RMM - Calculus Marathon 501 - 600

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501. Find all functions $f:\mathbb{R}\to(0,\infty)$ such that $\forall x,y\in\mathbb{R}$:

$$2(f(x)+f(y))\left(f^2(x)+f^2(y)+3f(x)+3f(y)\right)=3(f(x)+3)(f(y)+3)(f(x)+f(y)-2)$$

Proposed by Nguyen Van Canh-Vietnam & Daniel Sitaru-Romania

Solution 1 by Tran Hong-Vietnam

Let $x=y$ *we have:* $2\cdot 2f(x)\cdot[2f^2(x)+6f(x)]=3\cdot 2[f(x)+3]^2[f(x)-1]$

(Let $F=f(x)\Rightarrow F>0\ \forall x\in\mathbb{R}$*)*

$$4F(F^2+3F)=3[F+3]^2[F-1]$$

$$\Leftrightarrow (F+3)(F-3)^2=0\overset{F>0}{\Leftrightarrow}F=3\Leftrightarrow f(x)=3\ \forall x\in\mathbb{R}.$$

Answer: $f(x)=3\ (\forall x\in\mathbb{R})$

Solution 2 by Ravi Prakash-New Delhi-India

Let $x=y$ *and put* $f(x)=t>0$. *Equation gives us:*

$$2(t+t)(t^2+t^2+3t+3t)=3(t+3)(t+3)(t+t-2)$$

$$\Rightarrow 2(2t)(2t)(t+3)=3(t+3)^2(2t-2)\Rightarrow 4t^2=3(t+3)(t-1)$$

$$\Rightarrow 4t^2=3(t^2+2t-3)\Rightarrow t^2-6t+9=0\Rightarrow (t-3)^2=0\Rightarrow t=3$$

Thus, $f(x)=3\ \forall x\in\mathbb{R}$.

502. Find:

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_{\varepsilon}^{\frac{\pi^2}{4}-\varepsilon}\left(\frac{1}{1+\tan(\sqrt{x})+\cot(\sqrt{x})}\right)dx\right)$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Artan Ajredini-Presheva-Serbie

We put $x=t^2,\ dx=2t\,dt$

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{2t}{1+\tan t+\cot t}dt$$

Now, we use the formula

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

and we get:

$$\Omega = \int_0^{\frac{\pi}{2}} \frac{2\left(\frac{\pi}{2}-t\right)}{1+\tan t+\cot t}\,dt$$

Then,

$$2\Omega = \int_0^{\frac{\pi}{2}} \frac{\pi-2t+2t}{1+\tan t+\cot t}\,dt = \pi\int_0^{\frac{\pi}{2}} \frac{dt}{1+\tan t+\cot t} = \left|\begin{matrix}\tan t = u\\ t = \arctan u\\ dt = \dfrac{u}{1+u^2}\end{matrix}\right| =$$

$$= \pi\int_0^{\infty} \frac{\frac{du}{1+u^2}}{1+u+\frac{1}{u}} = \pi\int_0^{\infty} \frac{\frac{du}{1+u^2}}{\frac{u+u^2+1}{u}} =$$

$$= \pi\int_0^{\infty} \frac{u\,du}{(1+u^2)(u^2-u-1)} = \pi\int_0^{\infty} \frac{du}{u^2+1} - \pi\int_0^{\infty} \frac{du}{u^2+u+1} =$$

$$= \pi\arctan(u)\Big|_0^{\infty} - \pi\int_0^{\infty} \frac{du}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}} =$$

$$= \frac{\pi^2}{2} - \frac{2\pi}{\sqrt{3}}\arctan\left(\frac{u+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\Bigg|_0^{\infty} = \frac{\pi^2}{2} - \frac{2\pi}{\sqrt{3}}\arctan\left(\frac{2u+1}{\sqrt{3}}\right)\Bigg|_0^{\infty} =$$

$$= \frac{\pi^2}{2} - \frac{2\pi^2}{2\sqrt{3}} + \frac{2\pi^2}{6\sqrt{3}} \Rightarrow \Omega = \frac{\pi^2}{4}\left(1+\frac{2}{\sqrt{3}}+\frac{2}{3\sqrt{3}}\right) \Rightarrow \Omega = \frac{\pi^2}{4}\left(1-\frac{2}{\sqrt{3}}+\frac{2}{3\sqrt{3}}\right)$$

Solution 2 by Kelvin Hong-Rawang-Malaysia

$$\Omega = \int_0^{\frac{\pi^2}{4}} \frac{1}{1+\tan\sqrt{x}+\cot\sqrt{x}}\,dx \overset{u^2=x}{=} \int_0^{\frac{\pi}{2}} \frac{2u\,du}{1+\tan u+\cot u}$$

$$\overset{u=\frac{\pi}{2}-t}{=} \int_0^{\frac{\pi}{2}} \frac{\pi-2t}{1+\tan t+\cot t}\,dt = \int_0^{\frac{\pi}{2}} \frac{\pi}{1+\tan t+\cot t} - \frac{2t}{1+\tan t+\cot t}\,dt$$

$$=\int_0^{\frac{\pi}{2}}\frac{\pi}{1+\tan t+\cot t}dt-I=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan t+\cot t}dt$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{\tan t}{\tan^2 t+\tan t+1}dt\overset{k=\tan t}{=}\frac{\pi}{2}\int_0^{\infty}\frac{k}{(k^2+k+1)(k^2+1)}dk$$

$$=\frac{\pi}{2}\int_0^{\infty}\frac{1}{k^2+1}-\frac{1}{k^2+k+1}dk=\frac{\pi}{2}\int_0^{\infty}\frac{1}{k^2+1}-\frac{4}{3}\cdot\frac{1}{\left(\frac{2k+1}{\sqrt{3}}\right)^2+1}dk$$

$$=\frac{\pi}{2}\left[\tan^{-1}k-\frac{4}{3}\cdot\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2k+1}{\sqrt{3}}\right)\right]_0^{\infty}=\frac{\pi}{2}\left[\tan^{-1}k-\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2k+1}{\sqrt{3}}\right)\right]_0^{\infty}$$

$$=\frac{\pi}{2}\left[\left(\frac{\pi}{2}-0\right)-\frac{2}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)\right]=\frac{\pi}{2}\left[\frac{\pi}{2}-\frac{2\pi}{3\sqrt{3}}\right]=\left(\frac{9-4\sqrt{3}}{36}\right)\pi^2$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\Omega=\int_0^{\left(\frac{\pi}{2}\right)^2}\frac{dx}{1+\tan\sqrt{x}+\cot\sqrt{x}}$$

Put $\sqrt{x}=\theta$ ***or*** $x=\theta^2$

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{2\theta d\theta}{1\tan\theta+\cot\theta}d\theta=\int_0^{\frac{\pi}{2}}\frac{2\left(\frac{\pi}{2}-\theta\right)d\theta}{1+\tan\left(\frac{\pi}{2}-\theta\right)+\cot\left(\frac{\pi}{2}-\theta\right)}$$

$$=\int_0^{\frac{\pi}{2}}\frac{(\pi-2\theta)d\theta}{1+\tan\theta+\cot\theta}\Rightarrow\Omega=\frac{\pi}{2}I$$

where

$$I=\int_0^{\frac{\pi}{2}}\frac{d\theta}{1+\tan\theta+\cot\theta}$$

Put $\tan\theta=t$

$$I=\int_0^\infty \frac{t}{(1+t^2+t)(1+t^2)}dt=\int_0^\infty\left(\frac{1}{1+t^2}-\frac{1}{1+t^2+t}\right)dt$$

$$=\left(\tan^{-1}t-\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right)\Bigg]_0^\infty=\frac{\pi}{2}-\frac{2}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\frac{\pi}{2}-\frac{2\pi}{3\sqrt{3}}$$

$$\Omega=\frac{\pi^2}{2}\left(\frac{1}{2}-\frac{2}{3\sqrt{3}}\right)$$

503. Let $\Omega(x)=2\sum_{k=0}^{\infty}\frac{1}{2k+1}\tanh^{2k+1}\left(\frac{1}{2\Gamma(x)}\right)$

Prove that:

$$\int_{0.5}^{1.5} e^{-\Gamma(x)}\psi(x)\{1+\Omega(x)\}\,dx=\frac{e^{-\sqrt{\pi}}-2e^{-\frac{\sqrt{\pi}}{2}}}{\sqrt{\pi}}$$

Proposed by Obidah Al Sharafy-Sana'a-Yemen

Solution 1 by Kamel Benaicha-Algeirs-Algerie

$$\Omega(x)=2\sum_{k=0}^{+\infty}\frac{\tanh^{2k+1}\left(\frac{1}{2\Gamma(x)}\right)}{2k+1}$$

$$I=\int_{\frac{1}{2}}^{\frac{3}{2}}\exp(-\Gamma(x))(1+\Omega(x))\,\Psi(x)dx$$

$$\operatorname{arctanh}(t)=\frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)=\frac{1}{2}\sum_{p=1}^{+\infty}\left(\frac{(-1)^{p-1}+1}{p}\right)t^p=\sum_{p=0}^{+\infty}\frac{t^{2p+1}}{2p+1}$$

$$\therefore\ \Omega(x)=2\frac{1}{2\Gamma(x)}=\frac{1}{\Gamma(x)}$$

$$\therefore\ I=\int_{\frac{1}{2}}^{\frac{3}{2}}\exp(-\Gamma(x))\left(\frac{\Gamma(x)+1}{\Gamma(x)}\right)\psi(x)dx$$

$$t=\Gamma(x)\Rightarrow dt=\Gamma'(x)dx\ dt\ t\in\left[\frac{1}{2}\sqrt{\pi},\sqrt{\pi}\right]$$

$$\therefore I=-\int_{\frac{1}{2}\sqrt{\pi}}^{\sqrt{\pi}}\exp(-t)\left(1+\frac{1}{t}\right)\frac{1}{t}dt$$

$$\int\exp(-t)\left(1+\frac{1}{t}\right)\frac{1}{t}dt=\int\frac{\exp(-t)}{t}dt+\int\frac{\exp(-t)}{t^2}dt$$

$$=\int\frac{\exp(-t)}{t}dt-\frac{\exp(-t)}{t}-\int\frac{\exp(-t)}{t}dt=-\frac{\exp(-t)}{t}$$

$$\therefore I=\frac{\exp(-t)}{t}\Bigg|_{\frac{\sqrt{\pi}}{2}}^{\sqrt{\pi}}=\frac{\exp(-\sqrt{\pi})}{\sqrt{\pi}}-\frac{2\exp\left(-\frac{\sqrt{\pi}}{2}\right)}{\sqrt{\pi}}$$

$$\therefore\int_{\frac{1}{2}}^{\frac{3}{2}}\exp(-\Gamma(x))\left(1+\Omega(x)\right)\Psi(x)dx=\frac{e^{-\sqrt{\pi}}-2e^{-\frac{\sqrt{\pi}}{2}}}{\sqrt{\pi}}$$

Solution 2 by Shafiqur Rahman-Bangladesh

$$\Omega(x)=2\sum_{k=0}^{\infty}\frac{1}{2k+1}\tanh^{2k+1}\left(\frac{1}{2\Gamma(x)}\right)=2\tanh^{-1}\left(\tanh\left(\frac{1}{2\Gamma(x)}\right)\right)=\frac{1}{\Gamma(x)}$$

Now,

$$\int_{\frac{1}{2}}^{\frac{3}{2}}e^{-\Gamma(x)}\Psi_0(x)(1+\Omega(x))dx=\int_{\frac{1}{2}}^{\frac{3}{2}}e^{-\Gamma(x)}\left(1+\frac{1}{\Gamma(x)}\right)\frac{\Gamma'(x)}{\Gamma(x)}dx=$$

$$=-\int_{\frac{1}{2}}^{\frac{3}{2}}\left(\frac{d}{dx}\left(e^{-\Gamma(x)}\frac{1}{\Gamma(x)}\right)+e^{-\Gamma(x)}\frac{d}{dx}\left(\frac{1}{\Gamma(x)}\right)\right)dx=-\left[\frac{e^{-\Gamma(x)}}{\Gamma(x)}\right]_{\frac{1}{2}}^{\frac{3}{2}}=\frac{e^{-\sqrt{\pi}}}{\sqrt{\pi}}-\frac{e^{-\frac{\sqrt{\pi}}{2}}}{\frac{\sqrt{\pi}}{2}}$$

$$\therefore\int_{\frac{1}{2}}^{\frac{3}{2}}e^{-\Gamma(x)}\Psi_0(x)\left(1+\Omega(x)\right)dx=\frac{e^{-\sqrt{\pi}}-2e^{-\frac{\sqrt{\pi}}{2}}}{\sqrt{\pi}}$$

504. **Find:**

$$\Omega = \int (4\cot^3 x + \cot^2 x + \cot x - 2)\, e^x dx,\ x \in \left(0, \frac{\pi}{2}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Artan Ajredini-Preseva-Serbie

We have: $4\cot^3 x + \cot^2 x + \cot x - 2 = \cot x(\cot^2 x + 1) + 3\cot^3 x + \cot^2 x - 2 =$

$$= \cot x \csc^2 x + \cot^2 x + 1 + 3(\cot^3 x - 1) =$$

$$= \cot x \csc^2 x + \csc^2 x + 3(\cot x - 1)(\cot^2 x + \cot x + 1) =$$

$$= \cot x \csc^2 x + \csc^2 x + 3(\cot x - 1)(\csc^2 x + \cot x) =$$

$$= \cot x \csc^2 x + \csc^2 x + 3(\cot x \csc^2 x + \cot^2 x - \csc^2 x - \cot x) =$$

$$= \cot x \csc^2 x + \csc^2 x + 3\cot x \csc^2 x + 3(\cot^2 x + 1) - 3 - 3\csc^2 x - 3\cot x =$$

$$= \cot x \csc^2 x + \csc^2 x + 3\cot x \csc^2 x + 3\csc^2 x - 3\csc^2 x - 3\cot x - 3 =$$

$= 4\cot x \csc^2 x + \csc^2 x - 3\cot x - 3$. *Now, we have:*

$$\Omega = \int (4\cot x \csc^2 x + \csc^2 x - 3\cot x - 3)\, e^x dx =$$

$$= \int 4\cot x \csc^2 x \cdot e^x dx + \int e^x \csc^2 x\, dx - 3\int e^x \cot x\, dx - 3\int c^x\, dx \overset{IBP}{=}$$

$$= -2e^x \cot^2 x + 2\int \cot^2 x \cdot e^x\, dx + \int e^x \csc^2 x\, dx - 3\int e^x \cot x\, dx - 3\int e^x\, dx =$$

$$= -2e^x + \cot^2 x + 3\int e^x \csc^2 x\, dx - 3\int e^x \cot x\, dx - 5\int e^x\, dx \overset{IBP}{=}$$

$$= -2c^x \cot^2 x - 3e^x \cot x + 3\int e^x \cot x\, dx - 3\int e^x \cot x\, dx - 5\int e^x\, dx =$$

$$= -2c^x \cot^2 x - 3c^x \cot x - 5e^x + c$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

Suppose $F(x) = (\mathrm{A} \cdot \cot^2 x + B\cot x + C)e^x$ *such that*

$$F'(x) = (4\cot^3 x + \cot^2 x + \cot x - 2) = f(x), \forall x \in \left]0, \frac{\pi}{2}\right[$$

$$F'(x) = \left(-\mathrm{A} \cdot 2\cot x(1 + \cot^2 x) - B(1 + \cot^2 x)\right) \cdot e^x + e^x(\mathrm{A}\cot^2 x + B\cot x + C)$$

$$F'(x) = (-2\mathrm{A}(\cot x + \cot^3 x) - B - B\cot^2 x + \mathrm{A}\cot^2 x + B\cot x + C)e^x$$

$$F'(x) = (-2A\cot^3 x - 2A\cot x - B - B\cot^2 x + A\cot^2 x + B\cot x + C)\cdot e^x$$

$$F'(x) = \left(-2A\cot^3 x + (A-B)\cot^2 x + (-2A+B)\cot x + (C-B)\right)\cdot e^x$$

$$f(x) = 4\cot^3 x + \cot^2 x + \cot x - 2$$

$$-2A = 4 \Rightarrow A = -2$$

$$A - B = 1 \Rightarrow A - 1 = B \Rightarrow B = -3$$

$-2\Delta A + B = 1$ ***is true.***

$$C - B = -2$$

$$C + 3 = -2 \Rightarrow C = -5$$

So: $F(x) = (-2\cot^2 x - 3\cot x - 5)e^x$

$$\Omega = \int (4\cot^3 x + \cot^2 x + \cot x - 2)\, e^x dx = (-2\cot^2 x - 3\cot x - 5)e^x + C$$

Solution 3 by Avishek Mitra-West Bengal-India

$$\int (4\cot^3 x + \cot^2 x + \cot x - 2)\, e^x dx$$

$$= \int [4\cot x(\csc^2 x - 1) + (\csc^2 x - 1) + \cot x - 2]e^x\, dx$$

$$= \int [4\cot x\cdot \csc^2 x - 3\cot x - 2\csc^2 x + 3\csc^2 x - 3)]\, e^x dx$$

$$= \int e^x[-3(1+\cot x) + 3\csc^2 x]\, dx - \int e^x(2\csc^2 x - 4\cot x\cdot\csc^2 x)\, dx$$

$$= -3e^x(\cot x + 1) - 2e^x\csc^2 x + c$$

By applying: $\int e^x[f(x) + f'(x)] = e^x f(x) + c$

$$= -3e^x(\cot x + 1) - 2e^x(\cot^2 x + 1) + c$$

$$= -e^x(5 + 3\cot x + 2\cot^2 x) + c \quad \textit{(Answer)}$$

Solution 4 by Shafiqur Rahman-Bangladesh

$$\int (4\cot^3 x + \cot^2 x + \cot x - 2)\, e^x dx = \int e^x\left(4\cot x\csc^2 x + \cot^2 x - 3\cot x - 2\right) dx =$$

$$= -2e^x\cot^2 x + \int e^x\left(3\cot^2 x - 3\cot x - 2\right)$$

$$= -2e^x\cot^2 x + \int e^x\left(3\csc^2 x - 3\cot x - 5\right) dx = -2e^x\cot^2 x - 3e^x\cot x - 5\int e^x\, dx$$

$$\therefore \int (4\cot^3 x + \cot^2 x + \cot x - 2)\, e^x dx = -e^x(2\cot^2 x + 3\cot x + 5) + C$$

505. If $x < a$, $a \in \mathbb{R}$ then:

$$(1 + ax - x^2)e^{x^2} < \frac{1}{a - x}\int_x^a e^{x^2}\, dx < (a^2 - ax)e^{a^2} + e^{x^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$(1 + ax - x^2)e^{x^2} \overset{(1)}{<} \frac{1}{a - x}\int_x^a e^{x^2}\, dx \overset{(2)}{<} (a^2 - ax)e^{a^2} + e^{x^2}$$

Let $f(x) = e^{x^2}$***. Then*** $f''(x) = (4x^2 + 2)e^{x^2} > 0 \therefore f(x)$ ***is convexe***

$\therefore$ ***by Hermite – Hadamard inequality,***

$$\frac{1}{a - x}\int_x^a e^{x^2}\, dx \le \frac{e^{x^2} + e^{a^2}}{2} \overset{?}{<} (a^2 - ax)e^{a^2} + e^{x^2} \Leftrightarrow e^{x^2} + e^{a^2} < 2(a^2 - ax)e^{a^2} + 2e^{x^2}$$

$$\Leftrightarrow e^{a^2} - e^{x^2} < 2ae^{a^2}(a - x) \Leftrightarrow \frac{e^{a^2} - e^{x^2}}{a - x} \overset{(2a)}{<} 2ae^{a^2}(\because a - x > 0)$$

By Cauchy's MVT, there exists θ ***satisfying*** $a > \theta > x$***, such that*** $\frac{e^{a^2}-e^{x^2}}{a-x} = \frac{d}{dx}\left(e^{x^2}\right)_{x=\theta}$

$\therefore \frac{e^{a^2}-e^{x^2}}{a-x} = 2\theta e^{\theta^2}$***, where*** $a > \theta > x < 2ae^{a^2}(\because \theta < a) \Rightarrow$ ***(2a) is true*** $\Rightarrow$ ***(2) is true.***

Also, by Hermite – Hadamard's inequality,

$$\frac{1}{a - x}\int_x^a e^{x^2} dx \ge e^{\left(\frac{a+x}{2}\right)^2} \overset{?}{>} (1 + ax - x^2)e^{x^2} \Leftrightarrow \left(\frac{a + x}{2}\right)^2 \overset{(1a)}{>} \ln(1 + ax - x^2) + x^2$$

Now, $\ln\{1 + (ax - x^2)\} \overset{(i)}{\le} ax - x^2(\because \ln(1 + m) \le m)$

(i) $\Rightarrow$ ***RHS of (1a)*** $\le ax \overset{?}{<} \left(\frac{a+x}{2}\right)^2 \Leftrightarrow (a - x)^2 \overset{?}{>} 0 \to$ ***true*** $\therefore$ ***(1a) is true*** $\Rightarrow$***(1) is true***

(Done)

506. **Let $0 < a \leq b < \frac{\pi}{4}$. Prove:**

$$\frac{1}{a+b}\int_a^b\int_a^b sin(sin(sin(x+y)))\,dxdy \leq (b-a)^2$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Chris Kyriazis-Athens-Greece

$$\begin{matrix} 0 < a \leq x \leq b < \frac{\pi}{4} \\ 0 < a \leq y \leq b < \frac{\pi}{4} \end{matrix} \Rightarrow 0 < x + y < \frac{\pi}{2}$$

So, $0 < \sin(x+y) < 1 < \frac{\pi}{2} \Rightarrow$

So, $\sin(\sin(x+y)) > 0$ ***and final*** $\sin(\sin(\sin(x+y))) > 0$

So, we know, that for positive x ***holds that*** $\sin x < x$. ***This means that:***

$\sin(\sin(\sin(x+y))) < \sin(\sin(x+y)) < \sin(x+y) < x+y$.

Taking integrals, we have:

$$\int_a^b\int_a^b \sin(\sin(\sin(x+y)))\,dxdy < \int_a^b\int_a^b (x+y)\,dx\,dy =$$

$$= \int_a^b\left[\frac{x^2}{2} + yx\right]_a^b dy = \int_a^b\left(\frac{b^2}{2} - \frac{a^2}{2}\right) + y(b-a)dy$$

$$\left[\left(\frac{b^2-a^2}{2}\right)y + \frac{y^2}{2}(b-a)\right]_a^b = \frac{(b-a)(b+a)}{2}b - a + \frac{b^2-a^2}{2}(b-a) = \frac{(b-a)^2(b+a)}{2}\cdot 2.$$ ***So,***

$$\frac{1}{a+b}\int_a^b\int_a^b \sin(\sin(\sin(x+y)))\,dxdy < (b-a)^2$$

Solution 2 by Serban George Florin-Romania

$$\sin t \leq t \Rightarrow \sin(x+b) \leq x+y \Rightarrow \sin(\sin(x+y)) \leq \sin(x+y) \leq x+y,$$

$$\sin(\sin(x+y)) \leq x+y$$

$$\sin(\sin(\sin(x+y))) \leq \sin(x+y) \leq x+y \Rightarrow \sin(\sin(\sin(x+y))) \leq x+y$$

$$\frac{1}{a+b}\int_a^b\int_a^b \sin(\sin(\sin(x+y)))\,dxdy \le \frac{1}{a+b}\int_a^b\left(\int_a^b (x+y)dx\right)dy =$$

$$=\frac{1}{a+b}\int_a^b\left(\frac{x^2}{2}+yx\right)dy\Big|_a^b = \frac{1}{a+b}\int_a^b\left(\frac{b^2-a^2}{2}+y(b-a)\right)dy =$$

$$=\frac{1}{a+b}\left[\frac{b^2-a^2}{2}\cdot y+\frac{y^2}{2}(b-a)\right]_a^b = \frac{1}{a+b}\cdot\left[\frac{b^2-a^2}{2}\cdot(b-a)+\frac{b^2-a^2}{2}(b-a)\right] =$$

$$=\frac{1}{a+b}\cdot(b^2-a^2)(b-a) = \frac{(b-a)(b+a)(b-a)}{a+b} = (b-a)^2 \quad \textit{true.}$$

507. If $f:[0,1]\to\mathbb{R}, f(1)=3, f$ – continuous, f – convexe then:

$$\int_{\frac{1}{2}}^{1} f(x)\,dx < 1+\frac{1}{3}\int_0^{\frac{1}{2}} f(x)\,dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdallah El Farisi-Bechar-Algerie

We have:

$$\frac{4}{3}\int_{\frac{1}{2}}^{1} f(x)\,dx \overset{H-H}{\le} \frac{4}{3}\frac{f(1)+f\left(\frac{1}{2}\right)}{4} = 1+\frac{1}{3}f\left(\frac{1}{2}\right) \overset{H-H}{\le} 1+\frac{1}{3}\int_0^1 f(x)\,dx$$

then:

$$\int_{\frac{1}{2}}^{1} f(x)\,dx \le 1+\frac{1}{3}\int_0^{\frac{1}{2}} f(x)\,dx$$

Solution 2 by Artan Ajredini-Presheva-Serbie

Since f is a convex function, we have: $f(ta+(1-t)b)\le tf(a)+(1-t)f(b)$

For $a=0$ and $b=1$ we have: $f(1-t)\le(1-t)f(1)=3(1-t)$ (1)

By integrating the inequality a) side by side for t from $\frac{1}{2}$ to 1 we have.

$$\int_{\frac{1}{2}}^{1} f(1-t)\,dt \le 3\int_{\frac{1}{2}}^{1} (1-t)\,dt$$

We substitute $1-t = x$ ***and we have:*** $\int_0^1 f(x)\,dx \le 3\int_0^1 f(x)\,dx = \frac{3}{2}x^2\Big|_0^{\frac{1}{2}} = \frac{3}{8}$ ***(2)***

By integrating the inequality (1) side by side for t ***from 0 to*** $\frac{1}{2}$ ***we have:***

$$\int_0^1 f(1-t)\,dt \le 3\int_0^{\frac{1}{2}} (1-t)\,dt$$

Also, by substituting $1-t = x$ ***we have:*** $\int_{\frac{1}{2}}^1 f(x)\,dx \le 3\int_{\frac{1}{2}}^1 x\,dx = \frac{3}{2}x^2\Big|_{\frac{1}{2}}^{1} = \frac{9}{8}$ ***(3)***

From (2) and (3) we have:

$$\int_{\frac{1}{2}}^{1} f(x)\,dx \le \frac{9}{8} = 3\cdot\frac{3}{8} = 3\cdot\int_0^{\frac{1}{2}} f(x)\,dx \Rightarrow \int_{\frac{1}{2}}^{1} f(x)\,dx - 3\int_0^{\frac{1}{2}} f(x)dx \le 0 \Rightarrow$$

$$\Rightarrow 3\int_{\frac{1}{3}}^1 f(x)\,dx - \int_0^1 f(x)\,dx \le 2\int_{\frac{1}{2}}^1 f(x)\,dx + 2\int_0^{\frac{1}{2}} f(x)\,dx \quad \textit{(4)}$$

Substituting (1) and (3) to (4) and we have:

$$3\int_{\frac{1}{2}}^{1} f(x)\,dx - \int_0^{\frac{1}{2}} f(x)\,dx \le 2\int_{\frac{1}{2}}^{1} f(x) + 2\int_0^{\frac{1}{2}} f(x)\,dx \le 2\frac{9}{8} + 2\frac{3}{8} = \frac{9}{4} + \frac{3}{4} = \frac{12}{4} = 3$$

Deductively,

$$3\int_{\frac{1}{2}}^{1} f(x)\,dx - \int_0^{\frac{1}{2}} f(x)\,dx \le 3 \Rightarrow 3\int_{\frac{1}{2}}^{1} f(x)\,dx \le 3 + \int_0^{\frac{1}{2}} f(x)\,dx \Rightarrow$$

$$\Rightarrow \int_{\frac{1}{2}}^{1} f(x)\,dx \le 1 + \frac{1}{3}\int_0^{\frac{1}{2}} f(x)\,dx$$

Q.E.D.

508.

$$a,b,c\geq\frac{\pi}{4},\Omega(a)=\int_{\frac{\pi}{4}}^{a}\left(\frac{x^2+1+\tan^{-1}x}{x^4+2x^2+1+(\tan^{-1}x)^2}\right)dx$$

Prove that:

$$\left(1+2\Omega(a)\right)b^2+\left(1+2\Omega(b)\right)c^2+\left(1+2\Omega(c)\right)a^2\leq a^4+b^4+c^4$$

Proposed by Daniel Sitaru – Romania

Solution by Artan Ajredini-Presheva-Serbie

By Bergström inequality:

$$x^4+2x^2+1+(\tan^{-1}x)^2=(x^2+1)^2+(\tan^{-1}x)^2\geq\frac{(x^2+1+\tan^{-1}x)^2}{2}$$

Therefore $\frac{x^2+1+\tan^{-1}x}{x^4+2x^2+1+(\tan^{-1}x)^2}\leq 2\frac{x^2+1+\tan^{-1}x}{(x^2+1+\tan^{-1}x)^2}=\frac{2}{x^2+1+\tan^{-1}x}$ ***(1)***

Since $\frac{\pi}{4}\leq x\leq a\Rightarrow\tan^{-1}x>-1$. *So,* $\frac{x^2+1+\tan^{-1}x}{x^4+2x^2+1+(\tan^{-1}x)^2}\leq\frac{2}{x^2+1-1}=\frac{2}{x^2}$ ***(2)***

From (2) we have: $\Omega(a)\leq 2\int_{\frac{\pi}{4}}^{a}\frac{dx}{x^2}=-\frac{2}{x}=-\frac{2}{a}+\frac{2}{\frac{\pi}{4}}=-\frac{2}{a}+\frac{8}{\pi}<-\frac{2}{0}+8$ ***(3)***

From (3) we have: $\left(1+2\Omega(0)\right)b^2\leq\left(17-\frac{4}{a}\right)b^2=(17a-4)\frac{b^2}{a}$ ***(4)***

Since $a\geq\frac{\pi}{4}>0$ *we have:* $(a+1)^3\geq 0\Rightarrow$

$$a^3+3a^2+3a+1\geq a^3-2a+3a+1\geq 0\Rightarrow$$

$$\Rightarrow a^3\geq 17a-1>17a-4\quad\textbf{(5)}$$

By substituting (5) to (4) we have: $\left(1+2\Omega(a)\right)b^2\leq\frac{a^3}{a}b^2=a^2b^2$. *Hence,*

$LHS\leq a^2b^2+b^2c^2+a^2c^2$. *By Cauchy-Schwarz inequality we have:*

$$LHS\leq a^2b^2+b^2c^2+a^2c^2\leq a^4+b^4+c^4\quad \text{Q.E.D.}$$

509.

$$si\,(x)=-\int_x^\infty\left(\frac{\sin t}{t}\right)dt,\ x>0$$

$$\Omega_1=\int_\gamma^e\left(\frac{1}{x}\left(si\,(e^2x)-si(\pi x)\right)\right)dx,\ \Omega_2=\int_\pi^{e^2}\left(si(ex)-si(\gamma x)\right)dx$$

$$A\cdot\Omega_1<\Omega_2,\ B\cdot\Omega_1=\Omega_2,\ C\cdot\Omega_1>\Omega_2$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$si\,(x)=-\int_x^\infty\frac{\sin\theta}{\theta}d\theta$$

$$si'(x)=-\left[\lim_{\theta\to\infty}\frac{\sin\theta}{\theta}-\frac{\sin x}{x}\right]=\frac{\sin x}{x}$$

Now,

$$\Omega_1=\int_\gamma^e\frac{1}{x}\left(si\,(e^2x)-si\,(\pi x)\right)dx=\int_\gamma^e\left(\int_\pi^{e^2}si'(tx)dt\right)dx=$$

$$=\int_\pi^{e^2}\left(\int_\gamma^e si'(tx)dx\right)dt$$

[Interchange order of integration]

$$=\int_\pi^{e^2}\frac{si\,(tx)}{t}\Bigg]_\gamma^e dt=\int_\pi^{e^2}\frac{si\,(te)-si\,(\gamma t)}{t}dt=\int_\pi^{e^2}\frac{si(ex)-\sin(\gamma x)}{x}dx=\Omega_2$$

510. If $1<a\le b$ then:

$$\int_a^b\int_a^b\log\left(\frac{x+y}{2}\right)^{x+y}dx\,dy\ge(b-a)^2\log\left(\frac{a+b}{2}\right)^{a+b}$$

Proposed by Daniel Sitaru – Romania

Solution by Artan Ajredini-Presheva-Serbie

We consider the function $f(x) = x\ln x$, *for* $x > 0$

$f'(x) = \ln x + 1 > 0 \Rightarrow f$ *– convex*

Therefore, we apply the Hermite – Hadamard inequality for double integral:

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b \log\left(\frac{x+y}{2}\right)^{x+y} \geq \log\left(\frac{\frac{a+b}{2}+\frac{a+b}{2}}{2}\right)^{\left(\frac{a+b}{2}+\frac{a+b}{2}\right)} = \log\left(\frac{a+b}{2}\right)^{(a+b)} \Rightarrow$$

$$\Rightarrow \int_a^b\int_a^b \log\left(\frac{x+y}{2}\right)^{x+y} \geq (b-a)^2\log\left(\frac{a+b}{2}\right)^{(a+b)}$$

511.

$$0 < a,b,c \leq 1, \Omega(a) = \frac{8}{\pi}\int_0^a \frac{x\cos(10\cos^{-1}x)\,dx}{(x^2+a^2)(\cos(11\cos^{-1}x)+\cos(9\cos^{-1}x))}$$

Prove that:

$$\left(\Omega(a)+\Omega(b)+\Omega(c)\right)\left(6+\frac{1}{\Omega^3(a)}+\frac{1}{\Omega^3(b)}+\frac{1}{\Omega^3(c)}\right) \geq 27$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

For $\theta \in \mathbb{R}$,

$$\cos(11\theta)+\cos(9\theta) = 2\cos(10\theta)\cos\theta \Rightarrow \cos(11\cos^{-1}x)+\cos(9\cos^{-1}x) =$$

$$= 2\cos(10\cos^{-1}x)\cos(\cos^{-1}x) = 2x\cos(10\cos^{-1}x)$$

$$\therefore \Omega(a) = \frac{8}{\pi}\int_0^a \frac{x\cos(10\cos^{-1}x)\,dx}{(x^2+a^2)[\cos(11\cos^{-1}x+\cos(9\cos^{-1}x))]} =$$

$$= \frac{8}{\pi}\int_0^a \frac{x\cos(10\cos^{-1}x)}{(x^2+a^2)2x\cos(10\cos^{-1}x)}dx = \frac{4}{\pi}\cdot\frac{1}{a}\tan^{-1}\frac{x}{a}\Big]_0^a = \frac{1}{a}\cdot\frac{4}{\pi}\cdot\frac{\pi}{4} = \frac{1}{a}$$

Now, $[\Omega(a)+\Omega(b)+\Omega(c)]\left[6+\frac{1}{\Omega^3(a)}+\frac{1}{\Omega^3(b)}+\frac{1}{\Omega^3(c)}\right] = \left[\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right][6+a^3+b^3+c^3] \geq$

$$\geq 3\left(\frac{1}{abc}\right)^{\frac{1}{3}} 9(1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1)^{\frac{1}{9}} = 27\left(\frac{1}{abc}\right)^{\frac{1}{3}}(abc)^{\frac{1}{3}} = 27$$

512. $f:[a,b]\to(0,\infty)$ f – continuous, $0 < a \leq b$

$$\Omega(x,y,z) = \left(\frac{f^2(x)+f^2(y)}{f(x)+f(y)}\right)^3 + \left(\frac{f^2(y)+f^2(z)}{f(y)+f(z)}\right)^3 + \left(\frac{f^2(z)+f^2(x)}{f(z)+f(x)}\right)^3$$

Prove that:

$$\int_a^b\int_a^b\int_a^b \Omega(x,y,z)\,dx\,dy\,dz \geq 3(b-a)^2\int_a^b f^3(x)\,dx$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

For $a,b > 0, \left(\frac{a^2+b^2}{a+b}\right)^3 \geq \frac{1}{2}(a^3+b^3) \Leftrightarrow 2(a^2+b^2)^3 \geq (a^3+b^3)(a+b)^3 \Leftrightarrow$

$$\Leftrightarrow 2(a^6+3a^4b^2+3a^2b^4+b^6) \geq (a^3+b^3)(a^3+3a^2b+3ab^2+b^3) \Leftrightarrow$$

$$\Leftrightarrow a^6+b^6-3a^5b-2a^3b^3-3ab^5+3a^2b^4+3a^4b^2 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (a^3-b^3)^2-3ab(a^3-b^3)(a-b) \geq 0 \Leftrightarrow (a^3-b^3)[a^3-b^3-3ab(a-b)] \geq 0 \Leftrightarrow$$

$\Leftrightarrow (a^3-b^3)(a-b)^3 \geq 0 \Leftrightarrow (a-b)^4(a^2+ab+b^2) \geq 0$ ***which is true***

Thus, $\Omega(x,y,z) = \left(\frac{f(x)^2+f(y)^2}{f(x)+f(y)}\right)^3 + \left(\frac{f(y)^2+f(z)^2}{f(y)+f(z)}\right)^3 + \left(\frac{f(z)^2+f(x)^2}{f(z)+f(x)}\right)^3 \geq$

$$\geq \frac{1}{2}(f(x)^3+f(y)^3)+\frac{1}{2}(f(y)^3+f(z)^3)+\frac{1}{2}(f(z)^3+f(x)^3) = f(x)^3+f(y)^3+f(z)^3$$

$$\therefore \int_a^b\int_a^b\int_a^b \Omega(x,y,z)\,dx\,dy\,dz \geq \int_a^b\int_a^b\int_a^b [f(x)^3+f(y)^3+f(z)^3]\,dxdydz =$$

$$= 3(b-a)^2\int_a^b f(x)^3\,dx$$

513. If $f,g,h:[0,1]\to(0,\infty), f,g,h$ - continuous then:

$$27e^{\int_0^1(\log(f(x)\cdot g(x)\cdot h(x)))dx}\le\left(\int_0^1(f(x)+g(x)+h(x))dx\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

Integral Analogue of $AM\ge GM$. Suppose $f:[a,b]\to(0,\infty)$ be continuous function,

then $e^{\frac{1}{b-a}\int_a^b\ln f(x)dx}\le\frac{1}{b-a}\int_a^b f(x)\,dx$

Now, $e^{\frac{1}{1-0}\int_0^1\ln f(x)\cdot g(x)\cdot h(x)dx}=e^{\int_0^1(\ln f(x)+\ln g(x)+\ln h(x))dx}$

$$=e^{\int_0^1\ln f(x)dx+\int_0^1\ln g(x)dx+\int_0^1\ln h(x)dx}=\prod_{cyc}\left(\int_0^1 f(x)\,dx\right)\overset{AM\ge GM}{\le}\frac{1}{27}\left(\sum_{cyc}\int_0^1 f(x)\,dx\right)^3$$

$$27e^{\int_0^1\ln f(x)\cdot g(x)\cdot h(x)dx}\le\left(\sum_{cyc}\int_0^1 f(x)\,dx\right)^3$$

(proved)

514. Prove without softs:

$$\left(\int_0^1(\gamma^x\cdot e^{1-x})dx\right)\left(\int_0^1(e^x\cdot\pi^{1-x})dx\right)<e\int_0^1(\gamma^x\cdot\pi^{1-x})\,dx$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

According to Chebyshev's inequality

$$\left(\int_0^1 e^{1-x}\cdot\gamma^x\,dx\right)\left(\int_0^1 e^x\cdot\pi^{1-x}dx\right)<(1-0)^2\int_0^1 e^{1-x}\gamma^x e^x\pi^{1-x}\,dx=e\int_0^1\gamma^x\pi^{1-x}\,dx$$

(Proved)

515. If $0 < a \leq b$ then:

$$\frac{3}{2}\int_a^b\int_a^b\left(\frac{x^2+y^2}{x^4+x^2y^2+y^4}\right)dxdy \leq \left(log\left(\frac{b}{a}\right)\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$x^4+x^2y^2+y^4 = (x^4+y^4+2x^2y^2)-x^2y^2 = (x^2+y^2)^2-(xy)^2 \overset{(1)}{=}$$
$$= (x^2+y^2+xy)(x^2+y^2-xy)$$

Now, $(x-y)^2 \geq 0 \Rightarrow x^2+y^2-2xy \geq 0 \Rightarrow 2x^2+2y^2-2xy \geq x^2+y^2 \Rightarrow$

$$\Rightarrow x^2+y^2 \overset{(2)}{\leq} 2(x^2-xy+y^2)$$

(1), (2) $\Rightarrow$ *LHS* $\leq \frac{3}{2}\cdot 2\int_a^b\int_a^b \frac{(x^2-xy+y^2)dx\,dy}{(x^2+xy+y^2)(x^2-xy+y^2)} = 3\int_a^b\int_a^b \frac{dx\,dy}{x^2+xy+y^2} \overset{A-G}{\leq} 3\int_a^b\int_a^b \frac{dx\,dy}{3xy}$

$$= \int_a^b\int_a^b \frac{dx\,dy}{xy} = \int_a^b\left[\int_a^b \frac{dx}{x}\right]\frac{dy}{y} = \left(\ln\frac{b}{a}\right)\int_a^b \frac{dy}{y} = \left(\ln\frac{b}{a}\right)^2$$

(Proved)

Solution 2 by Soumitra Mandal-Chandar Nagore-India

We know, $x^4+x^2y^2+y^4 \geq \frac{3}{4}(x^2+y^2)^2$

$$\frac{3}{2}\int_a^b\int_a^b \frac{x^2+y^2}{x^4+x^2y^2+y^4}dx\,dy \leq \frac{3}{2}\cdot\frac{4}{3}\int_a^b\int_a^b \frac{dx\,dy}{x^2+y^2} \overset{AM\geq GM}{\leq} \left(\int_a^b \frac{dx}{x}\right)\left(\int_a^b \frac{dy}{y}\right) = \left(\log\frac{b}{a}\right)^2$$

(proved)

Solution 3 by Togrul Ehmedov-Baku-Azerbaijan

$$\frac{3}{2}\int_a^b\int_a^b \frac{x^2+y^2}{x^4+x^2y^2+y^4 \geq 3x^2y^2}dx\,dy \leq \frac{1}{2}\int_a^b\int_a^b\left(\frac{1}{x^2}+\frac{1}{y^2}\right)dx\,dy$$

$$\leq \int_a^b\int_a^b \frac{dx\,dy}{xy} = \int_a^b \frac{1}{y}\ln x\Big|_a^b dy = \int_a^b \frac{1}{y}\ln\frac{b}{a}dy = \ln^2\left(\frac{b}{a}\right)$$

516. If $0 < a \le b$ then:

$$\int_a^b \int_a^b \int_a^b \left(\frac{1}{1+\sqrt{xy}} + \frac{1}{1+\sqrt{yz}} + \frac{1}{1+\sqrt{zx}}\right) dx\,dy\,dz \le 3(b-a)^2 \log\left(\frac{1+b}{1+a}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Artan Ajredini-Presheva-Serbie

WLOG we suppose that $x \ge y \ge z$. Then

$$\frac{1}{1+\sqrt{xy}} \le \frac{1}{1+\sqrt{y^2}} = \frac{1}{1+y}, 0 < a \le y \le b$$

$$\frac{1}{1+\sqrt{yz}} \le \frac{1}{1+\sqrt{z^2}} = \frac{1}{1+z}, 0 < a \le z \le b$$

$$\frac{1}{1+\sqrt{zx}} \le \frac{1}{1+\sqrt{z^2}} = \frac{1}{1+z}, 0 < a \le z < b$$

By summing the above inequalities, we get: $\frac{1}{1+\sqrt{xy}} + \frac{1}{1+\sqrt{yz}} + \frac{1}{1+\sqrt{zx}} \le \frac{1}{1+y} + \frac{2}{1+z}$

Hence,

$$\int_a^b \int_a^b \int_a^b \left(\frac{1}{1+\sqrt{xy}} + \frac{1}{1+\sqrt{yz}} + \frac{1}{1+\sqrt{zx}}\right) dx\,dy\,dz \le \int_a^b \int_a^b \int_a^b \left(\frac{1}{1+y} + \frac{2}{1+z}\right) dx\,dy\,dz =$$

$$= \int_a^b \int_a^b \int_a^b \frac{dx\,dy\,dz}{1+y} + 2\int_a^b \int_a^b \int_a^b \frac{dx\,dy\,dz}{1+z} =$$

$$= (b-a)^2 \log(1+y)\Big|_0^b + 2(b-a)^2 \log(1+z)\Big|_0^b =$$

$$= (b-a)^2 \log\left(\frac{1+b}{1+a}\right) + 2(b-a)^2 \log\left(\frac{1+b}{1+a}\right) = 3(b-a)^2 \log\left(\frac{1+b}{1+a}\right)$$

517. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \left(\frac{\cos^2 x + \tan x}{\cos^4 x + \tan^2 x}\right)^2 dx \le \log\left|\frac{\tan b}{\tan a}\right|$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Artan Ajredini-Presheva-Serbie

By Bergström's inequality: $\cos^4 x+\tan^2 x\ge\frac{(\cos^2 x+\tan x)^2}{2}$. *Therefore*

$$\left(\frac{\cos^2 x+\tan x}{\cos^4 x+\tan^2 x}\right)^2\le 4\frac{(\cos^2 x+\tan x)^2}{(\cos^2 x+\tan x)^4}=\frac{4}{(\cos^2 x+\tan x)^2}\quad (1)$$

By AM-GM inequality we have: $(\cos^2 x+\tan x)^2\ge 4\cos^2 x\tan x\quad (2)$

We substitute (2) to (1) and we get: $\left(\frac{\cos^2 x+\tan x}{\cos^4 x+\tan^2 x}\right)^2\le\frac{1}{\cos^2 x\tan x}$. *Definetly,*

$$\int_a^b\left(\frac{\cos^2 x+\tan x}{\cos^4 x+\tan^2 x}\right)^2dx\le\int_a^b\frac{dx}{\cos^2 x\tan x}=\int_a^b\frac{\frac{dx}{\cos^2 x}}{\tan x}=\int_a^b\frac{d(\tan x)}{\tan x}=\ln\left|\frac{\tan b}{\tan a}\right|$$

Solution 2 by El Harati Youness-Morocco

$$A=\int_a^b\left(\frac{\cos^2 x+\tan x}{\cos^4 x+\tan^2 x}\right)^2dx$$

$$\left(\frac{x+y}{2}\right)^2\le\frac{x^2+y^2}{2}\therefore A\le 2\int_a^b\frac{\cos^4 x+\tan^2 x}{|\cos^4 x+\tan^2 x|^2}dx$$

$$=\int_a^b\frac{2\,dx}{\cos^4 x+\tan^2 x}\le\int_a^b\frac{dx}{\sqrt{\cos^4 x\tan^2 x}}=\int_a^b\frac{\frac{1}{\cos^2 x}}{\tan x}dx=[\log|\tan x|]_a^b=\log\left|\frac{\tan b}{\tan a}\right|$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\because a\le x\le b\ \&\ a>0\ \&\ b<\frac{\pi}{2}\therefore 0<x<\frac{\pi}{2}\therefore\tan x>0$$

Now, $\left(\frac{\cos^2 x+\tan x}{\cos^4 x+\tan^2 x}\right)^2=\frac{(\cos^4 x+\tan^2 x)+2\cos^2 x\tan x}{(\cos^4 x+\tan^2 x)^2}=\frac{1}{\cos^4 x+\tan^2 x}+\frac{2\cos^2 x\tan x}{(\cos^4 x+\tan^2 x)^2}$

$$\underset{(1)}{\overset{A-G}{\le}}\frac{1}{2\cos^2 x\tan x}+\frac{2\cos^2 x\tan x}{4\cos^4 x\tan^2 x}\quad(\because\tan x>0)$$

$$=\frac{\sec^2 x}{\tan x}$$

$$(1)\Rightarrow\int_a^b\left(\frac{\cos^2 x+\tan x}{\cos^4 x+\tan^2 x}\right)^2dx\le\int_a^b\frac{\sec^2 x\,dx}{\tan x}=[\ln|\tan x|]_a^b$$

$$=\ln|\tan b|-\ln|\tan a|=\ln\left|\frac{\tan b}{\tan a}\right|\ \textit{(Proved)}$$

518. If $f:[a,b]\to(0,\infty), a\le 5<7\le b, f$ continuous then:

$$\frac{\left(\int_a^b f^5(x)dx\right)^7}{\left(\int_a^b f^7(x)dx\right)^5}<(b-a)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Omran Kouba-Damascus-Syria

By Hölder's inequality we have for $g:[a,b]\to(0,\infty)$

$$\int_a^b g(x)\,dx\le\left(\int_a^b dx\right)^{\frac{2}{7}}\left(\int_a^b (g(x))^{\frac{7}{5}}dx\right)^{\frac{5}{7}}$$

