_a with r_a we propose:

5) In ΔABC

$$\mathbf{\frac{4r}{R}\left(4+\frac{r}{R}\right) \leq \sum \frac{(h_a+h_b)(h_a+h_c)}{bc} \leq 2\left(4+\frac{r}{R}\right)}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.
6) In ΔABC

$$\mathbf{\sum \frac{(h_a+h_b)(h_a+h_c)}{bc} = \frac{5s^2+r^2+4Rr}{4R^2}}$$

Proof.

Using $h_a = \frac{2S}{a}$ *we obtain:*

$$\sum \frac{(h_a+h_b)(h_a+h_c)}{bc} = \sum \frac{\left(\frac{2S}{a}+\frac{2S}{b}\right)\left(\frac{2S}{a}+\frac{2S}{c}\right)}{bc} = 4S^2 \sum \frac{\frac{a+b}{ab} \cdot \frac{a+c}{ac}}{bc} =$$

$$= \frac{4S^2}{a^2b^2c^2} \sum (a+b)(a+c) = \frac{4r^2s^2}{16R^2r^2s^2} \cdot (5s^2+r^2+4Rr) = \frac{5s^2+r^2+4Rr}{4R^2}$$

which follows from $\sum (a+b)(a+c) = 5s^2+r^2+4Rr.$

□

Let's get back to the main problem.

Left hand inequality.

*Using the **Lemma** we write the inequality:*

$$\frac{5s^2+r^2+4Rr}{4R^2} \geq \frac{4r}{R}\left(4+\frac{r}{R}\right) \Leftrightarrow s^2 \geq 12Rr+3r^2$$

which follows from Gerretsen's inequality: $s^2 \geq 16Rr-5r^2$

It remains to prove that: $16Rr-5r^2 \geq 12Rr+3r^2 \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

The right hand inequality.

*Using the **Lemma** the inequality can be written:*

$$\frac{5s^2+r^2+4Rr}{4R^2} \leq 2\left(4+\frac{r}{R}\right) \Leftrightarrow 5s^2 \leq 32R^2+4Rr-r^2$$

which follows from Gerretsen's inequality: $s^2 \leq 4R^2+4Rr+3r^2$.

It remains to prove that: $5(4R^2+4Rr+3r^2) \leq 32R^2+4Rr-r^2 \Leftrightarrow$

$\Leftrightarrow 3R^2-4Rr-4r^2 \geq 0 \Leftrightarrow (R-2r)(3R+2r) \geq 0$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum \frac{(h_a+h_b)(h_a+h_c)}{bc}$ *and*

$\sum \frac{(r_a+r_b)(r_a+r_c)}{bc}$ *the following inequality holds:*

7) In ΔABC:

$$\sum \frac{(h_a+h_b)(h_a+h_c)}{bc} \leq \sum \frac{(r_a+r_b)(r_a+r_c)}{bc}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using identities **2)** and **6)** the inequality can be written:*

$$\frac{5s^2+r^2+4Rr}{4R^2} \leq \frac{4R+r}{r} \Leftrightarrow 5s^2r \leq (4R+r)(4R^2-r^2)$$

which follows from Gerretsen's inequality: $s^2 \leq 4R^2+4Rr+3r^2$.

It remains to prove that: $5r(4R^2+4Rr+3r^2) \leq (4R+r)(4R^2-r^2) \Leftrightarrow$

$\Leftrightarrow 2R^3-2R^2r-3Rr^2-2r^3 \geq 0 \Leftrightarrow (R-2r)(2R^2+2Rr+r^2) \geq 0$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remarak.

The sequence of inequalities can be written:

8) In ΔABC:

$$\frac{4r}{R}\left(4+\frac{r}{R}\right)\le\sum\frac{(h_a+h_b)(h_a+h_c)}{bc}\le\sum\frac{(r_a+r_b)(r_a+r_c)}{bc}\le\frac{9R}{2r}$$

Proposed by Marin Chirciu - Romania

Proof.

*See inequalities **3)**, **5)** and **7)**.*
Equality holds if and only if the triangle is equilateral.

□

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MARIN CHIRCIU

1) In ΔABC

$$\sum \frac{l_a^2}{h_b h_c} \le \left(\frac{R}{r}\right)^2 - 1$$

Proposed by Mehmet Şahin - Ankara - Turkey

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC

$$\sum \frac{s(s-a)}{h_b h_c} = \frac{s^2 + r^2 - 8Rr}{4r^2}$$

Proof.

Using $h_a = \frac{2S}{a}$ *we obtain:*

$$\sum \frac{s(s-a)}{h_b h_c} = \sum \frac{s(s-a)}{\frac{2S}{b} \cdot \frac{2S}{c}} = \frac{s}{4S^2} \sum bc(s-a) = \frac{s^2 + r^2 - 8Rr}{4r^2},$$ *which follows from:*

$$\sum bc(s-a) = s(s^2 + r^2 - 8Rr)$$

□

Let's get back to the main problem.