Taking seventh power and applying this to $g(x)=\left(f(x)\right)^5$ we get

$$\left(\int_a^b \left(f(x)\right)^5dx\right)^7\le(b-a)^2\left(\int_a^b \left(f(x)\right)^7dx\right)^5$$

With equality if f is constant. The desired inequality follows, with no conditions on a and b.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{\left(\int_a^b f^5(x)dx\right)^7}{\left(\int_a^b f^7(x)\,dx\right)^5}=\frac{\left(\int_a^b f^5(x)dx\right)^5}{\int_a^b f^7(x)\,dx}\cdot\left(\int_a^b f^5(x)\,dx\right)^2$$

$$\le\left(\int_a^b \frac{f^5(x)}{f^7(x)}dx\right)^5\cdot\left(\int_a^b f^5(x)\,dx\right)^2=\left(\int_a^b \frac{1}{f^2(x)}dx\right)^5\cdot\left(\int_a^b f^5(x)\,dx\right)^2$$

$$=\left(\int_a^b \frac{1}{f^2(x)}dx\right)^2\cdot\left(\int_a^b f^5(x)\,dx\right)^2\cdot\left(\int_a^b \frac{1}{f^2(x)}dx\right)^3\le\left(\int_a^b \frac{f^5(x)}{f^2(x)}dx\right)^2\cdot\left(\int_a^b \frac{1}{f^2(x)}dx\right)^3$$

$$=\left(\int_a^b f^3(x)\,dx\right)^2\cdot\left(\int_a^b \frac{1}{f^2(x)}dx\right)^2\cdot\left(\int_a^b \frac{1}{f^2(x)}dx\right)$$

$$\le\left(\int_a^b \frac{f^3(x)}{f^2(x)}dx\right)^2\cdot\int_a^b \frac{1}{f^2(x)}dx=\left(\int_a^b f(x)\,dx\right)^2\cdot\left(\int_a^b \frac{1}{f^2(x)}dx\right)$$

$$\le\left(\int_a^b \frac{f(x)}{f^2(x)}dx\right)\left(\int_a^b f(x)\,dx\right)=\left(\int_a^b \frac{1}{f(x)}dx\right)\left(\int_a^b f(x)\,dx\right)$$

$$\le\int_a^b \frac{f(x)}{f(x)}dx=\int_a^b 1\,dx=x\Big|_a^b, a\le 5<7\le b=b-a\le(b-a)^2$$

519. If $a,b,c>0, a+b+c=2$ then:

$$b^3\Omega(a)+c^3\Omega(b)+a^3\Omega(c)\ge\frac{8}{25},\Omega(a)=\lim_{n\to\infty} n\int_0^1 \frac{x^n}{(x+a)^2}dx$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

Theorem: Let $f:[0,1]\to\mathbb{R}$, f continuos. Then:

$$\lim_{n\to\infty} n\int_0^1 x^n f(x)dx=f(1)$$

In our case $f(x)=\frac{1}{(x+a)^2}\Rightarrow\Omega(a)=\frac{1}{(1+a)^2}$ ***(1)***

From (1) we must show: $\frac{b^3}{(1+a)^2}+\frac{c^3}{(1+b)^2}+\frac{a^3}{(1+c)^2}\ge\frac{8}{25}$ ***(2)***

From Hölder's inequality we have: $\frac{b^3}{(1+a)^2}+\frac{c^3}{(1+b)^2}+\frac{a^3}{(1+c)^2}\ge\frac{(a+b+c)^3}{(a+b+c+3)^2}$ ***(3)***

We have $a+b+c=2$ ***(4)***

From (3) + (4) $\Rightarrow\frac{b^3}{(1+a)^2}+\frac{c^3}{(1+b)^2}+\frac{a^3}{(1+c)^2}\ge\frac{8}{25}\Rightarrow$ ***(2) its true.***

520.

$$a,b,c>0,2e(a+b+c)=3e+2,\Omega(a)=\lim_{n\to\infty} n\left(\left(\frac{n+1}{n}\right)^{n+a}-e\right)$$

Prove that:

$$\Omega(a)\cdot\Omega(b)\cdot\Omega(c)\le\frac{1}{27}$$

Proposed by Daniel Sitaru – Romania

Solution by Sagar Kumar-Patna Bihar-India

$$L(2)=\lim_{x\to0}\frac{(1+x)^{2+\frac{1}{x}}-e}{x}$$

$$L(a)=\lim_{x\to0}\frac{(1+x)^{\frac{1}{x}}(1+x)^{a}-e}{x}=\lim_{x\to0}\frac{e\left(1-\frac{x}{2}+\frac{11}{24}x^2\right)(1+ax+\cdots)-e}{x}$$

$$=\lim_{x\to0}\frac{e\left(1+ax-\frac{x}{2}+\frac{ax^2}{2}+\cdots-1\right)}{x}=\lim_{x\to0}\frac{e(2a-1)}{2}+0(x)$$

$$L(a)=\frac{e(2a-1)}{2};\ L(a)+L(b)+L(c)\ge3\big(L(a)L(b)L(c)\big)^{-\frac{1}{3}}$$

$$L(a)L(b)L(c)\le\frac{\big(L(a)+L(b)+L(c)\big)^3}{27}$$

$L(a)+L(b)+L(c)=\frac{e}{2}(2(a+b+c)-3)$ ***and*** $a+b+c=\frac{3e+2}{2e}$

$$2(a+b+c)=\frac{3e+2}{e}$$

$L(a)+L(b)+L(c)=\frac{e}{2}\left(3+\frac{2}{e}-3\right)=1.$ ***Hence*** $\prod L(a)\le\frac{1}{27}$ ***(proved)***

521.

$$\int_b^b\frac{\sqrt[4]{1+\sin2x}-\sqrt[4]{1-\sin2x}}{\sqrt[4]{1+\sin2x}+\sqrt[4]{1-\sin2x}}dx\le\log\left|\frac{\cos a}{\cos b}\right|,\frac{\pi}{4}<a\le b<\frac{\pi}{2}$$

Proposed by Daniel Sitaru – Romania

Solution by Amit Dutta-Jamshedpur-India

$$\frac{\pi}{4}\le x<\frac{\pi}{2}\Rightarrow\tan\left(\frac{\pi}{4}\right)\le\tan x<\tan\left(\frac{\pi}{2}\right)\Rightarrow1\le\tan x<\infty\Rightarrow\tan x\ge1\Rightarrow\sin x\ge\cos x$$

$$\int_a^b\frac{\sqrt[4]{\cos^2x+\sin^2x+2\sin x\cos x}-\sqrt[4]{\cos^2x+\sin^2x-2\sin x\cos x}}{\sqrt[4]{\cos^2x+\sin^2x+2\sin x\cos x}+\sqrt[4]{\cos^2x+\sin^2x-2\sin x\cos x}}dx$$

$$=\int_a^b \frac{\sqrt[4]{(\cos x+\sin x)^2}-\sqrt[4]{(\cos x-\sin x)^2}}{\sqrt[4]{(\cos x+\sin x)^2}+\sqrt[4]{(\cos x-\sin x)^2}}dx$$

$$=\int_a^b \frac{\sqrt{(\cos x+\sin x)}-\sqrt{\sin x-\cos x}}{\sqrt{\cos x+\sin x}+\sqrt{\sin x-\cos x}}dx;\ \{\sin x>\cos x\};$$

$$=\int_a^b \frac{\sqrt{\tan x+1}-\sqrt{\tan x-1}}{\sqrt{\tan x+1}+\sqrt{\tan x-1}}dx$$

Rationalizing

$$=\int_a^b \frac{\left(\sqrt{\tan x+1}-\sqrt{\tan x-1}\right)\left(\sqrt{\tan x+1}-\sqrt{\tan x-1}\right)}{\left(\sqrt{\tan x+1}+\sqrt{\tan x-1}\right)\left(\sqrt{\tan x+1}-\sqrt{\tan x-1}\right)}dx$$

$$\int_a^b \frac{\left(\sqrt{\tan x+1}-\sqrt{\tan x-1}\right)^2}{(\tan x+1)-(\tan x-1)}dx=\int_a^b \frac{\tan x+1+\tan x-1-2\sqrt{\tan^2 x-1}}{\tan x+1-\tan x+1}dx$$

$$=\int_a^b \frac{2\left(\tan x-\sqrt{\tan^2 x-1}\right)}{2}dx=\int_a^b \tan x-\sqrt{\tan^2 x-1}\,dx$$

$$\because \tan x\ge 1\Rightarrow \tan^2 x-1\ge 0\Rightarrow \sqrt{\tan^2 x-1}\ge 0\Rightarrow -\sqrt{\tan^2 x-1}\le 0$$

$$=\int_a^b \tan x-\sqrt{\tan^2 x-1}dx\le \int_a^b \tan x\,dx\le -[\ln|\cos x|]_a^b\le -\ln\left|\frac{\cos b}{\cos a}\right|\le \ln\left|\frac{\cos a}{\cos b}\right|$$

522. If $f:\mathbb{R}\to(0,\infty)$, f continuous then:

$$\int_0^a\int_0^a\int_0^a\left(\frac{f^4(x)+f^4(y)+f^4(z)}{f^3(x)+f^3(y)+f^3(z)}\right)^5 dx\,dy\,dz\ge a^2\int_0^a f^5(x)\,dx$$

Proposed by Daniel Sitaru – Romania

Solution by Sanong Huayrerai-Nakon Pathom-Thailand

$f:\mathbb{R}\to(0,\infty)$, ***continuous, we have***

$$3\left(f^4(x)+f^4(y)+f^4(z)\right)^5\ge\left(f^4(x)+f(x)f^3(y)+f(x)f^3(z)\right)^5+$$

$$+\left(f(y)f^3(x)+f^4(y)+f(y)f^3(z)\right)^5+\left(f(z)f^3(x)+f(z)f^3(y)+f^4(z)\right)^5\ge$$

$$\geq \left(f^5(x)+f^5(y)+f^5(z)\right)\left(f^3(x)+f^3(y)+f^3(z)\right)^5$$

Hence $\left(\frac{f^4(x)+f^4(y)+f^4(z)}{f^3(x)+f^3(y)+f^3(z)}\right)^5 \geq \frac{f^5(x)+f^5(y)+f^5(z)}{3}$

Hence $\int_0^a\int_0^a\int_0^a\left(\frac{f^4(x)+f^4(y)+f^4(z)}{f^3(x)+f^3(y)+f^3(z)}\right)^5 dx\,dy\,dz \geq \frac{1}{3}\int_0^a\int_0^a\int_0^a\left(f^5(x)+f^5(y)+f^5(z)\right)dx\,dy\,dz =$

$= a^2\int_0^a f^5(x)\,dx$. *Therefore, it is to be true.*

523.

$$\Omega(n) = \int_{-1}^{1} x\,log(1+n^{3x})\,dx, n\in\mathbb{N}^*$$

Prove that:

$$9\left(1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}\right)^2 > 4e^{2\Omega(n)}\left(1+e^{\Omega(n)}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$\Omega(n) = \int_{-1}^{1} x\log(1+n^{3x})\,dx$, *let* $x=-t; dx=-dt$, *when* $x=1, t=-1$

when $x=-1, t=1$ *then* $\Omega(n) = \int_1^{-1}(-t)\log(1+n^{-3t})(-dt) = \int_{-1}^{1} t\log\left(\frac{1+n^{3t}}{n^{3t}}\right)dt$

$$= -\Omega(n) + \int_{-1}^{1} 3t^2\log n\,dx \Rightarrow 2\Omega(n) = \log n\,[t^3]_{t=-1}^{t=1} \Rightarrow \Omega(n) = \log n$$

$4e^{2\Omega(n)}\left(1+e^{\Omega(n)}\right) = 4n^2(1+n)$. *We need to prove*

$9\left(1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}\right)^2 > 4n^2(1+n)$. *For* $n=1$, *the statement is true*

From $=2, 9\left(1+\sqrt{2}\right)^2 > 4\cdot 4\cdot 3 \Rightarrow 1+\sqrt{2} > \sqrt{5.33}$, *which is true.*

Let's us assume its true for $n=k$, *then* $9\left(\sum_{p=1}^{k}\sqrt{p}\right)^2 > 4k^2(1+k)$

Now, $9\left(\sum_{p=1}^{k+1}\sqrt{p}\right)^2 = 9\left(\sum_p^k\sqrt{p}\right)^2 + 18\left(\sum_{p=1}^{k}\sqrt{p}\right)\sqrt{k+1} + 9(k+1)$

$> 4k^2(1+k) + 12\sqrt{k^2(k+1)}\cdot\sqrt{k+1} + 9(k+1) = (k+1)(2k+3)^2$

$> 4(k+1)^2(1+(1+k))$, *hence by mathematical induction*

$$9\left(1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}\right)^2 > 4e^{2\Omega(n)}\left(1+\Omega(n)\right)$$ ***(proved)***

Solution 2 by Amit Dutta-Jamshedpur-India

$$\Omega(n) = \int_{-1}^{1} x\log(1+n^{3x})\,dx$$

Using $\int_0^a F(n)\,dx = \int_0^a F(a-x)\,dx \Rightarrow \Omega(n) = \int_{-1}^{1} -x\log(1+n^{-3x})\,dx$

$\Omega(n) = \int_{-1}^{1} -x[\log(1+n^{3x}) - \log(n^{3n})]\,dx$, $\Omega(n) = -\int_{-1}^{1} x\log(1+n^{3x})\,dx + \int_{-1}^{1} x\log(n^{3x})\,dx$

$$\Omega(n) = -\Omega(n) + \int_{-1}^{1}(3x^2)\log(n)\,dx,\ 2\Omega(n) = 3\int_{-1}^{1} x^2(\log n)\,dx$$

$$2\Omega(n) = (3\log n)\int_{-1}^{1} x^2\,dx,\ 2\Omega(n) = (3\log n)\times 2\int_0^1 x^2\,dx$$

$\Omega(n) = 3\log n \times \frac{1}{3} \Rightarrow \Omega(n) = \log_e n$. ***The Inequality is***

$9\left(1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}\right)^2 > 4e^{2\Omega(n)}\left(1+e^{\Omega(n)}\right)$. ***i.e., we have to prove***

$$9\left(1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}\right)^2 > 4e^{2\ln n}\left(1+e^{\ln n}\right)$$

$$9\left(1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}\right)^2 > 4n^2(n+1)$$

Or $\left(1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}\right) > \left(\frac{2n\sqrt{n+1}}{3}\right)$ ***(1)***

Using AM of m^th power ≥ m^th power of AM: $\frac{a_1^m+a_2^m+\cdots+a_n^m}{n} \le \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^m, \forall m\in(0,1)$

Using this, $\left(\frac{1+2+3+\cdots+n}{n}\right)^{\frac{1}{2}} \ge \left(\frac{1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n}\right) \Rightarrow$

$$\Rightarrow\left[\frac{n(n+1)}{2n}\right]^{\frac{1}{2}} \ge \left(\frac{1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n}\right) > \frac{2\sqrt{n+1}}{3}$$

$\Rightarrow\left[\frac{n(n+1)}{2n}\right]^{\frac{1}{2}} > \frac{2\sqrt{n+1}}{3} \Rightarrow \frac{\sqrt{n+1}}{\sqrt{2}} > \frac{2\sqrt{n+1}}{3} \Rightarrow 3 > 2\sqrt{2} \rightarrow$ ***which is true***

Hence the inequality in (1) is true.

Solution 3 by Marian Ursărescu-Romania

$$\left.\begin{array}{r}\Omega(n)=\int_{-1}^{1}x\ln(1+n^{3x})\,dx\\ x=-t|'\Rightarrow dx=-dt\ x=-1\Rightarrow t=1\ x=1\Rightarrow t=-1\end{array}\right\}\Rightarrow$$

$$\Omega(n)=\Omega(n)=\int_{1}^{-1}-t\ln(1+n^{-3t})\,(-dt)=-\int_{1}^{-1}t\ln\left(1+\frac{1}{n^{3t}}\right)dt=$$

$$=-\int_{-1}^{1}t\ln\left(\frac{n^{3t}+1}{n^{3t}}\right)dt=-\int_{-1}^{1}t\ln(1+n^{3t})\,dt+\int_{-1}^{1}t\ln n^{3t}dt\Rightarrow$$

$$2\Omega(n)=\int_{-1}^{1}3t^2\ln n\,dt\Rightarrow 2\Omega(n)=t^3\ln n\Big|_{-1}^{1}\Rightarrow\Omega(n)=\ln n\Rightarrow$$

we must show this: $9\left(1+\sqrt{2}+\dots+\sqrt{n}\right)^2>4n^2(n+1)\Leftrightarrow$

$$1+\sqrt{2}+\dots+\sqrt{n}>\frac{2n\sqrt{n+1}}{3},\forall n\geq 1$$

$P(1):1>\frac{2\sqrt{2}}{3}\Leftrightarrow 3>2\sqrt{2}$ ***true.***

Now: $P(k):1+\sqrt{2}+\dots+\sqrt{k}>\frac{2k\sqrt{k+1}}{3}$

$$P(k+1):1+\sqrt{2}+\dots+\sqrt{k+1}>\frac{2(k+1)\sqrt{k+2}}{3}$$

From $P(k)\Rightarrow 1+\sqrt{2}+\dots+\sqrt{k}+\sqrt{k+1}>\frac{2k\sqrt{k+1}}{3}+\sqrt{k+1}$

We must show this: $\frac{2k\sqrt{k+1}}{3}+\sqrt{k+1}>\frac{2(k+1)\sqrt{k+2}}{3}\Leftrightarrow$

$2k+3>2\sqrt{(k+1)(k+2)}\Leftrightarrow 4k^2+12k+9>4k^2+12k+8\Leftrightarrow 9>8$ ***true.***

524. If $0<a\leq b\leq\frac{\pi}{4}$ then:

$$\int_a^b\int_a^b\int_a^b\sin x\cdot\sin y\cdot\sin z\cdot\sin(x+y+z)\,dxdydz\leq\frac{(b-a)^3}{4}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$\sin(x+y+z) \le 1$ *then* $\int_a^b \int_a^b \int_a^b \sin x \cdot \sin y \cdot \sin z \cdot \sin(x+y+z) dxdydz$

$$\le \int_a^b \int_a^b \int_a^b \sin x \cdot \sin y \cdot \sin z \, dxdydz = \left(\int_a^b \sin x \, dx\right)\left(\int_a^b \sin y \, dy\right)\left(\int_a^b \sin z \, dz\right)$$

$= (\cos a - \cos b)^3$, *we need to prove* $(\cos a - \cos b)^3 \le \frac{(b-a)^3}{4}$

now $\sin x$ *is concave on* $\left[0, \frac{\pi}{4}\right]$ *so applying Hermite Hadamard*

$$(b-a)\frac{\sin a + \sin b}{2} \le \int_a^b \sin x \, dx \le (b-a)\sin\left(\frac{a+b}{2}\right) \Rightarrow$$

$\Rightarrow \cos a - \cos b \le \frac{b^2-a^2}{2}$ *need to show* $\frac{(b-a)^3(b+a)^3}{8} \le \frac{(b-a)^3}{4}$

$\Leftrightarrow (b-a)^3\{2-(a+b)^3\} \ge 0$, *which is true* $\begin{bmatrix} since, a+b \le \frac{\pi}{8} \\ \left(\frac{\pi}{8}\right)^3 < 2 \end{bmatrix}$

$\therefore \int_a^b \int_a^b \int_a^b \sin x \cdot \sin y \cdot \sin z \cdot \sin(x+y+z) \, dx \, dy \, dz \le \frac{(b-a)^3}{4}$ *(proved)*

Solution 2 by Amit Dutta-Jamshedpur-India

Let $P = \sin x \cdot \sin y \cdot \sin z \cdot \sin(x+y+z)$

Let $F(u) = \sin u \Rightarrow F'(u) = \cos u; F''(u) = -\sin u \le 0, \forall u \in \left(0, \frac{\pi}{4}\right]$

$\Rightarrow F(u)$ *is a concave function* $\forall u \in \left(0, \frac{\pi}{4}\right] \Rightarrow \frac{F(x)+F(y)+F(z)}{3} \le F\left(\frac{x+y+z}{3}\right) \Rightarrow$

$\Rightarrow \frac{\sin x + \sin y + \sin z}{3} \le \sin\left(\frac{x+y+z}{3}\right)$ *(1)*

Using GM $\le$ *AM:* $(\sin x \cdot \sin y \cdot \sin z)^{\frac{1}{3}} \le \frac{\sin x + \sin y + \sin z}{3}$

$\Rightarrow \sin x \cdot \sin y \cdot \sin z \le \left(\frac{\sin x + \sin y + \sin z}{3}\right)^3$ *(2)*

From (1) & (2): $\sin x \cdot \sin y \cdot \sin z \le \sin^3\left(\frac{x+y+z}{3}\right) \therefore P \le \left\{\sin\left(\frac{x+y+z}{3}\right)\right\}^3 \sin(x+y+z)$

Let $\left(\frac{x+y+z}{3}\right) = t \therefore P \le (\sin^3 t)(\sin 3t); P \le (\sin^3 t)(3\sin t - 4\sin^3 t)$

$P \le 3\sin^4 t - 4\sin^6 t$ *(3)*

Let $G(t) = 3\sin^4 t - 4\sin^6 t$

$G'(t) = 12\sin^3 t \cos t - 24\sin^5 t \cos t = 12\sin^3 t \cos t\,(1 - 2\sin^2 t) =$

$$= 12\sin^3 t\cos t\,[1 - 2(1 - \cos^2 t)]$$

$G'(t) = 12\sin^3 t\cos t\,(2\cos^2 t - 1)$. *We have* $0 < t \le \frac{\pi}{4} \Rightarrow \cos t \ge \cos\frac{\pi}{4}$

$\cos^2 t \ge \frac{1}{2} \Rightarrow (2\cos^2 t - 1) \ge 0 \Rightarrow G'(t) \ge 0\ \ \forall t \in \left(0, \frac{\pi}{4}\right] \Rightarrow G(t)$ *is an increasing function so,* $G(t)$ *is maximum at* $t = \frac{\pi}{4} \therefore [G(t)]_{\max} = (3\sin^4 t - 4\sin^6 t)$ *at* $t = \frac{\pi}{4}$

$$[G(t)]_{\max} = G\left(\frac{\pi}{4}\right) = \frac{1}{4} \therefore G(t) \le \frac{1}{4} \quad \textit{(4)}$$

From (3) & (4): $P \le 3\sin^4 t - 4\sin^6 t \le \frac{1}{4} \therefore P \le \frac{1}{4}$

$$\int_a^b\int_a^b\int_a^b P\,dx\,dy\,dz \le \int_a^b\int_a^b\int_a^b \frac{1}{4}dx\,dy\,dz \le \frac{1}{4}\left(\int_a^b dx\right)\left(\int_a^b dy\right)\left(\int_a^b dz\right) \le \frac{(b-a)^3}{4}$$

$$\Rightarrow \int_a^b\int_a^b\int_a^b \sin x\cdot\sin y\cdot\sin z\cdot\sin(x+y+z)\,dx\cdot dy\cdot dz \le \frac{(b-a)^3}{4}$$

525. Prove that:

$$\int_{-\infty}^{\infty} \frac{\sin\left(\theta + \frac{\pi}{2}\right)d\theta}{1+\theta^2} \le \frac{\pi}{2e\sqrt{e}}\left(e^{\sin^2\theta} + e^{\cos^2\theta}\right), \theta \in \mathbb{R}$$

Proposed by Surjeet Singhania-India

Solution by Shafiqur Rahman-Bangladesh

$$\int_{-\infty}^{\infty} \frac{\sin\left(\theta + \frac{\pi}{2}\right)d\theta}{1+\theta^2} = \int_{-\infty}^{\infty} \frac{\cos\theta d\theta}{1+\theta^2} = \int_{-\infty}^{\infty} \frac{e^{i\theta}d\theta}{1+\theta^2} = 2\pi \times \left(\frac{e^{i\theta}}{\theta+i}\right)_{\theta=i} = \frac{\pi}{e} =$$

$$= \frac{\pi}{e\sqrt{e}}\sqrt{e^{\sin^2\theta}\cdot e^{\cos^2\theta}} \le \frac{\pi}{2e\sqrt{e}}\left(e^{\sin^2\theta} + e^{\cos^2\theta}\right)$$

526. If $0 < a \le b$ then:

$$\int_a^b\int_a^b\left(\sqrt{\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}}\right)dx\,dy \geq \sqrt{3}\cdot log\left(\frac{b}{a}\right)^{b-a}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Togrul Ehmedov-Baku-Azerbaijan

$$\int_a^b\int_a^b\sqrt{\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}}\,dx\,dy \geq \int_a^b\int_a^b\frac{\sqrt{3}}{\sqrt{xy}}\,dx\,dy$$

$$\geq \sqrt{3}\int_a^b\int_a^b\frac{1}{\sqrt{xy}}dxdy \geq \frac{3}{2}\int_a^b\int_a^b\left(\frac{1}{x}+\frac{1}{y}\right)dxdy \geq \frac{3}{2}\int_a^b\left[\ln\left(\frac{b}{a}\right)+\frac{1}{y}(b-a)\right]dy$$

$$\geq 3(b-a)\ln\left(\frac{b}{a}\right) = 3\ln\left(\frac{b}{a}\right)^{b-a}$$

Solution 2 by Tran Hong-Vietnam

For $x, y > 0$ *we have* $\sqrt{\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}} \geq \frac{\sqrt{3}}{2}\left(\frac{1}{x}+\frac{1}{y}\right) \Leftrightarrow \frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2} \geq \frac{3}{4}\left(\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}\right)$

$$\Leftrightarrow \frac{1}{4x^2}-\frac{1}{2xy}+\frac{1}{4y^2} \geq 0 \Leftrightarrow \left(\frac{1}{2x}-\frac{1}{2y}\right)^2 \geq 0 \quad \textit{(true)}$$

$$\Rightarrow \int_a^b\int_a^b\sqrt{\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}}\,dx\,dy \geq \int_a^b\int_a^b\frac{\sqrt{3}}{2}\left(\frac{1}{x}+\frac{1}{y}\right)dx\,dy$$

$$= \frac{\sqrt{3}}{2}\int_a^b\int_a^b\left(\frac{1}{x}+\frac{1}{y}\right)dx\,dy = \frac{\sqrt{3}}{2}\int_a^b\left[\left(\ln x+\frac{1}{y}x\right)\Big|_a^b\right]dy$$

$$= \frac{\sqrt{3}}{2}\int_a^b\left[\ln\left(\frac{b}{a}\right)+\frac{b-a}{y}\right]dy = \frac{\sqrt{3}}{2}\left[\ln\left(\frac{b}{a}\right)y+(b-a)\ln y\right]\Big|_a^b$$

$$= \sqrt{3}(b-a)\ln\left(\frac{b}{a}\right) = \sqrt{3}\ln\left(\frac{b}{a}\right)^{b-a}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2} \geq \frac{3}{4}\left(\frac{1}{x}+\frac{1}{y}\right)^2 \Rightarrow \sqrt{\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}} \geq \frac{\sqrt{3}}{2}\left(\frac{1}{x}+\frac{1}{y}\right)$$

$$\int_a^b \int_a^b \sqrt{\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}}\,dx\,dy \ge \frac{\sqrt{3}}{2}\int_a^b \left(\int_a^b \frac{dx}{x}\right)dy + \frac{\sqrt{3}}{2}\int_a^b \left(\int_a^b \frac{dy}{y}\right)dx = \sqrt{3}\log\left(\frac{b}{a}\right)^{b-a}$$

Solution 4 by Artan Ajredini-Presheva-Serbie

$$\left(\frac{1}{x}-\frac{1}{y}\right)^2 \ge 0 \Rightarrow \frac{1}{x^2}+\frac{1}{y^2} \ge \frac{2}{xy} \Rightarrow \sqrt{\left(\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}\right)} \ge \sqrt{3}\sqrt{\frac{1}{xy}}$$

WLOG we suppose that $x \ge y$. So,

$$\int_a^b \int_a^b \left(\sqrt{\left(\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}\right)}\right)dx\,dy \ge \int_a^b \int_a^b \sqrt{3}\frac{1}{\sqrt{xy}}dx\,dy \ge$$

$$\ge \sqrt{3}\int_a^b \int_a^b \frac{dx\,dy}{x} = \sqrt{3}(b-a)\log\left(\frac{b}{a}\right) = \sqrt{3}\log\left(\frac{b}{a}\right)^{b-a}$$

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

For $0 < a \le b$, we have: $\sqrt{\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}} \ge \sqrt{\frac{3}{4}\left(\frac{1}{x}+\frac{1}{y}\right)^2} = \frac{\sqrt{3}}{2}\left(\frac{1}{x}+\frac{1}{y}\right)$. *Hence*

$$\int_a^b \int_a^b \left(\sqrt{\left(\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}\right)}\right)dx\,dy \ge \int_a^b \int_a^b \frac{\sqrt{3}}{2}\left(\frac{1}{x}+\frac{1}{y}\right)dx\,dy = \frac{\sqrt{3}}{2}(x\ln y + y\ln x)\Big|_a^b =$$

$$= \frac{\sqrt{3}}{2}(b-a)(\ln y + \ln x)\Big|_a^b = \frac{\sqrt{3}}{2}(b-a)(\ln b - \ln a) = \sqrt{3}(b-a)\ln\left(\frac{b}{a}\right)$$

$$\ge \sqrt{3}(b-a)\log\left(\frac{b}{a}\right) = \sqrt{3}\log\left(\frac{b}{a}\right)^{(b-a)}$$ *. Therefore, it is to be true.*

527.

$$Ci(x) = \gamma + \log x + \sum_{n=1}^{\infty} \frac{(-x^2)^n}{2n \cdot (2n)!}$$

Prove that:

$$\int_a^b \left(\frac{\cos x}{x}\right)^2 dx + \int_a^b Ci^2(x)dx \ge Ci^2(b) - Ci^2(a),\ 0 < a \le b$$

Proposed by Daniel Sitaru – Romania

Solution by Avishek Mitra-India

As we know

$$Ci(x)=\gamma+\log x+\sum_{n=1}^{\infty}\frac{(-x^2)^n}{2n(2n)!}\Rightarrow Ci'(x)=\frac{1}{x}+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot x^{2n-1}}{(2n)!}=$$

$$=\frac{1}{x}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\infty\right)=\frac{\cos x}{x}$$

Now, by AM ≥ AM: $\frac{\left(\frac{\cos x}{x}\right)^2+Ci^2(x)}{2}\ge\frac{\cos x}{x}\cdot Ci(x)\Rightarrow$

$$\Rightarrow\frac{1}{2}\int_a^b S\,dx\ge\int_a^b Ci(x)\frac{\cos x}{x}dx\ge\int_a^b Ci(x)\,d\{Ci(x)\}\ge\frac{Ci^2(b)-Ci^2(a)}{2}$$

Hence $\int_a^b\left(\frac{\cos x}{x}\right)^2dx+\int_a^b Ci^2(x)\ge Ci^2(b)-Ci^2(a)$

528. If $a\ge 0$ then:

$$\int_0^a\frac{x^2dx}{(e^x+1)^2}+\int_0^a\frac{(e^x(1-x)+1)^2}{(e^x+1)^4}dx\ge\frac{a^2}{(e^a+1)^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Avishek Mitra-West Bengal-India

By AM ≥ GM:

$$\frac{\frac{x^2}{(e^x+1)^2}+\frac{[e^x(1-x)+1]^2}{(e^x+1)^4}}{2}\ge\frac{x[e^x(1-x)+1]}{(e^x+1)^3}\Rightarrow$$

$$\Rightarrow\frac{S}{2}\ge\frac{ex^x(1-x)+x}{(e^x+1)^3}\Rightarrow\int_0^a\frac{S}{2}dx\ge\int_0^a\frac{x(e^x+1)-x^2e^x}{(e^x+1)^3}dx$$

Now, $I=\int_0^a\frac{x^2e^x}{(e^x+1)^3}dx=\left[-x^2\cdot\frac{1}{2(e^x+1)^2}\right]_0^a-2\int_0^a\frac{x}{-2(e^x+1)^2}dx=-\frac{a^2}{2(e^a+1)^2}+\int_0^a\frac{x}{(e^x+1)^2}dx$

Hence, we have $\Rightarrow\int_0^a\frac{S}{2}dx\ge\frac{a^2}{2(e^a+1)^2}$

$$\Rightarrow \int_0^a \frac{x^2}{(e^x+1)^2}dx + \int_0^a \frac{[e^x(1-x)+1]^2}{(e^x+1)^4}dx \geq \frac{a^2}{(e^a+1)^2}$$

Solution 2 by Tran Hong-Vietnam

$$\text{Let } f(a) = \int_0^a \frac{x^2 dz}{(e^x+1)^2} + \int_0^a \frac{[e^x(1-x)+1]^2}{(e^x+1)^4} - \frac{a^2}{(e^a+1)^2}; \forall a \geq 0$$

$$\Rightarrow f'(a) = \frac{a^2}{(e^a+1)^2} + \frac{(e^a(1-a)+1)^2}{(e^a+1)^4} + \frac{2a(e^a(a-1)-1)}{(e^a+1)^3}$$

$$= \frac{(a+e^a(2a-1)-1)^2}{(e^a+1)^4} \geq 0; \forall a \geq 0 \Rightarrow f(a) \nearrow [0;+\infty) \Rightarrow f(a) \geq f(0) = 0 \Rightarrow \textit{Proved.}$$

529. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\left(\int_0^{\sqrt{ab}} (\sqrt[3]{x}\cdot \sin x)\,dx\right)\left(\int_0^{\frac{a+b}{2}} (\sqrt[3]{x}\cdot \cos x)dx\right) \leq \left(\int_0^{\sqrt{ab}} (\sqrt[3]{x}\cdot \cos x)\,dx\right)\left(\int_0^{\frac{a+b}{2}} (\sqrt[3]{x}\cdot \sin x)\,dx\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Vietnam

$$\text{Let } u = \sqrt{ab}, v = \frac{a+b}{2} \Rightarrow 0 < u \leq v < \frac{\pi}{2}$$

$$f(v) = \left(\int_0^u \sqrt[3]{x}\cos x\,dx\right)\left(\int_0^v \sqrt[3]{x}\sin x\,dx\right) - \left(\int_0^u \sqrt[3]{x}\sin x\,dx\right)\left(\int_0^v \sqrt[3]{x}\cos x\,dx\right)$$

$$\Rightarrow f'(v) = \sqrt[3]{v}\sin v \cdot \int_0^u \sqrt[3]{x}\cos x\,dx - \sqrt[3]{v}\cos v \int_0^u \sqrt[3]{x}\sin x\,dx$$

$$= \sqrt[3]{v}\left(\sin v \int_0^u \sqrt[3]{x}\cos x\,dx - \cos v \int_0^u \sqrt[3]{x}\sin x\,dx\right)$$

$$g(v) = \sin v \int_0^u \sqrt[3]{x}\cos x\,dx - \cos v \int_0^u \sqrt[3]{x}\sin x\,dx$$

$$\Rightarrow g'(v) = \cos v \int_0^u \sqrt[3]{x}\cos x\,dx + \sin x \int_0^u \sqrt[3]{x}\sin x\,dx > 0$$

$$g(0)\cdot g\left(\frac{\pi}{2}\right)<0\Rightarrow\exists!\,v_0\in\left(0,\frac{\pi}{2}\right):g(v_0)=0\Rightarrow f(v)\ge f(v_0)\ge f(u)=0\Rightarrow \textit{Proved.}$$

530. If $a,b,c>1, a+b+c=6$ then:

$$\frac{\Gamma'(a)}{\Gamma(a)}+\frac{\Gamma'(b)}{\Gamma(b)}+\frac{\Gamma'(c)}{\Gamma(c)}+\frac{ab+bc+ca}{2abc}<3\log 2$$

Proposed by Daniel Sitaru – Romania

Solution by Avishek Mitra-West Bengal-India

$$\frac{\Gamma'(a)}{\Gamma(a)}+\frac{\Gamma'(b)}{\Gamma(b)}+\frac{\Gamma'(c)}{\Gamma(c)}=\psi(a)+\psi(b)+\psi(c)$$

$$=\psi(a+1)+\psi(b+1)+\psi(c+1)-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}$$

$$=\log a+\log b+\log c-\frac{1}{2a}-\frac{1}{2b}-\frac{1}{2c}-\sum_{n=1}^{\infty}\frac{B_{2n}}{2n\cdot a^{2n}}-\sum_{n=1}^{\infty}\frac{B_{2n}}{2n\cdot b^{2n}}-\sum_{n=1}^{\infty}\frac{B_{2n}}{2n\cdot c^{2n}}$$

Now, given $\Omega=\frac{\Gamma'(a)}{\Gamma(a)}+\frac{\Gamma'(b)}{\Gamma(b)}+\frac{\Gamma'(c)}{\Gamma(c)}+\frac{ab+bc+ca}{2abc}$

$$=\log(abc)-\frac{1}{2a}-\frac{1}{2b}-\frac{1}{2c}+\frac{ab+bc+ca}{2abc}-S_1-S_2-S_3$$

$$=\log(abc)-S_1-S_2-S_3$$

as $a,b,c>1$ ***and*** $a+b+c=6$, ***for equality we may take*** $a=b=c=2$, ***by putting values*** $\Leftrightarrow\Omega=\log(2\cdot2\cdot2)-S_1-S_2-S_3=3\log2-(S_1+S_2+S_3)$

Clearly $\Omega<3\log 2$ ***(proved)***

531. Prove that:

$$\ln\left(\int_0^{\frac{\pi}{2}}\left(\frac{8^{\sin x}}{3^{\sin x}+4^{\sin x}}+\frac{27^{\sin x}}{2^{\sin x}+4^{\sin x}}+\frac{64^{\sin x}}{2^{\sin x}3^{\sin x}}\right)dx\right)>\ln\left(\frac{9\left((4!)^{\frac{2}{3}}-1\right)}{4\ln(4!)}\right)$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Soumitra Mandal-Chandar Nagore-India

$$\ln\left(\int_0^{\frac{\pi}{2}}\left(\frac{8^{\sin x}}{3^{\sin x}+4^{\sin x}}+\frac{27^{\sin x}}{2^{\sin x}+4^{\sin x}}+\frac{64^{\sin x}}{2^{\sin x}+3^{\sin x}}\right)dx\right)$$

$$\overset{HOLDER'S\ INEQUALITY}{\geq} \ln\left(\int_0^{\frac{\pi}{2}} \frac{(2^{\sin x}+3^{\sin x}+4^{\sin x})^3}{6(2^{\sin x}+3^{\sin x}+4^{\sin x})}dx\right) = \ln\left(\frac{1}{6}\int_0^{\frac{\pi}{2}}\left(\sum_{cyc} 2^{\sin x}\right)^2 dx\right)$$

$$\overset{AM\geq GM}{\geq} \ln\left(\frac{9}{6}\int_0^{\frac{\pi}{2}}\left(\sqrt[3]{2^{\sin x}\cdot 3^{\sin x}\cdot 4^{\sin x}}\right)^2 dx\right) = \ln\left(\frac{3}{2}\int_0^{\frac{\pi}{2}}(4!)^{\frac{2}{3}\sin x}dx\right)$$

$$\geq \ln\left(\frac{3}{2}\int_0^{\frac{\pi}{2}}(4!)^{\frac{2}{3}\sin x}\cdot\cos x\,dx\right) = \ln\left(\frac{3}{2}\left[\frac{(4!)^{\frac{2}{3}\sin x}}{\frac{2}{3}\ln(4!)}\right]_{x=0}^{x=\frac{\pi}{2}}\right) = \ln\left(\frac{9\left((4!)^{\frac{2}{3}}-1\right)}{4\ln(4!)}\right)$$

532. $f:(0,\infty)\to(1,\infty)$, f **– continuous. Prove that if** $0<a\leq b$ **then:**

$$4(b-a)^3+6(b-a)^2\int_a^b \log(f(x))\,dx \leq 3(b-a)^2\int_a^b f(x)dx + \left(\int_a^b f(x)dx\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Vietnam

We have: $\log t \leq t-1, \forall t\geq 1$ ***(*). In fact: Let*** $\varphi(t)=t-1-\log t\ \ (t\geq 1)$

$\varphi'(t)=1-\frac{1}{t}=\frac{t-1}{t}\geq 0 \Rightarrow \varphi(t)\nearrow[1;+\infty)\Rightarrow\varphi(t)\geq\varphi(1)=0\Rightarrow$ ***(*) true.***

Now, let $m=\int_a^b f(x)\,dx\ \ (f(x)\geq 1\Rightarrow m\geq b-a)$; ***RHS*** $=3(b-a)^2m+m^3$

$$LHS \overset{(*)}{\leq} 4(b-a)^3+6(b-a)^2\left\{\int_a^b[f(x)-1]\,dx\right\}$$

$$=4(b-a)^3+6(b-a)^2[m-(b-a)]=6(b-a)^2m-2(b-a)^3$$

Must show that: $m^3+3(b-a)^2m\geq 6(b-a)^2m-2(b-a)^3$

$$\Leftrightarrow m^3-3(b-a)^2m+2(b-a)^3\geq 0\quad (**)$$

Let $f(m)=m^3-3(b-a)^2m+2(b-a)^3\quad (m\geq b-a)$

$$\because f'(m)=3m^2-3(b-a)^2\geq 3(b-a)^2-3(b-a)^2=0\Rightarrow f(m)\nearrow[b-a;+\infty]$$

$$\Rightarrow f(m)\geq f(b-a)=(b-a)^3-3(b-a)^3+2(b-a)^3=0$$

$\Rightarrow$ ***(**) true*** $\Rightarrow LHS\leq RHS.$

533. If $a, b \in \mathbb{R}$ then:

$$b^3 + 6\int_a^b (\tan^{-1} x)\, dx \geq 3\log\left(\frac{1+b^2}{1+a^2}\right) + a^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Khaled Abd Almuti-Damascus-Syria

If $a, b \in \mathbb{R}$, then: (suppose $a \leq b$) then: $b^3 + 6\int_a^b (\arctan x)\, dx \geq 3\log\left(\frac{1+b^2}{1+a^2}\right) + a^3$

$$l_1 = b^3 + 6\left[\arctan x - \frac{1}{2}\ln(1+x^2)\right]_a^b$$

$$l_1 = b^3 + 6\left[\left(\arctan b - \frac{1}{2}\ln(1+b^2)\right) - \left(\arctan a - \frac{1}{2}\ln(1+a^2)\right)\right]$$

$$l_1 = b^3 + 6\left[\arctan b - \arctan a + \frac{1}{2}\ln\left(\frac{1+a^2}{1+b^2}\right)\right]$$

$$l_1 = b^3 + 6(\arctan b - \arctan a) = 3\ln\left(\frac{1+a^2}{1+b^2}\right)$$

$$l_1 = b^3 + 6(\arctan b - \arctan a) - 3\ln\left(\frac{1+b^2}{1+a^2}\right)$$

$$b^3 + 6(\arctan b - \arctan a) - 3\ln\left(\frac{1+b^2}{1+a^2}\right) \overset{?}{\geq} 3\log\left(\frac{1+b^2}{1+a^2}\right) + a^3$$

$$6(\arctan b - \arctan a) \overset{?}{\geq} 6\log\left(\frac{1+b^2}{1+a^2}\right) + a^3 - b^3$$

$$\arctan b - \arctan a \overset{?}{\geq} \log\left(\frac{1+b^2}{1+a^2}\right) + \frac{1}{6}(a^3 - b^3)$$

$$\arctan b - \arctan a \geq \log(1+b^2) - \log(1+a^2) + \frac{1}{8}a^3 - \frac{1}{6}b^3$$

$$\arctan b - \log(1+b^2) + \frac{1}{6}b^3 \overset{?}{\geq} \arctan a - \log(1+a^2) + \frac{1}{6}a^3$$

Let be the function: $f(x) = \arctan x - \log(1+x^2) + \frac{1}{6}x^3$

f *is derivable on* $\mathbb{R} =]-\infty, \infty[$

$$f'(x)=\frac{1}{1+x^2}-\frac{2x}{1+x^2}+\frac{1}{2}x^2=\frac{1-2x}{1+x^2}+\frac{x^2}{2}$$

$$f'(x)=\frac{1-2x+x^2+x^4}{2(1+x^2)}=\frac{x^4+x^2-2x+1}{2(1+x^2)}=\frac{x^4+(x-1)^2}{2(1+x^2)}>0$$

So, f is completely increasing on $\mathbb{R}$. So: If $a\le b\Rightarrow f(b)\ge f(a)$

$$\arctan b-\log(1+b^2)+\frac{1}{6}b^3\ge\arctan a-\log(1+a^2)+\frac{1}{6}a^3$$

Solution 2 by Tran Hong-Vietnam

$$\because \textit{Suppose } b\ge a;\ \textit{let } I=\int_a^b(\tan^{-1}x)\,dx;$$

$$\begin{cases}u=\tan^{-1}x\\ dv=dx\end{cases}\Rightarrow\begin{cases}du=\dfrac{dx}{x^2+1}\\ v=x\end{cases}\Rightarrow I=x\tan^{-1}x\Big|_a^b-\int_a^b\frac{x\,dx}{x^2+1}$$

$$=(b\tan^{-1}b-a\tan^{-1}a)-\frac{1}{2}\log(x^2+1)\Big|_a^b$$

$$=(b\tan^{-1}b-a\tan^{-1}a)-\frac{1}{2}\log\left(\frac{b^2+1}{a^2+1}\right)$$

$$\textit{Inequality}\Leftrightarrow b^3+6(b\tan^{-1}b-a\tan^{-1}a)\ge 6\log\left(\frac{b^2+1}{a^2+1}\right)+a^3$$

$$\Leftrightarrow b^3+6[b\tan^{-1}b-\log(b^2+1)]\ge a^3+6[a\tan^{-1}a-\log(a^2+1)]\quad (*)$$

$$\textit{Let } f(x)=x^3+6[x\tan^{-1}x-\log(x^2+1)]\ \forall x\in\mathbb{R};\ f'(x)=3x^2+6\left[\tan^{-1}x-\frac{x}{x^2+1}\right]$$

$$f''(x)=6x\left[\frac{2x}{(x^2+1)^2}+1\right];f''(x)=0\Leftrightarrow x=0\ \left(\because\frac{2x}{(x^2+1)^2}+1\ge 1-\frac{3\sqrt{3}}{8}>0\right)$$

$$\Rightarrow f'(x)\ge f'(0)=0\Rightarrow f(x)\nearrow \textit{on } \mathbb{R}.$$

Hence, $b\ge a\Rightarrow f(b)\ge f(a)\Rightarrow$ (*) true $\Rightarrow$ proved.

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\textit{Let } f(x)=x^2+2(\tan^{-1}x)-\frac{2x}{1+x^2}\ \textit{for all } x\in\mathbb{R}$$

$$f'(x)=2x+\frac{4x^2}{(1+x^2)^2},f''(x)=2+\frac{8x}{(1+x^2)^2}-\frac{32x^2}{(1+x^2)^3}$$

For $f'(\alpha)=0\Rightarrow\alpha=0$ then $f''(\alpha)=2>0$. Hence f attains minimum at $x=0$.

$$\therefore f(x)\ge f(0)=0\Rightarrow x^2+2(\tan^{-1}x)\ge\frac{2x}{1+x^2}$$

$$\Rightarrow \int_a^b x^2 dx + 2\int_a^b \tan^{-1} x\, dx \geq \int_a^b \frac{2x}{1+x^2} dx$$

$$\Rightarrow \frac{b^3 - a^3}{3} + 2\int_a^b \tan^{-1} x\, dx \geq \ln\left(\frac{1+b^2}{1+a^2}\right)$$

$$\therefore b^3 + 6\int_a^b \tan^{-1} x\, dx \geq 3\ln\left(\frac{1+b^2}{1+a^2}\right) + a^3$$

534.

$$\alpha(x) = \frac{4}{3}\sum_{n=1}^{\infty}\left(\left(-\frac{1}{3}\right)^n \cdot \cos^3(3^{n-1}x)\right), x \in \left(0, \frac{\pi}{2}\right), \beta(x) = \alpha\left(\frac{\pi}{2} - x\right)$$

Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\lim_{\substack{x\to 0\\ x>0}} \left(\frac{\beta(x)\beta(3x)\cdot\ldots\cdot\beta((2n-1)x)}{\beta(2x)\beta(4x)\cdot\ldots\cdot\beta(2nx)}\right)}$$

Proposed by Daniel Sitaru – Romania

Solution by Sagar Kumar-Kolkata-India

$$\cos^3 x = \frac{1}{4}(\cos(3x) + 3\cos x);\ \cos^3(3^{n-1}x) = \frac{1}{4}\left((\cos 3^n x) + 3\cos(3^{n-1}x)\right)$$

$$\alpha(x) = \frac{1}{4} \times \frac{4}{3}\left(\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^n (\cos 3^n x + 3\cos(3^{n-1}x))\right)$$

$$\alpha(x) = \frac{1}{3}\left(-\frac{1}{3}(\cos 3x + 3\cos x) + \frac{1}{9}(\cos 9x + 3\cos 3x)\right) - \frac{1}{27}(\cos 27x + 3\cos(9x)) \ldots \alpha(x) = \frac{1}{3}\begin{Bmatrix} -\frac{\cos 3x}{3} - \cos x \\ \frac{\cos 9x}{9} + \frac{\cos 3x}{3} \\ -\frac{\cos 27x}{27} - \frac{\cos 9x}{9} \end{Bmatrix}$$

$$\alpha(x) = \frac{1}{3}\left(\lim_{n\to\infty}\left(-\frac{1}{3}\right)^n \cos(3^n x) - \cos x\right),\ \alpha(x) = -\frac{\cos x}{3}; \alpha\left(\frac{\pi}{2} - x\right) = \frac{-\sin x}{3}$$

$$\beta(x) = -\frac{\sin x}{3},\quad \frac{\beta(x)\beta(3x)\ldots\beta(2n-1)x}{\beta(2x)\beta(4x)\ldots\beta(nx)} = M$$

$$\lim_{x\to 0} M = \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\ldots\cdot(2n)} = N,\ \Omega = \lim_{n\to\infty}(N)^{\frac{1}{n}}$$

$$\Omega = \lim_{n\to\infty}\left(\frac{2n!}{(2\cdot 4\cdot 6\cdot 8\cdot\ldots\cdot(2n))^2}\right)^{\frac{1}{n}},\ \Omega = \lim_{n\to\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^{\frac{1}{n}};\ \Omega = \frac{1}{4}\lim_{n\to\infty}(2nC_n)^{\frac{1}{n}}$$

Using Stirling approximation $n!\sim\sqrt{2n\pi}\left(\frac{n}{e}\right)^n$, $\Omega=\frac{1}{4}\lim_{n\to\infty}\left(\frac{\sqrt{4\pi n}\cdot\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2n\pi}\right)\left(\frac{n}{e}\right)^{2n}}\right)^{\frac{1}{n}}$

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{\sqrt{n\pi}}\right)^{\frac{1}{n}}=1$$

535.

$$a,b,c>0, 2e(a+b+c)=3e+2, \Omega(a)=\lim_{n\to\infty}n\left(\left(\frac{n+1}{n}\right)^{n+a}-e\right)$$

Prove that:

$$\Omega(a)\cdot\Omega(b)\cdot\Omega(c)\le\frac{1}{27}$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

$$\Omega(a)=\lim_{n\to\infty}n\left(\left(\frac{n+1}{n}\right)^{n+a}-e\right)=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n+a}-e}{\frac{1}{n}}\overset{Heine}{=}$$

$$\lim_{n\to\infty}\frac{\left(1+\frac{1}{x}\right)^{x+a}-e}{\frac{1}{x}}=\lim_{y\to 0}\frac{(1+y)^{\frac{1}{y}+a}-e}{y}\overset{L'H}{=}$$

$$\lim_{y\to 0}(1+y)^{\frac{1}{y}+a}\left[-\frac{1}{y^2}\ln(1+y)+\left(\frac{1}{y}+a\right)\frac{1}{1+y}\right]=e\lim_{y\to 0}\left[-\frac{\ln(1+y)}{y^2}+\frac{1+ay}{y(1+y)}\right]=$$

$$=e\lim_{y\to 0}\frac{-(1+y)\ln(1+y)+y+ay^2}{y^3+y^2}\overset{L'H}{=}e\lim_{y\to 0}\frac{-\ln(1+y)-1+1+2ay}{3y^2+2y}\overset{L'H}{=}$$

$$=e\lim_{y\to 0}\frac{-1+2a}{6y+2}=\frac{e(2a-1)}{2}\qquad \textit{(1)}$$

From (1) we must show: $\frac{e^3}{8}(2a-1)(2b-1)(2c-1)\le\frac{1}{27}\Leftrightarrow$

$$\frac{e}{2}\sqrt[3]{(2a-1)(2b-1)(2c-1)}\le\frac{1}{3}\qquad \textit{(2)}$$

But $\sqrt[3]{(2a-1)(2b-1)(2c-1)}\le\frac{2(a+b+c)-3}{3}$ ***(3)***

From (2)+(3) we must show: $\frac{e}{2}\left[\frac{2(a+b+c)}{3}-3\right]\le\frac{1}{3}\Leftrightarrow 2e(a+b+c)-3e\le 2\Leftrightarrow$

$\Leftrightarrow 2e(a+b+c) \le 3e+2$ ***which its true.***

536. Find:

$$\Omega = \lim_{n\to\infty} \int_{\frac{1}{n}}^{n} \frac{1}{x}\left(\frac{\cos 3x}{1+3x} - \frac{\cos 2x}{1+2x}\right) dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Omran Kouba-Damascus-Syria

Consider a continuous function $f:(0,\infty)\to\mathbb{R}$ ***such that:***

$$\lim_{x\to 0} f(x) = \alpha, \lim_{x\to\infty} f(x) = \beta$$

Then, for positive a, b ***we can write:***

$$\int_{\frac{1}{n}}^{n} \frac{f(ax)-f(bx)}{x} dx = \int_{\frac{1}{n}}^{n} \frac{f(ax)}{x} dx - \int_{\frac{1}{n}}^{n} \frac{f(bx)}{x} dx = \int_{\frac{a}{n}}^{an} \frac{f(x)}{x} dx - \int_{\frac{b}{n}}^{bn} \frac{f(x)}{x} dx$$

$$= \int_{\frac{a}{n}}^{\frac{b}{n}} \frac{f(x)}{x} dx + \int_{bn}^{an} \frac{f(x)}{x} dx \quad \textbf{(1)}$$

$$= (\alpha-\beta)\ln\frac{b}{a} + \int_{\frac{a}{n}}^{\frac{b}{n}} \frac{f(x)-\alpha}{x} dx + \int_{bn}^{an} \frac{f(x)-\beta}{x} dx$$

But $\left|\int_{\frac{a}{n}}^{\frac{b}{n}} \frac{f(x)-\alpha}{x} dx\right| \le \sup\left\{|f(x)-\alpha| : 0 < x < \frac{\max(a,b)}{n}\right\}\left|\ln\frac{b}{a}\right|$ ***and***

$$\left|\int_{bn}^{\frac{a}{n}} \frac{f(x)-\beta}{x} dx\right| \le \sup\{|f(x)-\beta| : x > n\min(a,b)\}\left|\ln\frac{a}{b}\right|$$

Thus $\lim_{n\to\infty} \int_{\frac{a}{n}}^{\frac{b}{n}} \frac{f(x)-\alpha}{x} dx = 0, \lim_{n\to\infty} \int_{bn}^{\frac{a}{n}} \frac{f(x)-\beta}{x} dx = 0$

So, letting n ***tend to*** ∞ ***in (1) we get*** $\lim_{n\to\infty} \int_{\frac{1}{n}}^{n} \frac{f(ax)-f(bx)}{x} = (\alpha-\beta)\ln\frac{b}{a}$

In particular, if $a=3, b=2 f(x) = \frac{\cos x}{1+x}$ ***we get:*** $\lim_{n\to\infty} \int_{\frac{1}{n}}^{n} \frac{1}{x}\left(\frac{\cos 3x}{1+3x} - \frac{\cos 2x}{1+2x}\right) dx = \ln\frac{2}{3}$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty}\int_{\frac{1}{n}}^{n}\frac{1}{x}\left(\frac{\cos 3x}{1+3x}-\frac{\cos 2x}{1+2x}\right)dx=\int_0^{\infty}\frac{1}{x}\left(\frac{\cos 3x}{1+3x}-\frac{\cos 2x}{1+2x}\right)dx$$

$$=\int_0^{\infty}\frac{\frac{\cos 3x}{1+3x}}{x}dx-\int_0^{\infty}\frac{\frac{\cos 2x}{1+2x}}{x}dx=\lim_{\substack{\varepsilon\to 0+\\ X\to\infty}}\int_{\varepsilon}^{X}\frac{\frac{\cos 3x}{1+3x}}{x}dx-\lim_{\substack{\varepsilon\to 0+\\ X\to\infty}}\int_{\varepsilon}^{X}\frac{\frac{\cos 2x}{1+2x}}{x}dx$$

$$=\lim_{\substack{\varepsilon\to 0+\\ X\to\infty}}\int_{3\varepsilon}^{3X}\frac{\frac{\cos u}{1+u}}{u}du-\lim_{\substack{\varepsilon\to 0+\\ X\to\infty}}\int_{2\varepsilon}^{2X}\frac{\frac{\cos v}{1+v}}{v}dv\begin{bmatrix}let\ 3x=u,2x=v;dx=\frac{du}{3},dx=\frac{dv}{2}\\ when\ x=\varepsilon,u=3\varepsilon;x=X,u=3X\\ when\ x=\varepsilon,v=2\varepsilon,x=X,v=2X\end{bmatrix}$$

$$=\lim_{\substack{\varepsilon\to 0+\\ X\to\infty}}\left[\int_{3\varepsilon}^{2\varepsilon}\frac{\frac{\cos u}{1+u}}{u}du+\int_{2\varepsilon}^{2X}\frac{\frac{\cos u}{1+u}}{u}du+\int_{2X}^{3X}\frac{\frac{\cos u}{1+u}}{u}du\right]-\lim_{\substack{\varepsilon\to 0+\\ X\to\infty}}\int_{2\varepsilon}^{2X}\frac{\frac{\cos u}{1+u}}{u}du$$

$$=\lim_{\varepsilon\to 0+}\int_{3\varepsilon}^{2\varepsilon}\frac{\frac{\cos u}{1+u}}{u}-\lim_{X\to\infty}\int_{3X}^{2X}\frac{\frac{\cos u}{1+u}}{u}du=\lim_{\varepsilon\to 0+}\frac{\cos\xi}{1+\xi}\int_{3\varepsilon}^{2\varepsilon}\frac{du}{u}-\lim_{X\to\infty}\frac{\cos\zeta}{1+\zeta}\int_{3X}^{2X}\frac{du}{u}$$

[By First Mean value Theorem there exists $\xi\in[2\varepsilon,3\varepsilon]$ ***and*** $\zeta\in[2X,3X]$***]***

$$=\log\left(\frac{2}{3}\right)\quad \textit{(Ans:)}$$

Solution 3 by Shafiqur Rahman-Bangladesh

$$\lim_{n\to\infty}\int_{\frac{1}{n}}^{n}\frac{1}{x}\left(\frac{\cos 3x}{1+3x}-\frac{\cos 2x}{1+2x}\right)dx=\int_0^{\infty}\frac{\frac{\cos 3x}{1+3x}-\frac{\cos 2x}{1+2x}}{x}dx$$

$$\left[Let\ f(x)=\frac{\cos x}{1+x}\therefore f(0)=1,f(\infty)=0\right]=\int_0^{\infty}\frac{f(3x)-f(2x)}{x}dx=$$

$$=[f(0)-f(\infty)]\ln\left(\frac{2}{3}\right)\ \textit{[Using Frullani integrals]}$$

$$\lim_{n\to\infty}\int_{\frac{1}{n}}^{n}\frac{1}{x}\left(\frac{\cos 3x}{1+3x}-\frac{\cos 2x}{1+2x}\right)dx=\ln\left(\frac{2}{3}\right)$$

537. Find:

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{k+n^{14}}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$0 \le \sin\left(\frac{k}{n}\right) \le \frac{k}{n}$$

$$n^{14} \le k + n^{14} \le n + n^{14} \Rightarrow n^2 \le \sqrt[7]{k+n^{14}} \le \sqrt[7]{n+n^{14}}$$

$$\Rightarrow \frac{0}{(n+n^{14})^{\frac{1}{7}}} \le \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{k+n^{14}}} \le \frac{k}{n^3} \Rightarrow 0 \le \sum_{k=1}^{n}\frac{\sin\left(\frac{k}{n}\right)}{(k+n^{14})^{\frac{1}{7}}} \le \frac{1}{n^3}\cdot\sum_{k=1}^{n}k$$

$$\Rightarrow 0 \le \sum_{k=1}^{n}\frac{\sin\left(\frac{k}{n}\right)}{(k+n^{14})^{\frac{1}{7}}} \le \frac{n(n+1)}{2n^3}$$

As $\lim_{n\to\infty}\frac{n(n+1)}{2n^3} = \lim_{n\to\infty}\frac{n+1}{2n^2} = 0 \therefore$ *by the Sandwich theorem,*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin\left(\frac{k}{n}\right)}{[k+n^{14}]^{\frac{1}{7}}} = 0$$

Solution 2 by Artan Ajredini-Presheva-Serbie

We have: $-\frac{1}{\sqrt[7]{k+n^{14}}} \le \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{k+n^{14}}} \le \frac{1}{\sqrt[7]{k+n^{14}}}$ ***(1)***

From AM-GM inequality we have:

$$\frac{k+n^{14}}{2} \ge \sqrt[2]{k-n^{14}} \Rightarrow k+n^{14} \ge 2\sqrt[2]{k-n^{14}} \Rightarrow k+n^{14} \ge 2\cdot n^7\sqrt[2]{k} \Rightarrow$$

$$\Rightarrow \sqrt[7]{k+n^{14}} \ge \sqrt[7]{2}\cdot n\sqrt[14]{k} \Rightarrow \frac{1}{\sqrt[7]{k+n^{14}}} \le \frac{1}{\sqrt[7]{2}\cdot n\cdot\sqrt[3]{k}} \quad \textbf{(2)}$$

$$-\frac{1}{\sqrt[7]{k+n^{14}}} \ge -\frac{1}{\sqrt[7]{2}\cdot n\sqrt[14]{k}} \quad \textbf{(3)}$$

Substitution (2) and (3) to (1) and we have: $-\frac{1}{\sqrt[7]{2}\cdot n\sqrt[14]{k}} \le \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{k+n^{14}}} \le \frac{1}{\sqrt[7]{2}\cdot n\sqrt[14]{k}}$ ***(4)***

By summing side by side for $k = \overline{1,n}$ *and adding with* $\lim_{n\to\infty}$ *we have:*

$$-\frac{1}{\sqrt[7]{2}}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt[n]{k}} \le \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{k+n^{14}}} \le \frac{1}{\sqrt[7]{2}}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt[7]{k}}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt[14]{k}} = \lim_{n\to\infty}\frac{1}{n\sqrt[14]{n}}\sum_{k=1}^{n}\frac{1}{\sqrt[14]{\frac{k}{n}}} = \lim_{n\to\infty}\frac{1}{\sqrt[14]{n}}\cdot\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt[14]{\frac{k}{n}}} =$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt[14]{n}}\cdot\int_0^1\frac{dx}{\sqrt[14]{x}} = \lim_{n\to\infty}\frac{1}{\sqrt[14]{n}}\cdot\frac{x^{\frac{13}{14}}}{\frac{13}{14}}\Big|_0^1 = 0 \quad (6)$$

We substitute (6) to (5) and by sandwich theorem we have:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{k+n^{14}}} = 0$$

538.

$$\Omega(n) = \sum_{k=1}^{\infty}\frac{2k^2+2nk+k-1}{(2k+2n+2)!!}$$

Find:

$$\Omega = \lim_{n\to\infty}\left(n!\cdot\Omega(n)\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Shafiqur Rahman-Bangladesh

$$\Omega(n) = \sum_{k=1}^{\infty}\frac{2k^2+2nk+k-1}{(2k+2n+2)!!} = \sum_{k=1}^{\infty}\left(\frac{k}{(2k+2n)!!}-\frac{k+1}{(2k+2n+2)!!}\right) = \frac{1}{(2n+2)!!}$$

Now, $\lim_{n\to\infty}(n!\cdot\Omega(n)) = \lim_{n\to\infty}\left(\frac{n!}{(2n+2)!!}\right) = \lim_{n\to\infty}\left(\frac{1}{2^{n+1}(n+1)}\right) \therefore \lim_{n\to\infty}(n!\cdot$

$\Omega n = 0$

539. If $a_n = \int_0^{\frac{\pi}{4}}\tan^n x\,dx, n = 2,3,\ldots$ Find the limit

$$\lim_{n\to\infty}\left[\frac{1}{(n-2)(a_n+a_{n-2})}\right]^{2n}$$

Proposed by Dimitris Kastriotis-Athens-Greece

Solution 1 by Ravi Prakash-New Delhi-India

For $n \geq 3$, $a_n + a_{n-2} = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2}(1+\tan^2 x)dx = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2}\sec^2 x\,dx =$

$$= \left[\frac{1}{n-1}(\tan x)^{n-1}\right]_0^{\frac{\pi}{4}} = \frac{1}{n-1}$$

$$\therefore \lim_{n\to\infty}\left[\frac{1}{(n-2)(a_n+a_{n-2})}\right]^{2n} = \lim_{n\to\infty}\left[\frac{1}{(n-2)\left(\frac{1}{n-1}\right)}\right]^{2n} = \lim_{n\to\infty}\left[\frac{n-1}{n-2}\right]^{2n}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n-2}\right)^{n-2}\right]^2\left[\left(1+\frac{1}{n-2}\right)^n\right] = (e^2)(1) = e^2$$

Solution 2 by Shafiqur Rahman-Bangladesh

$$a_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x\,dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx = \frac{1}{n-1} - a_{n-2} \Rightarrow$$

$$\Rightarrow \frac{1}{a_n + a_{n-2}} = n-1$$

Now $\lim_{n\to\infty}\left(\frac{1}{(n-2)(a_n+a_{n-2})}\right)^{2n} = \lim_{n\to\infty}\left(\frac{n-1}{n-2}\right)^{2n} = \lim_{n\to\infty}\left(\frac{\left(1-\frac{1}{n}\right)^n}{\left(1-\frac{2}{n}\right)^n}\right)^2 = \left(\frac{e^{-1}}{e^{-2}}\right)^2$

$$\therefore \lim_{n\to\infty}\left(\frac{1}{(n-2)(a_n+a_{n-2})}\right)^{2n} = e^2$$

Solution 3 by Yen Tung Chung-Taichung-Taiwan

Since, $a_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x\,dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx$

$= \left(\frac{1}{n-1}\tan^{n-1} x\right)\Big|_0^{\frac{\pi}{4}} - a_{n-2} = \frac{1}{n-1} - a_{n-2}$ *we have* $a_n + a_{n-2} = \frac{1}{n-1}$ *so,*

$$\lim_{n\to\infty}\left[\frac{1}{(n-2)(a_n+a_{n-2})}\right]^{2n} = \lim_{n\to\infty}\left(\frac{n-1}{n-2}\right)^{2n} = \lim_{n\to\infty}\exp\left\{2n\ln\left(\frac{n-1}{n-2}\right)\right\} =$$

$$= \exp\left\{2\lim_{n\to\infty}\frac{\ln(n-1)-\ln(n-2)}{\frac{1}{n}}\right\} = \exp\left\{2\lim_{n\to\infty}\frac{\frac{1}{n-1}-\frac{1}{n-2}}{-\frac{1}{n^2}}\right\} =$$

$$= \exp\left\{2\lim_{n\to\infty}\frac{n^2}{(n-1)(n-2)}\right\} = \exp(2) = e^2$$

Solution 4 by Sagar Kumar-Kolkata-India

$$a_n = \int_0^{\frac{\pi}{4}} tan^n x\, dx;\ a_n = \int_0^{\frac{\pi}{4}} tan^{n-2}(x)\sec^2 x\, dx - a_{n-2}$$

$$a_n + a_{n-2} = \left(\frac{1}{n-1}\right) \Rightarrow \lim_{n\to\infty}\left[\frac{1}{(n-2)(a_n + a_{n-2})}\right]^{2n} \Rightarrow \lim_{n\to\infty}\left(\frac{n-1}{n-2}\right)^{2n} = e^2$$

540. Find:

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\frac{1}{n}\sum_{k=2}^{n}\frac{1+k^2}{(k-1)k(k+1)!}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu – Romania

$$\frac{k^2+1}{(k-1)k(k+1)!} = \frac{(k+1)^2-2k}{(k-1)k(k+1)!} = \frac{k+1}{(k-1)kk!} - \frac{2}{(k-1)(k+1)!} =$$

$$= \frac{1}{(k-1)k!} + \frac{1}{(k-1)kk!} - \frac{2}{(k-1)(k+1)!} = \frac{1}{(k-1)k!} - \frac{2}{(k-1)(k+1)!} + \frac{1}{(k-1)kk!}$$

$$= \frac{k-1}{(k-1)(k+1)!} + \frac{1}{(k-1)kk!} = \frac{1}{(k+1)!} + \frac{1}{(k-1)kk!} = \frac{1}{(k+1)!} + \frac{1-k+k}{(k-1)kk!} =$$

$$= \frac{1}{(k+1)!} - \frac{k-1}{(k-1)kk!} + \frac{k}{(k-1)kk!} = \frac{1}{(k+1)!} - \frac{1}{kk!} + \frac{1}{(k-1)k!} =$$

$$= \frac{1}{(k+1)!} + \frac{1-k+k}{(k-1)k!} - \frac{1}{kk!} = \frac{1}{(k+1)!} - \frac{k-1}{(k-1)k!} + \frac{k}{(k-1)k!} - \frac{1}{kk!} \Rightarrow$$

$$\Rightarrow \frac{k^2+1}{(k-1)k(k+1)!} = \frac{1}{(k+1)!} - \frac{1}{k!} + \frac{1}{(k-1)(k-1)!} - \frac{1}{kk!} \Rightarrow$$

$$\Rightarrow \sum_{k=2}^{n}\frac{1+k^2}{(k-1)k(k+1)!} = \sum_{k=2}^{n}\left(\frac{1}{(k+1)!} - \frac{1}{k!} + \frac{1}{(k-1)(k-1)!} - \frac{1}{kk!}\right)$$

$$=\frac{1}{(n+1)!}-\frac{1}{2!}+1-\frac{1}{nn!}=\frac{1}{2}+\frac{1}{(n+1)!}-\frac{1}{nn!}\Rightarrow$$

$$\sum_{k=2}^{n}\frac{1+k^2}{(k-1)k(k+1)!}=\frac{1}{2}-\frac{1}{n(n+1)!}\quad \textbf{(1)}$$

From (1) we must calculate: $\lim_{n\to\infty}\sqrt[n]{\frac{1}{n}\left(\frac{1}{2}-\frac{1}{n(n+1)!}\right)}=\lim_{n\to\infty}\frac{\frac{1}{n+1}\left(\frac{1}{2}-\frac{1}{(n+1)(n+2)!}\right)}{\frac{1}{n}\left(\frac{1}{2}-\frac{1}{n(n+1)!}\right)}=$

$$=\lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{\frac{1}{(n+1)(n+2)!}}{\frac{1}{2}-\frac{1}{n(n+1)!}}=1$$

Solution 2 by Sagar Kumar-Kolkata-India

$$S=\sum_{k=2}^{n}\left(\frac{k^2+1}{(k^2-1)kk!}\right)=\sum_{k=2}^{n}\left(\frac{1}{k\cdot k!}+\frac{(k+1)-(k-1)}{(k+1)!\,k(k-1)}\right)$$

$$=\sum_{k=2}^{n}\left(\frac{1}{kk!}+\frac{1}{kk!\,(k-1)}-\frac{1}{(k+1)!\,k}\right)=\sum_{k=2}^{n}\left(\frac{1}{k!\,(k-1)}-\frac{1}{(k+1)!\,k}\right)$$

$$=\left(\frac{1}{2!}-\frac{1}{(n+1)!\,n}\right);\ \Omega=\lim_{n\to\infty}\left(\left(\frac{1}{2!}-\frac{1}{n(n+1)!}\right)^{\frac{1}{n}}\right)^{\left(\frac{1}{n}\right)}$$

$$\log(\Omega)=\lim_{n\to\infty}\frac{\ln\left(\frac{1}{2!}-\frac{1}{n(n+1)!}\right)-\frac{\ln(n)}{n}}{n};\ \log(\Omega)=0;\ \Omega=1\quad \textit{(Answer)}$$

Solution 3 by Naren Bhandari-Nepal

Given that: $\lim_{n\to\infty}\sqrt[n]{\frac{1}{n}\sum_{k=2}^{n}\frac{k^2+1}{(k+1)!k(k-1)}}$***. Since***

$$A=\frac{k^2+1}{(k+1)!\,k(k-1)}=\frac{k}{(k+1)!\,(k+1)}-\frac{1}{(k+1)!}\left(\frac{1}{k-1}-\frac{1}{k}\right)$$

$$=\frac{1}{k!\,(k-1)}-\frac{1}{(k+1)!\,(k-1)}+\frac{1}{(k+1)!\,(k-1)}-\frac{1}{k(k+1)!}=\frac{1}{k!\,(k-1)}-\frac{1}{k(k+1)!}$$

Thus, partial sum of $\sum_{k=2}^{n}\frac{k^2+1}{k(k+1)!(k-1)}=\left(\frac{1}{2!}-\frac{1}{n(n+1)!}\right)$***. Thus, our required limit is***

$$L=\lim_{n\to\infty}\left[\frac{1}{2n}\left(1-\frac{2!}{n(n+1)!}\right)\right]^{\frac{1}{n}}=\lim_{n\to\infty}\frac{1}{\sqrt[n]{2n}}\left(\underbrace{1-\frac{2!}{n^2(n+1)!}+O(n^3)}_{\textit{Taylor series up to 3rd order}}\right)=1$$

Note $L = \lim_{n\to\infty} \exp\left[\frac{-\ln(2n)}{n} + \ln\left(1 - \frac{2}{n^2(n+1)!} + O(n^3)\right)\right] = e^0 = 1$

541. Find:

$$\Omega = \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{1}{n} + \frac{2k}{n^2} + \frac{k^2}{n^3}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\textit{Let } a_n = \prod_{k=1}^{n}\left(1 + \frac{1}{n} + \frac{2k}{n^2} + \frac{k^2}{n^3}\right)$$

$$\log a_n = \sum_{k=1}^{n} \log\left(1 + \frac{1}{n} + \frac{2k}{n^2} + \frac{k^2}{n^3}\right) = \sum_{k=1}^{n}\left[\begin{array}{c}\left(\frac{1}{n} + \frac{2k}{n^2} + \frac{k^2}{n^3}\right) - \frac{1}{2}\left(\frac{1}{n} + \frac{2k}{n^2} + \frac{k^2}{n^3}\right)^2 + \\ + \frac{1}{3}\left(\frac{1}{n} + \frac{2k}{n^2} + \frac{k^2}{n^3}\right)^3 \cdots\end{array}\right] =$$

$$= \sum_{k=1}^{n}\left[\frac{1}{n^3}(n+k)^2 - \frac{1}{2n^6}(n+k)^2 + \frac{1}{3n^a}(n+k)^3 + \cdots\right]$$

Note that $\sum_{k=1}^{n}(n+k)^r$ *is a polynomial of degree* $r+1$ *in*

$$\Rightarrow \frac{1}{n^{3r}}\sum_{k=1}^{n}(n+k)^r \to 0 \textit{ as } n \to \infty$$

Also, $\sum_{k=1}^{n}(n+k)^2 = \sum_{k=1}^{2n} k^2 - \sum_{k=1}^{n} k^2$

$$= \frac{1}{6}(2n)(2n+1)(4n+1) - \frac{1}{6}n(2n+1)(n+1) = \frac{1}{6}n(2n+1)[2(4n+1) - (n+1)]$$

$$= \frac{1}{6}n(2n+1)(7n+2) \Rightarrow \frac{1}{n^3}\sum_{k=1}^{n}(n+k)^2 \to \frac{7}{3} \textit{ as } n \to \infty$$

$$\therefore \lim_{n\to\infty}\log(a_n) = \frac{7}{3} \Rightarrow \lim_{n\to\infty} a_n = e^{\frac{7}{3}} \textit{ or } \Omega = e^{\frac{7}{3}}$$

Solution 2 by Marian Ursărescu-Romania

$$\textit{Let } a_n = \prod_{k=1}^{n}\left(1 + \frac{(n+k)^2}{n^3}\right) \Rightarrow \ln a_n = \sum_{k=1}^{n}\ln\left(1 + \frac{(n+k)^2}{n^3}\right)$$

$$\textit{Now, using: } x - \frac{x^2}{2} \le \ln(1+x) \le x, \forall x \ge 0$$

$$x=\frac{(n+k)^2}{n^3}\Rightarrow\frac{(n+k)^2}{n^3}-\frac{(n+1)^4}{2n^6}\le\ln\left(1+\frac{(n+k)^2}{n^3}\right)\le\frac{(n+k)^2}{n^3}\Rightarrow$$

$$\Rightarrow\sum_{k=1}^{n}\frac{(n+k)^2}{n^3}-\sum_{k=1}^{n}\frac{(n+1)^4}{2n^6}\le\sum_{k=1}^{n}\ln\left(1+\frac{(n+2)^2}{n^3}\right)\le\sum_{k=1}^{n}\frac{(n+k)^2}{n^3}\quad (1)$$

But $\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}(n+k)^2=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)^2=\int_0^1(1+x)^2\,dx=\frac{(1+x)^3}{3}\Big|_0^1=\frac{7}{3}$

(2)

$$\lim_{n\to\infty}\frac{1}{2n^6}\sum_{k=1}^{n}(n+k)^4=\lim_{n\to\infty}\frac{1}{2n}\left[\frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)^4\right]=\lim_{n\to\infty}\frac{1}{2n}\left(\int_0^1(1+x)^4dx\right)=0$$

(3)

From (1)+(2)+(3) $\Rightarrow\lim_{n\to\infty}\ln a_n=\frac{7}{3}\Rightarrow\Omega=\lim_{n\to\infty}a_n=e^{\frac{7}{3}}$

542. Find:

$$\Omega=\lim_{n\to\infty}\left(n^7\cdot\int_0^{\frac{1}{n^7}}\frac{x\sin x+\cos x}{2\sin x+3\cos x+6}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Dimitris Kastriotis-Athens-Greece

$$\Omega=\lim_{n\to\infty}\left(n^7\cdot\int_0^{\frac{1}{n^7}}\frac{x\sin x+\cos x}{2\sin x+3\cos x+6}dx\right)=\lim_{n\to\infty}\left(\frac{\int_0^{\frac{1}{n^7}}\frac{x\sin x+\cos x}{2\sin x+3\cos x+6}dx}{\frac{1}{n^7}}\right)$$

$$\overset{t=\frac{1}{n^7}}{=}\lim_{t\to 0}\left(\frac{\int_0^t\frac{x\sin x+\cos x}{2\sin x+3\cos x+6}dx}{t}\right)\overset{DLH}{=}\lim_{t\to 0}\left(\frac{\frac{d}{dt}\int_0^t\frac{x\sin x+\cos x}{2\sin x+3\cos x+6}dx}{\frac{d}{dt}(t)}\right)$$

$$=\lim_{t\to 0}\frac{t\sin t+\cos t}{2\sin t+3\cos t+6}=\frac{0+1}{0+3+6}=\frac{1}{9}$$

Solution 2 by Kartick Chandra Betal-India

$$\Omega=\lim_{n\to\infty}\left[n^7\int_0^{\frac{1}{n^7}}\frac{x\sin x+\cos x}{2\sin x+3\cos x+6}dx\right]=\lim_{n\to\infty}\left[\frac{\frac{\partial\left(\frac{1}{n^7}\right)}{\partial n}}{\frac{\partial\left(\frac{1}{n^7}\right)}{\partial n}}\cdot\left\{\frac{\left(\frac{1}{n^7}\right)\sin\left(\frac{1}{n^7}\right)+\cos\left(\frac{1}{n^7}\right)}{2\sin\left(\frac{1}{n^7}\right)+3\cos\left(\frac{1}{n^7}\right)+6}\right\}\right]=\frac{1}{3+6}=\frac{1}{9}$$

543.

$$\Omega_n=\binom{n}{7}+2\binom{n-1}{7}+3\binom{n-2}{7}+\cdots+(n-6)\binom{7}{7}, n\ge 7$$

Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\Omega_n}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Remus Florin Stanca-Romania

$$\lim_{n\to\infty}\Omega_n=\infty \textit{ because } \lim_{n\to\infty}(n-6)\binom{7}{7}=\infty$$

$$\Omega=\lim_{n\to\infty}(\Omega_n)^{\frac{1}{n}}=\lim_{n\to\infty}e^{\frac{\ln(\Omega_n)}{n}}=\lim_{n\to\infty}e^{\ln\frac{\Omega_{n+1}}{\Omega_n}}=\lim_{n\to\infty}\frac{\Omega_{n+1}}{\Omega_n}=$$

$$=\lim_{n\to\infty}\frac{\binom{n+1}{7}+2\binom{n}{7}+\cdots+(n-5)\binom{7}{7}}{\binom{n}{7}+2\binom{n-1}{7}+\cdots+(n-6)\binom{7}{7}}\overset{\frac{\infty}{\infty}}{\underset{Stolz-Cesaro}{=}}$$

$$\lim_{n\to\infty}\frac{\binom{n+2}{7}+\cdots+\binom{7}{7}}{\binom{n+1}{7}+\binom{n}{7}+\cdots+\binom{7}{7}}=\lim_{n\to\infty}\frac{\binom{n+3}{7}}{\binom{n+2}{7}}=\lim_{n\to\infty}\frac{\frac{(n+3)!}{(n-4)!}}{\frac{(n+2)!}{(n-5)!}}=$$

$$=\lim_{n\to\infty}\frac{(n-3)(n-2)(n-1)n(n+1)(n+2)(n+3)}{(n-4)(n-3)(n-2)(n-1)n(n+1)(n+2)}=1$$

$$\Omega=1.$$

Solution 2 by Rovsen Pirguliyev-Sumgait-Azerbaidjian

It is known that $\lim_{n\to\infty}\sqrt[n]{\Omega_n}=\lim_{n\to\infty}\frac{\Omega_{n+1}}{\Omega_n}$ ***(1)***

$$\Omega_{n+1}=\binom{n+1}{7}+2\binom{n}{7}+3\binom{n-1}{7}+\cdots+(n-5)\binom{7}{7}=$$

$$=\binom{n+1}{7}+\binom{n}{7}+\binom{n-1}{7}+\cdots+(n-5)\binom{7}{7}+\Omega_n \quad \textbf{(2)}$$

Using (2) in (1), we have:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\Omega_n}=\lim_{n\to\infty}\frac{\binom{n+1}{7}+\binom{n}{7}+\binom{n-1}{7}+\cdots+(n-5)\binom{7}{7}+\Omega_n}{\Omega_n}=$$

$$= \lim_{n\to\infty}\left(1 + \frac{\binom{n+1}{7} + \binom{n}{7} + \binom{n-1}{7} + \cdots + (n-5)\binom{7}{7}}{\binom{n}{7} + 2\binom{n-1}{7} + 3\binom{n-2}{7} + \cdots + (n-6)\binom{7}{7}}\right) = 1$$

Solution 3 by Michael Sterghiou-Greece

We have $\Omega_n \le (n-6)\cdot\sum_{k=0}^{n}\binom{k}{7} = (n-6)\binom{n+1}{8} = (n-6)\cdot\frac{(n+1)!}{8!(n-7)!} = S_n$ *that is*

$1 \le \Omega_n \le S_n \to 1 \le \sqrt[n]{\Omega_n} \le \sqrt[n]{S_n}$*. But*

$$\lim_{n\to\infty}(S_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{8!}\right)^{\frac{1}{n}}\cdot e^{\lim_{n\to\infty}\frac{\ln(n-6)(n-5)\ldots(n-1)n(n+1)}{n}} \quad (1)$$

But $(n-6)^2(n-5)\cdot\ldots\cdot(n-1)n(n+1) < (n+1)^9$ *and*

$$\lim_{n\to\infty}\frac{\ln(n+1)^9}{n} = 9\cdot\lim_{n\to\infty}\frac{\ln(n+1)}{n} = 9\cdot 0 = 0$$

Therefore from (1) $\lim_{n\to\infty} S_n = 1\cdot e^0 = 1$ *and therefore* $\lim_{n\to\infty}\Omega = 1$

544. Find:

$$\Omega = \lim_{n\to\infty}\frac{n - \sqrt[n]{5} - \sqrt[n]{15} - \sqrt[n]{25} - \cdots - \sqrt[n]{10n-5}}{n - \sqrt[n]{10} - \sqrt[n]{20} - \sqrt[n]{30} - \cdots - \sqrt[n]{10n}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michael Sterghiou-Greece

$$\Omega = \lim_{n\to\infty}\frac{n-\sqrt[n]{5}-\sqrt[n]{15}-\sqrt[n]{25}-\cdots-\sqrt[n]{10n-5}}{n-\sqrt[n]{10}-\sqrt[n]{20}-\sqrt[n]{30}-\cdots-\sqrt[n]{10n}} \quad (1)$$

$\lim_{n\to\infty}(10n-5)^{\frac{1}{n}} = 1$ *as* $e^{\lim_{n\to\infty}\frac{\ln(10n-5)}{n}} = e^0 = 1$. $\lim_{n\to\infty}(10n)^{\frac{1}{n}} = 1$ *(2)*

(1)$\to \Omega = \lim_{n\to\infty}\frac{n-\sum_1^n(10n-5)^{\frac{1}{n}}}{n-\sum_1^n(10n)^{\frac{1}{n}}} = \lim_{n\to\infty}\frac{n-(n+\xi_n)}{n-(n+\theta_n)} = \frac{0}{0}$ *for the following reason:*

(2) We can write $(10n-5)^{\frac{1}{n}}$ *as* $1 + a_n$ *where* a_n *is a sequence with zero limit and* $\sum_1^n(10n-5)^{\frac{1}{n}} = n + \xi$ *or* $= \sum 1 + a_n$ *where* $\xi_n \to 0$ *as sum of zero sequnces. The same applies to denominator:* $\sum(10n)^{\frac{1}{n}} = n + \theta_n, \theta_n \to 0$.

(2) now allows the application of DLH on (1) for $x \ge 1$.

$$\frac{d(10x-5)^{\frac{1}{x}}}{dx}=(10x-5)^{\frac{1}{x}}\cdot\left(\frac{10}{x(10x-5)}-\frac{\ln(10x-5)}{x^2}\right)\to 0,x\to\infty$$

Likewise, $\lim\frac{d(10x)^{\frac{1}{n}}}{dx}\to 0$, ***when*** $x\to\infty$

(1)$\to\ \Omega=\lim_{x\to\infty}\frac{(x)'-\sum\left[(10x-5)^{\frac{1}{x}}\right]'}{(x)'-\sum\left[(10x)^{\frac{1}{x}}\right]'}=\frac{1-\sum 0}{1-\sum 0}=1$. ***Therefore*** $\Omega=1$.

Solution 2 by Shafiqur Rahman-Bangladesh

$$\Omega=\lim_{n\to\infty}\frac{n-\sqrt[n]{5}-\sqrt[n]{15}-\sqrt[n]{25}-\cdots-\sqrt[n]{10n-5}}{n-\sqrt[n]{10}-\sqrt[n]{20}-\sqrt[n]{30}-\cdots-\sqrt[n]{10n}}=$$

$$=\lim_{n\to\infty}\frac{\left(\sqrt[n]{5}-1\right)+\left(\sqrt[n]{15}-1\right)+\left(\sqrt[n]{25}-1\right)+\cdots+\left(\sqrt[n]{10n-5}-1\right)}{\left(\sqrt[n]{10}-1\right)+\left(\sqrt[n]{20}-1\right)+\left(\sqrt[n]{30}-1\right)+\cdots+\left(\sqrt[n]{10n}-1\right)}=$$

$$=\lim_{n\to\infty}\frac{\frac{1}{n}(\ln 5+\ln 15+\ln 25+\cdots+\ln(10n-5))+O\left(\frac{1}{n^2}\right)}{\frac{1}{n}(\ln 10+\ln 20+\ln 30+\cdots+\ln(10n))+O\left(\frac{1}{n^2}\right)}=\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\ln(10n-5)}{\sum_{r=1}^{n}\ln(10n)}$$

$$\overset{Stolz-Cesaro\ th^m}{=}\lim_{n\to\infty}\frac{\ln(10n-5)}{\ln(10n)}$$

$$\therefore\ \Omega=\frac{n-\sqrt[n]{5}-\sqrt[n]{15}-\sqrt[n]{25}-\cdots-\sqrt[n]{10n-5}}{n-\sqrt[n]{10}-\sqrt[n]{20}-\sqrt[n]{30}-\cdots-\sqrt[n]{10n}}=1$$

545. Find:

$$\Omega=\lim_{n\to\infty}n\left(1-\frac{\log\left(1+\frac{\sqrt[10]{10}}{n}\right)^{n+1}}{\log\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)^{n}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

$$\Omega=\lim_{n\to\infty}n\left(1-\frac{(n+1)\ln\left(1+\frac{\sqrt[n]{10}}{n}\right)}{n\ln\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)}\right)=$$

$$= \lim_{n\to\infty} n\left(\frac{n\ln\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)-(n+1)\ln\left(1+\frac{\sqrt[10]{10}}{n}\right)}{n\ln\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)}\right) = \lim_{n\to\infty} \frac{n\left[\ln\left(1+\frac{\sqrt[10]{10}}{n+1}\right)-\ln\left(1+\frac{\sqrt[10]{10}}{n}\right)\right]}{\ln\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)} -$$

$$\frac{\ln\left(1+\frac{\sqrt[10]{10}}{n}\right)}{\ln\left(1+\frac{\sqrt[10]{10}}{n+1}\right)} \quad (1)$$

But $\lim_{n\to\infty} \frac{\ln\left(1+\frac{\sqrt[n]{10}}{n}\right)}{\ln\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)} = \lim_{n\to\infty} \frac{\frac{\ln\left(1+\frac{\sqrt[n]{10}}{n}\right)}{\frac{\sqrt[n]{10}}{n}}}{\frac{\ln\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)}{\frac{\sqrt[n+1]{10}}{n+1}}} \cdot \frac{\sqrt[n+1]{10}}{n+1} \cdot \frac{n}{\sqrt[n]{10}} = 1 \quad$ ***(2)***

$$\lim_{n\to\infty} \frac{\left[\ln\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)-\ln\left(1+\frac{\sqrt[n]{10}}{n}\right)\right]}{\frac{\ln\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)}{\frac{\sqrt[n+1]{10}}{n+1}}} = \lim_{n\to\infty} \frac{n\left[\ln\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)-\ln\left(1+\frac{\sqrt[n]{10}}{n}\right)\right]}{\frac{\sqrt[n+1]{10}}{n+1}} =$$

$$= \lim_{n\to\infty} \frac{1}{\frac{\sqrt[n+1]{10}}{n+1}\cdot\frac{n}{\sqrt[n]{10}}\cdot\frac{\sqrt[n]{10}}{n}} \cdot n\left[\ln\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)-\ln\left(1+\frac{\sqrt{10}}{n}\right)\right] =$$

$$= \lim_{n\to\infty} n^2\left[\ln\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)-\ln\left(1+\frac{\sqrt[n]{10}}{n}\right)\right] \quad (3)$$

Let $f:[n,n+1]\to\mathbb{R},\ f(x)=\ln\left(1+\frac{10^{\frac{1}{x}}}{x}\right)$***. From Lagrange's theorem***

$\Rightarrow \exists c\in(n,n+1)$ ***which*** $\frac{f(n+1)-f(n)}{n+1-n} = f'(c) \Rightarrow \ln\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)-\ln\left(1-\frac{\sqrt[n]{10}}{n}\right) = \frac{-10^{\frac{1}{c}}}{c^2}\cdot\frac{c+\ln 10}{c+10^{\frac{1}{c}}}$ ***(4)***

From (3)+(4)$\Rightarrow \lim_{n\to\infty} n^2\left[\ln\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)-\ln\left(1+\frac{\sqrt[n]{10}}{n}\right)\right] =$

$= \lim_{n\to\infty} n^2\cdot\frac{\left(-10^{\frac{1}{c}}\right)}{c^2}\cdot\frac{c+\ln 10}{c+10^{\frac{1}{c}}} = -1$ ***(5) because*** $c\in(n,n+1)$***.***

From (1)+(2)+(5)$\Rightarrow \Omega = -2.$

546. **Find:**

$$\Omega = \lim_{n\to\infty}\left((\pi + H_n)^{\frac{1+H_n}{H_n}} - (H_n)^{\frac{1+\pi+H_n}{\pi+H_n}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\Omega = \lim_{n\to\infty}(\pi + H_n)^{1+\frac{1}{H_n}} - (H_n)^{1+\frac{1}{\pi+H_n}} = \lim_{n\to\infty}(\pi + H_n)^{1+\frac{1}{H_n}} - (H_n)^{1+\frac{1}{H_n}} + \lim_{n\to\infty}(H_n)^{1+\frac{1}{H_n}} - (H_n)^{1+\frac{1}{\pi+H_n}} \quad (1)$$

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \lim_{n\to\infty} H_n = \infty$$

$$\Omega_1 = \lim_{n\to\infty}(\pi + H_n)^{1+\frac{1}{H_n}} - (H_n)^{1+\frac{1}{H_n}} = \lim_{n\to\infty}(H_n)^{1+\frac{1}{H_n}}\left(\left(\frac{\pi + H_n}{H_n}\right)^{1+\frac{1}{H_n}} - 1\right) =$$

$$= \lim_{n\to\infty} H_n^{\frac{1}{H_n}} \cdot H_n\left(e^{\left(1+\frac{1}{H_n}\right)\ln\left(1+\frac{\pi}{H_n}\right)} - 1\right) \quad (2)$$

$$\lim_{n\to\infty} H_n^{\frac{1}{H_n}} = e^{\lim_{n\to\infty}\frac{\ln H_n}{H_n}} \overset{C.S.}{=} e^{\lim_{n\to\infty}\frac{\ln H_{n+1} - \ln H_n}{H_{n+1}-H_n}} = e \cdot \frac{\lim_{n\to\infty}\ln\frac{H_n+1}{H_n}}{\frac{1}{n+1}} = e^{\lim(n+1)} \cdot \frac{\ln\left(1+\frac{H_{n+1}-H_n}{H_n}\right)}{\frac{H_{n+1}-H_n}{H_n}} \cdot \frac{H_{n+1}-H_n}{H_n}$$

$$= e^{\lim_{n\to\infty}(n+1)} \cdot \frac{1}{(n+1)H_n} = e^{\lim_{n\to\infty}\frac{1}{H_n}} = e^0 = 1 \quad (3)$$

$$\lim_{n\to\infty} H_n \cdot \frac{\left(e^{\left(1+\frac{1}{H_n}\right)\ln\left(1+\frac{\pi}{H_{n-1}}\right)}\right)}{\left(1+\frac{1}{H_n}\right)\ln\left(1+\frac{\pi}{H_n}\right)} \cdot \left(1+\frac{1}{H_n}\right)\ln\left(1+\frac{\pi}{H_n}\right) =$$

$$= \lim_{n\to\infty}(1 + H_n)\frac{\ln\left(1+\frac{\pi}{H_n}\right)}{\frac{\pi}{H_n}} \cdot \frac{\pi}{H_n} = \lim_{n\to\infty}(1 + H_n) \cdot \frac{\pi}{H_n} = \lim_{n\to\infty}\left(\frac{\pi}{1+n} + \pi\right) = \pi \quad (4)$$