Using $l_a^2 \le s(s-a)$ *and* ***Lemma*** *we obtain:*

$$\sum \frac{l_a^2}{h_b h_c} \le \sum \frac{s(s-a)}{h_b h_c} = \frac{s^2 + r^2 - 8Rr}{4r^2} \le \left(\frac{R}{r}\right)^2 - 1$$

where the last inequality is equivalent with: $s^2 \le 4R^2 + 8Rr - 5r^2$

which follows from Gerretsen's inequality: $s^2 \le 4R^2 + 4Rr + 3r^2$

and Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

The inequality can be strengthened:

3) In ΔABC:

$$\sum \frac{l_a^2}{h_b h_c} \le \left(\frac{R}{r}\right)^2 - \frac{R}{r} + 1$$

Proof.

Using $l_a^2 \le s(s-a)$ *and Lemma, we obtain:*

$$\sum \frac{l_a^2}{h_b h_c} \le \sum \frac{s(s-a)}{h_b h_c} = \frac{s^2 + r^2 - 8Rr}{4r^2} \le \left(\frac{R}{r}\right)^2 - \frac{R}{r} + 1$$

where the last inequality is equivalent with:

$s^2 \le 4R^2 + 4Rr + 3r^2$ *(Gerretsen's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **3)** is stronger then inequality **1)**:*

4) In ΔABC:

$$\sum \frac{l_a^2}{h_b h_c} \le \left(\frac{R}{r}\right)^2 - \frac{R}{r} + 1 \le \left(\frac{R}{r}\right)^2 - 1$$

Proof.

*See inequality **3)** and Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having an opposite sense:

Lemma.
5) In ΔABC

$$\sum \frac{l_a^2}{h_b h_c} \ge 3.$$

Proof.

Using $l_a \ge h_a$ *and* $\sum \frac{h_a^2}{h_b h_c} \ge 3$, *which follows from:* $\sum \frac{h_a^2}{h_b h_c} \ge \frac{(\sum h_a)^2}{\sum h_b h_c} \ge 3.$

Above we have used Bergström's inequality and the following inequality:

$(x+y+z)^2 \ge 3(xy+yz+zx)$

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the following sequence of inequalities:

6) In ΔABC:

$$3 \leq \sum \frac{l_a^2}{h_b h_c} \leq \left(\frac{R}{r}\right)^2 - \frac{R}{r} + 1 \leq \left(\frac{R}{r}\right)^2 - 1$$

Proof.

*See inequalities **3)** and **5)**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Regarding the above **Lemma** we propose:*

7) ΔABC

$$\frac{3R}{2r} \leq \sum \frac{s(s-a)}{h_b h_c} \leq \left(\frac{R}{r}\right)^2 - \frac{R}{r} + 1$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma** and Gerretsen's inequality* $16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$

*See inequalities **3)** and **5)**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Replacing h_a with r_a, we propose:

8) In ΔABC:

$$3 \leq \sum \frac{s(s-a)}{r_b r_c} \leq \frac{3R}{2r}$$

Proof.

We use $\sum \frac{s(s-a)}{r_b r_c} = 3$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum \frac{s(s-a)}{r_b r_c}$ *and* $\sum \frac{s(s-a)}{h_b h_c}$, *we can write the following relationship:*

9) In ΔABC

$$\sum \frac{s(s-a)}{r_b r_c} \leq \sum \frac{s(s-a)}{h_b h_c}$$

Proof.

Using the sums $\sum \frac{s(s-a)}{r_b r_c} = 3$ *and* $\sum \frac{s(s-a)}{h_b h_c} = \frac{s^2+r^2-8Rr}{4r^2}$ *we write the following inequality:* $3 \le \frac{s^2+r^2-8Rr}{4r^2} \Leftrightarrow s^2 \ge 8Rr + 11r^2$*, which follows from Gerretsen's inequality* $s^2 \ge 16Rr - 5r^2$ *and Euler's inequality* $R \ge 2r$.

□

Remark.

We can write the sequence of inequalities:

10) In ΔABC

$$\mathbf{3 \le \sum \frac{s(s-a)}{r_b r_c} \le \frac{3R}{2r} \le \sum \frac{s(s-a)}{h_b h_c} \le \left(\frac{R}{r}\right)^2 - \frac{R}{r} + 1 \le \left(\frac{R}{r}\right)^2 - 1}$$

Proposed by Marin Chirciu - Romania

Proof.

See inequalities ***7)*** *and* ***8)****.*

Equality holds if and only if the triangle is equilateral.

□

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