From (2)+(3)+(4) $\Rightarrow \Omega_1 = \pi$ *(5)*

$$\Omega_2 = \lim_{n\to\infty}(H_n)^{1+\frac{1}{H_n}} - (H_n)^{1+\frac{1}{\pi+H_n}} = \lim_{n\to\infty}(H_n)^{1+\frac{1}{\pi+H_n}}\left(H_n^{\frac{\pi}{H_n(1+H_n)}} - 1\right) =$$

$$= \lim_{n\to\infty} H_n^{\frac{1}{\pi+H_n}} \cdot H_n\left(H_n^{\frac{\pi}{H_n(1+H_n)}} - 1\right) \quad (6)$$

$$\lim_{n\to\infty} H_n^{\frac{1}{\pi+H_n}} = 1 \quad \text{(from 3)}$$

$$\lim_{n\to\infty}\frac{H_n\left(e^{\frac{\pi\ln H_n}{H_n(1+H_n)}}-1\right)}{\frac{\pi\ln H_n}{H_n(1+H_n)}}\cdot\frac{\pi\ln H_n}{H_n(1+H_n)}=\pi\lim_{n\to\infty}\frac{\ln H_n}{1+H_n}\overset{C.S}{=}\pi\lim_{n\to\infty}\frac{\ln H_{n+1}-\ln H_n}{H_{n+1}-H_n}=$$

$$\pi\lim_{n\to\infty}\frac{\ln\frac{H_{n+1}}{H_n}}{\frac{1}{n+1}}=0\quad \textit{(from (3)) (7)}$$

$$\textit{From (6)+(7)}\Rightarrow\Omega_2=0\quad \textit{(8)}$$

$$\textit{From (5)+(8)}\Rightarrow\Omega=\pi$$

Solution 2 by Remus Florin Stanca-Romania

$$\lim_{n\to\infty}(\pi+H_n)^{\frac{1+H_n}{H_n}}=\infty \text{ and } \lim_{n\to\infty}H_n^{\frac{1+\pi+H_n}{\pi+H_n}}=\infty$$

$$\Omega\overset{\infty-\infty}{=}\lim_{n\to\infty}\left(H_n^{\frac{1+\pi+H_n}{\pi+H_n}}\left(\frac{(\pi+H_n)^{\frac{1+H_n}{H_n}}}{H_n^{\frac{1+\pi+H_n}{\pi+H_n}}}\right)\right)=$$

$$\lim_{n\to\infty}\left(H_n^{\frac{1}{\pi+H_n}}\right)\cdot\lim_{n\to\infty}\left(H_n\left(\frac{(\pi+H_n)^{\frac{1+H_n}{H_n}}}{H_n^{\frac{1+\pi+H_n}{\pi+H_n}}}\right)\right)\quad (1)$$

$$\lim_{n\to\infty}H_n^{\frac{1}{\pi+H_n}}\overset{\infty^0}{=}\lim_{n\to\infty}e^{\frac{\ln H_n}{\pi+H_n}}\overset{Stolz-Cesaro}{=}\lim_{n\to\infty}e^{\frac{\ln\frac{H_{n+1}}{H_n}}{\frac{1}{n+1}}}=\lim_{n\to\infty}e^{\frac{\ln\left(1+\frac{1}{(n+1)H_n}\right)}{\frac{1}{(n+1)H_n}\cdot H_n}}=e^0=1$$

$$\underset{(1)}{\Rightarrow}\lim_{n\to\infty}H_n\left(\frac{(\pi+H_n)^{\frac{1+H_n}{H_n}}}{H_n^{\frac{1+\pi+H_n}{\pi+H_n}}}\right)=$$

$$\Omega=\lim_{n\to\infty}H_n\left(e^{\ln\left(\frac{(\pi+H_n)^{\frac{1+H_n}{H_n}}}{H_n^{\frac{1+\pi+H_n}{\pi+H_n}}}\right)}-1\right)=\lim_{n\to\infty}H_n\frac{e^{\ln\left(\frac{(\pi+H_n)^{\frac{1+H_n}{H_n}}}{H_n^{\frac{1+\pi+H_n}{\pi+H_n}}}\right)}-1}{\ln\left(\frac{(\pi+H_n)^{\frac{1+H_n}{H_n}}}{H_n^{\frac{1+\pi+H_n}{\pi+H_n}}}\right)}\cdot\frac{\ln\left(\frac{(\pi+H_n)^{\frac{1+H_n}{H_n}}}{H_n^{\frac{1+\pi+H_n}{\pi+H_n}}}\right)}{1}=$$

$$\lim_{n\to\infty} H_n \cdot \ln\left(\frac{(\pi+H_n)^{\frac{1}{H_n}}}{H_n^{\frac{1}{\pi+H_n}}} \cdot \frac{\pi+H_n}{H_n}\right) \textit{ because } \lim_{n\to\infty} \frac{(\pi+H_n)^{\frac{1+H_n}{H_n}}}{H_n^{\frac{1+\pi+H_n}{\pi+H_n}}} = 1$$

$$\Omega = \lim_{n\to\infty} \ln\left(\left(\frac{(\pi+H_n)^{\frac{1}{H_n}}}{H_n^{\frac{1}{\pi+H_n}}} \cdot \frac{\pi+H_n}{H_n}\right)^{H_n}\right)$$

$$\lim_{n\to\infty}\left(\frac{(\pi+H_n)^{\frac{1}{H_n}}}{H_n^{\frac{1}{\pi+H_n}}} \cdot \frac{\pi+H_n}{H_n}\right)^{H_n} = \lim_{n\to\infty}\left(\frac{\pi+H_n}{H_n}\right)^{H_n} \cdot \lim_{n\to\infty}\left(\frac{\pi+H_n}{\frac{H_n}{H_n^{\frac{\pi+H_n}{}}}}\right) = l$$

$$\lim_{n\to\infty}\left(\frac{\pi+H_n}{H_n}\right)^{H_n} = \lim_{n\to\infty}\left(\frac{\pi}{H_n}+1\right)^{\frac{H_n}{\pi}\cdot\frac{\pi}{H_n}\cdot H_n} = e^{\pi}$$

$$\lim_{n\to\infty}\frac{\pi+H_n}{\frac{H_n}{H_n^{\pi+H_n}}} = \lim_{n\to\infty}\frac{\pi}{\frac{H_n}{H_n^{\pi+H_n}}} + \lim_{n\to\infty}\frac{H_n}{\frac{H_n}{H_n^{\pi+H_n}}} = \lim_{n\to\infty} H_n^{\frac{\pi}{\pi+H_n}} = \lim_{n\to\infty} e^{\frac{\pi}{\pi+H_n}\cdot\ln H_n} =$$

$$e^0 = 1 \Rightarrow l = e^{\pi} \Rightarrow \Omega = \ln e^{\pi} = \pi \Rightarrow \Omega = \pi.$$

547.

$$\omega_n = \sin\left(\frac{1}{n+1}\right) + \sin\left(\frac{1}{n+2}\right) + \cdots + \sin\left(\frac{1}{2n}\right), n \in \mathbb{N}, n \geq 1$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)\cdot\omega_{n+1}^{n+1}} - \sqrt[n]{n\cdot\omega_n^n}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

$$x\cos x \leq \sin x \leq x, \forall x \geq 0 \Rightarrow \frac{1}{n+k}\cos\frac{1}{2n} \leq \frac{1}{n+k}\cos\frac{1}{n+k} \leq \sin\frac{1}{n+k} \leq \frac{1}{n+k} \Rightarrow$$

$$\Rightarrow \cos\frac{1}{2n}\sum_{k=1}^{n}\frac{1}{n+k} \leq \sum_{k=1}^{n}\sin\frac{1}{n+k} \leq \sum_{k=1}^{n}\frac{1}{n+k} \quad \textit{(1)}$$

$$\textit{But } \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}} = \int_0^1\frac{1}{1+x}dx = \ln 2 \quad \textit{(2)}$$

$$\textit{From (1)+(2)} \Rightarrow \lim_{n\to\infty}\omega_n = \ln 2 \quad \textit{(3)}$$

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{n+1}\cdot\omega_{n+1} - \sqrt[n]{n}\omega_n\right) =$$

$$= \lim_{n\to\infty}\left(\sqrt[n+1]{n+1}\cdot\omega_{n+1} - \sqrt[n]{n}\cdot\omega_{n+1} + \sqrt[n]{n}\omega_{n+1} - \sqrt[n]{n}\omega_n\right)$$

$$= \lim_{n\to\infty}\omega_{n+1}\left(\sqrt[n+1]{n+1} - \sqrt[n]{n}\right) + \lim_{n\to\infty}\sqrt[n]{n}\left(\omega_{n+1} - \omega_n\right) \quad (4)$$

$$\lim_{n\to\infty}\sqrt[n]{n}\left(\omega_{n+1} - \omega_n\right) = 1\cdot(\ln 2 - \ln 2) = 0 \quad (5)$$

Let $f:[n,n+1]\to\mathbb{R};\ f(x) = x^{\frac{1}{x}}$***. From Lagrange's theorem*** $\exists c\in(n,n+1)$***, so that:***

$$\frac{f(n+1)-f(n)}{n+1-n} = f'(c) \Rightarrow (n+1)^{\frac{1}{n+1}} - n^{\frac{1}{n}} = f'(c) \quad (6)$$

$$f'(x) = x^{\frac{1}{x}}\left(\frac{1-\ln x}{x^2}\right) \quad (7)$$

From (6)+(7)$\Rightarrow \sqrt[n+1]{n+1} - \sqrt[n]{n} = c^{\frac{1}{c}}\left(\frac{1-\ln c}{c^2}\right) \Rightarrow$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{n+1} - \sqrt[n]{n}\right) = \lim_{n\to\infty}c^{\frac{1}{c}}\left(\frac{1-\ln c}{c^2}\right) \quad (8)$$

Because $c\in(n,n+1),\ \lim_{n\to\infty}x^{\frac{1}{x}}\cdot\left(\frac{1-\ln x}{x^2}\right) = 0$ ***(9)***

$$\left.\begin{array}{r}\lim_{x\to\infty}x^{\frac{1}{x}} = \lim_{x\to\infty}e^{\frac{\ln x}{x}} = e^{\lim_{x\to\infty}\frac{1}{x}} = e^0 = 1\\ \lim_{x\to\infty}\frac{1-\ln x}{x^2} = \lim_{x\to\infty}\frac{-\frac{1}{x}}{2x} = 0\end{array}\right\}\Rightarrow (9)$$

From (9) $\Rightarrow \lim_{n\to\infty}\omega_{n+1}\left(\sqrt[n+1]{n+1} - \sqrt[n]{n}\right) = \ln 2\cdot 0 = 0$ ***(10)***

From (4)+(5)+(10)$\Rightarrow \Omega = 0.$

548.

$$H_n = \sum_{k=1}^{n}\frac{1}{k},\ G_n = \sum_{k=1}^{n}\frac{1}{k^2},\ T_n = \frac{H_n}{G_n}$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{log(1+2^{T_n})^{log(1+7^{T_n})}}{log(1+3^{T_n})^{log(1+5^{T_n})}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\Omega = \lim_{n\to\infty} \frac{\ln(1+7^{T_n})\cdot\ln(1+2^{T_n})}{\ln(1+5^{T_n})\cdot\ln(1+3^{T_n})} = \lim_{n\to\infty} \frac{\frac{\ln(1+7^{T_n})}{T_n}\cdot\frac{\ln(1+2^{T_n})}{T_n}}{\frac{\ln(1+5^{T_n})}{T_n}\cdot\frac{\ln(1+3^{T_n})}{T_n}} \quad (1)$$

$\lim_{n\to\infty}\frac{H_n}{G_n} = \infty$. *Let* $a > 1$.

$$\lim_{n\to\infty}\frac{\ln(1+a^{T_n})}{T_n} \overset{C.S.}{=} \lim_{n\to\infty}\frac{\ln(1+a^{T_{n+1}})-\ln(1+a^{T_n})}{T_{n+1}-T_n} = \lim_{n\to\infty}\frac{\ln\left(\frac{1+a^{T_{n+1}}}{1+a^{T_n}}\right)}{T_{n+1}-T_n} =$$

$$= \lim_{n\to\infty}\frac{\ln\left(1+\frac{1+a^{T_{n+1}}}{1+a^{T_n}}-1\right)}{T_{n+1}-T_n} = \lim_{n\to\infty}\frac{\ln\left(1+\frac{a^{T_{n+1}}-a^{T_n}}{1+a^{T_n}}\right)}{\frac{a^{T_{n+1}}-a^{T_n}}{1+a^{T_n}}}\cdot\frac{a^{T_{n+1}}-a^{T_n}}{(1+a^{T_n})(T_{n+1}-T_n)} =$$

$$= \lim_{n\to\infty}\frac{a^{T_n}(a^{T_{n+1}-T_n}-1)}{(1+a^{T_n})(T_{n+1}-T_n)} = \lim_{n\to\infty}\frac{a^{T_{n+1}-T_n}-1}{\left(\frac{1}{a^{T_n}}+1\right)(T_{n+1}-T_n)} = \ln a \quad (1)$$

From (1) $\Rightarrow \Omega = \frac{\ln 7\cdot\ln 2}{\ln 5\cdot\ln 3}$. *Observation:* $\lim_{n\to\infty} T_{n+1}-T_n = 0$, *because.*

$$\lim_{n\to\infty}\left(\frac{H_{n+1}}{G_{n+1}}-\frac{H_n}{G_n}\right) = \lim_{n\to\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n+1}}{1+\frac{1}{2^2}+\cdots+\frac{1}{(n^2+1)^2}} - \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}} =$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right)\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\left(1+\frac{1}{2^2}+\cdots+\frac{1}{(n+1)^2}\right)}{G_n\cdot G_{n+1}}$$

$$= \lim_{n\to\infty}\frac{\frac{1}{n+1}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)-\frac{1}{(n+1)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}{G_n\cdot G_{n+1}} =$$

$$= \lim_{n\to\infty}\frac{\frac{1}{n+1}\left[\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)-\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n+1}\right]}{G_n\cdot G_{n+1}} = 0$$

because $\lim_{n\to\infty} G_n = \frac{\pi^2}{6}$, *and* $\lim_{n\to\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n+1} \overset{(1)}{=} \lim_{n\to\infty}\frac{1}{n+1} = 0$

Solution 2 by Remus Florin Stanca-Romania

$$\Omega = \lim_{n\to\infty}\left(\frac{\ln(1+7^{T_n})\ln(1+2^{T_n})}{\ln(1+5^{T_n})\ln(1+3^{T_n})}\right) = \lim_{n\to\infty}\left(\frac{\ln(1+7^{T_n})}{\ln(1+5^{T_n})}\right)\lim_{n\to\infty}\left(\frac{\ln(1+2^{T_n})}{\ln(1+3^{T_n})}\right)$$

$l_1 = \lim_{n\to\infty}\left(\frac{\ln(1+7^{T_n})}{\ln(1+5^{T_n})}\right)$ *and* $l_2 = \lim_{n\to\infty}\left(\frac{\ln(1+2^{T_n})}{\ln(1+3^{T_n})}\right)$

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{\frac{1}{1^2}+\cdots+\frac{1}{n^2}} \overset{Stolz\ Cesaro}{\underset{\frac{\infty}{\infty}}{=}} \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{(n+1)^2}} = \infty \Rightarrow \lim_{n\to\infty} T_n = \infty$$

$l_1 \overset{Stolz\ Cesaro}{\underset{\frac{\infty}{\infty}}{=}} \lim_{n\to\infty}\frac{\ln\left(\frac{1+7^{T_{n+1}}}{1+7^{T_n}}\right)}{\ln\left(\frac{1+5^{T_{n+1}}}{1+5^{T_n}}\right)}$. ***Let*** $\alpha\in\mathbb{R}_+$***, we need to compute the limit:***

$$\lim_{n\to\infty}\frac{1+\alpha^{T_{n+1}}}{1+\alpha^{T_n}} = \lim_{n\to\infty}\alpha^{T_{n+1}-T_n}\frac{\frac{1}{\alpha^{T_{n+1}}}+1}{\frac{1}{\alpha^{T_n}}+1} = \lim_{n\to\infty}\alpha^{T_{n+1}}-T_n$$

$$\lim_{n\to\infty}(T_{n+1}-T_n) \overset{\infty-\infty}{=} \lim_{n\to\infty} T_n\left(\frac{T_{n+1}}{T_n}-1\right) =$$

$$\frac{6}{\pi^2}\ln\left(\lim_{n\to\infty}\left(\frac{1+\cdots+\frac{1}{n+1}}{1+\cdots+\frac{1}{n}}\right)^{1+\cdots+\frac{1}{n}}\cdot\lim_{n\to\infty}\left(\left(\frac{1+\cdots+\frac{1}{(n+1)^2}}{1+\cdots+\frac{1}{n^2}}\right)^{1+\cdots+\frac{1}{n}}\right)\right)^{-1}$$

$$\lim_{n\to\infty}\left(\frac{1+\cdots+\frac{1}{n+1}}{1+\cdots+\frac{1}{n}}\right)^{1+\cdots+\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{(n+1)\left(1+\ldots+\frac{1}{n}\right)}\right)^{(n+1)\left(1+\ldots+\frac{1}{n}\right)\cdot\frac{1+\cdots+\frac{1}{n}}{(n+1)\left(1+\cdots+\frac{1}{n}\right)}} = e^0 = 1$$

$$\lim_{n\to\infty}\left(\frac{1+\cdots+\frac{1}{(n+1)^2}}{1+\cdots+\frac{1}{n^2}}\right)^{1+\cdots+\frac{1}{n}} =$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{(n+1)^2\left(1+\cdots+\frac{1}{n^2}\right)}\right)^{(n+1)^2\left(1+\cdots\frac{1}{n^2}\right)\cdot\frac{1}{(n+1)^2\left(1+\cdots+\frac{1}{(n+1)^2}\right)}\cdot\left(1+\cdots+\frac{1}{n}\right)} = e^0 = 1 \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}\left(\left(\frac{1+\cdots+\frac{1}{(n+1)^2}}{1+\cdots+\frac{1}{n^2}}\right)^{1+\cdots+\frac{1}{n}}\right)^{-1} = 1$$

$$\succ \lim_{n\to\infty} T_{n+1}-T_n = \ln(1) = 0 \succ \lim_{n\to\infty}\frac{1+\alpha^{T_{n+1}}}{1+\alpha^{T_n}} = \alpha^0 = 1$$

$$\Rightarrow \lim_{n\to\infty}\frac{\ln\left(\frac{7^{T_{n+1}+1}}{7^{T_n}+1}\right)}{\ln\left(\frac{5^{T_{n+1}}+1}{5^{T_n}+1}\right)} = \lim_{n\to\infty}\frac{\dfrac{\ln\left(\frac{7^{T_{n+1}}}{7^{T_n}+1}-1+1\right)}{\frac{7^{T_{n+1}}-7^{T_n}}{7^{T_n}+1}}\cdot\dfrac{7^{T_{n+1}}-7^{T_n}}{7^{T_n}+1}}{\dfrac{\ln\left(\frac{5^{T_{n+1}}+1}{5^{T_n}+1}-1+1\right)}{\frac{5^{T_{n+1}}-5^{T_n}}{5^{T_n}+1}}}$$

$$= \lim_{n\to\infty}\frac{7^{T_{n+1}}-7^{T_n}}{7^{T_n}+1}\cdot\frac{5^{T_n}+1}{5^{T_{n+1}}-5^{T_n}} = \lim_{n\to\infty}\frac{7^{T_{n+1}}-7^{T_n}}{5^{T_{n+1}}-5^{T_n}}\cdot\left(\frac{5}{7}\right)^{T_n} =$$

$$= \lim_{n\to\infty}\left(\frac{7}{5}\right)^{T_n}\cdot\left(\frac{5}{7}\right)^{T_n}\cdot\frac{7^{T_{n+1}-T_n}-1}{5^{T_{n+1}-T_n}-1} = \lim_{n\to\infty}\frac{\frac{7^{T_{n+1}-T_n}-1}{T_{n+1}-T_n}}{\frac{5^{T_{n+1}-T_n}-1}{T_{n+1}-T_{n+1}}} = \frac{\ln 7}{\ln 5} = l_1$$

Because we proved that $\lim_{n\to\infty}(T_{n+1}-T_n) = 0$ ***we can prove in the same way that***

$$\lim_{n\to\infty}\frac{\ln(1+2^{T_n})}{\ln(1+3^{T_n})} = \frac{\ln 2}{\ln 3} = l_2 \Rightarrow \Omega = l_1 l_2 = \frac{\ln 7}{\ln 5}\cdot\frac{\ln 2}{\ln 3} \Rightarrow \Omega = \frac{\ln 7}{\ln 5}\cdot\frac{\ln 2}{\ln 3}$$

549. Find:

$$\Omega = \lim_{n\to\infty}\left(1 - \log 2 + \sum_{k=1}^{n}\sin\frac{1}{n+k}\right)^n$$

Proposed by Daniel Sitaru – Romania

Solution by Michael Sterghiou-Greece

$$\Omega = \lim_{n\to\infty}\left(1 - \ln 2 + \sum_{k=1}^{n}\sin\frac{1}{k+n}\right)^n \quad (1)$$

Lemma: $\forall x \in [0,\Gamma]: x\left(1-\frac{x}{\Gamma}\right) \le \sin x \le x$. ***LHS is easily proven by considering the function***

$f(x) = x\left(1-\frac{x}{\Gamma}\right) - \sin x$ ***over*** $[0,\Gamma]$.

$x = \frac{1}{k+n} \to \frac{1}{n+k}\left[1-\frac{1}{\Gamma(n+k)}\right] \le \sin\frac{1}{n+k} \le \frac{1}{n+k}$ ***(2). We know that***

$\sum_1^\infty\frac{1}{n} = \infty$ ***and that*** $\sum_1^n\frac{1}{k} = \ln 2 - \gamma + \varepsilon_n$ ***where*** γ = ***constant and*** $\varepsilon \simeq \frac{1}{2} \to 0, k \to \infty$.

Also $\sum_1^{2n}\frac{1}{k} = \ln(2n) - \gamma + \varepsilon_{2n}, \varepsilon_{2n} \simeq \frac{1}{4n}$. ***This means that***

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$\sum_{k=1}^{n}\frac{1}{n+k}=\left(\sum_{1}^{2n}\frac{1}{k}\right)-\left(\sum_{1}^{n}\frac{1}{k}\right)=\ln 2-\frac{1}{4n}$. *Now,* $\sum_{1}^{\infty}\frac{1}{n^2}=\frac{\Gamma^2}{6},\sum_{1}^{\infty}\frac{1}{(2n)^2}=\frac{\Gamma^2}{6}$ *therefore*

$\sum_{k=1}^{n}\frac{1}{(n+k)^2}\to 0$ *when* $n\to\infty$. *Taking sums to infity in (2)* $\to$

$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k+n}-\sum_{k=1}^{n}\frac{1}{\Gamma(k+n)^2}\right)=\ln 2-0$ *and in the RHS* $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\ln 2.$

We can write also $\ln 2-\frac{1}{4n}<\sum_{k=1}^{n}\frac{1}{k+n}<\ln 2-\frac{1}{4n}$ *and in turn:*

$$\left(1-\frac{1}{4n}\right)^n<\left(1-\ln 2+\sum_{k=1}^{n}\sin\frac{1}{k+n}\right)^n<\left(1-\frac{1}{4n}\right)^n$$

Taking limits $e^{-\frac{1}{4}}<\Omega<e^{-\frac{1}{4}}$ *hence* $\Omega=e^{-\frac{1}{4}}$*. Done!*

550. Find:

$$\Omega=\lim_{n\to\infty}\frac{\left(1+2^p\cdot\sqrt{2}+3^p\cdot\sqrt[3]{3}+\cdots+n^p\cdot\sqrt[n]{n}\right)^{q+1}}{(1^q+3^q+5^q+\cdots+(2n-1)^q)^{p+1}},p,q\in\mathbb{N},p,q\geq 1$$

Proposed by Marian Ursărescu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\Omega_n=\lim_{n\to\infty}\frac{\left(1+2^p\sqrt{2}+3^p\sqrt[3]{3}+\cdots+n^p\sqrt[n]{n}\right)^{q+1}}{(1^q+3^q+\cdots+(2n-1)^q)^{p+1}}$$

$=\lim_{n\to\infty}(x_n\cdot y_n)$ *where* $x_n=\frac{\left(1+2^p\sqrt{2}+3^p\sqrt[3]{3}+\cdots+n^p\sqrt[n]{n}\right)^{q+1}}{n}$ *and*

$y_n=\frac{n}{(1+3^q+\cdots+(2n-1)^q)^{p+1}}$ *for all* $n\in\mathbb{N},p,q\geq 1$

$$\lim_{n\to\infty}\sqrt[q+1]{x_n}=\lim_{n\to\infty}\frac{1+2^p\sqrt{2}+3^p\sqrt[3]{3}+\cdots+n^p\sqrt[n]{n}}{\sqrt[q+1]{n}}\overset{CAESARO\ STOLZ}{=}\lim_{n\to\infty}\frac{(n+1)^p\sqrt[n+1]{n+1}}{\sqrt[q+1]{n+1}-\sqrt[q+1]{n}}$$

$$=\lim_{n\to\infty}\left(n^{\frac{q}{q+1}}\frac{(n+1)^p\sqrt[n+1]{n+1}}{\frac{\left(1+\frac{1}{n}\right)^{\frac{1}{q+1}}-1}{\frac{1}{n}}}\right)=(q+1)\lim_{n\to\infty}n^{\frac{q}{1+q}}(n+1)^p \text{ where } \lim_{n\to\infty}\sqrt[n]{n}=1$$

$$\lim_{n\to\infty}x_n=(q+1)^{q+1}\lim_{n\to\infty}n^q(n+1)^{p(q+1)}$$

$$\lim_{n\to\infty}\sqrt[p+1]{y_n}=\lim_{n\to\infty}\frac{\sqrt[p+1]{n}}{1+3^q+\cdots+(2n-1)^q}=\lim_{n\to\infty}\frac{\sqrt[p+1]{n+1}-\sqrt[p+1]{n}}{(2n+1)^q}$$

$$=\lim_{n\to\infty}\left(n^{-\frac{p}{1+p}}\frac{\frac{\left(1+\frac{1}{n}\right)^{\frac{1}{p+1}}-1}{\frac{1}{n}}}{(2n+1)^q}\right)=\frac{1}{p+1}\lim_{n\to\infty}\frac{1}{n^{\frac{p}{1+p}}(2n+1)^q}$$

$$\lim_{n\to\infty}y_n=\frac{1}{(p+1)^{p+1}}\lim_{n\to\infty}\frac{1}{n^p(2n+1)^{q(p+1)}}$$

$$\therefore \lim_{n\to\infty}\Omega=\lim_{n\to\infty}(x_ny_n)=\frac{(q+1)^{q+1}}{(p+1)^{p+1}}\lim_{n\to\infty}\frac{n^q(n+1)^{p(q+1)}}{n^p(2n+1)^{q(p+1)}}=\frac{(q+1)^{q+1}}{2^{pq+q}(p+1)^{p+1}}$$ ***(Answer)***

551. Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=0}^{2n}(-1)^k\cdot\frac{1}{\binom{2n}{k}}$$

Proposed by Daniel Sitaru – Romania

Solution by Sagar Kumar-Patna Bihar-India

$$L=\lim_{n\to\infty}\sum_{k=0}^{2n}\frac{(-1)^k(k)!\,(2n-k)!}{(2n)!};\ L=\lim_{n\to\infty}\frac{1}{(2n)!}\sum_{k=0}^{2n}(2n-k)!\,(k)!\,(-1)^k$$

$$=\lim_{n\to\infty}\frac{1}{(2n)!}\left((2n)!-(2n-1)!+(2n-2)!\,(2)!-(2n-3)!\,(3)!\ldots\right)+(2n-1)!+(2n)!$$

$$=\lim_{n\to\infty}\frac{2\left((2n)!-(2n-1)!+(2n-2)!\,(2)!\right)}{(2n)!}+\frac{(n!)^2(-1)^n}{(2n)!}=2+\lim_{n\to\infty}\frac{(-1)^n(n!)^2}{(2n)!}$$

$$=2+\lim_{n\to\infty}\frac{(-1)^n2n\pi\left(\frac{n}{e}\right)^{2n}}{\sqrt{2\pi}\sqrt{2n}\left(\frac{2n}{e}\right)^{2n}}$$

$$L=2+L_1;\ L_1=\lim_{n\to\infty}\frac{(-1)^n\sqrt{n\pi}}{(2)^{2n}}\Rightarrow L_1=0;\ L=2$$ ***(Answer)***

552. Find:

$$\Omega = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\zeta(n+1)-1\right)$$

Proposed by Khalef Ruhemi-Jarash-Jordan

Solution by proposer

$$I := \sum_{n=1}^{\infty}(-1)^{n-1}\left(\zeta(n+1)-1\right) = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\sum_{n=1}^{\infty}\frac{1}{j^{n+1}}-1\right)$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\left(\sum_{j=2}^{\infty}\frac{1}{j^{n+1}}\right) = \sum_{j=2}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{j^{n+1}}$$

$$= \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{j}\right)^{n+1} = \sum_{j=2}^{\infty}\frac{1}{j^2}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{j}\right)^{n-1}$$

$$= \sum_{j=2}^{\infty}\frac{1}{j^2}\left(\frac{1}{1+\frac{1}{j}}\right) = \sum_{j=2}^{\infty}\frac{1}{j^2+j} = \sum_{j=2}^{\infty}\left(\frac{1}{j}-\frac{1}{j+1}\right) = \frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\cdots = \frac{1}{2}$$

$$\therefore I := \sum_{n=1}^{\infty}(-1)^{n-1}\left(\zeta(n+1)-1\right) = \frac{1}{2}$$

553. **Find:**

$$\Omega = \lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(\frac{n^2+n+k^2}{n^2+k^2}\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Șerban George Florin – Romania

$$a_n = \prod_{k=1}^{n}\frac{n^2+n+k^2}{n^2+k^2} = \prod_{k=1}^{n}\left(1+\frac{n}{n^2+k^2}\right) > 1$$

$$\ln a_n = \sum_{k=1}^{n}\ln\left(1+\frac{n}{n^2+k^2}\right)$$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1,\ (\forall)x_n \to 0,\ (\forall)\varepsilon > 0,\ \left|\frac{\ln(1+x_n)}{x_n}-1\right| < \varepsilon$$

$$-\varepsilon < \frac{\ln(1+x_n)}{x_n} - 1 < \varepsilon, -\varepsilon x_n < \ln(1+x_n) - x_n < \varepsilon x_n$$

$$(1-\varepsilon)x_n < \ln(1+x_n) < (\varepsilon+1)x_n, x_n = \frac{n}{n^2+k^2} \to 0; n \to \infty$$

$$(1-\varepsilon)\frac{n}{n^2+k^2} < \ln\left(1+\frac{n}{n^2+k^2}\right) < (1+\varepsilon)\frac{n}{n^2+k^2}$$

$$(1-\varepsilon)n\sum_{k=1}^{n}\frac{1}{n^2+k^2} < \sum_{k=1}^{n}\ln\left(1+\frac{n}{n^2+k^2}\right) < (1+\varepsilon)n\sum_{k=1}^{n}\frac{1}{n^2+k^2}$$

$$\frac{(1-\varepsilon)n}{n^2}\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2} < \ln a_n < \frac{(1+\varepsilon)^n}{n^2}\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2}$$

$$\underbrace{(1-\varepsilon)\cdot\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2}}_{\downarrow\ \int_0^1\frac{dx}{1+x^2}} < \ln a_n < \underbrace{(1+\varepsilon)\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2}}_{\downarrow\ \int_0^1\frac{dx}{1+x^2}}$$

$$\Rightarrow \lim_{n\to\infty}\ln a_n = \int_0^1\frac{dx}{1+x^2} = \arctan x\Big|_0^1 = \frac{\pi}{4} \Rightarrow \lim_{n\to\infty}a_n = e^{\frac{\pi}{4}}$$

Solution 2 by Nassim Nicholas Taleb – USA

$$\Omega = e^{\frac{\pi}{4}}$$

Proof.

$$f = \prod_{k=1}^{n}\frac{k^2+n^2+n}{k^2+n^2}$$

Since $\frac{x-1}{x} < \log(x) < x-1$,

$$\sum_{k=1}^{n}\frac{n}{k^2+n+n^2} < \log(f) = \sum_{k=1}^{n}\left(\log\left(\frac{k^2+n^2+n}{k^2+n^2}\right)\right) < \sum_{k=1}^{n}\frac{n}{k^2+n^2}$$

We have various ways to go about it. Note that by removing the logs we have something resembling Dirichlet series.

We have the upper bound taken continuously $\int_0^n \frac{n}{k^2+n^2}\mathrm{d}k = \frac{\pi}{4}$

and the lower bound $\int_0^n \frac{n}{k^2+n^2+n}dk = \frac{\sqrt{n}\tan^{-1}\left(\sqrt{\frac{n}{n+1}}\right)}{\sqrt{n+1}}$

$\lim_{n\to\infty}\frac{\sqrt{n}\tan^{-1}\left(\sqrt{\frac{n}{n+1}}\right)}{\sqrt{n+1}} = \frac{\pi}{4}$*, which completes the proof.*

554. Evaluate:

$$\Omega = \lim_{n\to\infty}\int_{2n}^{3n}\frac{\tan^{-1}(x)}{x}dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

As $\tan^{-1}x$ *is an increasing function,* $\tan^{-1}(2n) \le \tan^{-1}x \le \tan^{-1}(3n)$,

$$\forall x \in [2n, 3n] \Rightarrow \frac{\tan^{-1}(2n)}{x} \le \frac{\tan^{-1}x}{x} \le \frac{\tan^{-1}(3n)}{x} \quad \forall x \in [2n, 3n]$$

$$\Rightarrow \tan^{-1}(2n)\int_{2n}^{3n}\frac{1}{x}dx \le \int_{2n}^{3n}\frac{\tan^{-1}(x)}{x}dx \le \tan^{-1}(3n)\int_{2n}^{3n}\frac{1}{x}dx$$

$$\Rightarrow (\tan^{-1}(2n))\ln\left(\frac{3}{2}\right) \le \int_{2n}^{3n}\frac{\tan^{-1}(x)}{x}dx \le \tan^{-1}(3n)\ln\left(\frac{3}{2}\right)$$

Taking limit as $n \to \infty$*, we get*

$$\frac{\pi}{2}\ln\left(\frac{3}{2}\right) \le \lim_{n\to\infty}\int_{2n}^{3n}\frac{\tan^{-1}(x)}{x}dx \le \frac{\pi}{2}\ln\left(\frac{3}{2}\right) \therefore \lim_{n\to\infty}\int_{2n}^{3n}\frac{\tan^{-1}x}{x}dx = \frac{\pi}{2}\ln\left(\frac{3}{2}\right)$$

Solution 2 by Shafiqur Rahman-Bangladesh

$$\lim_{n\to\infty}\int_{2n}^{3n}\frac{\tan^{-1}x}{x}dx[x\to nx] = \lim_{n\to\infty}\int_{2}^{3}\frac{\tan^{-1}nx}{x}dx = \frac{\pi}{2}\int_{2}^{3}\frac{dx}{x} \therefore \lim_{n\to\infty}\int_{2n}^{3n}\frac{\tan^{-1}x}{x}dx = \frac{\pi}{2}\ln\left(\frac{3}{2}\right)$$

Solution 3 by Shivam Sharma-New Delhi-India

Let $\frac{x}{n} = y$; $dx = n\,dy \Rightarrow \lim_{n\to\infty}\frac{\tan^{-1}(ny)}{y}dy \Rightarrow \int_2^3\frac{1}{y}\lim_{n\to\infty}(\tan^{-1}(ny))\,dy$

$\Rightarrow \frac{\pi}{2}\int_2^3\frac{1}{y}dy \Rightarrow \frac{\pi}{2}[\ln(y)]_2^3$ ***(OR)*** $\Omega = \frac{\pi}{2}\left(\ln\left(\frac{3}{2}\right)\right)$ ***(Q.E.D)***

Solution 4 by Amir Sofi-France

By integration par parts

$$\int_{2n}^{3n}\frac{\arctan(x)}{x}dx = \arctan 3n\ln 3n - \arctan 2n\ln 2n - \int_{2n}^{3n}\frac{1}{1+x^2}\ln x\,dx$$

$$\lim_{n\to\infty}(\arctan 3n\ln 3n - \arctan 2n\ln 2n) = \frac{1}{2}\pi\ln\left(\frac{3}{2}\right) \text{ and}$$

$$\frac{\ln 2n}{1+9n^2} \le \int_{2n}^{3n}\frac{1}{1+x^2}\ln x\,dx \le \frac{\ln 3n}{1+4n^2} \Rightarrow \lim_{n\to\infty}\int_{2n}^{3n}\frac{1}{1+x^2}\ln x\,dx = 0$$

$$\text{Hence } \int_{2n}^{3n}\frac{\arctan(x)}{x}dx \underset{n\nearrow\infty}{\to} \frac{1}{2}\pi\ln\left(\frac{3}{2}\right)$$

555. Find:

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_{\varepsilon}^{1}\left(\frac{log(1+x)}{x(1+x^2)}\right)dx\right)$$

Proposed by Vasile Mircea Popa – Romania

Solution by Togrul Ehmedov-Baku-Azerbaijan

$$I = \int_0^1 \frac{\ln(1+x)}{x(1+x^2)}dx = \int_0^1 \ln(1+x)\left[\frac{A}{x}+\frac{Bx+C}{1+x^2}\right]dx$$

$$A = 1; B = -1; C = 0$$

$$I = \int_0^1 \ln(1+x)\left[\frac{1}{x}-\frac{x}{1+x^2}\right]dx = \int_0^1 \frac{\ln(1+x)}{x}dx - \int_0^1 \frac{x\ln(1+x)}{1+x^2}dx$$

$$N = \int_0^1 \frac{\ln(1+x)}{x}dx; M = \int_0^1 \frac{x\ln(1+x)}{1+x^2}dx$$

$$N = \int_0^1 \frac{\ln(1+x)}{x}dx = \ln(x)\ln(1+x)\Big|_0^1 - \int_0^1 \frac{\ln x}{1+x}dx = -\int_0^1 \frac{\ln x}{1+x}dx = -Li_2(-1) = \frac{\pi^2}{12}$$

$$M = \int_0^1 \frac{x\ln(1+x)}{1+x^2}dx; M(a) = \int_0^1 \frac{x\ln(1+ax)}{1+x^2}dx$$

$$M'(a)=\int_0^1\frac{x^2}{(1+x^2)(1+ax)}dx=\int_0^1\left[\frac{Ax+B}{1+x^2}+\frac{C}{1+ax}\right]dx$$

$$A=\frac{a}{1+a^2};B=-\frac{1}{1+a^2};C=\frac{1}{1+a^2}$$

$$M'(a)=\int_0^1\left[\frac{\frac{a}{1+a^2}x-\frac{1}{1+a^2}}{1+x^2}+\frac{\frac{1}{1+a^2}}{1+ax}\right]dx=\frac{1}{1+a^2}\int_0^1\frac{ax-1}{1+x^2}dx+\frac{1}{1+a^2}\int_0^1\frac{dx}{1+ax}$$

$$=\frac{a}{1+a^2}\int_0^1\frac{x}{1+x^2}dx-\frac{1}{1+a^2}\int_0^1\frac{1}{1+x^2}dx+\frac{1}{1+a^2}\int_0^1\frac{dx}{1+ax}=\frac{1}{2}\ln\frac{a}{1+a^2}-\frac{\pi}{4}\cdot\frac{1}{1+a^2}+\frac{\ln(1+a)}{a(1+a^2)}$$

$$M(a)=M(1)-M(0)=M(1)$$

$$M=\frac{1}{2}\ln 2\int_0^1\frac{a}{1+a^2}da-\frac{\pi}{4}\int_0^1\frac{1}{1+a^2}da+\int_0^1\frac{\ln(1+a)}{a(1+a^2)}da=\frac{1}{4}\ln^2 2-\frac{\pi^2}{16}+I$$

$$I=N-M=\frac{\pi^2}{12}-\left(\frac{1}{4}\ln^2 2-\frac{\pi^2}{16}+I\right);\ I=\frac{7\pi^2}{96}-\frac{1}{8}\ln^2 2$$

556. Find:

$$\Omega=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(\frac{1}{(25k^2+5k-6)(n-k+1)^2}\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Naren Bhandari-Nepal

Since $\frac{1}{25k^2+5k-6}=\frac{1}{(5k+3)(5k-2)}=\frac{1}{5}\left(\frac{1}{5k-2}-\frac{1}{5k+3}\right)$*. The sum can be written as:*

$$S=\sum_{n=1}^{\infty}\left(\frac{1}{5}\sum_{k=1}^{n}\left(\frac{1}{5k-2}-\frac{1}{5k+3}\right)\frac{1}{(n-k+1)^2}\right)$$

$$=\frac{1}{5}\left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{5k-2}-\frac{1}{5k+3}\right)\frac{1}{(n-k+1)^2}\right)$$

$$=\frac{1}{5}\left(\sum_{k=1}^{\infty}\left(\frac{1}{5k-2}-\frac{1}{5k+3}\right)\sum_{n=1}^{\infty}\frac{1}{n^2}\right)=\frac{1}{5}\left(\frac{1}{3}\cdot\frac{\pi^2}{6}\right)=\frac{\pi^2}{90}$$

Solution 2 by Tran Hong-Vietnam

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\left[\frac{1}{25k^2+5k-6}\cdot\frac{1}{(n-k+1)^2}\right]\right)=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{25k^2+5k-6}\cdot\frac{1}{(n-k+1)^2}\right]$$

$$=\sum_{k=1}^{\infty}\frac{1}{25k^2+5k-6}\cdot\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$=\sum_{k=1}^{\infty}\frac{1}{25k^2+5k-6}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{6}\cdot\frac{1}{5}\lim_{m\to\infty}\sum_{k=1}^{m}\left(\frac{1}{5n-2}-\frac{1}{5n+3}\right)$$

$$=\frac{\pi^2}{30}\cdot\lim_{m\to\infty}\left(\frac{1}{3}-\frac{1}{8}+\frac{1}{8}-\cdots-\frac{1}{5m-2}+\frac{1}{5m-2}-\frac{1}{5m+3}\right)=\frac{\pi^2}{30}\cdot\lim_{m\to\infty}\left(\frac{1}{3}-\frac{1}{5m+3}\right)=\frac{\pi^2}{90}$$

Solution 3 by Shivam Sharma-New Delhi-India

$$\frac{1}{5}\left[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{1}{5k-2}-\frac{1}{5k+3}\right)\left(\frac{1}{(n-k+1)^2}\right)\right]$$

$$\Rightarrow\frac{1}{5}\left[\left(\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)\right)\left(\sum_{k=1}^{\infty}\left(\frac{1}{5k-2}-\frac{1}{5k+3}\right)\right)\right]\Rightarrow\frac{\pi^2}{30}\left[\sum_{k=1}^{\infty}\left(\frac{1}{5k+3}-\frac{1}{5k+8}\right)\right]$$

$$\Rightarrow\frac{\pi^2}{30}\left[\frac{1}{5}\sum_{k=1}^{\infty}\left(\frac{1}{k+\frac{3}{5}}-\frac{1}{k+\frac{8}{5}}\right)\right]\Rightarrow\frac{\pi^2}{150}\left[\psi\left(\frac{8}{5}\right)-\psi\left(\frac{3}{5}\right)\right]\Rightarrow\frac{\pi^2}{150}\left[\frac{5}{3}\right]$$

(OR) $\Omega=\frac{\pi^2}{90}$ ***(Answer)***

557. Find:

$$\Omega=\lim_{n\to\infty}\left(\int_0^{\frac{1}{2}}\left(\frac{arcsec(nx)\cdot log(1-x)}{2x^2-2x+1}\right)dx\right)$$

Proposed by Abdul Mukhtar-Nigeria

Solution 1 by Sagar Kumar-Patna Bihar-India

$$\Omega=\lim_{n\to\infty}\int_0^{\frac{1}{2}}\frac{\sec^{-1}(nx)\ln(1-x)}{2\left(x^2-x+\frac{1}{2}\right)}dx,\qquad\Omega=\frac{\pi}{4}\int_0^{\frac{1}{2}}\frac{\ln(1-x)}{\left(x-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}dx$$

$$\Omega = \pi \int_0^{\frac{1}{2}} \frac{\ln\left(\frac{1}{2} + x\right) dx}{(2x)^2 + 1}, \quad \Omega = \pi \int_0^{\frac{1}{2}} \frac{\ln(2x+1) - \ln 2}{(2x)^2 + 1} dx$$

$$\Omega = \pi \int_0^{\frac{1}{2}} \frac{\ln(2x+1)}{(2x)^2 + 1} dx - \left(\frac{\pi^2 \ln 2}{8}\right)$$

Put $2x = t$, $I = \frac{\pi}{2}\int_0^1 \frac{\ln(1+t)}{t^2+1} dt = \frac{\pi^2}{16}\ln 2$; $\Omega = \frac{\pi^2 \ln 2}{16}$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\Omega = \lim_{n\to\infty} \int_0^{\frac{1}{2}} \frac{\operatorname{arcsec}(nx)\log(1-x)}{2x^2 - 2x + 1} dx$$

$$= \int_0^{\frac{1}{2}} \frac{\lim_{n\to\infty} \operatorname{arcsec}(nx)\log(1-x)}{2\left\{\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}\right\}} dx = \frac{\pi}{4}\int_0^{\frac{1}{2}} \frac{\log(1-x)}{\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}} dx$$

$$= -\frac{\pi}{4}\int_{\frac{1}{2}}^{0} \frac{\log\left(z + \frac{1}{2}\right)}{z^2 + \frac{1}{4}} dz \left[\begin{array}{c} where\, \frac{1}{2} - x = z \Rightarrow dx - dz \\ when\ x = 0, z = \frac{1}{2}; when\ x = \frac{1}{2}, z = 0 \end{array}\right]$$

$$= \frac{\pi}{4}\int_0^{\frac{1}{2}} \frac{\log\left(z + \frac{1}{2}\right)}{z^2 + \frac{1}{4}} dz = \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \log\left(\frac{\tan\theta + 1}{2}\right) d\theta \left[\begin{array}{l} where\ z = \frac{1}{2}\tan\theta \Rightarrow dz = \frac{1}{2}\sec^2\theta\, d\theta \\ when\ z = 0, \theta = 0; when\ z = \frac{1}{2}, \theta = \frac{\pi}{4} \end{array}\right]$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \log\left\{\frac{\tan\left(\frac{\pi}{4} - \theta\right) + 1}{2}\right\} d\theta = \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \log\left(\frac{\frac{1 - \tan\theta}{1 + \tan\theta} + 1}{2}\right) d\theta$$

$$= \frac{\pi^2}{8}\log 2 - \int_0^{\frac{\pi}{2}} \log\left(\frac{\tan\theta + 1}{2}\right) d\theta \Rightarrow 2\Omega = \frac{\pi^2 \log 2}{8} \Rightarrow \Omega = \frac{\pi^2 \log 2}{16}$$

558. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{2^n}\prod_{k=1}^{n}\left(2-\frac{3}{3k-1}\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

For $0\le x<1$, let $f(x)=e^{-x}-(1-x); f'(x)=-e^{-x}+1=1-e^{-x}>0$

$\Rightarrow f(x)$ increases on $[0,1]\Rightarrow f(x)>f(0)$ for $0<x<1\Rightarrow e^{-x}>1-x$ for $0<x<1$

Let $a_k=\frac{3}{6k-2}$. Note $0<a_k<1$. Thus, $1-a_k<e^{-a_k}\Rightarrow$

$0<\prod_{k=1}^{n}(1-a_k)<e^{-\sum_{k=1}^{n}a_k}$. As $\sum_{n=1}^{\infty}a_n$ diverges, $e^{-\sum_{k=1}^{n}a_k}\to e^{-\infty}=0$

$$\therefore\lim_{n\to\infty}\prod_{k=1}^{n}(1-a_k)=0$$

Solution 2 by Tran Hong-Vietnam

$$U_n=\frac{1}{2^n}\prod_{k=1}^{n}\left(2-\frac{3}{3k-1}\right)=\prod_{k=1}^{n}\frac{6k-5}{6k-2};$$

$$U_1=\frac{1}{4};$$

$$U_2=\frac{1}{4}\cdot\frac{7}{10}=\frac{7}{40}<\frac{1}{\sqrt{6}}=\frac{1}{\sqrt{6\cdot2-6}};$$

$$U_3=\frac{1}{4}\cdot\frac{7}{10}\cdot\frac{13}{16}=\frac{91}{640}<\frac{1}{\sqrt{12}}=\frac{1}{\sqrt{6\cdot3-6}};$$

$$U_4=\frac{1}{4}\cdot\frac{7}{10}\cdot\frac{13}{16}\cdot\frac{19}{22}=\frac{1729}{14080}<\frac{1}{\sqrt{18}}=\frac{1}{\sqrt{6\cdot4-6}}$$

…

$$U_n=\prod_{k=1}^{n}\frac{6k-5}{6k-2}<\frac{1}{\sqrt{6n-6}}$$

$$\Rightarrow 0<U_n<\frac{1}{\sqrt{6n-6}}\ \forall n\ge2$$

$$\Rightarrow\lim_{n\to\infty}U_n=0$$

Solution 3 by Marian Ursărescu-Romania

$$\text{Let } a_n=\frac{1}{2^n}\prod_{k=1}^{n}\left(2-\frac{3}{3k-1}\right)=\prod_{k=1}^{n}\left(1-\frac{3}{6k-2}\right)=\prod_{k=1}^{n}\frac{6k-5}{6k-2}\quad (1)$$

$$\left.\begin{array}{l}\sqrt{1\cdot 7}<\frac{1+7}{2}=4\\ \sqrt{7\cdot 13}<\frac{7+13}{2}=10\\ \sqrt{13\cdot 19}<\frac{13+19}{2}=16\\ \vdots\\ \sqrt{(6n-11)(6n-5)}<6n-8\end{array}\right\}\Rightarrow$$

$$\Rightarrow 1\cdot 7\cdot 13\cdot\ldots\cdot(6n-11)\sqrt{6n-5}<4\cdot 10\cdot\ldots\cdot(6n-8)\Rightarrow$$

$$\Rightarrow\frac{1\cdot 7\cdot 13\cdot\ldots\cdot(6n-11)\sqrt{6n-5}}{4\cdot 10\cdot\ldots\cdot(6n-8)}<1\Rightarrow$$

$$\Rightarrow\frac{1\cdot 7\cdot 13\cdot\ldots\cdot(6n-5)}{4\cdot 10\cdot\ldots\cdot(6n-2)}<\frac{\sqrt{6n-5}}{6n-2}\quad (2)$$

$$\text{From (1)+(2)}\Rightarrow 0<a_n<\frac{\sqrt{6n-5}}{6n-2}\Rightarrow\lim_{n\to\infty}a_n=0$$

559. $x_0>0,\ x_{n+1}=\frac{1}{x_1^p}+\frac{1}{x_2^p}+\cdots+\frac{1}{x_n^p},\ p\in\mathbb{N}^*$

Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{\sqrt[p+1]{n^{p+2}}}\cdot\sqrt{\sum_{1\le i<j\le n}x_ix_j}\right)$$

Proposed by Marian Ursărescu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\text{We know, }\lim_{u\to 0}\frac{(1+u)^r-1}{u}=r$$

$$x_{n+1}=\sum_{i=1}^{n}\frac{1}{x_i^p}=x_n+\frac{1}{x_n^p},\ p\in\mathbb{N}^*\text{ and } x_0>0$$

Now, $\{x_n\}_{n=1}^{\infty}$ *is an increasing function hence let* $\lim_{n\to\infty}x_n=l$

$$\therefore l=l+\frac{1}{l^p}\Rightarrow l\to\infty,\text{ which is a contradiction, }\therefore\lim_{n\to\infty}x_n=\infty$$

$$\therefore \Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[p+1]{n^{p+2}}}\cdot\sqrt{\sum_{1\le i<j\le n} x_i x_j}\right) \Rightarrow \sqrt{2}\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[p+1]{n^{p+2}}}\cdot\sqrt{2\sum_{1\le i<j\le n} x_i x_j}\right)$$

$$\Rightarrow \sqrt{2}\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[p+1]{n^{p+2}}}\cdot\sqrt{\left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2}\right) = \lim_{n\to\infty}\frac{x_1 + x_2 + \cdots + x_n}{\sqrt[p+1]{n^{p+2}}}$$

$$\left[\because \lim_{n\to\infty}\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n^{\frac{2p+4}{p+1}}} = 0\right]$$

$$\overset{\substack{CAESARO\\STOLZ}}{=} \lim_{n\to\infty}\frac{x_{n+1}}{(n+1)^{\frac{p+2}{p+1}} - n^{\frac{p+2}{p+1}}} = \lim_{n\to\infty}\frac{\frac{x_{n+1}}{\sqrt[p+1]{n}}}{\frac{\left(1+\frac{1}{n}\right)^{\frac{p+2}{p+1}} - 1}{\frac{1}{n}}} = \frac{p+1}{p+2}\lim_{n\to\infty}\frac{x_{n+1}}{\sqrt[p+1]{n}}$$

$$= \frac{p+1}{p+2}\sqrt[p+1]{\lim_{n\to\infty}\frac{x_{n+1}^{p+1}}{n}} \overset{\substack{CAESARO\\STOLZ}}{=} \frac{p+1}{p+2}\sqrt[p+1]{\lim_{n\to\infty}\left(x_{n+2}^{p+1} - x_{n+2}^{p+1}\right)}$$

$$= \frac{p+1}{p+2}\sqrt[p+1]{\lim_{n\to\infty}\left\{\left(x_{n+1} + \frac{1}{x_{n+1}^{p}}\right)^{p+1} - x_{n+1}^{p+1}\right\}} = \frac{p+1}{p+2}\sqrt[p+1]{\lim_{x_{n+1}\to\infty}\frac{\left(1+\frac{1}{x_{n+1}^{p+1}}\right)^{p+1} - 1}{\frac{1}{x_{n+1}^{p+1}}}}$$

$$= \frac{(p+1)^{p+1}\sqrt{p+1}}{p+2} = \frac{\sqrt[p+1]{(p+1)^{p+2}}}{p+2} \Rightarrow \Omega = \frac{\sqrt[p+1]{(p+1)^{p+2}}}{\sqrt{2}(p+2)}$$ ***(Answer)***

560. Find:

$$\Omega = \lim_{n\to\infty}\int_{ne}^{n\pi}\frac{\log(1+5x)}{x\log(1+2x)}dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Shafiqur Rahman-Bangladesh

$$\lim_{n\to\infty}\int_{ne}^{n\pi}\frac{\ln(1+5x)}{x\ln(1+2x)}dx\ [x\to nx] = \lim_{n\to\infty}\int_{e}^{\pi}\frac{\ln(1+5nx)}{x\ln(1+2nx)}dx =$$

$$= \int_e^{\pi} \frac{1}{x} \lim_{n\to\infty} \frac{\ln(1+5nx)}{\ln(1+2nx)} dx = \int_e^{\pi} \frac{1}{x} dx \therefore \lim_{n\to\infty} \int_{ne}^{n\pi} \frac{\ln(1+5x)}{x\ln(1+2x)} dx = \ln\pi - 1$$

Solution 2 by Igor Soposki-Skopje-Macedonia

$$\Omega = \lim_{n\to\infty} \int_{ne}^{n\pi} \frac{\ln(1+5x)}{x\ln(1+2x)} dx = \begin{Bmatrix} \frac{x}{n} = t \\ x = nt \\ dx = ndt \end{Bmatrix} = \lim_{n\to\infty} \int_e^{\pi} \frac{\ln(1+5nt)}{nt\ln(1+2nt)} \cdot ndt =$$

$$= \int_e^{\pi} \frac{1}{t} \cdot \underbrace{\left[\lim_{n\to\infty} \frac{n\cdot\ln(1+5nt)}{n\cdot\ln(1+2nt)}\right]}_{L} dt = \int_e^{\pi} \frac{dt}{t} = \ln t = \ln\frac{\pi}{e}$$

$$L = \lim_{n\to\infty} \frac{\ln(1+5nt)}{\ln(1+2nt)} \overset{LR}{=} \lim_{n\to\infty} \frac{\frac{1}{1+5nt}\cdot 5t}{\frac{1}{1+2nt}\cdot 2t} = \lim_{n\to\infty} \frac{5}{2} \cdot \frac{1+2nt}{1+5nt} \overset{LR}{=} \frac{5}{2} \lim_{n\to\infty} \frac{2t}{5t} = 1$$

Solution 3 by Sagar Kumar-Patna Bihar-India

$$I = \lim_{n\to\infty} \int_{ne}^{n\pi} \frac{\ln(1+5x)}{x\ln(1+2x)} dx$$

Put $x = ny;\ dx = ndy$

$$I = \int_e^{\pi} \lim_{n\to\infty} \frac{\ln(1+5ny)}{y\ln(1+2ny)} dy = \int_e^{\pi} \frac{dy}{y} = \ln\left(\frac{\pi}{e}\right)$$

561. Find:

$$\Omega = \lim_{n\to\infty} \left(1 - \frac{\pi^2}{6} + \sum_{k=1}^{n} \frac{1}{k^2}\right)^n$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6}$ ***and*** $\lim_{n\to\infty} n\left(\sum_{k=1}^{n} \frac{1}{k^2} - \frac{\pi^2}{6}\right) = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{1}{k^2} - \frac{\pi^2}{6}}{\frac{1}{n}}$

$$\overset{\substack{Caesaro\\Stolz}}{=} \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n+1} - \frac{1}{n}} = -\lim_{n\to\infty} \frac{n}{n+1} = -1$$

Let us consider a sequence $u_n:\mathbb{N}\to\mathbb{R}$ *defined by* $u_n=\sum_{k=1}^{n}\frac{1}{k^2}-\frac{\pi^2}{6}$ *for all* $n\ge 1$

$$\therefore \lim_{n\to\infty}u_n=\frac{\pi^2}{6}\text{ now }\lim_{n\to\infty}\left(1-\frac{\pi^2}{6}+\sum_{k=1}^{n}\frac{1}{k^2}\right)^n=\lim_{\frac{1}{u_n}\to\infty}\left\{1+\frac{1}{\frac{1}{u_n}}\right\}^{nu_n}$$

$$=e^{\lim\limits_{n\to\infty}n\left(\sum_{k=1}^{n}\frac{1}{k^2}-\frac{\pi^2}{6}\right)}=\frac{1}{e}$$

Solution 2 by Marian Ursărescu-Romania

Because $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}=\frac{\pi^2}{6}$, *we have* $1^\infty\Rightarrow$

$$\Omega=\lim_{n\to\infty}\left(1+\sum_{k=1}^{n}\frac{1}{k^2}-\frac{\pi^2}{6}\right)^n=e^{\lim_{n\to\infty}n\left(\sum_{k=1}^{n}\frac{1}{k^2}-\frac{\pi^2}{6}\right)}\quad (1)$$

Now, using Cesaro-Stolz for $\frac{0}{0}\Rightarrow$

$$\lim_{n\to\infty}n\left(\sum_{k=1}^{n}\frac{1}{k^2}-\frac{\pi^2}{6}\right)=\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{k^2}-\frac{\pi^2}{6}}{\frac{1}{n}}=$$

$$=\lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n+1}-\frac{1}{n}}=\lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{n-n-1}{n(n+1)}}=\lim_{n\to\infty}\frac{-n}{n+1}=-1\quad (2)$$

From (1)+(2)$\Rightarrow\Omega=e^{-1}=\frac{1}{e}$.

562. **Find:**

$$\Omega=\lim_{n\to\infty}\frac{1\cdot n+3\cdot(n-1)+5\cdot(n-2)+\cdots+(2n-1)\cdot 1}{(n+1)^4-n^4}$$

Proposed by Daniel Sitaru – Romania

Solution by Amit Dutta-Jamshedpur-India

$$\Omega=\lim_{n\to\infty}\left[\frac{\sum_{r=1}^{n}(2r-1)(n-r+1)}{(n+1)^4-n^4}\right];\ \Omega=\lim_{n\to\infty}\left[\frac{\sum_{r=1}^{n}(2rn-2r^2+2r-n+r-1)}{(n+1)^4-n^4}\right]$$

$$\Omega=\lim_{n\to\infty}\left[\frac{2n\sum_{r=1}^{n}r-2\sum_{r=1}^{n}r^2+2\sum_{r=1}^{n}r-(1+n)\sum_{r=1}^{n}1+\sum_{r=1}^{n}r}{(n+1)^4-n^4}\right]$$

$$\Omega=\lim_{n\to\infty}\left[\frac{\frac{2n\cdot n(n+1)}{2}-\frac{2n(n+1)(2n+1)}{6}+3\sum_{r=1}^{n}r-n(n+1)}{(n+1)^4-n^4}\right]$$

$$\Omega = \lim_{n\to\infty}\left[\frac{n^2(n+1) - \frac{2n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} - n(n+1)}{[(n+1)^2+n^2][(n+1)^2-n^2]}\right]$$

$$\Omega = \lim_{n\to\infty}\frac{n(n+1)\left[n - \frac{(2n+1)}{3} + \frac{3}{2} - 1\right]}{[(n+1)^2+n^2][(2n+1)]};\ \Omega = \lim_{n\to\infty}\left[\frac{n(n+1)(2n+1)}{6(2n+1)(2n^2+2n+1)}\right]$$

$$\Omega = \lim_{n\to\infty}\left[\frac{(n^2+n)}{6(2n^2+2n+1)}\right];\ \Omega = \lim_{n\to\infty}\left[\frac{1}{6\left[\frac{2(n^2+n)+1}{n^2+n}\right]}\right]$$

$$\Omega = \lim_{n\to\infty}\left[\frac{1}{6\left[2+\frac{1}{n^2+n}\right]}\right];\ \Omega = \frac{1}{12}$$

563. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{\log\left(\Gamma\left(\frac{1}{n}\right)\right) + \log\left(\Gamma\left(\frac{2}{n}\right)\right) + \dots + \log\left(\Gamma\left(\frac{n-1}{n}\right)\right)}{n}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Farid Khelili-Algerie

Let $\Omega = \lim_{n\to\infty}\frac{1}{n}\left\{\ln\Gamma\left(\frac{1}{n}\right) + \ln\Gamma\left(\frac{2}{n}\right) + \dots + \ln\Gamma\left(\frac{n-1}{n}\right)\right\}$ *then*

$$\Omega = \lim_{n\to\infty}\frac{1}{n}\ln\left\{\Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(\frac{2}{n}\right)\cdot\ldots\cdot\Gamma\left(\frac{n-1}{n}\right)\right\} = \lim_{n\to\infty}\frac{\ln(a_n)}{n}$$

where $a_n = \Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(\frac{2}{n}\right)\cdot\ldots\cdot\Gamma\left(\frac{n-1}{n}\right) = \prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)$

Since $a_n = \prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right) = \prod_{k=1}^{n-1}\Gamma\left(1-\frac{k}{n}\right)$ *then* $a_n^2 = \prod_{k=1}^{n-1}\left\{\Gamma\left(\frac{k}{n}\right)\Gamma\left(1-\frac{k}{n}\right)\right\} = \prod_{k=1}^{n-1}\frac{\pi}{\sin\left(\frac{k\pi}{n}\right)}$

where we have used the reflection formula $\Gamma\left(\frac{k}{n}\right)\Gamma\left(1-\frac{k}{n}\right) = \frac{\pi}{\sin\left(\frac{k\pi}{n}\right)}$

Using the factorization $\frac{x^n-1}{x-1} = \prod_{k=1}^{n-1}\left(x - \exp\left(\frac{2k\pi i}{n}\right)\right);\ n \ge 1$

to get $\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$; *then* $a_n^2 = \prod_{k=1}^{n-1}\frac{\pi}{\sin\left(\frac{k\pi}{n}\right)} = \frac{1}{n}(2\pi)^{n-1};\ n \ge 1$

It follows that $\Omega = \lim_{n\to\infty}\frac{\ln(a_n)}{n} = \lim_{n\to\infty}\frac{\ln(a_n^2)}{2n} = \lim_{n\to\infty}\frac{\ln\left(\frac{1}{n}(2\pi)^{n-1}\right)}{2n}$

$$\Omega = \lim_{n\to\infty}\frac{1}{2n}\{(n-1)\ln(2\pi) - \ln n\} = \frac{1}{2}\ln(2\pi) = \ln\left(\sqrt{2\pi}\right)$$

Solution 2 by Naren Bhandari-Nepal

$L = \lim_{n\to\infty}\frac{1}{n}\left(\sum_{k=1}^{n}\log\Gamma\left(\frac{k}{n}\right)\right) = \int_0^1 \log\Gamma(x)\,dx$. *We have that:*

$$\int_0^z \log\Gamma(x)\,dx = \frac{z}{2}\log(2\pi) + \frac{z(z-1)}{2} + z\log(z) - \log G(z+1)$$

Setting limits gives us $L = \frac{1}{2}\log(2\pi)$

Note: $G(z+1) = \Gamma(z)G(z)$ *where* $G(z)$ *is the* G *– function and* $G(0) = 0, G(1) = 1$.

Solution 3 by Abdul Mukhtar-Nigeria

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1}\ln\left(\Gamma\left(\frac{k}{n}\right)\right) = \int_0^1 \ln(\Gamma(x))\,dx = \int_0^{n+1} \log\Gamma(x)\,dx$$

Set $f(a) = \int_0^{a+1}\log\Gamma(x)\,dx$. *Differentiate both sides*

$$f'(a) = \log(1+a) - \log\Gamma(a) = \log(a)\,;\ f'(a) = \log(a)$$

Integrate both sides $f(a) = a\log(a) - a + c$. *Set* $a = 0$

$f(0) = 0 + c \Rightarrow c = \int_0^1 \log\Gamma(x)\,dx$. *By reflection formula*

$$\int_0^1 \log\Gamma(x)\,dx = \int_0^1 \log(\pi)\,dx - \int_0^1 \log\sin(\pi x)\,dx - \int_0^1 \log\Gamma(1-x)\,dx$$

$$= 2\int_0^1 \log\Gamma(x) = \int_0^1 \log(\pi)\,dx - \int_0^1 \log\sin(\pi x)\,dx = \log(2\pi) - \int_0^1 \log|2\sin(\pi x)|\,dx$$

$\int_0^1 \log|2\sin(\pi x)|\,dx \Rightarrow$ *Clausen integral. So,* $\frac{2}{\pi}\int_0^{2\pi}\log\left|2\sin\left(\frac{x}{2}\right)\right|dx = \frac{2}{\pi}Cl_2(2\pi) = 0$

$$\therefore \int_0^1 Log\Gamma(x)dx = \frac{1}{2}\log(2\pi) = \log\sqrt{2\pi}$$

Solution 4 by Sagar Kumar-Patna Bihar-India

$$\Omega = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n-1}\left(\log\left(\Gamma\left(\frac{r}{n}\right)\right)\right)$$

$$\Gamma = \int_0^1 \log(\Gamma(x))\,dx \quad \textit{(1)},\ \Omega = \int_0^1 \log(\Gamma(1-x))\,dx \quad \textit{(2)}$$

(1)+(2): $\Omega = \frac{1}{2}\int_0^1 \log(\pi\csc(x\pi))\,dx$

$$\Omega = \frac{1}{2}\left(\log\pi + \int_0^1 \ln(\csc\pi x)\,dx\right) \quad \Omega = \ln\sqrt{\pi} + \frac{1}{2\pi}\int_0^{\pi}\ln(\csc x)\,dx$$

$$\Omega = \ln\sqrt{\pi} - \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\ln(\sin x)\,dx$$

$$\Omega = \ln\sqrt{\pi} + \frac{\pi}{2\pi}(\ln 2) = \ln\left(\sqrt{2\pi}\right) \quad \textit{(Answer)}$$

Solution 5 by Shivam Sharma-New Delhi-India

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1}\ln\left(\Gamma\left(\frac{k}{n}\right)\right) \Rightarrow \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1}\ln\left(\Gamma\left(\frac{k}{n}\right)\right) \Rightarrow \int_0^1 \ln(\Gamma(t))\,dt \quad \textit{(1)}$$

Let $I(x) = \int_x^{x+1}\ln(\Gamma(t))\,dt$

$$I'(x) = \ln(\Gamma(x+1)) - \ln(\Gamma(x)) \quad \textit{(2)}$$

$$I'(x) = \ln(x) \quad \textit{(3)}$$

Now, integrate equation (3), w.r.t. x*, we get,* $I(x) = x\ln(x) - x + C$ *(4)*

Now, $I(0) = \int_0^1 \ln(\Gamma(t))\,dt \Rightarrow \frac{1}{2}\ln(\pi)\,[t]_0^1 - \frac{1}{2}\int_0^1 \ln(\sin(\pi t))\,dt$

$$\Rightarrow \frac{\ln(\pi)}{2} - \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\ln(\sin t)\,dt \Rightarrow \frac{\ln(\pi)}{2} + \frac{\ln(2)}{2}$$

(OR) $I(0) = \ln\left(\sqrt{2\pi}\right)$*. Put* $x = 0$ *in equation (4), we get,* $\ln\left(\sqrt{2\pi}\right) = I(0) = C + 0$ *(5)*

Put $x = 0$*, we get,* $\ln\left(\sqrt{2\pi}\right) = x\ln(x) - x + C$*. We get,* $C = \ln\left(\sqrt{2\pi}\right)$ *(6)*

Putting the value of C *in equation (4), we get,* $I(x) = x\ln(x) - x + \ln\left(\sqrt{2\pi}\right)$ *(7)*

Now, coming back to the (1). Put $x = 0$ *in equation (8), we get,* $\Omega = I(0) = \ln\left(\sqrt{2\pi}\right)$

(OR) $\Omega = \ln\left(\sqrt{2\pi}\right)$ *(Answer)*

Solution 6 by Nassim Nicholas Taleb-USA

First, $\exp\left(\frac{\sum_{k=1}^{n}\log\left(\Gamma\left(\frac{k}{n}\right)\right)}{n}\right) = \left(\prod_{k=1}^{n}\Gamma\left(\frac{k}{n}\right)\right)^{\frac{1}{n}}$

We can use 2 identities:

First identity, $\Gamma(x)\Gamma(1-x) = \pi\cos[\pi x]$

Second, $\prod_{k=1}^{n-1}\cos\left(\frac{\pi k}{n}\right) = \frac{2^{-1+n}}{n}$

We have

$$\left(\prod_{k=1}^{n}\Gamma\left(\frac{k}{n}\right)\right)^{2} = \prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)\Gamma\left(1-\frac{k}{n}\right) = \prod_{k=1}^{n-1}\pi\cos\left(\frac{\pi k}{n}\right) = \frac{2^{n-1}\pi^{n-1}}{n}$$

$$\left(\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)\Gamma\left(\frac{n}{n}\right)\right)^{\frac{1}{n}} = \left(\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)\right)^{\frac{1}{n}} = \left(\frac{(2\pi)^{-1+n}}{n}\right)^{\frac{1}{2}/n}$$

Taking logs, we need the $\lim_{n\to\infty}\log\left(\left(\frac{(2\pi)^{n-1}}{n}\right)^{\frac{1}{2n}}\right) =$

$$\lim_{n\to\infty}\left(\frac{\log[2]}{2}-\frac{\log[2]}{2n}-\frac{\log[n]}{2n}+\frac{\log[\pi]}{2}-\frac{\log[\pi]}{2n}\right) = \frac{1}{2}\log[2\pi]$$

Solution 7 by Nassim Nicholas Taleb-USA

Looking for $\int_0^1\log(\Gamma(x))\,dx$

We have, for x *real,*

$\int\frac{\Gamma'(x)}{\Gamma(x)}dx = \log(\Gamma(x))$. $\int_0^z\log(\Gamma(x))\,dx$ *is therefore by definition the Polygamma function with negative second order derivative*

$\psi^{(-2)}(z)$. *For* $z = 1$, $\psi^{(-2)}(1) = \frac{1}{2}\log(2\pi)$.

564. Find:

$$\Omega = \lim_{n\to\infty} n^2\left(\sqrt[n+5]{7}-\sqrt[n+8]{7}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

$$\Omega = \lim_{n\to\infty}\left[\frac{7^{\frac{1}{n+5}} - 7^{\frac{1}{n+8}}}{\left(\frac{1}{n^2}\right)}\right]$$

$$\textit{Put } n = \frac{1}{t}, n \to \infty, t \to 0 \Rightarrow \Omega = \lim_{t\to 0}\left(\frac{7^{\frac{t}{1+5t}} - 7^{\frac{t}{1+8t}}}{t^2}\right)$$

$$\Omega = \lim_{t\to 0} 7^{\frac{t}{1+8t}}\frac{\left[7^{\left(\frac{t}{1+5t}-\frac{t}{1+8t}\right)} - 1\right]}{t^2}, \quad \Omega = \lim_{t\to 0} 7^{\frac{t}{1+8t}}\frac{\left[7^{\frac{3t^2}{(1+5t)(1+8t)}} - 1\right]}{t^2}$$

$$\Omega = \lim_{t\to 0} 7^{\frac{t}{1+8t}}\left[\frac{7^{\frac{3t^2}{(1+5t)(1+8t)}} - 1}{\frac{3t^2}{(1+5t)(1+8t)}}\right]\cdot\frac{3t^2}{t^2(1+5t)(1+8t)}$$

$$\Omega = 1 \times \ln 7 \times \lim_{t\to 0}\frac{3t^2}{t^2(1+5t)(1+8t)}\left\{\because \lim_{t\to 0}\frac{a^t - 1}{t} = \ln a\right\}$$

$$\Omega = 3\ln 7$$

Solution 2 by Sagar Kumar-Patna Bihar-India

$$\lim_{x\to 0}\left[\frac{7^{\frac{x}{5x+1}} - 7^{\frac{x}{8x+1}}}{x^2}\right] \Rightarrow \lim_{x\to 0}\ln 7\,\frac{\left(\frac{x}{5x+1} - \frac{x}{8x+1}\right)}{x^2}$$

$$\Rightarrow \lim_{x\to 0}\ln 7\left(\frac{3x^2}{x^2(40x^2+13x+1)}\right) \Rightarrow 3\ln 7 = \ln(343)$$

Solution 3 by Marian Ursărescu-Romania

$$\Omega = \lim_{n\to\infty} n^2\left(7^{\frac{1}{n+5}} - 7^{\frac{1}{n+8}}\right) = \lim_{n\to\infty} n^2\, 7^{\frac{1}{n+8}}\left(7^{\frac{1}{n+5}-\frac{1}{n+8}} - 1\right) =$$

$$= \lim_{n\to\infty} n^2\cdot 7^{\frac{1}{n+8}}\cdot\left(7^{\frac{3}{(n+5)(n+8)}} - 1\right) = \lim_{n\to\infty} 7^{\frac{1}{n+8}}\frac{\left(7^{\frac{3}{(n+5)(n+8)}} - 1\right)}{\frac{3}{(n+5)(n+8)}}\cdot\frac{3n^2}{(n+5)(n+8)} = 3\ln 7$$

Solution 4 by Kartick Chandra Betal-India

$$\Omega = \lim_{n\to\infty} n^2\left[7^{\frac{1}{n+5}} - 7^{\frac{1}{n+8}}\right] = \lim_{n\to\infty} n^2 e^{\frac{1}{n+8}}\left[e^{\left(\frac{1}{n+5}-\frac{1}{n+8}\right)\ln 7} - 1\right]$$

$$= \lim_{n\to\infty}\left[e^{\frac{1}{n+8}}\right]\cdot\lim_{n\to\infty}\left[\frac{e^{\frac{3\ln 7}{(n+5)(n+8)}}-1}{\frac{3\ln 7}{(n+5)(n+8)}}\right]\cdot\lim_{n\to\infty}\left[\frac{3\ln 7}{\left(1+\frac{5}{n}\right)\left(1+\frac{8}{n}\right)}\right] = 1\cdot 1\cdot 3\ln 7 = 3\ln 7$$

565.

$$x_0 > 0, \sqrt[p]{x_n} = \frac{1+x_n-x_{n+1}}{x_{n+1}-x_n}, n\in\mathbb{N}^*, p\in\mathbb{N}^*, p\geq 2$$

Find:

$$\Omega = \lim_{n\to\infty}\frac{x_n^{p+1}}{n^p}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

We know, $\lim_{u\to 0}\frac{(1+u)^r-1}{u} = r$ *now* $\sqrt[p]{x_n} = \frac{1+x_n-x_{n+1}}{x_{n+1}-x_n} \Rightarrow x_{n+1} = x_n + \frac{1}{\sqrt[p]{x_n}+1}$ *for all* $x_0 > 0$ *where* $n\to\infty$*. Since,* $x_0 > 0$ *then* $x_1 > x_0, x_2 > x_1, \dots, x_{k+1} > x_k$*, hence* $\{x_n\}_{n=0}^{\infty}$ *is increasing hence it converges to a limit. So, let* $\lim_{n\to\infty} x_n = l$ *then* $l = l + \frac{1}{\sqrt[p]{l}+1} \Rightarrow$

$$l\to\infty$$

Hence the claim is contradictory. So $\lim_{n\to\infty} x_n = \infty$

$$\Omega = \lim_{n\to\infty}\frac{x_n^{p+1}}{n^p} \Rightarrow \sqrt[p]{\Omega} = \lim_{n\to\infty}\frac{x_n^{\frac{p+1}{p}}}{n} \overset{\substack{CAESARO\\STOLZ}}{=} \lim_{n\to\infty}\frac{x_{n+1}^{\frac{p+1}{p}}-x_n^{\frac{p+1}{p}}}{n+1-n}$$

$$= \lim_{n\to\infty}\left(\frac{x_n^{\frac{p+1}{p}}}{x_n^{\frac{p+1}{p}}+x_n}\right)\frac{\left(1+\frac{1}{x_n^{\frac{p+1}{p}}+x_n}\right)^{\frac{p+1}{p}}-1}{\frac{1}{x_n^{\frac{p+1}{p}}+x_n}} = \frac{p+1}{p}\lim_{x_n\to\infty}\frac{1}{1+\frac{1}{\sqrt[p]{x_n}}}$$

$$= \frac{p+1}{p} \Rightarrow \Omega = \left(1+\frac{1}{p}\right)^p \quad \textit{(Answer)}$$

Solution 2 by Shafiqur Rahman-Bangladesh

$$\lim_{n\to\infty}\frac{x_n^{p+1}}{n^p}=\left\{\lim_{n\to\infty}\frac{x_n^{\frac{p+1}{p}}}{n}\right\}^p=\left\{\lim_{n\to\infty}\left(x_{n+1}^{\frac{p+1}{p}}-x_n^{\frac{p+1}{p}}\right)\right\}^p\ [Stolz-Cesaro\ th^m]=$$

$$=\lim_{n\to\infty}\left\{x_n^{\frac{p+1}{p}}\left(\left(\frac{x_{n+1}}{x_n}\right)^{\frac{p+1}{p}}-1\right)\right\}^p$$

$$=\lim_{n\to\infty}\left\{x_n^{\frac{p+1}{p}}\left(\left(1+\frac{1}{x_n+x_n^{\frac{p+1}{p}}}\right)^{\frac{p+1}{p}}-1\right)\right\}^p=\lim_{n\to\infty}\left\{x_n^{\frac{p+1}{p}}\left(\frac{\frac{p+1}{p}}{x_n+x_n^{\frac{p+1}{p}}}\right)\right\}^p=$$

$$=\lim_{n\to\infty}\left\{\left(\frac{\frac{p+1}{p}}{x_n^{-\frac{1}{p}}+1}\right)\right\}^p\ \therefore\ \lim_{n\to\infty}\frac{x_n^{p+1}}{n^p}=\left(\frac{p+1}{p}\right)^p$$

$$\left[Here\ \lim_{n\to\infty}x_n=\infty\ \therefore\ \lim_{n\to\infty}\frac{1}{x_n+x_n^{\frac{p+1}{p}}}=0\ \therefore\ \lim_{n\to\infty}\left(1+\frac{1}{x_n+x_n^{\frac{p+1}{p}}}\right)^{\frac{p+1}{p}}-1=\lim_{n\to\infty}\left(\frac{\frac{p+1}{p}}{x_n+x_n^{\frac{p+1}{p}}}\right)\&\lim_{n\to\infty}x_n^{-\frac{1}{p}}=0\right]$$

566.

$$x_0\in(0,1),x_{n+1}=x_n\sqrt[p]{1-x_n},y_0>0,y_{n+1}=y_n+\frac{1}{y_n^{p-1}},n,p\in\mathbb{N},p\geq 2.$$

Find:

$$\Omega=\lim_{n\to\infty}\left(y_n\sqrt[p]{x_n}\right)$$

Proposed by Marian Ursărescu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

We know, $\lim_{u\to 0}\frac{(1+u)^r-1}{u}=r$ *and* $\lim_{u\to 0}\frac{(1-u)^{-r}-1}{u}=r$

$x_{n+1}=x_n\sqrt[p]{1-x_n}$ *where* $x_0\in(0,1)$,

then we have $1>x_0\geq x_1\geq x_2\geq x_3\geq\cdots\geq x_n\geq\cdots>0$

$\therefore\{x_n\}_{n=1}^{\infty}$ *is a decreasing function similarly,* $\{y_n\}_{n=1}^{\infty}$ *defined by*

$y_{n+1}=y_n+\frac{1}{y_n^{p-1}}$ *is an increasing function.*

Let $\lim_{n\to\infty}x_n=l$ *and* $\lim_{n\to\infty}y_n=m$ *then* $l=l\sqrt[p]{1-l}\Rightarrow l=0$ *and*

$$m=m+\frac{1}{m^{p-1}}\Rightarrow m=\infty\therefore \lim_{n\to\infty}x_n=0 \textit{ and } \lim_{n\to\infty}y_n=\infty$$

$$\Omega=\lim_{n\to\infty}\left(y_n\sqrt[p]{x_n}\right)\Rightarrow\Omega^p=\lim_{n\to\infty}\left(\frac{y_n^p}{n}\cdot\frac{n}{\frac{1}{x_n}}\right)\overset{\substack{CAESARO\\STOLZ}}{=}\lim_{n\to\infty}\frac{y_{n+1}^p-y_n^p}{n+1-n}\cdot\frac{n+1-n}{\frac{1}{x_{n+1}}-\frac{1}{x_n}}$$

$$=\lim_{n\to\infty}\left(\left(y_n+\frac{1}{y_n^{p-1}}\right)^p-y_n^p\right)\cdot\lim_{n\to\infty}\frac{1}{\frac{1}{x_n\sqrt[p]{1-x_n}}-\frac{1}{x_n}}$$

$$=\lim_{y_n\to\infty}\frac{\left(1+\frac{1}{y_n^p}\right)^p-1}{\frac{1}{y_n^p}}\cdot\lim_{x_n\to 0}\frac{1}{\frac{(1-x_n)^{-\frac{1}{p}}-1}{x_n}}=p^2\Rightarrow\Omega=\sqrt[p]{p^2}$$

567. Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\sqrt[p]{\frac{n^2}{i\cdot j}},p\in\mathbb{N},p\ge 2$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Artan Ajredini-Presheva-Serbie

$$\Omega=\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le 1<j\le n}\sqrt[p]{\frac{n^2}{ij}}=\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\sqrt[p]{\frac{n}{i}}\sqrt[p]{\frac{n}{j}}=\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=2}^{n}\sqrt[p]{\frac{n}{i}}\sqrt[p]{\frac{n}{j}}=$$

$$=\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\sqrt[p]{\frac{n}{i}}\sum_{j=2}^{n}\sqrt[p]{\frac{n}{j}}=\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\sqrt[p]{\frac{n}{i}}\left(\sum_{j=1}^{n}\sqrt[p]{\frac{n}{j}}-\sqrt[p]{n}\right)=$$

$$=\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\sqrt[p]{\frac{n}{i}}\sum_{j=1}^{n}\sqrt[p]{\frac{n}{j}}-\lim_{n\to\infty}\frac{\sqrt[p]{n}}{n^2}\sum_{ij=1}^{n}\sqrt[p]{\frac{n}{i}}=L_1-L_2\quad\textit{(1)}$$

$$L_1=\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\sqrt[p]{\frac{n}{i}}\sum_{j=1}^{n}\sqrt[p]{\frac{n}{j}}=\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{-\frac{1}{p}}\sum_{j=1}^{n}\left(\frac{j}{n}\right)^{-\frac{1}{p}}$$

From the definition of double integral, we have:

$$L_1=\int_0^1\int_0^1(xy)^{\frac{1}{p}}dx\,dy=\int_0^1x^{-\frac{1}{p}}dx\int_0^1y^{-\frac{1}{p}}dy=\frac{x^{1-\frac{1}{p}}}{1-\frac{1}{p}}\Bigg|_0^1\frac{y^{1-\frac{1}{p}}}{1-\frac{1}{p}}\Bigg|_0^1=\frac{1}{1-\frac{1}{p}}\cdot\frac{1}{1-\frac{1}{p}}=\frac{p^2}{(p-1)^2}\quad\textit{(2)}$$

$$L_2 = \lim_{n\to\infty} \frac{\sqrt[p]{n}}{n^2}\sum_{i=1}^{n}\sqrt[p]{\frac{n}{i}} = \lim_{n\to\infty}\frac{1}{n^{1-\frac{1}{p}}}\cdot\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{-\frac{1}{p}} = \lim_{n\to\infty}\frac{1}{n^{1-\frac{1}{p}}}\int_0^1 x^{-\frac{1}{p}}dx = 0 \quad (3)$$

We substitute (2) and (3) to (1) and we have: $\Omega = \frac{p^2}{(p-1)^2}$

Solution 2 by Ravi Prakash-New Delhi-India

For $p > 2$***. Let*** $a_i = \left(\frac{n}{i}\right)^{\frac{1}{p}} = \frac{1}{\left(\frac{i}{n}\right)^{\frac{1}{p}}}$

$$2\sum_{1\le i<j\le n} a_i a_j = \left(\sum_{i=1}^{n} a_i\right)^2 - \sum_{i=1}^{n} a_i^2 \Rightarrow 2\frac{1}{n^2}\sum_{1\le i<j\le n} a_i a_j = \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^2 - \frac{1}{n}\cdot\frac{1}{n}\sum_{i=1}^{n} a_i^2$$

$$2\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n} a_i a_j = \left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} a_i\right)^2 - \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n}\sum_{i} a_i^2\right)$$

$$= \left(\int_0^1 \frac{1}{x^{\frac{1}{p}}}dx\right)^2 - \left(\lim_{n\to\infty}\frac{1}{n}\right)\int_0^1 \frac{1}{x^{\frac{2}{p}}}dx = \left(\left.\frac{x^{-\frac{1}{p}+1}}{1-\frac{1}{p}}\right]_0^1\right)^2 - (0)\left[\frac{x^{1-\frac{2}{p}}}{1-\frac{2}{p}}\right]_0^1 = \frac{p^2}{(p-1)^2} - 0$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n} a_i a_j = \frac{p^2}{2(p-1)^2}$$

568. Find:

$$\Omega = \lim_{n\to\infty}\int_{\frac{1}{n^2}}^{n^2}\frac{e^{(bx)^2} - e^{(ax)^2}}{x\cdot e^{(a^2+b^2)x^2}}dx, a < b$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Igor Soposki-Skopje-Macedonia

$$I = \lim_{n\to\infty}\int_{\frac{1}{n^2}}^{n^2}\frac{e^{(bx)^2} - e^{(ax)^2}}{xe^{(a^2+b^2)x^2}} = \left\{\begin{matrix} x^2 = t \\ 2x\,dx = dt\end{matrix}\right\} = \lim_{n\to\infty}\frac{1}{2}\int_{\frac{1}{n}}^{n}\left[\frac{e^{-a^2t} - e^{-b^2t}}{t}\right]dt = \frac{1}{2}\int_0^t \frac{e^{-a^2t} - e^{-b^2t}}{t}dt$$

$$I_1(\alpha) = \int_0^t \underbrace{\frac{e^{-\alpha t}}{t}}_{f(\alpha,t)}dt =? f'_\alpha(\alpha,t) = \frac{-te^{-\alpha t}}{t} = -e^{-\alpha t}$$

$$I_1'(\alpha)=\int_0^t -e^{-\alpha t}\,dt=+\frac{1}{\alpha}\cdot e^{-\alpha t}\Big|_0^t=-\frac{1}{\alpha}\Rightarrow I_1(\alpha)=\int\left(-\frac{1}{\alpha}\right)d\alpha=-\ln\alpha$$

$$I=\frac{1}{2}[-\ln(a^2)+\ln(b^2)]=\frac{1}{2}\ln\left(\frac{b}{a}\right)^2=\ln\frac{b}{a}$$

Solution 2 by Sagar Kumar-Patna Bihar-India

$$I=\lim_{n\to\infty}\int_{\frac{1}{n^2}}^{n^2}\frac{e^{b^2x^2}-e^{a^2x^2}}{xe^{(a^2+b^2)x^2}}dx;a<b;I=\int_0^\infty\frac{e^{-a^2x^2}-e^{-b^2x^2}}{x}dx$$

Let $I(a)=\int_0^\infty\frac{e^{-a^2x^2}-1}{x}dx;\ I'(a)=\int_0^\infty\frac{e^{-a^2x^2}(-2ax^2)}{x}dx=-2a\int_0^\infty xe^{-a^2x^2}dx$

Put $a^2x^2=t\quad 2xdx=\frac{dt}{a^2}=\frac{-a}{a^2}\int_0^\infty e^{-t}\,dt=\frac{1}{a}(e^{-t})\Big|_0^\infty=-\frac{1}{a}$

$I(a)=\ln\left(\frac{1}{a}\right)+C_1\quad I(1)=C_1$. Similarly, $I(b)=\ln\left(\frac{1}{b}\right)+C_2;\ I(1)=C_2=C_1$

$$I(a)-I(b)=I=\ln\left(\frac{b}{a}\right)$$

Solution 3 by Shafiqur Rahman-Bangladesh

$$\lim_{n\to\infty}\int_{\frac{1}{n^2}}^{n^2}\frac{e^{(bx)^2}-e^{(ax)^2}}{x\cdot e^{(a^2+b^2)x^2}}dx=\int_0^\infty\frac{e^{-(ax)^2}-e^{-(bx)^2}}{x}dx$$

[Using Frullani integral]

$$=[e^0-e^{-\infty}]\ln\left(\frac{b}{a}\right)\ \therefore\lim_{n\to\infty}\int_{\frac{1}{n^2}}^{n^2}\frac{e^{(bx)^2}-e^{(ax)^2}}{x\cdot e^{(a^2+b^2)x^2}}dx=\ln\left(\frac{b}{a}\right)$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty}\int_{\frac{1}{n}}^{n}\frac{e^{-a^2x^2}-e^{-b^2x^2}}{x}dx=\int_0^\infty\frac{e^{-a^2x^2}-e^{-b^2x^2}}{x}dx$$

$$=\lim_{\substack{\varepsilon\to0+\\X\to\infty}}\int_\varepsilon^X\frac{e^{-a^2x^2}}{x}dx-\lim_{\substack{\varepsilon\to0+\\X\to\infty}}\int_\varepsilon^X\frac{e^{-b^2x^2}}{x}dx=\lim_{\substack{\varepsilon\to0+\\X\to\infty}}\int_{a\varepsilon}^{aX}\frac{e^{-u^2}}{u}du-\lim_{\substack{\varepsilon\to0+\\X\to\infty}}\int_{b\varepsilon}^{bX}\frac{e^{-v^2}}{v}dv$$

$$=\lim_{\substack{\varepsilon\to0+\\X\to\infty}}\left[\int_{a\varepsilon}^{b\varepsilon}\frac{e^{-u^2}}{u}du+\int_{b\varepsilon}^{bX}\frac{e^{-u^2}}{u}du+\int_{bX}^{aX}\frac{e^{-u^2}}{u}du\right]-\lim_{\substack{\varepsilon\to0+\\X\to\infty}}\int_{b\varepsilon}^{bX}\frac{e^{-u^2}}{u}du$$

$$= \lim_{\varepsilon\to 0+}\int_{a\varepsilon}^{b\varepsilon}\frac{e^{-u^2}}{u}du - \lim_{X\to\infty}\int_{aX}^{bX}\frac{e^{-u^2}}{u}du = \lim_{\varepsilon\to 0+} e^{-\xi^2}\int_{a\varepsilon}^{b\varepsilon}\frac{du}{u} - \lim_{X\to\infty} e^{-\zeta^2}\int_{bX}^{bX}\frac{du}{u}$$

[By First Mean value Theorem There Exists** $\xi \in [a\varepsilon, b\varepsilon]$ **and** $\zeta \in [aX, bX]$**]

$$= \lim_{\varepsilon\to 0+} e^{-\xi^2}\ln\left(\frac{b}{a}\right) - \lim_{X\to\infty} e^{-\zeta^2}\ln\left(\frac{b}{a}\right) = \ln\left(\frac{b}{a}\right)$$

569. $f: [-1,1] \to \mathbb{R}, f(x) = (\sin^{-1} x)^2, f^{(n)} - n'th\ derivative$**. Find:**

$$\Omega = \lim_{n\to\infty}\frac{\sqrt[2n]{f^{(2n)}(0)}}{n}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $y = (\sin^{-1} x)^2 = f(x)$***. Find*** $\lim_{n\to\infty}\frac{\left(f^{2n}(0)\right)^{\frac{1}{2n}}}{n}$***. Let*** $y = (\sin^{-1} x)^2;\ y_1 = \frac{2(\sin^{-1} x)}{\sqrt{1-x^2}}$

$$\sqrt{1-x^2}\,y_1 = 2(\sin^{-1} x) \Rightarrow \sqrt{1-x^2}\,y_2 - \frac{x}{\sqrt{1-x^2}}y_1 = \frac{2}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)y_2 - xy_1 = 2 \quad (1)$$

By the Lebniz's rule for nth derivative: $(1-x^2)y_{n+2} + n_{C_1(-2x)}y_{n+1} + n_{C_2(2)(-1)y_n}$

$$-[xy_{n+1} + ny_n] = 0 \Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$$

When $x = 0,\ y_{n+2}(0) = n^2y_n(0)$ ***(2)***

From (1): $y_2(0) = 2$***, From (2):*** $y_{n+2}(0) = n^2y_n(0)$

$$y_4(0) = 2^2y_2(0) = 2^2(2)$$

$$y_6(0) = 4^2y_4(0) = (4^2)(2^2)(2) = (2!)^2 2^5$$

$$y_8(0) = 6^2y_6(0) = 6^2(4^2)(2^2)(2) = (3!)^2 2^7$$

$$y_{10}(0) = 8^2y_8(0) = (4!)^2 2^9$$

$$y_{2n+2}(0) = (n!)^2 2^{2n+1} \Rightarrow y_{2n}(0) = \left((n-1)!\right)^2 2^{2n-1}$$

$$\frac{\left(y_{2n(0)}\right)^{\frac{1}{2n}}}{n} = \frac{\left((n-1)!\right)^{\frac{1}{n}}2^{1-\frac{1}{n}}}{n} = \left(\frac{(n-1)!}{n^n}\right)^{\frac{1}{n}}2^{1-\frac{1}{n}}$$

But

$$\lim_{n\to\infty}\left(\frac{(n-1)!}{n^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n!}{(n+1)^{n+1}}\cdot\frac{n^n}{(n-1)!} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{e} \therefore \lim_{n\to\infty}\frac{\left(f^{2n}(0)\right)^{\frac{1}{2n}}}{n} = \frac{2}{e}$$

Solution 2 by Shafiqur Rahman-Bangladesh

$$f(x) = (\sin^{-1}x)^2 \Rightarrow f'(x) = \frac{2\sin^{-1}x}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}f'(x) = 2\sin^{-1}x \Rightarrow$$

$$\Rightarrow \sqrt{1-x^2}f''(x) - \frac{x}{\sqrt{1-x^2}}f'(x) = \frac{2}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)f''(x) - xf'(x) = 2 \Rightarrow (1-x^2)f^{n+2}(x) - (2n+1)xf^{n+1}(x) - n^2f^n(x) = 0$$

[Using leibnits th^m]

$$\Rightarrow f^{n+2}(0) - n^2f^n(0) = 0[x=0] \Rightarrow \frac{f^{2n+2}(0)}{f^{2n}(0)} = 4n^2[n\to 2n] \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}\sqrt{\frac{\frac{f^{2n+2}(0)}{(n!)^2}}{\frac{f^{2n}(0)}{\{(n-1)!\}^2}}} = 2 \Rightarrow \lim_{n\to\infty}\sqrt{\sqrt[n]{\frac{f^{2n}(0)}{\{(n-1)!\}^2}}} = 2$$

$$\Rightarrow \lim_{n\to\infty}\sqrt[2n]{\frac{f^{2n}(0)}{\frac{n^{2n}}{e^{2n}}}} = 2\left[\because \lim_{n\to\infty}\{(n-1)!\}^2 \approx \lim_{n\to\infty}(n!)^2 = \lim_{n\to\infty}\frac{n^{2n}}{e^{2n}}\right] \therefore \lim_{n\to\infty}\frac{\sqrt[2n]{f^{2n}(0)}}{n} = \frac{2}{e}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

We know, $(\arcsin x)^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{\left(n^2\binom{2n}{n}\right)}$ ***where*** $x\in[-1,1]$

$$f^{(2n)}(x) = \frac{2^{2n-1}(n!)^2}{n^2} + \frac{1}{2}\sum_{m=n+1}^{\infty}\frac{2^{2m}x\cdot(2m)!}{\left(m^2\binom{2m}{m}\right)} \Rightarrow f^{(2n)}(0) = \frac{2^{2n-1}(n!)^2}{n^2}$$

$$\Omega = \lim_{n\to\infty}\frac{\sqrt[2n]{f^{(2n)}(0)}}{n} = \lim_{n\to\infty}\sqrt{\frac{\sqrt[n]{\frac{2^{2n-1}(n!)^2}{n^2}}}{n^2}} = \sqrt{\lim_{n\to\infty}\sqrt[n]{\frac{2^{2n-1}\cdot(n!)^2}{n^{2(n+1)}}}}$$

$$\overset{CAUCHY-D'ALEMBERT}{=}\sqrt{\lim_{n\to\infty}\left(\frac{2^{2n+1}\cdot\{(n+1)!\}^2}{(n+1)^{2(n+2)}}\cdot\frac{n^{2(n+1)}}{2^{2n-1}\cdot(n!)^2}\right)}=\sqrt{\lim_{n\to\infty}\frac{4}{\left(1+\frac{1}{n}\right)^{2n}}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^4}}=\frac{2}{e}$$

570. $x_0=0, x_1=2, x_{n+1}=(2n+3)\left(x_n+(2n+1)!!\,(n+2)\right), a_n>0,$

$$n\in\mathbb{N}, \lim_{n\to\infty}\frac{a_{n+1}}{na_n}=a>0$$

Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}^2}{x_{n+1}}}-\sqrt[n]{\frac{a_n^2}{x_n}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu and Neculai Stanciu – Romania

Solution by Shafiqur Rahman-Bangladesh

$$x_{n+1}=(2n+3)\left(x_n+(2n+1)!!\,(n+2)\right)\Rightarrow\frac{x_{n+1}}{(2n+3)!!}-\frac{x_n}{(2n+1)!!}=n+2\Rightarrow$$

$$\Rightarrow\frac{x_n}{(2n+1)!!}-x_0=\frac{n(n+3)}{2}\Rightarrow x_n=\frac{n(n+3)}{2}(2n+1)!!$$

$$\therefore\frac{x_{n+1}}{x_n}=\frac{(n+1)(n+4)(2n+3)}{n(n+3)}$$

Now, $\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}^2}{x_{n+1}}}-\sqrt[n]{\frac{a_n^2}{x_n}}\right)=\lim_{n\to\infty}\left((n+1)\cdot\sqrt[n+1]{\frac{a_{n+1}^2}{(n+1)^{n+1}\cdot x_{n+1}}}-n\sqrt[n]{\frac{a_n^2}{n^n\cdot x_n}}\right)=$

$$=\lim_{n\to\infty}\left(\frac{\frac{a_{n+1}^2}{(n+1)^{n+1}\cdot x_{n+1}}}{\frac{a_n^2}{n^n\cdot x_n}}\right)=\lim_{n\to\infty}\left(\frac{n^3(n+3)}{(n+1)^2(n+4)(2n+3)\left(1+\frac{1}{n}\right)^n}\left(\frac{a_{n+1}}{na_n}\right)^2\right)$$

$$\therefore\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}^2}{x_{n+1}}}-\sqrt[n]{\frac{a_n^2}{x_n}}\right)=\frac{a^2}{2e}$$

571. Find:

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$$\Omega = \lim_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=1}^{n}\left(\binom{n}{k}\cdot\sin^{-1}\left(\frac{k}{n}\right)\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

We shall use the following results:

1. $\sum_{k=0}^{n} k\binom{n}{k}\frac{1}{2^n} = \frac{n}{2}$

2. $\sum_{k=0}^{n} k(k-1)\binom{n}{k}\frac{1}{2^n} = \frac{n(n-1)}{4}$

3. (1), (2) $\Rightarrow \sum_{k=0}^{n} k^2\binom{n}{k}\frac{1}{2^n} = \frac{n(n+1)}{4}$

4. $\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{n}{2}-k\right)^2 = \frac{1}{4}n^2 - \frac{n^2}{2} + \frac{n(n+1)}{4} = \frac{n}{4}$

Main question

Let $f(x) = \sin^{-1} x, 0 \le x \le 1$. *As* $f(x)$ *is continuous at* $x = \frac{1}{2}$*, given* $\varepsilon > 0\ \exists\delta > 0$ *such that:* $\left|f(x) - \frac{\pi}{6}\right| = \left|f(x) - f\left(\frac{1}{2}\right)\right| < \frac{\varepsilon}{2}$ *whenever,* $\left|x - \frac{1}{2}\right| < \delta, 0 \le x \le 1$.

Now, $\frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k}\sin^{-1}\left(\frac{k}{n}\right) - \frac{\pi}{6} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left[\sin^{-1}\left(\frac{k}{n}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right]$

$$\left[\because \sum_{k=0}^{n}\binom{n}{k} = 2^n\right]$$

$$= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left[f\left(\frac{k}{n}\right) - f\left(\frac{1}{2}\right)\right]$$

$\Rightarrow \left|\sum_{k=0}^{n}\frac{1}{2^n}\binom{n}{k}\left\{\sin^{-1}\left(\frac{k}{n}\right) - \frac{\pi}{6}\right\}\right| \le \sum_{k=0}^{n}\frac{1}{2^n}\binom{n}{k}\left|\sin^{-1}\left(\frac{k}{n}\right) - \frac{\pi}{6}\right|$ *(1)*

We split the set $\{0, 1, 2, \dots, n\}$ *into two subsets* A *and* B.

$0 \le k \le n, k \in A$ *if* $\left|\frac{k}{n} - \frac{1}{2}\right| < \delta$ *and* $k \in B$ *if* $\left|\frac{k}{n} - \frac{1}{2}\right| \ge \delta$.

Now, $\sum_{k\in A}\frac{1}{2^n}\binom{n}{k}\left|f\left(\frac{k}{n}\right) - f\left(\frac{1}{2}\right)\right| \le \frac{\varepsilon}{2}\sum_{k\in A}\frac{1}{2^n}\binom{n}{k} \le \frac{\varepsilon}{2}(1) = \frac{\varepsilon}{2}$ *(2)*

If $k \in B$*, then* $\left|\frac{k}{n} - \frac{1}{2}\right| \ge \delta \Rightarrow \left(k - \frac{n}{2}\right)^2 \ge n^2\delta^2$

Now, $\sum_{k\in B}\frac{1}{2^n}\binom{n}{k}\left|f\left(\frac{k}{n}\right) - f\left(\frac{1}{2}\right)\right| \le \left(\frac{\pi}{2} + \frac{\pi}{2}\right)\frac{1}{n^2\delta^2}\sum_{k\in B}\left(k - \frac{n}{2}\right)^2\binom{n}{k}\frac{1}{2^n} \le \frac{\pi}{n^2\delta^2}\cdot\frac{n}{4}$ *[using (4)]*

$$\Rightarrow \sum_{k\in B}\frac{1}{2^n}\binom{n}{k}\left|f\left(\frac{k}{n}\right)-f\left(\frac{1}{2}\right)\right| \le \frac{\pi}{4n\delta^2} \quad (3)$$

We choose n sufficiently large, so that $\frac{\pi}{4n\delta^2} < \frac{\varepsilon}{2}$ *(4)*

[This is possible as $\frac{1}{n} \to 0$ *as* $n \to \infty$*]*

Using (1), (2), (3), (4) we get: $\left|\frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k}\sin^{-1}\left(\frac{k}{n}\right)-\frac{\pi}{6}\right| < \varepsilon$

for sufficiently large values of n. $\Rightarrow \lim_{n\to\infty}\frac{1}{2^n}\sum_k^n\binom{n}{k}\sin^{-1}\left(\frac{k}{n}\right) = \frac{\pi}{6}$

572. Find:

$$\Omega = \lim_{n\to\infty}\left(\sinh\frac{1}{n+1}+\sinh\frac{1}{n+2}+\cdots+\sinh\frac{1}{2n}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

$$\because \sinh(x) = \left(\frac{e^x - e^{-x}}{2}\right) \therefore \sinh\left(\frac{1}{n+1}\right) = \left(\frac{e^{\frac{1}{n+1}} - e^{-\frac{1}{n+1}}}{2}\right)$$

$$\therefore \lim_{n\to\infty}\sinh\left(\frac{1}{n+1}\right) = \lim_{n\to\infty}\left(\frac{e^{\frac{1}{n+1}} - e^{-\frac{1}{n+1}}}{2}\right)$$

$$= \left\{\lim_{n\to\infty}\frac{e^{\frac{1}{n+1}}-1}{\left(\frac{1}{n+1}-1\right)} \times \frac{\left(\frac{1}{n+1}-1\right)}{2}\right\} - \left\{\lim_{n\to\infty}\frac{e^{-\frac{1}{n+1}}-1}{\left(-\frac{1}{n+1}-1\right)} \times \left(\frac{-\frac{1}{n+1}-1}{2}\right)\right\}$$

Using $\lim_{x\to 0}\frac{e^x-1}{n} = 1 \Rightarrow$

$$\left\{1 \times \lim_{n\to\infty}\left(\frac{\frac{1}{n+1}-1}{2}\right)\right\} - \left\{1 \times \lim_{n\to\infty}\frac{\left(-\frac{1}{n+1}-1\right)}{2}\right\}$$

$$\Rightarrow \lim_{n\to\infty}\left(-\frac{n}{2(n+1)}\right) - \lim_{n\to\infty}\left(-\frac{(n+2)}{2(n+1)}\right) \Rightarrow \lim_{n\to\infty}\frac{-n+n+2}{2(n+1)} = \lim_{n\to\infty}\frac{1}{(n+1)}$$

So, $\lim_{n\to\infty}\sinh\left(\frac{1}{n+1}\right) = \lim_{n\to\infty}\frac{1}{(n+1)}$

Similarly, $\lim_{n\to\infty}\sinh\left(\frac{1}{n+2}\right) = \lim_{n\to\infty}\frac{1}{(n+2)}$

⋮

$$\lim_{n\to\infty}\sinh\left(\frac{1}{2n}\right)=\lim_{n\to\infty}\left(\frac{1}{2n}\right)$$

$$\Rightarrow\lim_{n\to\infty}\left\{\sinh\left(\frac{1}{n+1}\right)+\sinh\left(\frac{1}{n+2}\right)+\cdots+\sin\left(\frac{h}{2n}\right)\right\}=$$

$$=\lim_{n\to\infty}\left\{\frac{1}{(n+1)}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right\}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\left(\frac{r}{n}\right)}$$

$$=\int_0^1\frac{1}{(1+x)}dx=[\log_e(1+x)]_0^1=\ln 2-\ln 1=\ln 2$$

Solution 2 by Marian Ursărescu-Romania

It's easy to prove: $x\le\sinh x\le x\cosh x,\forall x\ge 0$

$$x=\frac{1}{n+1}\Rightarrow\frac{1}{n+1}\le\sinh\frac{1}{n+1}\le\frac{1}{n+1}\cosh\frac{1}{n+1}$$

$$x=\frac{1}{n+2}\Rightarrow\frac{1}{n+2}\le\sinh\frac{1}{n+2}\le\frac{1}{n+2}\cosh\frac{1}{n+2}$$

⋮

$$x=\frac{1}{2n}\Rightarrow\frac{1}{2n}\le\sinh\frac{1}{2n}\le\frac{1}{2n}\cosh\frac{1}{2n}$$

$$\Rightarrow\sum_{k=1}^{n}\frac{1}{n+k}\le a_n\le\sum_{k=1}^{n}\frac{1}{n+k}\cdot\cosh\frac{1}{n+k}\le\sum_{k=1}^{n}\frac{1}{n+k}\cosh\frac{1}{2n}$$

$$\Rightarrow\sum_{k=1}^{n}\frac{1}{n+k}\le a_n\le\sum_{k=1}^{n}\frac{1}{n+k}\cosh\frac{1}{2n}\quad\text{(1)}$$

But $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{n+\frac{k}{n}}=\int_0^1\frac{1}{1+x}dx=\ln(1+x)\Big|_0^1=\ln 2$ ***(2)***

$$\lim_{n\to\infty}\cosh\frac{1}{2n}=1\quad\text{(3)}$$

From (1)+(2)+(3)$\Rightarrow\Omega=\ln 2.$

573.

$$x_0>0,x_{n+1}=x_n+\frac{1}{\sqrt{x_n^4+x_n^2+1}}$$

Find:

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$$\Omega = \lim_{n\to\infty} \frac{\sum_{1\le i<j\le n} x_i x_j}{n^2 \cdot \sqrt[3]{n^2}}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Shafiqur Rahman-Bangladesh

$$\Omega = \lim_{n\to\infty} \frac{\sum_{1\le 1<j\le n} x_i x_j}{n^2 n^{\frac{2}{3}}} = \frac{1}{2}\left[\lim_{n\to\infty} \frac{(\sum_{i=1}^n x_i)^2}{n^{\frac{8}{3}}} - \lim_{n\to\infty} \frac{\sum_{i=1}^n x_i^2}{n^{\frac{8}{3}}}\right] =$$

$$= \frac{1}{2}\left[\lim_{n\to\infty} \left(\frac{\sum_{i=1}^n x_i}{n^{\frac{4}{3}}}\right)^2 - \lim_{n\to\infty} \frac{\sum_{i=1}^n x_i^2}{n^{\frac{8}{3}}}\right] =$$

$$= \frac{1}{2}\left[\lim_{n\to\infty} \left(\frac{x_{n+1}}{(n+1)^{\frac{4}{3}} - n^{\frac{4}{3}}}\right)^2 - \lim_{n\to\infty} \frac{x_{n+1}^2}{(n+1)^{\frac{8}{3}} - n^{\frac{8}{3}}}\right] \quad \textit{[Stolz – Cesaro th}^m\textit{]}$$

$$= \frac{1}{2}\left[\lim_{n\to\infty} \left(\frac{x_{n+1}}{n^{\frac{4}{3}}\left(\left(1+\frac{1}{n}\right)^{\frac{4}{3}} - 1\right)}\right)^2 - \lim_{n\to\infty} \frac{x_{n+1}^2}{n^{\frac{8}{3}}\left(\left(1+\frac{1}{n}\right)^{\frac{8}{3}} - 1\right)}\right] =$$

$$= \frac{1}{2}\left[\lim_{n\to\infty} \left(\frac{x_{n+1}}{n^{\frac{4}{3}}\left(\frac{4}{3n}\right)}\right)^2 - \lim_{n\to\infty} \frac{x_{n+1}^2}{n^{\frac{8}{3}}\left(\frac{8}{3n}\right)}\right] = \frac{9}{32}\lim_{n\to\infty}\left(\frac{x_{n+1}}{n^{\frac{1}{3}}}\right)^2 - \frac{3}{16}\lim_{n\to\infty}\frac{x_{n+1}^2}{n^{\frac{5}{3}}} =$$

$$= \frac{9}{32}\lim_{n\to\infty}\left(\frac{x_{n+1}^3}{n}\right)^{\frac{2}{3}} - \frac{3}{16}\lim_{n\to\infty}\left(\frac{x_{n+1}^{\frac{6}{5}}}{n}\right)^{\frac{5}{3}}$$

$$= \frac{9}{32}\lim_{n\to\infty}\left(x_{n+2}^3 - x_{n+1}^3\right)^{\frac{2}{3}} - \frac{3}{16}\lim_{n\to\infty}\left(x_n^{\frac{6}{5}} - x_n^{\frac{6}{5}}\right)^{\frac{5}{3}} \quad \textit{[Stolz – Cesaro th}^{th}\textit{]}$$

$$= \frac{9}{32}\lim_{n\to\infty}\left(x_{n+1}^3\left(\left(\frac{x_{n+2}}{x_{n+1}}\right)^3 - 1\right)\right)^{\frac{2}{3}} - \frac{3}{16}\lim_{n\to\infty}\left(x_{n+1}^{\frac{6}{5}}\left(\left(\frac{x_{n+2}}{x_{n+1}}\right)^{\frac{6}{5}} - 1\right)\right)^{\frac{5}{3}}$$

$$= \frac{9}{32}\lim_{n\to\infty}\left(x_{n+1}^3\left(\left(1+\frac{1}{x_{n+1}^3\sqrt{1+\frac{1}{x_{n+1}^2}+\frac{1}{x_{n+1}^4}}}\right)-1\right)\right)^{\frac{2}{3}}-$$

$$-\frac{3}{16}\lim_{n\to\infty}\left(x_{n+1}^6\left(\left(1+\frac{1}{x_{n+1}^3\sqrt{1+\frac{1}{x_{n+1}^2}+\frac{1}{x_{n+1}^4}}}\right)^{\frac{6}{5}}-1\right)\right)^{\frac{5}{3}}$$

[Here $\lim_{n\to\infty} x_n = \infty$ *&* $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^k - 1 = \lim_{n\to\infty}\frac{k}{n}$*]*

$$= \frac{9}{32}\lim_{n\to\infty}\left(\frac{3}{\sqrt{1+\frac{1}{x_{n+1}^2}+\frac{1}{x_{n+1}^4}}}\right)^{\frac{2}{3}} - \frac{3}{16}\lim_{n\to\infty}\left(\frac{6}{5x_{n+1}^{\frac{9}{5}}\sqrt{1+\frac{1}{x_{n+1}^2}+\frac{1}{x_{n+1}^4}}}\right)^{\frac{5}{3}} =$$

$$= \frac{9}{32}3^{\frac{2}{3}} - 0 \therefore \Omega = \lim_{n\to\infty}\frac{\sum_{1\le i<j\le n} x_i x_j}{n^2 n^{\frac{2}{3}}} = \frac{3^{\frac{8}{3}}}{32}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

We know, $\lim_{u\to 0}\frac{(1+u)^r-1}{u} = r$ *and* $x_{n+1} = x_n + \frac{1}{\sqrt{x_n^4+x_n^2+1}}$

Now, $x_0 > 0$ *then* $x_1 > x_0, x_2 > x_1, \dots, x_{k+1} > x_k$*, ... hence* $\{x_n\}_n^{\infty} = 0$

Is monotone increasing. Let $\lim_{n\to\infty} x_n = l$ *so,* $l = l + \frac{1}{\sqrt{l^4+l^2+1}} \Rightarrow l \to \infty$

Hence our claim is contradictory, we have, $\lim_{n\to\infty} x_n = \infty$ *then*

$$\Omega = \lim_{n\to\infty}\frac{\sum_{1\le i<j\le n} x_i x_j}{n^2\sqrt[3]{n^2}} \Rightarrow 2\Omega = \lim_{n\to\infty}\frac{(x_1+x_2+\cdots+x_n)^2}{n^{\frac{8}{3}}} - \lim_{n\to\infty}\frac{\sum_{i=1}^n x_i^2}{n^{\frac{8}{3}}}$$

$$\Rightarrow \sqrt{2\Omega} = \lim_{n\to\infty}\frac{x_1+x_2+\cdots+x_n}{n^{\frac{4}{3}}} \text{ and } \lim_{n\to\infty}\frac{\sum_{i=1}^n x_i^2}{n^{\frac{8}{3}}} = 0$$

$$\Rightarrow \sqrt{2\Omega} \overset{CAESARO\ STOLZ}{=} \lim_{n\to\infty} \frac{x_{n+1}}{(n+1)^{\frac{4}{3}} - n^{\frac{4}{3}}} = \left(\lim_{n\to\infty} \frac{x_{n+1}}{\sqrt[3]{n}}\right) \left(\lim_{n\to\infty} \frac{1}{\frac{\left(1+\frac{1}{n}\right)^{\frac{4}{3}} - 1}{\frac{1}{n}}} \right)$$

$$= \frac{3}{4} \lim_{n\to\infty} \sqrt[3]{\frac{x_{n+1}^3}{n}} \overset{CAESARO\ STOLZ}{=} \frac{3}{4} \sqrt[3]{\lim_{n\to\infty} \left(x_{n+2}^3 - x_{n+1}^3\right)}$$

$$= \frac{3}{4} \sqrt[3]{\lim_{x_n\to\infty} \left\{ \left(x_{n+1} + \frac{1}{\sqrt{x_{n+1}^4 + x_{n+1}^2 + 1}} \right)^3 - x_{n+1}^3 \right\}}$$

$$= \frac{3}{4} \sqrt[3]{\lim_{x_n\to\infty} \frac{x_{n+1}^3}{\sqrt[3]{x_{n+1}^6 + x_{n+1}^4 + x_{n+1}^2}} \cdot \frac{\left(1 + \frac{1}{\sqrt{x_{n+1}^6 + x_{n+1}^4 + x_{n+1}^2}}\right)^3 - 1}{\frac{1}{\sqrt{x_{n+1}^6 + x_{n+1}^4 + x_{n+1}^2}}}}$$

$$= \frac{3}{4} \sqrt[3]{3 \lim_{x_n\to\infty} \frac{1}{\sqrt{1 + \frac{1}{x_{n+1}^2} + \frac{1}{x_{n+1}^4}}}} = \frac{3\sqrt[3]{3}}{4} \Rightarrow \Omega = \frac{3^{\frac{8}{3}}}{32}$$

574. Find:

$$\Omega = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \left(\frac{3^{n-k-1}(4n-4k-1)}{n-k+1} \cdot \binom{n+1}{k+1} \right) \right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Remus Florin Stanca-Romania

$$\sum_{k=0}^{n} \frac{3^{n-k-1}(4n-4k-1)}{n-k+1} \binom{n+1}{k+1} = \sum_{k=0}^{n} 3^{n-k-1}(4n-4k-1) \cdot \frac{(n+1)!}{(n-k)!\,(k+1)!} \cdot \frac{1}{n-k+1}$$

$$=\sum_{k=0}^{n}3^{n-k-1}(4n-4k-1)\cdot\frac{(n+2)!}{(n-k+1)!\,(k+1)!}\cdot\frac{1}{n+2}=$$

$$=\sum_{k=0}^{n}3^{n-k-1}(4n-4k-1)\binom{n+2}{k+1}\cdot\frac{1}{n+2}=\frac{1}{n+2}\sum_{k=0}^{n}3^{n-k-1}(4n-4k-1)\binom{n+2}{k+1}=$$

$$=\frac{1}{n+2}\left(\sum_{k=0}^{n}4(n-k)3^{n-k-1}\binom{n+2}{k+1}-\sum_{k=0}^{n}3^{n-k-1}\binom{n+2}{k+1}\right)$$

$$4\sum_{k=0}^{n}(n-k)3^{n-k-1}\cdot\frac{(n+2)!}{(n+1-k)!\,(k+1)!}=$$

$$=4\sum_{k=0}^{n}(n-k+1)3^{n-k-1}\frac{(n+2)!}{(n-k+1)!\,(k+1)!}-4\sum_{k=0}^{n}3^{n-k-1}\cdot\frac{(n+2)!}{(n-k+1)!\,(k+1)!}=s_1$$

$$4\sum_{k=0}^{n}3^{n-k-1}\cdot\frac{(n+2)!}{(n-k)!\,(k+1)!}=4\sum_{k=0}^{n}(n+2)\,3^{n-k-1}\cdot\frac{(n+1)!}{(n-k)!\,(k+1)!}=$$

$$=\frac{4(n+2)}{3}\sum_{k=0}^{n}\binom{n+1}{k+1}3^{n+1-(k+1)}\cdot 1^{k+1}=\frac{4(n+2)}{3}(4^{n+1}-3^{n+1})$$

$$4\sum_{k=0}^{n}3^{n-k-1}\frac{(n+2)!}{(n-k+1)!\,(k+1)!}=\frac{4}{9}\sum_{k=0}^{n}\binom{n+2}{k+1}3^{n+2-(k+1)}\cdot 1^{k+1}=\frac{4}{9}(4^{n+2}-3^{n+2})$$

$$\Rightarrow s_1=\frac{4(n+2)}{3}(4^{n+1}-3^{n+1})-\frac{4}{9}(4^{n+2}-3^{n+2})$$

$$\sum_{k=0}^{n}3^{n-k-1}\binom{n+2}{k+1}=\sum_{k=0}^{n}3^{n-k-1}\cdot\frac{(n+2)!}{(n-k+1)!\,(k+1)!}=\frac{1}{9}\sum_{k=0}^{n}3^{(n+2)-(k+1)}\cdot\binom{n+2}{k+1}$$

$$=\frac{1}{9}(4^{n+2}-3^{n+2})>s=\frac{1}{n+2}\left(\frac{4(n+2)(4^{n+1}-4^{n+1})}{3}-\frac{4}{9}(4^{n+2}-3^{n+2})-\frac{1}{9}(4^{n+2}-3^{n+2})\right)$$

$$=\frac{4(4^{n+1}-3^{n+1})}{3}-\frac{4}{9(n+2)}(4^{n+2}-3^{n+2})-\frac{1}{9(n+2)}(4^{n+2}-3^{n+2})$$

$$=\frac{4(4^{n+1}-3^{n+1})}{3}-5\cdot\frac{4^{n+2}-3^{n+2}}{9(n+2)}=$$

$$=\frac{(3n+6)(4^{n+2}-4\cdot 3^{n+1})-5\cdot 4^{n+2}+5\cdot 3^{n+2}}{9(n+2)}=$$

$$= \frac{4^{n+2}(3n+1) - 3^{n+2}(4n+3)}{9(n+2)} = \frac{3^{n+2}(4n+3)}{9(n+2)} \cdot \frac{\left(\frac{4}{3}\right)^{n+2} \cdot \frac{3n+1}{4n+3} - 1}{1} \succ$$

$$\Rightarrow \Omega = \infty \cdot \infty = \infty.$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\text{Let } a_n = \sum_{k=0}^{n} \frac{3^{n-k-1}(4n-4k-1)}{n-k+1}\binom{n+1}{k+1} = \sum_{k=0}^{n} \frac{3^{n-k-1}(4n-4k+4-5)}{n-k+1}\binom{n+1}{k+1}$$

$$= \frac{4}{3}\sum_{k=0}^{n} 3^{(n+1)-(k+1)}\binom{n+1}{k+1} - \frac{5}{9}\sum_{k=0}^{n} 3^{n-k+1}\frac{1}{n-k+1}\binom{n+1}{k+1}$$

$$= \frac{4}{3}[(1+3)^{n+1} - 3^{n+1}] - \frac{5}{9}\sum_{k=0}^{n}\int_0^3 x^{n-k}\binom{n+1}{k+1}dx$$

$$= \frac{4}{3}(4^{n+1} - 3^{n+1}) - \frac{5}{9}\int_0^3 \left[\sum_{k=0}^{n}\binom{n+1}{k+1}x^{n+1-(k+1)}\right]dx$$

$$= \frac{4}{3}(4^{n+1} - 3^{n+1}) - \frac{5}{9}\int_0^3 [(1+x)^{n+1} - x^{n+1}]\,dx$$

$$= \frac{4}{3}(4^{n+1} - 3^{n+1}) - \frac{5}{9(n+2)}[(4^{n+2} - 1) - 3^{n+2}]$$

$$= \left[\frac{1}{3} - \frac{5}{9(n+2)}\right]4^{n+2} - \left(4 - \frac{5}{n+2}\right)3^n + \frac{5}{9(n+2)}$$

$$a_n = \frac{(3n+1)4^{n+2}}{9(n+2)} - \frac{(4n+3)}{n+2}(3^n) + \frac{5}{9(n+2)}$$

$$a_n \to \infty \text{ as } n \to \infty$$

575.

$$x_n = \sum_{k=0}^{n}\left(\frac{2^k}{5^n}\binom{2n-k}{n}\right)$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{x_n} - \log(1+x_n)^{\frac{1}{x_n^2}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Naren Bhandari-Bajura-Nepal

$$x_n=\sum_{k=0}^{n}\left(\frac{2^k}{5^n}\binom{2n-k}{n}\right)=\frac{1}{5^n}\sum_{k=1}^{n}\left(2^k\binom{2n-k}{n}\right)$$

By induction, it can be shown that: $\sum_{k=1}^{n}\left(2^k\binom{2n-k}{n}\right)=4^n\Rightarrow x_n=\frac{4^n}{5^n}$

Now, $\Omega=\lim_{n\to\infty}\left(\frac{5^n}{4^n}-\left(\frac{5^n}{4^n}\right)^2\log\left(1+\frac{4^n}{5^n}\right)\right)=$

$$=\lim_{n\to\infty}\left(\frac{5^n}{4^n}-\left(\frac{5^n}{4^n}\right)^2\left(\frac{4^n}{5^n}-\frac{1}{2}\left(\frac{4^n}{5^n}\right)^2+O\left(\left(\frac{4^n}{5^n}\right)^3\right)\right)\right)=\frac{1}{2}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$5^nx_n=\sum_{k=0}^{n}2^k\binom{2n-k}{n}=a_n\ \textit{(say)}$$

$$\therefore a_n=2^n\binom{n}{n}+2^{n-1}\binom{n+1}{n}+2^{n-2}\binom{n+2}{n}+\cdots+2\binom{2n-1}{n}+\binom{2n}{n}$$

$$=2^n\binom{n}{0}+2^{n-1}\binom{n+1}{1}+2^{n-2}\binom{n+2}{2}+\cdots+2\binom{2n-1}{n-1}+\binom{2n}{n}$$

$$a_{n+1}=2^{n+1}\binom{n+1}{0}+2^n\binom{n+2}{1}+2^{n-1}\binom{n+3}{2}+\cdots+2^2\binom{2n}{n-1}$$

$$2\binom{2n+1}{n}+\binom{2n+2}{n+1}$$

$$=2^{n+1}\left[\binom{n}{0}\right]+2^n\left[\binom{n+1}{0}+\binom{n+1}{1}\right]+2^{n-1}\left[\binom{n+2}{1}+\binom{n+2}{2}\right]+$$

$$+2^{n-2}\left[\binom{n+3}{2}+\binom{n+3}{3}\right]+$$

$$+\cdots+2^2\left[\binom{2n-1}{n-1}+\binom{2n-1}{n-2}\right]+2\left[\binom{2n}{n-1}+\binom{2n}{n}\right]+\left[\binom{2n+1}{n}+\binom{2n+1}{n+1}\right]$$

$$=2\left[2^n\binom{n}{0}+2^{n-1}\binom{n+1}{1}+2^{n-2}\binom{n+2}{2}+\cdots+\binom{2n}{n}\right]$$

$$+\frac{1}{2}\left[2^{n+1}\binom{n+1}{0}+2^n\binom{n+2}{1}+2^{n-1}\binom{n+3}{2}+\cdots+\binom{2n+1}{n}(2)+\binom{2n+2}{n+1}\right]$$

$$\left[\because\binom{2n+1}{n+1}=\frac{1}{2}\binom{2n+2}{n+1}\right]$$

$\therefore a_{n+1}=2a_n+\frac{1}{2}a_{n+1}\Rightarrow a_{n+1}=4a_n\Rightarrow a_n=4^{n-1}a_1$ *where* $a_1=\sum_{k=0}^{1}2^k\binom{2-k}{1}=4$

$$\therefore a_n = 4^n \Rightarrow x_n = \left(\frac{4}{5}\right)^n$$

Note $\lim_{n\to\infty} x_n = 0$. *Now,* $\Omega = \lim_{n\to\infty}\left[\frac{1}{x_n} - \log(1+x_n)^{\frac{1}{x_n^2}}\right]$

$$= \lim_{n\to\infty}\left[\frac{1}{x_n} - \frac{1}{x_n^2}\left(x_n - \frac{1}{2}x_n^2 + \frac{1}{3}x_n^3 \dots\right)\right] = \lim_{n\to\infty}\left[\frac{1}{2} - \frac{1}{3}x_n + \frac{1}{4}x_n^2 \dots\right] = \frac{1}{2} + 0 = \frac{1}{2}$$

576.

$$\Omega_1(n) = \left(\sum_{k=1}^{n}\frac{k^4}{(1+k^2)^2}\right)\left(\sum_{k=1}^{n}\frac{k^5}{(1+k^2)^2}\right)\left(\sum_{k=1}^{n}\frac{k^6}{(1+k^2)^2}\right),$$

$$\Omega_2(n) = \left(\sum_{k=1}^{n}k\right)\left(\sum_{k=1}^{n}k^2\right)\left(\sum_{k=1}^{n}k^3\right)$$

Find:

$$\Psi = \lim_{n\to\infty}\left(\frac{\Omega_1(n)}{n\cdot\Omega_2(n)}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Serban George Florin-Romania

$$\Omega_1(n) = \left(\sum_{k=1}^{n}\frac{k^4}{k^4\left(1+\frac{1}{k^2}\right)^2}\right)\cdot\left(\sum_{k=1}^{n}\frac{k^5}{k^4\left(1+\frac{1}{k^2}\right)^2}\right)\cdot\left(\sum_{k=1}^{n}\frac{k^6}{k^4\left(1+\frac{1}{k^2}\right)^2}\right)$$

$$\Omega_1(n) = \left(\sum_{k=1}^{n}\frac{1}{\left(1+\frac{1}{k^2}\right)^2}\right)\cdot\left(\sum_{k=1}^{n}\frac{k}{\left(1+\frac{1}{k^2}\right)^2}\right)\cdot\left(\sum_{k=1}^{n}\frac{k^2}{\left(1+\frac{1}{k^2}\right)^2}\right)$$

$$\Omega_2(n) = \frac{n(n+1)}{2}\cdot\frac{n(n+1)(2n+1)}{6}\cdot\frac{n^2(n+1)^2}{4} = \frac{n^4(n+1)^4(2n+1)}{48}$$

$$n\Omega_2(n) = \frac{n^5(n+1)^4(2n+1)}{48}$$

$$0<\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{\left(1+\frac{1}{k^2}\right)^2}<\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{\left(1+\frac{1}{n^2}\right)^2}=\frac{1}{n^2}\cdot\frac{n}{\left(1+\frac{1}{n^2}\right)^2}=\frac{1}{n}\cdot\frac{n^4}{(n^2+1)^2}\overset{n\to\infty}{\rightarrow}0$$

$$\Rightarrow\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{\left(1+\frac{1}{k^2}\right)^2}\overset{n\to\infty}{\rightarrow},\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{\left(1+\frac{1}{k^2}\right)^2}=0$$

$$0<\frac{1}{n^3}\sum_{k=1}^{n}\frac{k}{\left(1+\frac{1}{k^2}\right)^2}<\frac{1}{n^3}\sum_{k=1}^{n}\frac{k}{\left(1+\frac{1}{n^2}\right)^2}=\frac{1}{n^3}\cdot\frac{n(n+1)}{2}\cdot\frac{n^4}{(n^2+1)^2}=\frac{n^4(n+1)}{2n^2(n^2+1)^2}$$

$$\overset{n\to\infty}{\rightarrow}\Rightarrow\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}\frac{k}{\left(1+\frac{1}{k^2}\right)^2}=0$$

$$0<\frac{1}{n^5}\sum_{k=1}^{n}\frac{k^2}{\left(1+\frac{1}{k^2}\right)^2}<\frac{1}{n^5}\sum_{k=1}^{n}\frac{k^2}{\left(1+\frac{1}{n^2}\right)^2}=\frac{1}{n^5}\cdot\frac{n(n+1)(2n+1)}{6}\cdot\frac{n^4}{(n^2+1)^2}=$$

$$=\frac{(n+1)(2n+1)}{6(n^2+1)^2}\to 0\Rightarrow\lim_{n\to\infty}\frac{1}{n^5}\sum_{k=1}^{n}\frac{k^2}{\left(1+\frac{1}{k^2}\right)^2}=0$$

$$\Psi=\lim_{n\to\infty}\frac{\Omega_1(n)}{n\Omega_2(n)}=\lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{\left(1+\frac{1}{k^2}\right)^2}\right)\left(\frac{1}{n^3}\sum_{k=1}^{n}\frac{k}{\left(1+\frac{1}{k^2}\right)^2}\right)\left(\frac{1}{n^5}\sum_{k=1}^{n}\frac{k^2}{\left(1+\frac{1}{k^2}\right)^2}\right)\cdot$$

$$\cdot\left(\frac{48n^{10}}{n^5(n+1)^4(2n+1)}\right)=\lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{\left(1+\frac{1}{k^2}\right)^2}\right)\left(\frac{1}{n^3}\sum_{k=1}^{n}\frac{k}{\left(1+\frac{1}{k^2}\right)^2}\right)\left(\frac{1}{n^5}\sum_{k=1}^{n}\frac{k^2}{\left(1+\frac{1}{k^2}\right)^2}\right)$$

$$\left(\frac{48n^{10}}{n^{10}\left(1+\frac{1}{n}\right)^4\left(2+\frac{1}{n}\right)}\right)=0\cdot 0\cdot 0\cdot 24=0$$

Solution 2 by Remus Florin Stanca-Romania

$$\sum_{k=1}^{n}\frac{k^4}{(1+k^2)^2}=\sum_{k=1}^{n}\left(\frac{k^2}{1+k^2}\right)^2=\sum_{k=1}^{n}\left(1-\frac{1}{k^2+1}\right)^2=\sum_{k=1}^{n}\left(1+\frac{1}{(k^2+1)^2}-\frac{2}{k^2+1}\right)$$

Let $a_k=\frac{1}{(k^2+1)^2}\succ\sum_{k=1}^{n}\frac{k^4}{(k^2+1)^2}=\sum_{k=1}^{n}(1+a_k-2(k^2+1)a_k)=\sum_{k=1}^{n}\left(1-a_k(2k^2+1)\right)$

In the same way $\sum_{k=1}^{n}\frac{k^5}{(k^2+1)^2}=\sum_{k=1}^{n}\left(k-ka_k(2k^2+1)\right)$

and $\sum_{k=1}^{n}\frac{k^6}{(k^2+1)^2}=\sum_{k=1}^{n}\left(k^2-k^2a_k(2k^2+1)\right)$

$$\Psi=\lim_{n\to\infty}\left(\frac{n-\sum_{k=1}^{n}a_k(2k^2+1)}{n}\cdot\frac{\sum_{k=1}^{n}k-\sum_{k=1}^{n}ka_k(2k^2+1)}{\sum_{k=1}^{n}k}\cdot\frac{\sum_{k=1}^{n}k^2-\sum_{k=1}^{n}k^2a_k(2k^2+1)}{\sum_{k=1}^{n}k^2}\cdot\frac{1}{\sum_{k=1}^{n}k^3}\right)=$$

$$=\lim_{n\to\infty}\left(1-\frac{\sum_{k=1}^{n}(2k^2+1)}{n}\right)\cdot\lim_{n\to\infty}\left(1-\frac{\sum_{k=1}^{n}ka_k(2k^2+1)}{\sum_{k=1}^{n}k}\right)\cdot\lim_{n\to\infty}\left(1-\frac{\sum_{k=1}^{n}k^2a_k(2k^2+1)}{\sum_{k=1}^{n}k^2}\right)\cdot$$

$$\lim_{n\to\infty}\frac{1}{\sum_{k=1}^{n}k^3}\quad (1)$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}a_k(2k^2+1)}{n}\overset{Stolz\ Cesaro}{=}\lim_{n\to\infty}\frac{a_{n+1}(2(n+1)^2+1)}{n+1-n}=\lim_{n\to\infty}\frac{1}{((n+1)^2+1)^2}\cdot(2(n+1)^2+1)=0$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}ka_k(2k^2+1)}{\sum_{k=1}^{n}k}\overset{Stolz\ Cesaro}{=}\lim_{n\to\infty}\frac{(n+1)a_{n+1}(2(n+1)^2+1)}{n+1}=$$

$$=\lim_{n\to\infty}\frac{1}{((n+1)^2+1)^2}\cdot(2(n+1)^2+1)=0$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}k^2a_k(2k^2+1)}{\sum_{k=1}^{n}k^2}\overset{Stolz\ Cesaro}{=}0\ \overset{(1)}{\Rightarrow}\ \Psi=1\cdot1\cdot1\cdot\frac{1}{\infty}=0\Rightarrow\Psi=0$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\sum_{k=1}^{n}\frac{k^4}{(1+k^2)^2}<\sum_{k=1}^{n}(1)=n,\ \sum_{k=1}^{n}\frac{k^5}{(1+k^2)^2}<\sum_{k=1}^{n}k\ \text{and}\ \sum_{k=1}^{n}\frac{k^6}{(1+k^2)^2}<\sum_{k=1}^{n}k^2$$

$$\therefore 0<\Omega_1(n)<n\left(\sum_{k=1}^{n}k\right)\left(\sum_{k=1}^{n}k^2\right)\Rightarrow 0<\frac{\Omega_1(n)}{n\Omega_2(n)}<\frac{1}{\sum_{k=1}^{n}k^3}=\frac{4}{n^2(n+1)^2}\to 0\ \text{as}\ n\to 0$$

$$\therefore\lim_{n\to\infty}\frac{\Omega_1(n)}{n\Omega_2(n)}=0$$

577. Find the limit:

$$\lim_{k\to\infty}\left(1+\frac{1}{k^2}\left(\sum_{n=1}^{\infty}\frac{n^{10}}{10^n\cdot n!}\right)\right)^{k^4}$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution 1 by Ravi Prakash-New Delhi-India

Let $a_n=\frac{n^{10}}{(10^n)(n!)}$, *then* $\frac{a_n}{a_{n+1}}=\frac{n^{10}}{(10^n)(n!)}\cdot\frac{10^{n+1}(n+1)!}{(n+1)^{10}}=\left(1-\frac{1}{n+1}\right)^{10}(10)(n+1)$

$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=\infty \therefore \sum a_n$ ***converges. Let*** $S=\sum_{n=1}^{\infty}a_n>0$***. Now,***

$$\lim_{k\to\infty}\left(1+\frac{1}{k^2}\left(\sum_{n=1}^{\infty}a_n\right)\right)^{k^4}=\lim_{k\to\infty}\left(1+\frac{S}{k^2}\right)^{k^4}=\lim_{k\to\infty}\left[\left(1+\frac{S}{k^2}\right)^{k^2}\right]^{k^2}=(e^S)^{\infty}=\infty$$

Solution 2 by Remus Florin Stanca – Romania

$$l=\lim_{k\to\infty}\left(1+\frac{1}{k^2}\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{i^{10}}{10^i\cdot i!}\right)\right)^{k^4}$$

Let $k=n\cdot p, p>0\succ l=\lim_{n\to\infty}\left(1+\frac{1}{n^2p^2}\sum_{i=1}^{n}\frac{i^{10}}{10^i\cdot i!}\right)^{(np)^4}$

$$\lim_{n\to\infty}\frac{1}{n^2p^2}\left(\sum_{i=1}^{n}\frac{i^{10}}{10^i\cdot i!}\right)=\frac{1}{p^2}\cdot\lim_{n\to\infty}\frac{\sum_{i=1}^{n}\frac{i^{10}}{10^i\cdot i!}}{n^2}\overset{Stolz\ Cesaro}{=}\lim_{k\to\infty}\frac{\frac{(n+1)^{10}}{10^{n+1}(n+1)!}}{2n+1}=$$

$=\lim_{k\to\infty}\frac{1}{10^{n+1}(2n+1)}\cdot\frac{(n+1)^{10}}{(n+1)!}=0$ ***because the limit of*** $\frac{x_{n+1}}{x_n}=0<1$

where $x_n=\frac{n^{10}}{n!}$ ***and*** $\lim_{k\to\infty}\frac{(n+1)^{10}}{(n+1)!}\cdot\frac{n!}{n^{10}}=0$

so $\lim_{k\to\infty}\left(1+\frac{1}{n^2p^2}\left(\sum_{i=1}^{n}\frac{i^{10}}{10^i\cdot i!}\right)\right)^{(np^4)}\overset{1^{\infty}}{=}e^{\lim_{n\to\infty}n^2p^2\cdot\sum_i^n\left(\frac{i^{10}}{10^i\cdot i!}\right)}=l$ ***(1)***

$\sum_i^n\left(\frac{i^{10}}{10^i\cdot i!}\right)\geq\frac{1}{10}$ ***and*** $\sum_{i=1}^{n+1}\left(\frac{i^{10}}{10^i\cdot i!}\right)>\sum_{i=1}^{n}\left(\frac{i^{10}}{10^i\cdot i!}\right)\Rightarrow\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^{10}}{10^i\cdot i!}\right)=\infty\overset{(1)}{\Rightarrow}l=e^{\infty}=\infty\Rightarrow$

$$\Rightarrow\lim_{k\to\infty}\left(1+\frac{1}{k^2}\left(\sum_{n=1}^{\infty}\frac{n^{10}}{10^n\cdot n!}\right)\right)^{k^4}=\infty$$

578. Find:

$$\Omega=\lim_{n\to\infty}n^8\int_0^{\frac{1}{n^5}}\frac{x^2+1}{x^4+x^2+1}dx$$

Proposed by Daniel Sitaru – Romania

Solution by Igor Soposki-Skopje-Macedonia

$$I=\int\frac{x^2+1}{x^4+x^2+1}dx=\int\frac{\frac{x^2+1}{x^2}}{\frac{x^4+x^2+1}{x^2}}dx=\int\frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+\left(\sqrt{3}\right)^2}dx=$$

$$=\left\{\begin{matrix}x-\frac{1}{x}=t\\\left(1+\frac{1}{x^2}\right)dx=dt\end{matrix}\right\}=\int\frac{dt}{t^2+\left(\sqrt{3}\right)^2}=\frac{1}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}}=\frac{1}{\sqrt{3}}\arctan\frac{x^2-1}{\sqrt{3}x}$$

$$I=\int_0^{\frac{1}{n^5}}\frac{x^2+1}{x^4+x^2+1}dx=\frac{1}{\sqrt{3}}\arctan\frac{x^2-1}{\sqrt{3}x}\Bigg|_0^{\frac{1}{n^5}}=\frac{1}{\sqrt{3}}\arctan\frac{n^5\sqrt{3}}{n^{10}-1}$$

$$L=\lim_{n\to\infty}n^8\cdot\frac{1}{\sqrt{3}}\cdot\arctan\frac{n^5\sqrt{3}}{n^{10}-1}=\frac{1}{\sqrt{3}}\lim_{n\to\infty}\frac{\arctan\frac{n^5\sqrt{3}}{n^{10}-1}}{\frac{1}{(n)^8}}=\frac{0}{0}$$

$$\lim_{n\to\infty}\frac{\frac{1}{1+\left(\frac{n^5\sqrt{3}}{n^{10}-1}\right)^2}\cdot\frac{5\sqrt{3}n^4(n^{10}-1)-10\sqrt{3}n^5n^9}{(n^{10}-1)^2}}{-\frac{8n^7}{n^{16}}}=$$

$$=\lim_{n\to\infty}\frac{\frac{5\sqrt{3}n^4(n^{10}-1)-10\sqrt{3}n^{14}}{(n^{10}-1)^2+3n^{10}}}{-\frac{8}{n^9}}=-\lim_{n\to\infty}\frac{n^9\left[5\sqrt{3}n^{14}-5\sqrt{3}n^4-10\sqrt{3}n^{14}\right]}{8[n^{20}-2n^{10}+1+3n^{10}]}=$$

$$=\lim_{n\to\infty}\frac{n^9 5\sqrt{3}[n^{14}+n^4]}{8[n^{20}+n^{10}+1]}=\frac{5\sqrt{3}}{8}\lim_{n\to\infty}\frac{n^9n^{14}\left[1+\frac{1}{n^{10}}\right]}{n^{20}\left[1+\frac{1}{n^{10}}+\frac{1}{n^{20}}\right]}=\frac{5\sqrt{3}}{8}\lim_{n\to\infty}n^3=+\infty$$

$$L=\frac{1}{\sqrt{3}}\cdot(+\infty)=+\infty$$

579.

$$\Omega=\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n+2)!!}$$

Find:

$$\Omega = \lim_{n\to\infty}\left((\pi + n\omega)^{1+\frac{1}{n\omega}} - (n\omega)^{1+\frac{1}{\pi+n\omega}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Srinivasa Raghava-AIRMC-India

$\Omega = \sum_{m=1}^{\infty}\frac{(2m-1)!!}{(2m+2)!!}$. *We know that:* $(2m-1)!! = \frac{\Gamma\left(m+\frac{1}{2}\right)2^n}{\sqrt{\pi}} = \frac{2}{2m+1}\left(\frac{2m+1}{2}\right)!$

$$(2m+2)!! = 2^m(2m+2)m!$$

then the above sum becomes

$$\omega = \sum_{m=1}^{\infty}\frac{1}{2^{m+1}} = \frac{1}{2}$$

$\lim_{n\to\infty}(\pi + n\omega)^{1+\frac{1}{n\omega}} - (n\omega)^{1+\frac{1}{\pi+n\omega}} = \lim_{n\to\infty}\left(\pi + \frac{n}{2}\right)^{1+\frac{2}{n}} - \left(\frac{n}{2}\right)^{1+\frac{1}{\pi+\frac{n}{2}}}$ *(substituting* $\omega = \frac{1}{2}$*)*

$= \lim_{n\to\infty}\pi + \frac{2\pi - 4\pi\ln(2) - 4\pi\ln\left(\frac{1}{2}\right)}{n} + O\left(\frac{1}{n^2}\right)$ *(series expansion around* $n = \infty$*)*

hence the answer is π*.*

Solution 2 by Naren Bhandari-Bajura-Nepal

Call $\omega = \lim_{N\to\infty}\sum_{k=1}^{n}\frac{(2k-1)!!}{(2k+2)!!}$. *Since* $\frac{(2k-1)!!}{(2k+2)!!} = \left(\frac{(2k)!}{2^k k!}\right)\left(\frac{1}{2(k+1)\cdot 2^k\cdot k!}\right)$

$= \frac{(2k!)}{2^{2k+1}\cdot k!(k+1)!} = \frac{1}{2^{2k}}\binom{2k}{k} - \frac{1}{2^{2k+2}}\binom{2(k+1)}{k+1}$. *Thus,*

$$\Omega = \sum_{k=1}^{N}\left(\frac{1}{2^{2k}}\binom{2k}{k} - \frac{1}{2^{2k+2}}\binom{2(k+1)}{k+1}\right)$$

Giving us the N_{th} *partial sum of the series as* $\omega_{N\to\infty} = \frac{1}{2} - \frac{1}{2^{2N+2}}\binom{2(N+1)}{N+1} = \frac{1}{2}$. *Note*

Setting $n = (N+1) \Rightarrow 2n = 2(N+1)$ *which follows as*

$$A = \frac{1}{2^{2N+2}}\binom{2(N+1)}{N+1} = \frac{1}{4^n}\binom{2n}{n} = \frac{1}{4^n}\left(\frac{4^n}{\sqrt{\pi n}}\right)$$

by asymptotic form for central binomial coefficient and setting $n \to \infty$ *we prove*

$A = 0$. *setting* $\omega = \frac{1}{2}$

$$\Omega = \lim_{n\to\infty}\left(\pi+\frac{n}{2}\right)^{1+\frac{2}{n}} - \left(\frac{n}{2}\right)^{1+\frac{2}{2\pi+n}} = \lim_{n\to\infty}\left(f(x)\right)^{g(x)} - \lim_{n\to\infty}\left(h(x)\right)^{j(x)}$$

for sufficiently large value of n, $g(x)=j(x)=1$ *implies* $\Omega = \lim_{n\to\infty}\left(\pi+\frac{n}{2}-\frac{n}{2}\right)=\pi$.

Thus, the limit is π.

Solution 3 by Pierre Mounir-Cairo-Egypt

$$\omega = \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n+2)!!}$$

Whether ω *is convergent or divergent, we have:* $n\omega \to \infty$ *as* $n\to\infty$ $(\omega>0)$

$$\therefore \Omega = \lim_{n\to\infty}\left\{(\pi+n)^{1+\frac{1}{n}} - n^{1+\frac{1}{\pi+n}}\right\} \quad (\omega n \to n)$$

$$= \lim_{n\to\infty}\left\{\left[(\pi+n)^{1+\frac{1}{n}} - n^{1+\frac{1}{n}}\right] + \left[n^{1+\frac{1}{n}} - n^{1+\frac{1}{\pi+n}}\right]\right\}$$

$$= \lim_{n\to\infty} n^{1+\frac{1}{n}} \times \left[\left(1+\frac{\pi}{n}\right)^{1+\frac{1}{n}} - 1\right] + \lim_{n\to\infty} n^{1+\frac{1}{\pi+n}} \times \left[n^{\frac{\pi}{n^2+\pi n}} - 1\right]$$

$$= \lim_{n\to\infty} n^{\frac{1}{n}} \times n\left[\frac{\left(1+\frac{\pi}{n}\right)^{1+\frac{1}{n}} - 1}{\left(1+\frac{1}{n}\right)\ln\left(1+\frac{\pi}{n}\right)}\right]\left(1+\frac{1}{n}\right)\ln\left(1+\frac{\pi}{n}\right)$$

$$+ \lim_{n\to\infty} n^{\frac{1}{\pi+n}} \times n\left[\frac{n^{\frac{\pi}{n^2+\pi n}} - 1}{\left(\frac{\pi}{n^2+\pi n}\right)\ln n}\right]\left(\frac{\pi}{n^2+\pi n}\right)\ln n$$

$$= 1 \times \lim_{x=\left(1+\frac{\pi}{n}\right)^{1+\frac{1}{n}}\to 1}\left[\frac{x-1}{\ln x}\right] \times \lim_{n\to\infty}\ln\left(1+\frac{\pi}{n}\right)^{n+1}$$

$$+1 \times \lim_{y=n^{\frac{\pi}{n^2+\pi n}}\to 1}\left[\frac{y-1}{\ln y}\right] \times \pi\lim_{n\to\infty}\ln(n)^{\frac{1}{n+\pi}}$$

$$= 1 \times 1 \times \ln e^{\pi} + 1 \times 1 \times \pi \times \ln 1$$

$$\therefore L = \pi$$

580. Find:

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to 0}\left(\frac{\underbrace{(\sin x)^{(\sin x)^{(\sin x)^{\cdots}}}}_{for\ "n"\ times}}{\underbrace{x^{x^{x^{\cdots}}}}_{for\ "n"\ times}}\right)\right)$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Feti Sinani-Kosovo

When $x\to 0^+$ **we have:** $x^x=e^{x\ln x}=e^{\sqrt{x}+o(\sqrt{x})}=1+\sqrt{x}+o(\sqrt{x})$

$$x^{x^x}=x^{1+\sqrt{x}+o(\sqrt{x})}=xe^{(\sqrt{x}+o(\sqrt{x}))\ln(x)}=x\left(1+\sqrt[3]{x}+o(\sqrt[3]{x})\right)=x+o(x)$$

$$x^{x^{x^x}}=x^{x+o(x)}=e^{(x+o(x))\ln x}=e^{\sqrt{x}+o(\sqrt{x})}=1+\sqrt{x}+o(\sqrt{x})$$

$$\left.x^{x^{\cdots^{x^x}}}\right\}n\ \text{times}=\begin{cases}1+\sqrt{x}+o(\sqrt{x}) & n-even\\ x+o(x) & n-odd\end{cases}$$

$$(\sin x)^{(\sin x)^{\cdots (\sin x)^{(\sin x)}}}=(x+o(x))^{(x+o(x))^{\cdots (x+o(x))^{(x+o(x))}}}\qquad x\to 0^+$$

$$(x+o(x))^{(x+o(x))}=e^{(x+o(x))\ln(x+o(x))}=e^{(x+o(x))\ln(x)+o(x)}=1+\sqrt{x}+o(\sqrt{x})$$

$$(x+o(x))^{(x+o(x))^{(x+o(x))}}=(x+o(x))^{1+\sqrt{x}+o(\sqrt{x})}=x^{1+\sqrt{x}+o(\sqrt{x})}e^{(1+\sqrt{x}+o(\sqrt{x}))\ln(1+o(1))}$$

$$=(x+o(x))(1+o(1))=x+o(x)$$

$$\left.(\sin x)^{(\sin x)^{\cdots (\sin x)^{(\sin x)}}}\right\}n\ \text{times}=\begin{cases}1+\sqrt{x}+o(\sqrt{x}) & n-even\\ x+o(x) & n-odd\end{cases}$$

$$\lim_{n\to\infty}\left(\lim_{x\to 0^+}\frac{(\sin x)^{(\sin x)^{\cdots (\sin x)^{(\sin x)}}}}{x^{x^{\cdots x^x}}}\right)=\lim_{n\to\infty}\left(\lim_{x\to 0^+}(1+o(1))\right)=1$$

581. Find:

$$\Omega_1=\lim_{n\to\infty}\left(1-\frac{\pi^4}{90}+\sum_{k=1}^{n}\frac{1}{k^4}\right)^n$$

$$\Omega_2 = \lim_{n\to\infty}\left(4-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}\right)^n$$

$$\Omega_3 = \lim_{n\to\infty}\left(5-4\log 2-\frac{\pi^2}{6}+\sum_{k=1}^{n}\frac{1}{k^2(2k+1)}\right)^n$$

$$\Omega_4 = \lim_{n\to\infty}\left(1-\frac{\pi^2}{8}+\sum_{k=1}^{n}\frac{1}{(2k-1)^2}\right)^n$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Remus Florin Stanca-Romania

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^4}=\frac{\pi^4}{90}\Rightarrow \Omega_1 \overset{1^\infty}{=} \lim_{n\to\infty}\left(1+\sum_{k=1}^{n}\frac{1}{k^4}-\frac{\pi^4}{90}\right)^{\frac{1}{\sum_{k=1}^{n}\frac{1}{k^4}-\frac{\pi^4}{90}}\cdot\left(\sum_{k=1}^{n}\frac{1}{k^4}-\frac{\pi^4}{90}\right)n} =$$

$$= e^{\lim_{n\to\infty} n\left(\sum_{k=1}^{n}\frac{1}{k^4}-\frac{\pi^4}{90}\right)} \overset{Stolz\ Cesaro}{=} e^{\lim_{n\to\infty}\frac{\frac{1}{(n+1)^4}}{\frac{1}{n+1}-\frac{1}{n}}} = e^{\lim_{n\to\infty} -\frac{1}{(n+1)^4}\cdot(n^2+n)} = e^0 = 1$$

$$\succ \Omega_1 = 1$$

$$\Omega_2 = \lim_{n\to\infty}\left(4-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}\right)^n$$

$$\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)^2=\sum_{k=1}^{n}\frac{1}{k^2}+\sum_{k=1}^{n}\frac{1}{(k+1)^2}-2\sum_{k=1}^{n}\frac{1}{k(k+1)}=$$

$$\Rightarrow \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}=\frac{\pi^2}{6}+\frac{\pi^2}{6}-1-2=\frac{\pi^2}{3}-3\Rightarrow$$

$$\succ \Omega_2 \overset{1^\infty}{=} \lim_{n\to\infty}\left(1+3-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}\right)^{\frac{1}{\left(3-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}\right)}\left(3-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}\right)n} =$$

$$= e^{\lim_{n\to\infty} n\left(3-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}\right)} \overset{Stolz\ Cesaro}{=} e^{\lim_{n\to\infty}\frac{\frac{1}{(n^2+3n+2)^2}}{\frac{1}{n+1}-\frac{1}{n}}} = e^{\lim_{n\to\infty} -\frac{1}{(n^2+3n+2)^2}\cdot(n^2+n)} = e^0 = 1$$

$$\Rightarrow \Omega_2 = 1$$

$$\Omega_3 = \lim_{n\to\infty}\left(5-4\ln 2-\frac{\pi^2}{6}+\sum_{k=1}^{n}\frac{1}{k^2(2k+1)}\right)^n$$

$$\frac{1}{k^2(2k+1)} = \frac{Ak+B}{k^2}+\frac{C}{2k+1} = \frac{2Ak^2+Ak+2Bk+B+Ck^2}{k^2(2k+1)}\ \forall k$$

$$\Leftrightarrow 2A=-C, A=-2B, B=1 \Rightarrow A=-2, C=4$$

$$\Leftrightarrow \frac{1}{k^2(2k+1)} = \frac{-2k+1}{k^2}+\frac{4}{2k+1} = -\frac{2}{k}+\frac{1}{k^2}+\frac{4}{2k+1}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2(2k+1)} = \frac{\pi^2}{6}-4\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2k(2k+1)} = l$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2k}-\frac{1}{2k+1}\right) = -\lim_{n\to\infty}\left(-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n+1}\right) =$$

$$= -\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n+1}-2\left(\frac{1}{2}+\cdots+\frac{1}{2n}\right)\right) =$$

$$= -\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n+1}-1-\cdots-\frac{1}{n}\right) = -\lim_{n\to\infty}\left(\frac{1}{n+1}+\cdots+\frac{1}{2n+1}-1\right) =$$

$$= -\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{2n+1}-\ln(2n+1)+\ln(2n+1)-\left(1+\cdots+\frac{1}{n}-\ln(n)+\ln(n)\right)-1\right) =$$

$$= -4\ln(2)+4 \Rightarrow l = \frac{\pi^2}{6}+4\ln(2)-4$$

$$\Rightarrow \Omega_3 \overset{1^\infty}{=} \lim_{n\to\infty} e^{n\left(4-4\ln(2)-\frac{\pi^2}{6}+\sum_{k=1}^{n}\frac{1}{k^2(2k+1)}\right)} \overset{Stolz\ Cesaro}{=} e^{\lim_{n\to\infty}\frac{\frac{1}{(n+1)^2(2n+3)}}{-\frac{1}{n(n+1)}}} = e^0 = 1$$

$$\Rightarrow \Omega_3 = 1$$

$$\Omega_4 = \lim_{n\to\infty}\left(1-\frac{\pi^2}{8}+\sum_{k=1}^{n}\frac{1}{(2k-1)^2}\right)^n$$

$$\lim_{n\to\infty}\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots+\frac{1}{(2n-1)^2}\right) = \lim_{n\to\infty}\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(2n)^2}-\frac{1}{2^2}-\cdots-\frac{1}{4n^2}\right) =$$

$$= \frac{\pi^2}{6}-\lim_{n\to\infty}\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right) = \frac{\pi^2}{8}$$

$$\Rightarrow \Omega_4 \overset{1^\infty}{=} \lim_{n\to\infty} e^{n\left(1-\frac{\pi^2}{8}+\sum_{k=1}^{n}\frac{1}{(2k-1)^2}\right)} \overset{Stolz\ Cesaro}{=} e^{\lim_{n\to\infty}\frac{\frac{1}{(2n+1)^2}}{\frac{1}{n+1}-\frac{1}{n}}} = e^{\lim_{n\to\infty}-\frac{n^2+n}{4n^2+4n+1}} = e^{\frac{-1}{4}}$$

$$\succ \Omega_4 = \frac{1}{\sqrt[4]{e}}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^4} = \frac{\pi^4}{90}$ ***and*** $\lim_{n\to\infty} n\left(\sum_{k=1}^{n}\frac{1}{k^4} - \frac{\pi^4}{90}\right) = \lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{k^4}-\frac{\pi^4}{90}}{\frac{1}{n}}$

$$\overset{\substack{CESARO\\STOLZ}}{=} \lim_{n\to\infty}\frac{\frac{1}{(n+1)^4}}{\frac{1}{n+1}-\frac{1}{n}} = -\lim_{n\to\infty}\frac{n}{(n+1)^3} \overset{\substack{CESARO\\STOLZ}}{=} -\lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{\left(1+\frac{1}{n+1}\right)^3-1}{\frac{1}{(n+1)}}}$$

$= -\frac{1}{3}\lim_{n\to\infty}\frac{1}{(n+1)^2} = 0$. ***Let*** $u_n = \sum_{k=1}^{n}\frac{1}{k^4} - \frac{\pi^4}{90}$ ***for all*** $n \in \mathbb{N}$, $\lim_{n\to\infty} u_n = 0$

$$\Omega_1 = \lim_{n\to\infty}\left(1 - \frac{\pi^4}{90} + \sum_{k=1}^{n}\frac{1}{k^4}\right)^n = \lim_{\frac{1}{u_n}\to\infty}\left\{\left(1+\frac{1}{\frac{1}{u_n}}\right)^{\frac{1}{u_n}}\right\}^{nu_n} = e^{\lim_{n\to\infty} nu_n}$$

$$= 1 \textit{ (Answer)}$$

$$\log 2 = 1 - \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2k} + \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2k+1} \Rightarrow 4 - 4\log 2 = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2}{k} - \frac{4}{2k+1}\right)$$

$$\lim_{n\to\infty}\frac{1}{k^2(2k+1)} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2} - \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2}{k} - \frac{4}{2k+1}\right) = \frac{\pi^2}{6} - 4 + 4\log 2$$

$$\lim_{n\to\infty} n\left(4 - 4\log 2 - \frac{\pi^2}{6} + \sum_{k=1}^{n}\frac{1}{k^2(2k+1)}\right) = \lim_{n\to\infty}\frac{4 - 4\log 2 - \frac{\pi^2}{6} + \sum_{k=1}^{n}\frac{1}{k^2(2k+1)}}{\frac{1}{n}}$$

$$\overset{\substack{CESARO\\STOLZ}}{=} \lim_{n\to\infty}\frac{\frac{1}{(n+1)^2(2n+3)}}{\frac{1}{n+1}-\frac{1}{n}} = -\lim_{n\to\infty}\frac{\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(2+\frac{3}{n}\right)} = 0$$

Let $u_n = \sum_{k=1}^{n}\frac{1}{k^2(2k+1)} + 4 - 4\log 2 - \frac{\pi^2}{6}$ ***for all*** $n \in \mathbb{N}$

$\lim_{n\to\infty} u_n = 0$ ***then*** $\Omega_3 = \lim_{n\to\infty}\left(5 - 4\log 2 - \frac{\pi^2}{6} + \sum_{k=1}^{n}\frac{1}{k^2(2k+1)}\right)^n$

$$=\frac{1}{\frac{1}{u_n}\to\infty}\left\{\left(1+\frac{1}{\frac{1}{u_n}}\right)^{\frac{1}{u_n}}\right\}^{nu_n}=e^{\lim_{n\to\infty}nu_n}=e^0=1\ \textit{(Answer)}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}\ \textit{and}\ \lim_{n\to\infty}n\left(\sum_{k=1}^{n}\frac{1}{(2k-1)^2}-\frac{\pi^2}{8}\right)=\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{(2k-1)^2}-\frac{\pi^2}{8}}{\frac{1}{n}}$$

$$\overset{\substack{CESARO\\STOLZ}}{=}\lim_{n\to\infty}\frac{\frac{1}{(2n+1)^2}}{\frac{1}{n+1}-\frac{1}{n}}=-\lim_{n\to\infty}\frac{n(n+1)}{(2n+1)^2}\overset{\substack{CESARO\\STOLZ}}{=}-\frac{1}{4}.\ \textit{Let}\ u_n=\sum_{k=1}^{n}\frac{1}{(2k-1)^2}-\frac{\pi^2}{8}\ \textit{for all}$$

$$n\in\mathbb{N},\ \lim_{n\to\infty}u_n=0;\ \Omega_4=\lim_{n\to\infty}\left(1-\frac{\pi^2}{8}+\sum_{k=1}^{n}\frac{1}{(2k-1)^2}\right)^n=$$

$$\lim_{\frac{1}{u_n}\to\infty}\left\{\left(1+\frac{1}{\frac{1}{u_n}}\right)^{\frac{1}{u_n}}\right\}^{nu_n}=e^{\lim_{n\to\infty}nu_n}=\frac{1}{\sqrt[4]{e}}\quad\textit{(Answer)}$$

582. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{\sqrt[n]{1+\sin x}+\sqrt[n]{1-\sin x}-2}{n\left(\sqrt[n+1]{1-\sin x}+\sqrt[n+1]{1+\sin x}-2\right)}\right),x\in\mathbb{R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Avishek Mitra-India

$$\lim_{n\to\infty}\left[\frac{\sqrt[n]{1+\sin x}+\sqrt[n]{1-\sin x}-2}{n\left(\sqrt[n+1]{(1-\sin x)}+\sqrt[n+1]{(1+\sin x)}-2\right)}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\left[\frac{(1+\sin x)^{\frac{1}{n}}\cdot\log(1+\sin x)\left(-\frac{1}{n^2}\right)+(1-\sin x)^{\frac{1}{n}}\cdot\log(1-\sin x)\left(-\frac{1}{n^2}\right)}{(1-\sin x)^{\frac{1}{n+1}}\cdot\log(1-\sin x)\left(-\frac{1}{(n+1)^2}\right)+(1+\sin x)^{\frac{1}{n+1}}\cdot\log(1+\sin x)\left(-\frac{1}{(n+1)^2}\right)}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{(n+1)^2}{n^2}\cdot\lim_{n\to\infty}\frac{(1+\sin x)^{\frac{1}{n}}\cdot\log(1+\sin x)+(1-\sin x)^{\frac{1}{n}}\cdot\log(1-\sin x)}{(1-\sin x)^{\frac{1}{n+1}}\cdot\log(1-\sin x)+(1+\sin x)^{\frac{1}{n+1}}\cdot\log(1+\sin x)}$$

$$=\lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\cdot\frac{\log(1-\sin^2 x)}{\log(1-\sin^2 x)}$$

$$=0\cdot(1+0)^2\cdot 1=0\ \textit{(Answer)}$$

Solution 2 by Remus Florin Stanca-Romania

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n]{1+\sin x}+\sqrt[n]{1-\sin x}-2\right)\cdot\lim_{n\to\infty}\frac{1}{n\left(\sqrt[n+1]{1+\sin x}+\sqrt[n+1]{1-\sin x}-2\right)}$$

$$\lim_{n\to\infty} n\left(\sqrt[n+1]{1+\sin x}+\sqrt[n+1]{1-\sin x}-2\right)=$$

$$=\lim_{n\to\infty} n\left(\sqrt[n+1]{1+\sin x}-1\right)+\lim_{n\to\infty} n\left(\sqrt[n+1]{1-\sin x}-1\right)=$$

$$=\lim_{n\to\infty} n\frac{e^{\frac{\ln(1+\sin x)}{n+1}}-1}{\frac{\ln(1+\sin x)}{n+1}}\cdot\frac{\ln(1+\sin x)}{n+1}+\lim_{n\to\infty} n\frac{e^{\frac{\ln(1-\sin x)}{n+1}}}{\frac{\ln(1-\sin x)}{n+1}}\cdot\frac{\ln(1-\sin x)}{n+1}=$$

$$=\ln(1+\sin x)+\ln(1-\sin x)=\ln(\cos^2 x)=2\ln|\cos x|\succ\Omega=0\cdot\frac{1}{2\ln|\cos x|}=0\succ\Omega=0$$

Solution 3 by Naren Bhandari-Bajura-Nepal

Given that $\lim_{n\to\infty}\left(\frac{\left(\sqrt[n]{1+\sin x}+\sqrt[n]{1-\sin x}-2\right)}{n\left(\sqrt[n+1]{1+\sin x}+\sqrt[n+1]{1-\sin x}-2\right)}\right)$

For all $x\in\mathbb{R}$, $-1\le\sin x\le 1$. *Set* $y=\sin x$ *and for* $|y|<1$ *we have the following:*

$$\sqrt[n]{1+y}+\sqrt[n]{1-y}=1+\frac{y}{n}+\frac{y^2(1-n)}{2!\,n\cdot n}+O(y^3)+1-\frac{y}{n}+\frac{y^2(1-n)}{2!\,n\cdot n}-O(y^3)$$

$$\sqrt[n+1]{1+y}+\sqrt[n+1]{1-y}=1+\frac{y}{n+1}+\frac{y^2 n}{2!\,(n+1)^2}+O(y^3)+1-\frac{y}{n+1}+\frac{y^2 n}{2!\,(n+1)^2}-O(y^3)$$

Multiplying by n *is* 2nd *series we get*

$1+\frac{yn}{n+1}+\frac{y^2n^2}{2!(n+1)^2}+O(y^3)+1-\frac{yn}{n+1}-\frac{y^2n^2}{2!(n+1)^2}-O(y^3)$. *Thus, we have then*

$$L=\lim_{n\to\infty}\left(\frac{2+\frac{2y^2(1-n)}{2!\,(n)^2}+O(y^4)-2}{2n-\frac{2y^2n^2}{2!\,(n+1)^2}-O(y^4)-2n}\right)$$

$$=\lim_{n\to\infty}\left(-\frac{(n+1)^{2(1-n)}}{n^4}\right)=\lim_{n\to\infty}\left(\frac{1}{n^3}-\frac{1}{n^2}\right)\left(1+\frac{1}{n}\right)^2=0$$

583. **Find:**

$$\Omega=\lim_{n\to\infty}\left(\frac{\left[\sqrt[6]{n(n+1)(n+2)}\right]}{n\left[\sqrt{n}\right]}\right),\ [*]\text{ - great integer function}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Naren Bhandari-Bajura-Nepal

We note that $x-1\le\lfloor x\rfloor<x$ *for* $x>0$. *Here, then:*

$$\frac{\sqrt[6]{n(n+1)(n+2))}-1}{n(\sqrt{n}-1)} \leq \frac{\lfloor\sqrt[6]{n(n+1)(n+2))}\rfloor}{n\lfloor\sqrt{n}\rfloor} < \frac{\sqrt[6]{n(n+1)(n+2)}}{n\sqrt{n}}$$

When the $n \to \infty$ the left and right hand sides approaches to 0. And hence by squeeze theorem we deduce that:

$$\lim_{n\to\infty} \frac{\lfloor\sqrt[6]{n(n+1)(n+2))}\rfloor}{n\lfloor\sqrt{n}\rfloor} = 0$$

Solution 2 by Remus Florin Stanca-Romania

Let $[\sqrt{n}] = k \Rightarrow k \leq \sqrt{n} < k+1 \Rightarrow k^2 \leq n < (k+1)^2 \Rightarrow k^2+1 \leq n+1 < (k+1)^2+1$

$$\succ k^2+2 \leq n+2 < (k+1)^2+2$$

$$k^2 \leq n < (k+1)^2$$

$$k^2+1 \leq n+1 < (k+1)^2+1$$

$$k^2+2 \leq n+2 < (k+1)^2+2$$

$$\text{—————————————— “}\cdot\text{”}$$

$$\succ k^2(k^2+1)(k^2+2) \leq n(n+1)(n+2) < (k+1)^2((k+1)^2+1)((k+1)^2+2)$$

$$\Rightarrow \sqrt[6]{k^2(k^2+1)(k^2+2)} \leq \sqrt[6]{n(n+1)(n+2)} < \sqrt[6]{(k+1)^2((k+1)^2+1)((k+1)^2+2)}$$

$$\sqrt[6]{k^2(k^2+1)(k^2+2)} = \sqrt[6]{k^6\left(1+\frac{1}{k^2}\right)\left(1+\frac{2}{k^2}\right)} = k\sqrt[6]{\left(1+\frac{1}{k^2}\right)\left(1+\frac{2}{k^2}\right)}$$

We prove that $k\sqrt[6]{\left(1+\frac{1}{k^2}\right)\left(1+\frac{2}{k^2}\right)} < k+1\ \forall k \in \mathbb{N} \Leftrightarrow \left(1+\frac{1}{k^2}\right)\left(1+\frac{2}{k^2}\right) < \left(\frac{k+1}{k}\right)^6$

$1+\frac{1}{k^2} < \left(\frac{k+1}{k}\right)^3\ (?) \Leftrightarrow \frac{k^2+1}{k^2} < \frac{(k+1)^3}{k^3} \Leftrightarrow k^3+k < k^3+3k^2+3k+1$ ***(true)*** $\succ$

$\Rightarrow \left(1+\frac{1}{k^2}\right) < \left(\frac{k+1}{k}\right)^3$ ***(proved)***

$1+\frac{2}{k^2} < \left(\frac{k+1}{k}\right)^3\ (?) \Leftrightarrow \frac{k^2+2}{k^2} < \frac{(k+1)^3}{k^3} \Leftrightarrow k^3+2k < k^3+3k^2+3k+1$ ***(true)*** $\Rightarrow$

$\Rightarrow 1+\frac{2}{k^2} < \left(\frac{k+1}{k}\right)^3$ ***(proved), so:***

$$1+\frac{1}{k^2} < \left(\frac{k+1}{k}\right)^3$$

$$1+\frac{2}{k^2} < \left(\frac{k+1}{k}\right)^3$$

$$\Rightarrow\left(1+\frac{1}{k^2}\right)\left(1+\frac{2}{k^2}\right)<\left(\frac{k+1}{k}\right)^6\Leftrightarrow k\sqrt[6]{\left(1+\frac{1}{k^2}\right)\left(1+\frac{2}{k^2}\right)}<k+1\quad \textit{(proved)}$$

$$k\le k\sqrt[6]{\left(1+\frac{1}{k^2}\right)\left(1+\frac{2}{k^2}\right)}<k+1\Rightarrow\left[\sqrt[6]{k^2(k^2+1)(k^2+2)}\right]=k\forall k\in\mathbb{N}$$

$$\Rightarrow\left[\sqrt[6]{(k+1)^2((k+1)^2+1)((k+1)^2+2)}\right]=k+1\Rightarrow$$

$$\left[\sqrt[6]{n(n+1)(n+2)}\right]\in\{k;k+1\}$$

$$\textit{If}\left[\sqrt[6]{n(n+1)(n+2)}\right]=k\Rightarrow\left[\sqrt[6]{n(n+1)(n+2)}\right]=\left[\sqrt{n}\right]$$

$$\Rightarrow\Omega=\lim_{n\to\infty}\frac{[\sqrt{n}]}{n[\sqrt{n}]}=\frac{1}{\infty}=0\quad (1)$$

$$\textit{if}\left[\sqrt[6]{n(n+1)(n+2)}\right]=\left[\sqrt{n}\right]+1\Rightarrow\Omega=\lim_{n\to\infty}\frac{[\sqrt{n}]+1}{n[\sqrt{n}]}=\frac{1}{\infty}+\frac{1}{\infty}=0+0=0\quad (2)$$

from (1) and (2) we obtain that $\Omega=0$.

584. $f:\mathbb{R}\to\mathbb{R}, f(x)=\frac{1}{x^2+2x+3}, f^{(0)}(x)=f(x), f^{(n)}(x)$ – $\mathrm{n^{th}}$ derivative

Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{f^{(n)}(0)}{n!}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Avishek Mitra-India

$$f(x)=\frac{1}{(x^2+2x+3)}\Rightarrow[f(x)\cdot f(x^2+2x+3)]_{n+1}=0$$

$$\Rightarrow f^{(n+1)}(x)\cdot(x^2+2x+3)+(n+1)\cdot f^n(x)(2x+2)+{}^{n+1}C_2\cdot f^{(n-1)}(x)\cdot 2=0$$

$$\Rightarrow 3f^{(n+1)}(0)+2(n+1)\cdot f^n(0)+2\cdot{}^{n+1}C_2 f^{(n-1)}(0)=0$$

$$\Rightarrow 2(n+1)f^n(0)=-2\cdot{}^{n+1}C_2 f^{(n-1)}(0)-3f^{(n+1)}(0)$$

$$\Rightarrow\lim_{n\to\infty}\frac{f^n(0)}{n!}=-\lim_{n\to\infty}\frac{{}^{n+1}C_2\cdot f^{(n-1)}(0)}{n!}-\frac{3}{2}\cdot\lim_{n\to\infty}\frac{f^{(n+1)}(0)}{n!}$$

$$= -\lim_{n\to\infty}\frac{(n+1)!\,f^{(n-1)}(0)}{2!\,(n-1)!\,(n)!} - 0 = -\lim_{n\to\infty}\frac{(n+1)f^{(n-1)}(0)}{2(n-1)(n-2)!} - 0$$

$$= -\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)f^{(n-1)}(0)}{2\left(1-\frac{1}{n}\right)(n-2)!} - 0 = 0$$

Answer

Solution 2 by Pierre Mounir-Cairo- Egypt

$$L = \lim_{n\to\infty}\frac{f^{(n)}(0)}{n!}, f(x) = \frac{1}{x^2+2x+3}$$

$$f(x) = \frac{1}{(x+1)^2+\left(\sqrt{2}\right)^2} = \frac{1}{2\sqrt{2}i}\left(\frac{1}{x+1-i\sqrt{2}} - \frac{1}{x+1+i\sqrt{2}}\right)$$

$$f^{(n)}(x) = \frac{1}{2\sqrt{2}i}\left[\frac{(-1)^n n!}{\left(x+1-i\sqrt{2}\right)^{n+1}} - \frac{(-1)^n n!}{\left(x+1+i\sqrt{2}\right)^{n+1}}\right]$$

$$\therefore f^{(n)}(0) = \frac{1}{2\sqrt{2}i}\left[\frac{(-1)^n n!}{\left(1-i\sqrt{2}\right)^{n+1}} - \frac{(-1)^n n!}{\left(1+i\sqrt{2}\right)^{n+1}}\right]$$

$$f^{(n)}(0) = \frac{(-1)^n n!}{2\sqrt{2}i}\left[\frac{\left(1+i\sqrt{2}\right)^{n+1} - \left(1-i\sqrt{2}\right)^{n+1}}{(3)^{n+1}}\right]$$

$$f^{(n)}(0) = \frac{(-1)^n n!\left(\sqrt{3}\right)^{n+1}}{2\sqrt{2}i}\left[\frac{e^{(n+1)\tan^{-1}\left(\sqrt{2}\right)i} - e^{-(n+1)\tan^{-1}\left(\sqrt{2}\right)i}}{(3)^{n+1}}\right]$$

$$f^{(n)}(0) = \frac{(-1)^n n!}{\sqrt{2}\left(\sqrt{3}\right)^{n+1}}\left[\frac{e^{(n+1)\tan^{-1}\left(\sqrt{2}\right)i} - e^{-(n+1)\tan^{-1}\left(\sqrt{2}\right)i}}{2i}\right]$$

$$f^{(n)}(0) = \frac{(-1)^n n!\sin\left[(n+1)\tan^{-1}\left(\sqrt{2}\right)\right]}{\sqrt{2}\left(\sqrt{3}\right)^{n+1}}$$

$$\therefore L = \lim_{n\to\infty}\frac{(-1)^n\sin\left[(n+1)\tan^{-1}\sqrt{2}\right]}{\sqrt{2}\left(\sqrt{3}\right)^{n+1}}$$

We have: $0 \le \left|\frac{(-1)^n\sin\left[(n+1)\tan^{-1}\sqrt{2}\right]}{\sqrt{2}\left(\sqrt{3}\right)^n}\right| \le \frac{1}{\sqrt{2}\left(\sqrt{3}\right)^{n+1}}$

$$0 \le \lim_{n\to\infty}\left|\frac{(-1)^n\sin\left[(n+1)\tan^{-1}\sqrt{2}\right]}{\sqrt{2}\left(\sqrt{3}\right)^n}\right| \le \lim_{n\to\infty}\frac{1}{\sqrt{2}\left(\sqrt{3}\right)^{n+1}}$$

$0 \leq |L| \leq 0$, *by squeeze theorem:* $|L| = 0 \therefore L = 0$

Solution 3 by Ravi Prakash-New Delhi-India

$$f(x) = \frac{1}{x^2+2x+3} = \frac{1}{(x+1)^2+2} = \frac{1}{(x+1+\sqrt{2}i)(x+1-\sqrt{2}i)}$$

$$= \frac{1}{2\sqrt{2}i}\left[\frac{1}{x+1-\sqrt{2}i} - \frac{1}{x+1+\sqrt{2}i}\right]$$

$$f^n(x) = \frac{1}{2\sqrt{2}i}\left[\frac{(-1)^n n!}{(x+1-\sqrt{2}i)^{n+1}} - \frac{(-1)^n n!}{(x+1+\sqrt{2}i)^{n+1}}\right]$$

$$\frac{f^n(0)}{n!} = \frac{(-1)^n}{2\sqrt{2}i}\left[\frac{1}{(1-\sqrt{2}i)^{n+1}} - \frac{1}{(1+\sqrt{2}i)^{n+1}}\right]$$

Put $1 = r\cos\alpha;\ \sqrt{2} = r\sin\alpha, r^2 = 3, \tan\alpha = \sqrt{2}$

$$\Rightarrow \frac{f^n(0)}{n!} = \frac{(-1)^n}{2\sqrt{2}i}\left[\frac{r^{n+1}(\cos(n+1)\alpha + i\sin(n+\alpha) - \cos(n+1)\alpha + i\sin(n+1)\alpha)}{(1+2)^{n+1}}\right]$$

$= \frac{(-1)^n}{2\sqrt{2}i}\cdot\frac{3^{\frac{n+1}{2}}}{3^{n+1}}\cdot 2i\sin(n+1)\alpha = \frac{(-1)^n}{\sqrt{2}}\cdot\frac{\sin(n+1)\alpha}{3^{\frac{n+1}{2}}}$. *As* $\sin(n+1)\alpha$ *is bounded,*

$$\lim_{n\to\infty}\frac{f^n(0)}{n!} = \lim_{n\to\infty}\frac{(-1)^n\sin(n+1)\alpha}{\sqrt{2}\,e^{\frac{n+1}{2}}} = 0$$

585.

$$x_1, y_1 > 0, a \in \mathbb{R}, a > 1, n \in \mathbb{N}, n \geq 1,$$

$$x_{n+1} = a^{-(x_1+x_2+\cdots+x_n)}, y_{n+1} = a^{\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_n}}$$

Find:

$$\Omega = \lim_{n\to\infty}(x_n\cdot y_n)$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$x_{n+1} = a^{-(x_1+x_2+\cdots+x_n)} = \frac{x_n}{a^{x_n}} \Rightarrow x_{n+1} - x_n = \frac{x_n(1-a^{x_n})}{a^{x_n}} < 0$ *since,* $a > 1$

So, $x_{n+1} < x_n$ *for all* $n \in \mathbb{N}$, *hence* $\{x_n\}_{n=1}^{\infty}$ *is decreasing. Let* $\lim_{n\to\infty} x_n = l$

then $l = \frac{l}{a^l} \Rightarrow l = 0, \therefore x_n \to 0$ *as* $n \to \infty$

$$y_{n+1} = a^{\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_n}} = y_n a^{\frac{1}{y_n}} \Rightarrow y_{n+1} - y_n = y_n\left(\sqrt[y_n]{a} - 1\right) > 0 \text{ since, } a > 1$$

So, $y_{n+1} > y_n$ *for all* $n \in \mathbb{N}$*, hence* $\{y_n\}_{n=1}^{\infty}$ *is increasing. Let* $\lim_{n\to\infty} y_n = m$

then $m = m\sqrt[m]{a} \Rightarrow m \to \infty$

$$\Omega = \lim_{n\to\infty}(x_n y_n) = \lim_{n\to\infty}\left(\frac{n}{\frac{1}{x_n}} \cdot \frac{y_n}{n}\right) \overset{\substack{CAESARO\\STOLZ}}{=} \lim_{n\to\infty}\left(\frac{n+1-n}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} \cdot \frac{y_{n+1} - y_n}{n+1-n}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{a^{\frac{1}{y_n}}-1}{\frac{1}{y_n}}}{\frac{a^{x_n}-1}{x_n}} = \frac{\log_e a}{\log_e a} = 1 \text{ (Answer)}$$

Solution 2 by Remus Florin Stanca-Romania

$$x_1, y_1 > 0, a \in \mathbb{R}, a > 1, n \in \mathbb{N}, n \geq 1,$$

$$x_{n+1} = a^{-(x_1+\cdots+x_n)}, y_{n+1} = a^{\frac{1}{y_1}+\cdots+\frac{1}{y_n}}$$

Find: $\Omega = \lim_{n\to\infty}(x_n \cdot y_n)$

$$x_{n+1} = a^{-(x_1+\cdots x_{n-1})} \cdot a^{-x_n} \succ x_{n+1} = x_n \cdot a^{-x_n}$$

$$y_{n+1} = a^{\frac{1}{y_1}+\cdots+\frac{1}{y_{n-1}}} \cdot a^{\frac{1}{y_n}} \succ y_{n+1} = y_n \cdot a^{\frac{1}{y_n}}$$

we prove by using the Mathematical induction that $x_n > 0$*:*

1) we prove that $P(1)$: *"*$x_1 > 0$*" is true (true).*

2) we suppose that $P(n)$: *"*$x_n > 0$*" is true.*

3) we prove that $P(n+1)$: *"*$x_{n+1} > 0$*" is true by using* $P(n)$*:*

$x_{n+1} = x_n \cdot a^{-x_n}, a > 1, x_n > 0 \succ x_{n+1} > 0 \succ x_n > 0 \forall n \in \mathbb{N}$ *(proved)*

$x_{n+1} = x_n \cdot a^{-x_n} \Rightarrow \frac{x_{n+1}}{x_n} = a^{-x_n}$ *(1). Also, we know that* $a > 1$ *and* $x_n > 0$.

$\overset{(1)}{\Rightarrow} \frac{x_{n+1}}{x_n} < 1 \Rightarrow x_{n+1} < x_n \Rightarrow (x_n)_{n\in\mathbb{N}}$ *is a decreasing sequence.*

$x_n > 0$ *and* $(x_n)_{n\in\mathbb{N}}$ *is decreasing sequence* $\succ |l \in \mathbb{R}$ *such that* $\lim_{n\to\infty}(x_n) = l$

$$x_{n+1} = x_n \cdot a^{-x_n} \succ l = l \cdot a^{-l} \succ l\left(1 - a^{-l}\right) = 0 \succ l = 0 \succ \lim_{n\to\infty}(x_n) = 0$$

we prove by using the Mathematical induction that $y_n > 0$*:*

1) we prove that $P(1)$: *"*$y_1 > 0$*" is true (true)*

2) we suppose that $P(n)$: "$x_n > 0$" is true

3) we prove that $P(n+1)$: "$x_{n+1} > 0$" is true by using $P(n)$:

$$y_{n+1} = y_n \cdot a^{\frac{1}{y_n}}, y_n > 0 \succ y_{n+1} > 0 \succ y_n > 0 \forall n \in \mathbb{N} \text{ (proved)}$$

$$y_{n+1} = y_n \cdot a^{\frac{1}{y_n}} \succ \frac{y_{n+1}}{y_n} = a^{\frac{1}{y_n}}, \text{ we know also that } a > 1 \text{ and } y_n > 0 \succ \frac{y_{n+1}}{y_n} > 1$$

$$\Rightarrow y_{n+1} > y_n \Rightarrow (y_n)_{n\in\mathbb{N}} \text{ is an increasing sequence (2)}$$

$$\text{we suppose that } y_n \text{ is verged} \overset{(2)}{\Rightarrow} \exists l \in \mathbb{R} \text{ such that } \lim_{n\to\infty}(y_n) = l$$

$$y_{n+1} = y_n \cdot a^{\frac{1}{y_n}} \Rightarrow l = l \cdot a^{\frac{1}{l}} \Rightarrow l\left(1 - a^{\frac{1}{l}}\right) = 0 \Rightarrow l = 0, \text{ but } y_n > 0 \text{ and}$$

$$\text{increasing} \Rightarrow \text{contradiction} \Rightarrow \lim_{n\to\infty} y_n = \infty$$

$$\Omega = \lim_{n\to\infty}(nx_n) \cdot \lim_{n\to\infty}\left(\frac{y_n}{n}\right) (a)$$

$$\lim_{n\to\infty}\frac{y_n}{n} \overset{\text{Stolz Cesaro}}{=} \lim_{n\to\infty}(y_{n+1} - y_n) = \lim_{n\to\infty}\left(y_n\left(a^{\frac{1}{y_n}} - 1\right)\right) = \lim_{n\to\infty}\frac{a^{\frac{1}{y_n}} - 1}{\frac{1}{y_n}} = \ln(a) (3)$$

$$\lim_{n\to\infty}(nx_n) = \lim_{n\to\infty}\frac{n}{\frac{1}{x_n}} \overset{\text{Stolz Cesaro}}{=} \lim_{n\to\infty}\frac{1}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} = \lim_{n\to\infty}\frac{1}{\frac{1}{a^{x_n}x_n} - \frac{1}{x_n}} = \lim_{n\to\infty}\frac{1}{\frac{a^{x_n-1}}{x_n}} =$$

$$\frac{1}{\ln(a)} \text{ (4)}$$

$$\overset{(a);(3);(4)}{\Rightarrow} \lim_{n\to\infty}(x_n y_n) = \ln(a) \cdot \frac{1}{\ln(a)} = 1 \Rightarrow \Omega = 1.$$

586.

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, n \geq 1$$

Find:

$$\Omega = \lim_{n\to\infty}\left(H_n^2\left(\left(\frac{1 + H_n}{H_n}\right)^{H_n} - \log\left(\frac{1 + H_n}{H_n}\right)^{eH_n}\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Pierre Mounir-Cairo-Egypt

$$\Omega = \lim_{n\to\infty}\left\{\left[\left(\frac{1+H_n}{H_n}\right)^{H_n} - \ln\left(\frac{1+H_n}{H_n}\right)^{eH_n}\right]\right\}$$

$\because (H_n)_{n\geq 1}$ ***is a divergent sequence (partial sums).***

$\therefore x = \frac{1}{H_n} \to 0^+$ **as** $n \to \infty \Rightarrow \Omega = \lim_{x\to 0^+}\frac{(1+x)^{\frac{1}{x}} - e\ln(1+x)^{\frac{1}{x}}}{x^2} = e\lim_{x\to 0^+}\frac{\frac{(1+x)^{\frac{1}{x}}}{e} - 1 - \ln\left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]}{x^2}$

Let $y = \frac{(1+x)^{\frac{1}{x}}}{e} - 1 \to 0^+$ ***as*** $x \to 0^+ \Rightarrow \Omega = e\lim_{y\to 0^+}\frac{y - \ln(y+1)}{y^2} \times \lim_{x\to 0^+}\frac{\left[\frac{(1+x)^{\frac{1}{x}}}{e} - 1\right]^2}{x^2}$

$$= e\lim_{y\to 0^+}\frac{y-\ln(y+1)}{y^2} \times \left[\lim_{x\to 0^+}\frac{\frac{(1+x)^{\frac{1}{x}}}{e} - 1}{x}\right]^2$$

Let $M = \lim_{x\to 0^+}\frac{\frac{(1+x)^{\frac{1}{x}}}{e} - 1}{x} = \lim_{x\to 0^+}\frac{\frac{(1+x)^{\frac{1}{x}}}{e} - 1}{\ln\left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]} \times \frac{\ln\left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]}{x}$***. Let*** $z = \frac{(1+x)^{\frac{1}{x}}}{e} \to 1^-$ ***as*** $x \to 0^+ \Rightarrow$

$$M = \lim_{z\to 1^-}\frac{z-1}{\ln z} \times \lim_{x\to 0^+}\frac{\frac{1}{x}\ln(1+x) - 1}{x}$$

$$M = 1 \times \lim_{x\to 0^+}\frac{\ln(1+x) - x}{x^2} = -L \therefore \Omega = e \times L \times (-L)^2 = eL^3$$

$L = \lim_{x\to 0^+}\frac{x-\ln(1+x)}{x^2}$***. We have:*** $1 - y^2 < 1 < 1 + y^3\ \forall y > 0 \Rightarrow$

$$1 - y < \frac{1}{1+y} < 1 - y + y^2 \Rightarrow \int_0^x (1-y)\,dy < \int_0^x \frac{1}{1+y}dy < \int_0^x (1-y+y^2)dy \Rightarrow$$

$$x - \frac{x^2}{2} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}\quad \forall x > 0 \Rightarrow \frac{1}{2} - \frac{x}{3} < \frac{x - \ln(1+x)}{x^2} < \frac{1}{2} \Rightarrow$$

$\frac{1}{2} < \lim_{x\to 0^+}\frac{x-\ln(1+x)}{x^2} < \frac{1}{2} \Rightarrow$***By squeeze theorem:*** $L = \frac{1}{2} \Rightarrow \Omega = eL^3 = \frac{e}{8}$

587. Find:

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$$\Omega=\lim_{n\to\infty}\left(n\left(\left(\sum_{k=1}^{n}\frac{1}{k^2}\right)^{\frac{\pi^2}{6}}-\left(\frac{\pi^2}{6}\right)^{\sum_{k=1}^{n}\frac{1}{k^2}}\right)\right)$$

Proposed by Marian Ursărescu-Romania

Solution by Yubian Andres Bedoya Henao-Medellin-Colombia

Let $f(n)=\sum_{k=1}^{n}\frac{1}{k^2}$ ***and*** $L=\frac{\pi^2}{6}\Rightarrow\lim_{n\to\infty}f(n)=L\Rightarrow\Omega=\lim_{n\to\infty}n\left[f(n)^L-L^{f(n)}\right]=?$

$$\Rightarrow\Omega=\lim_{n\to\infty}\frac{f(n)^L-L^{f(n)}}{\frac{1}{n}}\to\frac{0}{0}\quad\textit{(LH)}$$

$$=\lim_{n\to\infty}\frac{Lf(n)^{L-1}f'(n)-L^{f(n)}\ln(L)\,f'(n)}{-\frac{1}{n^2}}$$

$$=\lim_{n\to\infty}n^2f'(n)\left[L^{f(n)}\ln(L)-Lf(n)^{L-1}\right]=L^L(\ln(L)-1)\lim_{n\to\infty}n^2f'(n)$$

$$\therefore f'(n)=\frac{d}{dn}\sum_{k=1}^{n}\frac{1}{k^2}=\frac{d}{dn}\sum_{k=1}^{n}\int_0^1\int_0^1(xy)^{k-1}\,dx\,dy=\frac{d}{dn}\int_0^1\int_0^1\frac{1-(xy)^n}{1-xy}\,dx\,dy$$

$$=-\int_0^1\int_0^1\frac{(xy)^n\ln(xy)}{1-xy}\,dx\,dy\quad(z=x^n\ \ w=y^n)=-\int_0^1\int_0^1\frac{zw\ln(zw)^{\frac{1}{n}}}{1-(zw)^{\frac{1}{n}}}\,\frac{(zw)^{\frac{1}{n}}dz\,dw}{n^2zw}$$

$$=\frac{1}{n^3}\int_0^1\int_0^1\left(\ln(zw)-\frac{\ln(zw)}{1-(zw)^{\frac{1}{n}}}\right)dz\,dw$$

$$\therefore\lim_{n\to\infty}n^2f'(n)=\lim_{n\to\infty}\frac{1}{n}\int_0^1\int_0^1\left(\ln(zw)-\frac{\ln(zw)}{1-(zw)^{\frac{1}{n}}}\right)dz\,dw$$

$$=\lim_{n\to\infty}\frac{1}{n}\int_0^1\int_0^1\left(\ln(zw)-\frac{\ln(zw)}{1-(zw)^{\frac{1}{n}}}\right)dz\,dw=-\lim_{n\to\infty}\frac{1}{n}\int_0^1\int_0^1\left(\frac{\ln(zw)}{1-(zw)^{\frac{1}{n}}}\right)dz\,dw\quad\textit{(L'H)}$$

$$=\lim_{n\to\infty}\int_0^1\int_0^1\left(\frac{\left(\frac{1}{n^2}\right)\ln(zw)}{\left(\frac{1}{n^2}\right)(zw)^{\frac{1}{n}}\ln(zw)}\right)dz\,dw=\lim_{n\to\infty}\int_0^1\int_0^1\left(\frac{1}{(zw)^{\frac{1}{n}}}\right)dz\,dw=\int_0^1\int_0^1dz\,dw=1$$

$$\Rightarrow \Omega = L^L(\ln(L)-1) = \left(\frac{\pi^2}{6}\right)^{\frac{\pi^2}{6}}\left[\ln\left(\frac{\pi^2}{6}\right)-1\right]$$

588. Let $n \in \mathbb{N} \geq 0$. Find:

$$\Phi = \sum_{n=0}^{\infty}\frac{(2n)!!}{(2n)!}\left[\int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty}\binom{n+1}{k}\frac{x^{2(n+1)}}{x^{2k}}\right)^{-1}dx\right]$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Zaharia Burghelea-Romania

$$\Phi = \sum_{n=0}^{\infty}\frac{(2n)!!}{(2n)!}\left[\int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty}\binom{n+1}{k}\frac{x^{2(n+1)}}{x^{2k}}\right)^{-1}dx\right]$$

Since $\sum_{k=0}^{\infty}\binom{n+1}{k}x^{-2k} = (1+x^{-2})^{n+1}$. *We have that:*

$$\left(\sum_{k=1}^{\infty}\binom{n+1}{k}\frac{x^{2(n+1)}}{x^{2k}}\right)^{-1} = \left(\frac{x^2}{x^2+1}\right)^{n+1}\frac{1}{x^{2(n+1)}} = \frac{1}{(x^2+1)^{n+1}}$$

$$I_n = \int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty}\binom{n+1}{k}\frac{x^{2(n+1)}}{x^{2k}}\right)^{-1}dx = \int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^{n+1}}$$

Substituting $x^2 = t$ *we get:* $I_n = \int_{-\infty}^{\infty}\frac{t^{-\frac{1}{2}}}{(t+1)^{n+1}}dt =$

$$= B\left(\frac{1}{2}, n+\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} = \frac{\sqrt{\pi}}{n!}\cdot\frac{(2n)!\sqrt{\pi}}{4^n n!} = \pi\frac{(2n)!}{4^n(n!)^2}$$

Where $B(x,y)$ *and* $\Gamma(x)$ *are Euler's beta and gamma function.*

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \text{ and } \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$$

Also, since $(2n)!! = [2n][2(n-1)][2(n-2)]\cdots 2 = 2^n n!$.

$$\Phi = \pi\sum_{n=0}^{\infty}\frac{2^n n!}{(2n)!}\cdot\frac{(2n)!}{4^n(n!)^2} = \pi\sum_{n=0}^{\infty}\frac{1}{2^n n!} = \pi\sqrt{e}$$

589. $\omega(n)=\sum_{i=1}^{n}\left[\frac{i^2+i+1}{i^2-i+1}\right]$, $[*]$ - great integer function

Find:

$$\Omega=\lim_{n\to\infty}\left(\log(3n+1)-\sum_{k=1}^{n}\frac{1}{\omega(k)}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Pierre Mounir-Cairo-Egypt

Given: $\omega(n)=\sum_{i=1}^{n}\left[\frac{i^2+i+1}{i^2-i+1}\right]$. *Find:* $\Omega=\lim_{n\to\infty}\left\{\ln(3n+1)-\sum_{k=1}^{n}\frac{1}{\omega(k)}\right\}$

$$\omega(n)=\sum_{i=1}^{n}\left[1+\frac{2i}{i^2-i+1}\right]=n+\sum_{i=1}^{n}\left[\frac{2i}{i^2-i+1}\right]$$

We have: $0<\frac{2i}{i^2-i+1}<1\ \forall i\geq 3\ \therefore\ \left[\frac{2i}{i^2-i+1}\right]=0\ \forall i\geq 3$

$$\omega(n)=n+[2]+\left[\frac{4}{3}\right]+\sum_{i=3}^{n}\left[\frac{2i}{i^2-i+1}\right]=n+3$$

$$\Omega=\lim_{n\to\infty}\left\{\ln(3n+1)-\sum_{k=1}^{n}\frac{1}{k+3}\right\}$$

$$=\lim_{n\to\infty}\left\{\ln(3n+1)-\ln n+\left(\ln n-\sum_{k=1}^{n}\frac{1}{k}\right)+1+\frac{1}{2}+\frac{1}{3}\right\}$$

$$=\lim_{n\to\infty}\ln\left(3+\frac{1}{n}\right)+\lim_{n\to\infty}\left(\ln n-\sum_{k=1}^{n}\frac{1}{k}\right)+\frac{11}{6}=\ln 3-\gamma+\frac{11}{6}$$

590. If $a_1, a_2, a_n, b_1, b_2, b_n$ denotes the distinct or identical digits and

$$17\times a_1a_2\cdots a_n=b_1b_2\ldots b_n\times 17=b_1+b_2+\cdots b_n$$

i.e. there exist the multiple of 17 such that the sum of the digits is 17. Find such smallest and largest four digit multiple of 17.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by proposer

Call the four - digit $n = \overline{abcd}$ *as per the divisibility test if* 17. *We have new number* $m = \overline{abc} - 5d = 10^2a + 10b + c - 5d$. *Assume the number is divisible by* 17. *Then we get:* $17m = 10^2a + 10b + c - 5d = 99a + 9b - 6d + \overbrace{(a + b + c + d)}^{17}$

$$17(m - 1) = 3(33a + 3b - 2d) \Rightarrow 17\left(\frac{m - 1}{3}\right) = 33a + 3b - 2d$$

For the smallest number we observe for the smallest four-digit number $m = 4$. *Then*

$$33a + 3b - 2d = 1 \times 17 = 17$$

Notice $a \neq 0$ *so, the smallest possible value of* $a = 1$ *then the last equation becomes:*

$$33 + 3b - 2d = 17 \Rightarrow 2d - 3b = 16 \Rightarrow d = \frac{16 + 3b}{2}$$

For $b = 0, 2$ *then,* $d = 8, 11$ *respectively. Since* $d \leq 9, \therefore d = 8, b = 0, a = 1$ *&* $c = 17 - 9 = 8$. *Which makes* 1088 *as smallest required number:*

For the largest number

Further, simplifying we see the formula: $b = \frac{1}{3}\left[17\left(\frac{m-1}{3}\right) - 33a + 2d\right] < 30, m \geq 4$

For the largest value of $\overline{abcd}$ *we want to maximize the value of* $a = 9$. *Also, we notice that* $0 \leq b \leq 8$ *and:* $281 \leq 17\left(\frac{m-1}{3}\right) \leq 305 \Rightarrow 50 \leq m \leq 54$ *or*

$$297 \leq 17\left(\frac{m - 1}{3}\right) \leq 321 \Rightarrow 53 \leq m \leq 57$$

Hence, we have $50 \leq m \leq 57$, *the largest possible for* m *is* 55 *such that* $b \in \mathbb{N} \leq 8$. *Plugging back to the above equation:* $b = \frac{306-297+2d}{3} = \frac{9+2d}{3} \Rightarrow d = 0, 3, 6$ *which solves for* $b = 5, 5, 7$ *and the largest* 4 *digits numbers we have* $\overline{abcd} = 9530, 9503, 97c6$ *(rejected). Therefore, the largest possible value of* $\overline{abcd} = 9503$ *since it's the multiple* 17 *only less a* 1000.

591. **Let x_1, x_2, x_3 be the roots of $x^3 + 3x^2 + 2x - 5 = 0$. Find:**

$$\Omega = \frac{1}{x_1 + 3} + \frac{1}{x_2 + 3} + \frac{1}{x_3 + 3}$$

Proposed by Boris Colakovic-Belgrade-Serbie

Solution 1 by Daniel Sitaru-Romania

$$y=\frac{1}{x+3}\to x+3=\frac{1}{y}\to x=\frac{1}{y}-3$$

$$\left(\frac{1}{y}-3\right)^3+3\left(\frac{1}{y}-3\right)^2+2\left(\frac{1}{y}-3\right)-5=0$$

$$\frac{1}{y^3}-\frac{9}{y^2}+\frac{27}{y}-27+\frac{3}{y^2}-\frac{18}{y}+27+\frac{2}{y}-6-5=0$$

$$1-9y+27y^2+3y-18y^2+5y^2-11y^3=0$$

$$-11y^3+11y^2-6y+1=0$$

$$11y^3-11y^2+6y-1=0$$

$$\Omega=\frac{1}{x_1+3}+\frac{1}{x_2+3}+\frac{1}{x_3+3}=y_1+y_2+y_3=-\frac{-11}{11}=1$$

Solution 2 by Sagar Kumar-Kolkata-India

$$x_1^3+3x_1^2+2x_1+6=11$$

$$\frac{x_1^2+2}{11}=\frac{1}{x_1+3}$$

$$\Omega=\frac{x_1^2+x_2^2+x_3^2+6}{11}\quad (1)$$

$$x_1+x_2+x_3=-3,\ x_1x_2+x_2x_3+x_1x_3=2$$

$$(x_1+x_2+x_3)^2=(11\Omega-6)+4\Rightarrow\frac{9-4+6}{11}=\Omega=1$$

Solution 3 by Marian Ursărescu-Romania

$$p(x)=x^3+3x^2+2x-5=(x-x_1)(x-x_2)(x-x_3)$$

$$p'(x)=3x^2+6x+2=(x-x_2)(x-x_3)+(x-x_1)(x-x_2)+(x-x_1)(x-x_3)$$

$$\frac{p'(x)}{p(x)}=\frac{1}{x-x_1}+\frac{1}{x-x_2}+\frac{1}{x-x_3}\Rightarrow\frac{p'(-3)}{P(-3)}=\frac{1}{-3-x_1}+\frac{1}{-3-x_2}+\frac{1}{-3-x_3}\Rightarrow$$

$$\Omega=\frac{-p'(-3)}{p(-3)}=-\frac{11}{-11}=1$$

592. Solve for real numbers:

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$$\begin{cases}2x\sqrt{1-y^2}+2y\sqrt{1-x^2}=\sqrt{3}\\2y\sqrt{1-z^2}+2z\sqrt{1-y^2}=\sqrt{3}\\2z\sqrt{1-x^2}+2x\sqrt{1-z^2}=\sqrt{3}\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Naren Bhandari-India

$$\begin{cases}2y\sqrt{1-x^2}+2y\sqrt{1-y^2}=\sqrt{3}\\2y\sqrt{1-z^2}+2y\sqrt{1-y^2}=\sqrt{3}\\2z\sqrt{1-x^2}+2z\sqrt{1-x^2}=\sqrt{3}\end{cases}$$

These systems of equations can be further expressed as:

$$\begin{cases}\sin^{-1}x+\sin^{-1}y=\frac{\pi}{3},\frac{2\pi}{3} & (1)\\\sin^{-1}y+\sin^{-1}z=\frac{\pi}{3},\frac{2\pi}{3} & (2)\\\sin^{-1}x+\sin^{-1}z=\frac{\pi}{3},\frac{2\pi}{3} & (3)\end{cases}$$

Adding these equations, we have that: $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{\pi}{2},\pi$ *(4)*

Notice that: $\frac{-\pi}{2}\le\phi\le\frac{\pi}{2}\Rightarrow-1\le\phi\le1$ *this implies the inequality above hold true for the following cases ie.*

(1). If $x=y=z$ *then:* $\sin^{-1}x=\sin^{-1}y=\sin^{-1}z=\frac{\pi}{6},\frac{\pi}{3}$ *which implies that*

$x=y=z=\frac{1}{2},\frac{\sqrt{3}}{2}$ *from equation (4)*

(2). If any two of them are equal and third one is different. WLOG, let $x=y$ *which directly implies* $z=0,x=y=1$ *which further follows as either* $x=y=\frac{\sqrt{3}}{2}$ *or* $x=y=\frac{1}{2}$ *and the same value will correspond either* $x=0$ *case or* $y=0$*. Thus, the solutions are* $0,1,\frac{1}{2},\frac{\sqrt{3}}{2}$*. Thus,* $x=0,y=z=\frac{\sqrt{3}}{2}$ $x=y=z=\frac{1}{2}$

$$x=1,y=z=\frac{1}{2}\quad y=1,x=z=\frac{1}{2}$$

$$z=1,x=y=\frac{1}{2}\quad y=0,x=z=\frac{\sqrt{3}}{2}$$

$$x = 0, x = y = \frac{\sqrt{3}}{2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$2x\sqrt{1-y^2} + 2\sqrt{1-x^2} \overset{(1)}{=} \sqrt{3}$$

$$2y\sqrt{1-z^2} + 2z\sqrt{1-y^2} \overset{(2)}{=} \sqrt{3}$$

$$2z\sqrt{1-x^2} + 2x\sqrt{1-z^2} \overset{(3)}{=} \sqrt{3}$$

(1) $\Rightarrow 2x\sqrt{1-y^2} = \sqrt{3} - 2y\sqrt{1-x^2} \Rightarrow 4x^2(1-y^2) = 3 - 4\sqrt{3}y\sqrt{1-x^2} + 4y^2(1-x^2)$

$$\Rightarrow 4y^2 - 4\sqrt{3(1-x^2)}y + 3 - 4x^2 = 0 \Rightarrow$$

$$\Rightarrow y = \frac{4\sqrt{3(1-x^2)} \pm \sqrt{48(1-x^2) - 16(3-4x^2)}}{8} = \frac{\sqrt{3(1-x^2)} \pm x}{2}$$

Similarly, (3) $\Rightarrow z = \frac{\sqrt{3(1-x^2)} \pm x}{2}$

Case 1) $y = z = \frac{\sqrt{3(1-x^2)} + x}{2}$

Then, using (2), $4y\sqrt{1-y^2} = \sqrt{3} \Rightarrow 16y^4 - 16y^2 + 3 = 0 \Rightarrow y^2 = \frac{16 \pm 8}{32} = \frac{3}{4}, \frac{1}{4} \Rightarrow$

$$\Rightarrow y = z = \pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}$$

Case 1a) $y = z = \frac{\sqrt{3}}{2} \therefore \sqrt{3(1-x^2)} + x = \sqrt{3} \Rightarrow 3(1-x^2) = \left(\sqrt{3} - x\right)^2 \Rightarrow x^2 = \frac{\sqrt{3}}{2}x$

$\Rightarrow x = 0, \frac{\sqrt{3}}{2} \therefore \left(x = 0, y = \frac{\sqrt{3}}{2}, z = \frac{\sqrt{3}}{2}\right)$ ***&*** $\left(x = \frac{\sqrt{3}}{2}, y = \frac{\sqrt{3}}{2}, z = \frac{\sqrt{3}}{2}\right)$ ***are two solutions.***

Case 2b) $y = z = \frac{1}{2} \therefore \sqrt{3(1-x^2)} + x = 1 \Rightarrow 3(1-x^2) = (1-x)^2 \Rightarrow$

$\Rightarrow (x-1)(2x+1) = 0 \Rightarrow x = 1, -\frac{1}{2}$***. Easy to verify that*** $x = -\frac{1}{2}$ ***is inadmissible and that*** $\left(x = 1, y = \frac{1}{2}, z = \frac{1}{2}\right)$ ***is a solution. Easy to note that at most are among*** x, y, z ***can be negative and*** $\therefore y = z = -\frac{\sqrt{3}}{2}, -\frac{1}{2}$ ***are impossible cases.***

Case 2) $y = z = \frac{\sqrt{3(1-x^2)} - x}{2}$

using (2), $y = z = \pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}$

Case 2a) $y = z = \frac{\sqrt{3}}{2} \therefore \sqrt{3(1-x^2)} - x = \sqrt{3} \Rightarrow 3(1-x^2) = \left(x+\sqrt{3}\right)^2 \Rightarrow$

$\Rightarrow x^2 = -\frac{\sqrt{3}}{2}x \Rightarrow x = 0, -\frac{\sqrt{3}}{2}$*. Easy to verify that* $x = -\frac{\sqrt{3}}{2}$ *is inadmissible and that*

$\left(x = 0, y = \frac{\sqrt{3}}{2}, z = \frac{\sqrt{3}}{2}\right)$ *is a solution*

Case 2b) $y = z = \frac{1}{2} \therefore \sqrt{3(1-x^2)} - x = 1 \Rightarrow 3(1-x^2) = x^2 + 2x + 1 \Rightarrow$

$\Rightarrow (x+1)(2x-1) = 0$*. Easy to verify that* $x = -1$ *is inadmissible and that*

$\left(x = y = z = \frac{1}{2}\right)$ *is a solution. Of course,* $y = z = -\frac{\sqrt{3}}{2}; y = z = -\frac{1}{2}$ *are impossible.*

Case 3) $y = \frac{\sqrt{3(1-x^2)}+x}{2}, z = \frac{\sqrt{3(1-x^2)}-x}{2}$

$$1 - y^2 = \frac{1+2x^2-2x\sqrt{3(1-x^2)}}{4} = \frac{1-x^2+3x^2-2\sqrt{3(1-x^2)}x}{4} = \left(\frac{\sqrt{1-x^2}-\sqrt{3}x}{2}\right)^2 \text{ and}$$

$$1 - z^2 = \left(\frac{\sqrt{1-x^2}+\sqrt{3}x}{2}\right)^2$$

Case 3a) $-1 \le x < -\frac{1}{2} \because x < -\frac{1}{2}, 1 - x^2 < 3x^2 \Rightarrow \sqrt{1-x^2} < -\sqrt{3}x \Rightarrow$

$\Rightarrow \sqrt{1-x^2} + \sqrt{3x} < 0 \therefore \sqrt{1-z^2} = \left|\frac{\sqrt{1-x^2}+\sqrt{3}x}{2}\right| = -\left(\frac{\sqrt{1-x^2}+\sqrt{3}x}{2}\right)$ *and of course,*

$$\sqrt{1-y^2} = \frac{\sqrt{1-x^2}-\sqrt{3}x}{2} \quad (\because x < 0)$$

$\therefore$ *(2) becomes:* $-8x\sqrt{1-x^2} = 2\sqrt{3} \Rightarrow 16x^4 - 16x^2 + 3 = 0 \Rightarrow x^2 = \frac{3}{4}, \frac{1}{4} \Rightarrow$

$\Rightarrow x = -\frac{1}{2}, -\frac{\sqrt{3}}{2}$*. But* $-1 \le x < -\frac{1}{2} \therefore x = -\frac{\sqrt{3}}{2}$

$\therefore x = -\frac{\sqrt{3}}{2}, y = 0, z = \frac{\sqrt{3}}{2}$*. But this solution doesn't satisfy given system.*

Case 3b) $-\frac{1}{2} \le x < 0 \because x \ge -\frac{1}{2}, \therefore \sqrt{1-x^2} + \sqrt{3}x \ge 0 \Rightarrow \sqrt{1-z^2} = \frac{\sqrt{1-x^2}+\sqrt{3}x}{2}$

Of course, $\sqrt{1-y^2} = \frac{\sqrt{1-x^2}-\sqrt{3}x}{2} \quad (\because x < 0)$

(1) becomes: $x\sqrt{1-x^2} = \sqrt{3}x^2 \Rightarrow x = 0, -\frac{1}{2}$

But $\therefore -\frac{1}{2} \le x < 0, \therefore x = -\frac{1}{2}, y = \frac{1}{2}, z = 1$*. Easy to check that this solution doesn't satisfy the system.*

Case 3c) $0 \le x < \frac{1}{2}$ $\sqrt{1-x^2} > \sqrt{3}x \Rightarrow \sqrt{1-y^2} = \frac{\sqrt{1-x^2}-\sqrt{x}}{2}$ *and of course,*

$$\sqrt{1-z^2} = \frac{\sqrt{1-x^2}+\sqrt{3}x}{2}$$

(1) becomes: $2x\sqrt{1-x^2} = 2\sqrt{3}x^2 \Rightarrow x = 0, \frac{1}{2}$

$\because 0 \le x < \frac{1}{2}, \therefore \left(x = 0, y = \frac{\sqrt{3}}{2}, z = \frac{\sqrt{3}}{2}\right)$ *which is a solution.*

Case 3d) $\frac{1}{2} \le x \le 1$*. Then* $\sqrt{3}x \ge \sqrt{1-x^2} \therefore \sqrt{1-y^2} = \frac{\sqrt{3}x-\sqrt{1-x^2}}{2}$ *and of course,*

$$\sqrt{1-z^2} = \frac{\sqrt{1-x^2}+\sqrt{3}x}{2}$$

(2) becomes: $8x\sqrt{1-x^2} = 2\sqrt{3} \Rightarrow 16x^4 - 16x^2 + 3 = 0 \Rightarrow x^2 = \frac{3}{4}, \frac{1}{4} \Rightarrow x = \frac{\sqrt{3}}{2}, \frac{1}{2}$

$\therefore \left(x = \frac{\sqrt{3}}{2}, y = \frac{\sqrt{3}}{2}, z = 0\right)$ *and* $\left(x = \frac{1}{2}, y = 1, z = \frac{1}{2}\right)$ *are two solutions.*

Case 4) $y = \frac{\sqrt{3(1-x^2)}-x}{2}, z = \frac{\sqrt{3(1-x^2)}+x}{2}$*. In this case, roles of* ? *and* ? *are simply interchanged. We'll obtain two more solutions:*

$\left(x = \frac{\sqrt{3}}{2}, y = 0, z = \frac{\sqrt{3}}{2}\right)$ *and* $\left(x = \frac{1}{2}, y = \frac{1}{2}, z = 1\right)$

$\therefore$ *all possible real solutions are:*

$$\begin{pmatrix} x = 0 \\ y = \frac{\sqrt{3}}{2} \\ z = \frac{\sqrt{3}}{2} \end{pmatrix}^{**}, \begin{pmatrix} x = \frac{\sqrt{3}}{2} \\ y = \frac{\sqrt{3}}{2} \\ z = \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} x = 1 \\ y = \frac{1}{2} \\ z = \frac{1}{2} \end{pmatrix}^{*}, \begin{pmatrix} x = \frac{1}{2} \\ y = \frac{1}{2} \\ z = \frac{1}{2} \end{pmatrix},$$

$$\begin{pmatrix} x = \frac{\sqrt{3}}{2} \\ y = \frac{\sqrt{3}}{2} \\ z = 0 \end{pmatrix}^{**}, \begin{pmatrix} x = \frac{1}{2} \\ y = 1 \\ z = \frac{1}{2} \end{pmatrix}^{*}, \begin{pmatrix} x = \frac{\sqrt{3}}{2} \\ y = 0 \\ z = \frac{\sqrt{3}}{2} \end{pmatrix}^{**}, \begin{pmatrix} x = \frac{1}{2} \\ y = \frac{1}{2} \\ z = 1 \end{pmatrix}^{*}$$

Solution 3 by Santos Martins Junior-Brusels-Belgium

Let $x\sqrt{(1-y^2)} = a$ *(1)*

$y\sqrt{(1-x^2)} = b$ **(2)**

$y\sqrt{(1-z)^2} = c$ **(3)**

$z\sqrt{(1-y^2)} = d$ **(4)**

$z\sqrt{(1-x^2)} = e$ **(5)**

$x\sqrt{(1-z^2)} = f$ **(6)**

The system becomes: $(a+b) = \frac{\sqrt{3}}{2}$ ***(7);*** $(c+d) = \frac{\sqrt{3}}{2}; (e+f) = \frac{\sqrt{3}}{2}$ ***(9)***

Doing $(1)^2 - (2)^2: (a^2 - b^2) = (x^2 - y^2)$

Doing: $(3)^2 - (4)^2: (c^2 - d^2) = (y^2 - z^2)$

Doing: $(5)^2 - (6)^2: (e^2 - f^2) = (z^2 - x^2)$

Adding up: $(a^2 + c^2 + e^2) = (b^2 + d^2 + f^2)$ ***(10)***

Doing $(7)^2 + (8)^2 + (9)^2: (a^2 + b^2 + 2ab + c^2 + d^2 + 2cd + e^2 + f^2 + 2ef) = \frac{9}{4}$

using $(10): 2(a^2 + c^2 + e^2) + 2(ab + cd + ef) = \frac{9}{4}$

$: \quad a(a+b) + c(c+d) + e(e+f) = \frac{9}{8}$

using $(7),(8),(9): \left\{\frac{\sqrt{3}}{2}\right\}^* (a+c+e) = \frac{9}{8} \Rightarrow (a+c+e) = \frac{3\sqrt{3}}{4}$ ***(11)***

Doing (7) + (8) + (9): $(a+c+e) + (b+d+f) = \frac{3\sqrt{3}}{2}$

using (11): $(b+d+f) = \frac{3\sqrt{3}}{4}$ ***(12)***

Also we know from equations (1) to (6) that: $(a^*c^*e) = (b^*d^*f)$ ***(13)***

We got from (11), (12): $(a+c+e) = (b+f+f)$ ***(14)***

We got from (10): $(a^2 + c^2 + e^2) = (b^2 + d^2 + f^2)$

Implying from (14) that $(ac + ce + ea) = (bd + df + fb)$ ***(15)***

We got from (13): $(ace) = (bdf) \Rightarrow (a,c,e)$ ***and*** (b,d,f) ***are the 3 roots of the same 3rd degree polynomial*** $\Rightarrow \{a,c,e\} = \{b,d,f\}$

1) Let's pick: $a = b \Rightarrow x\sqrt{(1-y^2)} = y\sqrt{(1-x^2)} \Rightarrow xy$ ***have the same sign***

$\Rightarrow$ ***squarring:*** $x^2 = y^2 \Rightarrow (x+y)(x-y) = 0$

$x = -y$ ***is rejected*** $\Rightarrow x = y: x\sqrt{(1-x^2)} = \frac{\sqrt{3}}{4} \Rightarrow x^4 - x^2 + \frac{3}{16} \Rightarrow$

$x=\frac{1}{2}$ *or* $\frac{\sqrt{3}}{2}\Rightarrow y=\frac{1}{2}$ *or* $\frac{\sqrt{3}}{2}$

$\Rightarrow$ *substituting in (8) or (9): when* $x=y=\frac{1}{2},z=1$ *or* $z=\frac{1}{2}$ *when*

$x=y=\frac{\sqrt{3}}{2},z=0$ *or* $z=\frac{\sqrt{3}}{2}$

$$(x,y,z)=\left(\frac{1}{2},\frac{1}{2},1\right)\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2},0\right)\left(\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right)$$

2) picking $a=d\Rightarrow x\sqrt{(1-y^2)}=z\sqrt{(1-y^2)}$

$y=1$ *solution* $\Rightarrow$ *from (7) and (8) we get:* $x=z=\frac{1}{2}\Rightarrow(x,y,z)=\left(\frac{1}{2},\frac{1}{2}\right)$

otherwise $x=z\Rightarrow$ *from (9):* $x\sqrt{(1-x^2)}=\frac{\sqrt{3}}{4}\Rightarrow x=z=\frac{1}{2}$ *or* $\frac{\sqrt{3}}{2}$

substituting in (7) or (8): when $x=z=\frac{1}{2},y=1$ *or* $y=\frac{1}{2}$

when $x=z=\frac{\sqrt{3}}{2},y=0$ *or* $y=\frac{\sqrt{3}}{2}$

$$(x,y,z)=\left(\frac{1}{2},1,\frac{1}{2}\right)\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\left(\sqrt{\frac{3}{2}},0,\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right)$$

3) picking $a=f\Rightarrow x\sqrt{(1-y^2)}=x\sqrt{(1-z^2)}$

$x=0$ *is solution* $\Rightarrow$ *from (7) and (9) we get:* $y=z=\frac{\sqrt{3}}{2}$

Otherwise $y=z$ *proceeding similarly we get*

$$(x,y,z)=\left(1,\frac{1}{2},\frac{1}{2}\right)\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\left(0,\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right)$$

Solution 4 by Shahlar Maharramov-Azerbaijan

$$\alpha+\beta+\gamma=\frac{\pi}{6}\left((-1)^n+(-1)^k+(-1)^n\right)+\frac{\pi}{2}(m+n+k)$$

Rearrangement, take places the cases: $(m,n,k)=(0,0,0),(0,0,1),(0,1,0),(1,0,0)$

$(0,1,1),(1,0,1),(1,1,0),(1,1,1)$

1) take $(0,0,0)\Rightarrow\alpha+\beta+\gamma=\frac{\pi}{2}$ *and* $a+\beta=b+\gamma=\alpha+\gamma=\frac{\pi}{3}\Rightarrow$

$$\Rightarrow(\alpha,\beta,\gamma)=\left(\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{6}\right)\Rightarrow(x,y,z)=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$$

2) take $(0,1,1)\Rightarrow\alpha+\beta+\gamma=\frac{\pi}{6}+\frac{\pi}{2}=\frac{2\pi}{3}\Rightarrow\alpha+\beta=\frac{\pi}{3};\beta+\gamma=\frac{\pi}{3},\gamma+\alpha=\frac{2\pi}{3}\Rightarrow$

$$\Rightarrow (\alpha,\beta,\gamma) = \left(\frac{\pi}{3}:0:\frac{\pi}{3}\right) \Rightarrow (x,y,z) = \left(\frac{\sqrt{3}}{2}:0:\frac{\sqrt{3}}{2}\right)$$

3) $(0,1,0)$ ***same process*** $(\alpha,\beta,\gamma) = \left(0:\frac{\pi}{3}:\frac{\pi}{3}\right)$

$$(x,y,z) = \left(0,\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right)$$

4) $(1,0,0) \Rightarrow (\alpha,\beta,\gamma) = \left(\frac{\pi}{3}:\frac{\pi}{3}:0\right), (x,y,z) = \left(\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}:0\right)$

5) $(0,1,1) \Rightarrow \alpha+\beta+\gamma = -\frac{\pi}{6}+\pi = \frac{5\pi}{6}$

$$(\alpha,\beta,\gamma) = \left(\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{2}\right) \quad (x,y,z) = \left(\frac{1}{2}:\frac{1}{2}:0\right)$$

6) $(1,0,1) \Rightarrow \alpha+\beta+\gamma = \frac{5\pi}{6}:\Rightarrow (\alpha,\beta,\gamma) = \left(\frac{\pi}{2},\frac{\pi}{6},\frac{\pi}{6}\right) \Rightarrow (x,y,z) = \left(0:\frac{1}{2}:\frac{1}{2}\right)$

7) $(1,1,0) \Rightarrow \alpha+\beta+\gamma = \frac{5\pi}{6} \Rightarrow (\alpha,\beta,\gamma) = \left(\frac{\pi}{6}:\frac{\pi}{2}:\frac{\pi}{6}\right) \Rightarrow (x,y,z) = \left(\frac{1}{2}:0:\frac{1}{2}\right)$

8) $(1,1,1) \Rightarrow \alpha+\beta+\gamma = \frac{3\pi}{2}-\frac{\pi}{2} = \pi$

$$(\alpha,\beta,\gamma) = \left(\frac{2\pi}{3}:\frac{2\pi}{3}:\frac{2\pi}{3}\right) \Rightarrow (x,y,z) = \left(\frac{\sqrt{3}}{2}:\frac{\sqrt{3}}{2}:\frac{\sqrt{3}}{2}\right)$$

593. Solve for real numbers:

$$\begin{cases} 0 < x,y,z < 1 \\ \dfrac{x}{1-x^2}+\dfrac{y}{1-y^2}+\dfrac{z}{1-z^2} = \dfrac{4xyz}{(1-x^2)(1-y^2)(1-z^2)} \\ x^2+y^2+z^2 = 1 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum x^2 \overset{(1)}{=} 1 \ \& \ \frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2} \overset{(2)}{=} \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

$$(2) \Rightarrow x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2) = 4xyz$$

$$\Rightarrow x(1-(1-x^2)+y^2z^2)+y(1-(1-y^2)+z^2x^2)+z(1-(1-z^2)+x^2y^2) =$$

$$= 4xyz\left(\because \sum x^2 = 1\right) \Rightarrow \sum x^3 + xyz\left(\sum xy\right) = 4xyz \Rightarrow 3xyz+\sum x\left(1-\sum xy\right)+$$

$$+xyz\left(\sum xy\right) = 4xyz\left(\because \sum x^2 = 1\right) \Rightarrow \sum x\left(1-\sum xy\right) = xyz\left(1-\sum xy\right) = 0$$

$$\Rightarrow (1-\sum xy)(\sum x - xyz) = 0 \quad (1)$$

Now, $\sum x^2 \overset{A-G}{\geq} 3\sqrt[3]{x^2y^2z^2} \Rightarrow 1 \geq 27x^2y^2z^2 \Rightarrow 27x^2y^2z^2 \overset{(2)}{\leq} 1$. *If* $\sum x = xyz$*, then,*

$$\left(\sum x\right)^2 = x^2y^2z^2 \Rightarrow 27\left(1+2\sum xy\right) = 27x^2y^2z^2 \left(\because \sum x^2 = 1\right) \overset{by\,(2)}{\leq} 1 \Rightarrow$$

$\Rightarrow 54\sum xy \leq -26$. *But* $54\sum xy > 0$ $(\because x,y,z>0)$ $\therefore \sum x \neq xyz \therefore$ *(1)* $\Rightarrow 1 = \sum xy$

Now, $\sum x^2 \geq \sum xy$ *(equality when* $x=y=z$*)* $\Rightarrow 1 \geq \sum xy$*,*

with equality when $x=y=z$*.*

& $\because 1 = \sum xy \therefore x=y=z$ *&* $\because \sum x^2 = 1, \therefore x^2 = \frac{1}{3} \Rightarrow x=y=z=\frac{1}{\sqrt{3}}$ *(ans)*

Solution 2 by Amit Dutta-Jamshedpur-India

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2} = \frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)} \Rightarrow \left(\frac{2x}{1-x^2}\right)+\left(\frac{2y}{1-y^2}\right)+$$

$$+\left(\frac{2z}{1-z^2}\right) = \left(\frac{2x}{1-x^2}\right)\left(\frac{2y}{1-y^2}\right)\left(\frac{2z}{1-z^2}\right)$$

Put $x=\tan\alpha, y=\tan\beta, z=\tan\gamma \Rightarrow \tan 2\alpha+\tan 2\beta+\tan 2\gamma =$

$$= (\tan 2\alpha)(\tan 2\beta)(\tan 2\gamma) \Rightarrow 2\alpha+2\beta+2\gamma = \pi$$

$$\alpha+\beta+\gamma = \frac{\pi}{2}$$

$$\tan(\alpha+\beta+\gamma) = \frac{\sum\tan\alpha - \prod\tan\alpha}{1-\sum\tan\alpha\tan\beta} = \tan\left(\frac{\pi}{2}\right) \Rightarrow \sum\tan\alpha\tan\beta = 1$$

$$\because x^2+y^2+z^2 = 1 \Rightarrow \tan^2\alpha+\tan^2\beta+\tan^2\gamma = 1$$

$$(\tan\alpha+\tan\beta+\tan\gamma)^2 = \sum\tan^2\alpha + 2\sum\tan\alpha\tan\beta = 1+2 = 3$$

$$\tan\alpha+\tan\beta+\tan\gamma = \sqrt{3} \quad \left\{\begin{matrix}\because x,y,z\in(0,1) \\ \Rightarrow \tan\alpha > 0 \\ \tan\beta > 0 \\ \tan\gamma > 0\end{matrix}\right\}$$

Now, by Cauchy's Schwarz Inequality

$$(\tan^2\alpha+\tan^2\beta+\tan^2\gamma)(1^2+1^2+1^2) \geq (\tan\alpha+\tan\beta+\tan\gamma)^2 \quad (1)$$

$$\because \tan^2\alpha+\tan^2\beta+\tan^2\gamma = 1$$

$$\tan\alpha+\tan\beta+\tan\gamma = \sqrt{3}$$

$\therefore$ *Equality holds in (1) {Cauchy's Inequality}*

$$\Rightarrow \frac{\tan\alpha}{1} = \frac{\tan\beta}{1} = \frac{\tan\gamma}{1} \because \tan\alpha + \tan\beta + \tan\gamma = \sqrt{3} \Rightarrow 3\tan\alpha = \sqrt{3} \Rightarrow \tan\alpha = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \tan\alpha = \tan\beta = \tan\gamma = \frac{1}{\sqrt{3}}$$

$$\left.\begin{aligned} x &= \tan\alpha = \frac{1}{\sqrt{3}} \\ \because y &= \tan\beta = \frac{1}{\sqrt{3}} \\ z &= \tan\gamma = \frac{1}{\sqrt{3}} \end{aligned}\right\} \Rightarrow (x,y,z) \equiv \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

594. Solve for real numbers:

$$x^3 + x^2 + 2x + log(x^3 + 2x + 1) = y$$
$$y^3 + y^2 + 2y + log(y^3 + 2y + 1) = x$$

Proposed by Amit Dutta-Jamshedpur-India

Solution 1 by Marian Ursărescu-Romania

Observation: Let the symmetric and circular system:

$$(1)\begin{cases} f(x_1) = x_2 \\ f(x_2) = x_3 \\ \vdots \\ f(x_n) = x_1 \end{cases}$$

Theorem 1. The system (1) is equivalent with the equation $f \circ f \circ \dots \circ f(x) = x$.

Theorem 2. If $f: D \to D$ is strictly increasing the equation

$$f \circ f \circ \dots \circ f(x) = x \Leftrightarrow f(x) = x$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Let $f(x) = x^3 + x^2 + 2x + \log(x^3 + 2x + 1)$

$f'(x) = \frac{3x^2+2}{x^3+2x+1} + 3x^2 + 2x + 2$. ***We must have*** $x^3 + 2x + 1 > 0$ ***for*** $\log(x^3 + 2x + 1)$ ***to be defined on*** $\mathbb{R} \Rightarrow x^3 + 2x + 1 > 0 \therefore f'(x) = \frac{3x^2+2}{x^3+2x+1} + (x+1)^2 + 2x^2 + 1 > 0 \Rightarrow$

$\Rightarrow f(x)$ ***is an increasing f^n. Let us assume $x \geq y$. Then $f(x)$ is an increasing f^n,***

$\therefore f(x) \geq f(y) \Rightarrow y \geq x (\because y = f(x) \,\&\, x = f(y)) \therefore x \geq y \,\&\, y \geq x$, *we must have* $x = y$.

If we assume $x \leq y$, *then* $f(y) \geq f(x) \Rightarrow x \geq y \Rightarrow x = y$. *So, we conclude combining both cases that* $x = y \therefore x^3 + x^2 + x + \log(x^3 + 2x + 1) = 0$. *Let* $g(x) = x^3 + x^2 + x +$

$$+\log(x^3 + 2x + 1)\,;\, g'(x) = \frac{3x^2 + 2}{x^3 + 2x + 1} + (x + 1)^2 + 2x^2 > 0 \Rightarrow g(x)$$

is an increasing f^n *&*

$\because g(0) = 0, \therefore g(x) = 0$ *iff* $x = 0 \therefore$ *only possible pair satisfying given equation is*

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \text{ (answer)}$$

595. Find $x, y, z \geq 2, t \geq 1$ such that:

$$\begin{cases} \sqrt{x-2} + \sqrt[3]{y-2} + \sqrt[4]{z-2} = 3\sqrt{t} \\ x + y + z = 6 + 3t \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

First if $\begin{matrix} x < 3 \\ y < 3 \\ z < 3 \end{matrix} \Rightarrow x + y + t < 9$, *but* $t \geq 1 \Rightarrow 6 + 3t \geq 9 \Rightarrow$ *its false* $\Rightarrow x, y, z \geq 3$.

Let $x - 2 = a, y - 2 = b, z - 2 = c, a, b, c \geq 1$

$$\begin{cases} \sqrt{a} + \sqrt[3]{b} + \sqrt[4]{c} = 3\sqrt{t} \\ a + b + c = 3t \end{cases} \Rightarrow \sqrt{a} + \sqrt[3]{b} + \sqrt[4]{c} = \sqrt{3(a + b + c)} \quad (1)$$

Because $a, b, c \geq 1 \Rightarrow \sqrt[3]{b} \leq \sqrt{b}$ *and* $\sqrt[4]{c} \leq \sqrt{c}$ *with equality for* $b = c = 1 \Rightarrow$

$$\sqrt{a} + \sqrt[3]{b} + \sqrt[4]{c} \leq \sqrt{a} + \sqrt{b} + \sqrt{c} \quad (2). \text{ From } (1)+(2) \Rightarrow$$

$$\sqrt{3(a + b + c)} \leq \sqrt{a} + \sqrt{b} + \sqrt{c} \Rightarrow 3(a + b + c) \leq \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 \quad (3)$$

From Cauchy's inequality $3(a + b + c) \geq \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2$ *(4)*

From (3)+(4) $\Rightarrow$ *in Cauchy's inequality we have equality* $\Rightarrow a = b = c = 1 \Rightarrow \begin{matrix} x = 3 \\ y = 3 \\ z = 3 \end{matrix} \Rightarrow t = 1$.

596. Solve for real numbers:

$$\begin{cases} \begin{vmatrix} x & y & 2 & 3 \\ y & x & 3 & 2 \\ 2 & 3 & x & y \\ 3 & 2 & y & x \end{vmatrix} = 0 \\ x + y - \sqrt{xy} = \sqrt{\dfrac{x^2 + y^2}{2}} \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

We have $x + y - \sqrt{xy} = \sqrt{\frac{x^2+y^2}{2}}$*. Put* $(x + y) = a, \sqrt{xy} = b \Rightarrow a - b = \sqrt{\frac{a^2-2b^2}{2}}$

Squaring both sides $\Rightarrow (a - b)^2 = \left(\frac{a^2-2b^2}{2}\right) \Rightarrow 2(a^2 + b^2 - 2ab) = a^2 - 2b^2 \Rightarrow$

$\Rightarrow 2a^2 + 2b^2 - 4ab = a^2 - 2b^2 \Rightarrow a^2 + 4b^2 - 4ab = 0 \Rightarrow (a - 2b)^2 = 0 \Rightarrow a = 2b$ *or*

$$x + y = 2\sqrt{xy} \Rightarrow \left(\sqrt{x} - \sqrt{y}\right)^2 = 0 \Rightarrow \sqrt{x} = \sqrt{y} \Rightarrow x = y$$

$$\begin{vmatrix} x & y & 2 & 3 \\ y & x & 3 & 2 \\ 2 & 3 & x & y \\ 3 & 2 & y & x \end{vmatrix} = 0 \Rightarrow x = y \Rightarrow \begin{vmatrix} x & x & 2 & 3 \\ x & x & 3 & 2 \\ 2 & 3 & x & x \\ 3 & 2 & x & x \end{vmatrix} = 0$$

Applying $R_2 \to (R_2 - R_1); R_4 \to (R_4 - R_3) \Rightarrow \begin{vmatrix} x & x & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 2 & 3 & x & x \\ 1 & -1 & 0 & 0 \end{vmatrix} = 0$

Applying $C_2 \to C_2 - C_1; C_4 \to C_4 - C_3 \Rightarrow \begin{vmatrix} x & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 \\ 2 & 1 & x & 0 \\ 1 & -2 & 0 & 0 \end{vmatrix} = 0$

Applying $R_3 \to R_3 - 2R_4 \Rightarrow \begin{vmatrix} x & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 5 & x & 0 \\ 1 & -2 & 0 & 0 \end{vmatrix} = 0$

Expanding this determinant $\Rightarrow x \begin{vmatrix} 0 & 1 & -2 \\ 5 & x & 0 \\ -2 & 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} 0 & 2 & 1 \\ 0 & 1 & -2 \\ 5 & x & 0 \end{vmatrix} = 0$

Again expanding: $-4x^2 + 25 = 0$

$$4x^2 = 25 \Rightarrow x = \pm\frac{5}{2} \because x = y \Rightarrow x = y = \pm\frac{5}{2}$$

But, $x + y - \sqrt{xy} = \sqrt{\frac{x^2+y^2}{2}}$

$RHS > 0 \Rightarrow LHS > 0 \Rightarrow x + y > \sqrt{xy} > 0 \therefore x + y > 0$***. So,*** $x \neq -\frac{5}{2}, y \neq -\frac{5}{2}$

$\therefore$ ***The only soluiton is*** $\left(x = y = \frac{5}{2}\right)$

Solution 2 by Ravi Prakash-New Delhi-India

As $\sqrt{xy}$ ***is involved, either both*** $x, y \leq 0$ ***or both*** $x, y \geq 0$***. If*** $x, y < 0$***, then***

$x + y - \sqrt{xy} < 0$ ***and*** $\sqrt{\frac{x^2+y^2}{2}} > 0 \therefore$ ***both*** $x, y \not< 0$***. Thus,*** $x, y \geq 0$***. Let***

$\Delta = \begin{vmatrix} x & y & 2 & 3 \\ y & x & 3 & 2 \\ 2 & 3 & x & y \\ 3 & 2 & y & x \end{vmatrix}$***. Applying*** $C_1 \to C_1 + C_2 + C_3 + C_4$***, we get*** $\Delta = (x + y + 5)\Delta_1$***,***

where

$$\Delta_1 = \begin{vmatrix} 1 & y & 2 & 3 \\ 1 & x & 3 & 2 \\ 1 & 3 & x & y \\ 1 & 2 & y & x \end{vmatrix} = \begin{vmatrix} 1 & y & 2 & 3 \\ 0 & x-y & 1 & -1 \\ 0 & 3-y & x-2 & y-3 \\ 0 & 2-y & y-2 & x-3 \end{vmatrix} = \begin{vmatrix} x-y & 1 & -1 \\ 3-y & x-2 & y-3 \\ 2-y & y-2 & x-3 \end{vmatrix}$$

$C_1 \to C_1 + C_3, C_2 \to C_2 + C_3$ ***gives*** $\Delta_1 = \begin{vmatrix} x-y-1 & 0 & -1 \\ 0 & x+y-5 & y-3 \\ x-y-1 & x+y-5 & x-3 \end{vmatrix} =$

$$= (x-y-1)(x+y-5)\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & y-3 \\ 1 & 1 & x-3 \end{vmatrix} = (x-y-1)(x+y-5)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & y-3 \\ 1 & 1 & x-2 \end{vmatrix} =$$

$$= (x-y-1)(x+y-5)(x-2-y+3) = (x-y-1)(x-y+1)(x+y-5)$$

Now, $\Delta = 0 \Rightarrow (x + y + 5)\Delta_1 = 0$***. As*** $x, y \geq 0$***, we get*** $\Delta_1 = 0 \Rightarrow$

$\Rightarrow (x-y-1)(x-y+1)(x+y-5) = 0 \Rightarrow x - y = 1$ ***or*** $x - y = -1$ ***or*** $x + y = 5$***.***

Case 1: $x - y = 1$***. Let*** $x = t + \frac{1}{2}, y = t - \frac{1}{2}, t \geq \frac{1}{2}$***. The second equation becomes***

$$2t - \sqrt{t^2 - \frac{1}{4}} = \sqrt{t^2 + \frac{1}{4}} \Rightarrow 2t = \sqrt{t^2 - \frac{1}{4}} + \sqrt{t^2 + \frac{1}{4}}$$

$$\Rightarrow 4t^2 = t^2 + \frac{1}{4} + t^2 - \frac{1}{4} + 2\sqrt{t^4 - \frac{1}{4}} \Rightarrow t^2 = \sqrt{t^4 - \frac{1}{16}} < t^2. \text{ Not possible.}$$

Similarly, $y - x = 1$ is not possible. Thus, we consider

Case 2: $x + y = 5$

Let $x = \frac{5}{2} - t$, $y = \frac{5}{2} + t$ where $-\frac{5}{2} \le t \le \frac{5}{2}$

Second equation now becomes: $5 - \sqrt{\frac{25}{4} - t^2} = \sqrt{\frac{25}{4} + t^2} \Rightarrow t = 0$. *Thus,* $x = \frac{5}{2}, y = \frac{5}{2}$.

597. Solve for real positive numbers:

$$\begin{cases} \frac{x+y}{1+xy} + \frac{xy}{1+x} + \frac{xy}{1+y} + \frac{x+y+2xy}{(1+x)(1+y)xy} = 3 \\ \sin x = \cos y \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Hoang La Nhat Tung-Hanoi-Vietnam

$$\frac{x+y}{1+xy} + \frac{xy}{1+x} + \frac{xy}{1+y} + \frac{x+y+2xy}{(1+x)(1+y)xy} = 3$$

$$\frac{(x+1)(y+1)}{xy+1} + xy\left(\frac{1}{1+x} + \frac{1}{1+y}\right) + \frac{x(y+1) + y(x+1)}{xy(1+x)(1+y)} = 4$$

AM-GM: $LHS \ge \frac{(x+1)(y+1)}{xy+1} + \frac{2xy}{\sqrt{(1+x)(1+y)}} + \frac{2\sqrt{xy(1+x)(1+y)}}{xy(1+x)(1+y)} = \frac{(x+1)(y+1)}{xy+1} + \frac{2\left(\sqrt{(xy)^3}+1\right)}{\sqrt{xy(1+x)(1+y)}}$

$$\ge 2\sqrt{\frac{(x+1)(y+1)\cdot 2\left(\sqrt{(xy)^3}+1\right)}{(xy+1)\sqrt{xy(x+1)(y+1)}}} = 2\sqrt{\frac{2\left(\sqrt{(xy)^3}+1\right)\sqrt{(x+1)(y+1)}}{(xy+1)\sqrt{xy}}} \quad (1)$$

$$\sqrt{(xy)^3} + 1 \ge \sqrt{\frac{(xy+1)^3}{2}} = (xy+1)\sqrt{\frac{xy+1}{2}} \ge (xy+1)\cdot\sqrt[4]{xy} \quad (2)$$

$$(1), (2) \Rightarrow LHS \ge 2\sqrt{\frac{2(xy+1)\sqrt[4]{xy}\cdot\sqrt{2\sqrt{x}\cdot 2\sqrt{y}}}{(xy+1)\sqrt{xy}}} = 2\sqrt{\frac{4\cdot\sqrt[4]{xy}\cdot\sqrt[4]{xy}}{\sqrt{xy}}} = 2\sqrt{4} = 4$$

$\Rightarrow$ *Equality occurs if* $x = y = 1 \Rightarrow \sin x = \sin 1 \neq 0 > 1 = 0 > y \rightarrow$ *(absurd)*

$\Rightarrow$ *system has no solution.*

598. Solve for real numbers:

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$$\begin{cases}\left(\frac{xy}{z}\right)^4+\left(\frac{yz}{x}\right)^4+\left(\frac{zx}{y}\right)^4=xyz\sqrt[4]{27\sum x^4}\\ x^4-4y^3+6z^2-4x+1=0\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\left(\frac{xy}{z}\right)^4+\left(\frac{yz}{x}\right)^4+\left(\frac{zx}{y}\right)^4\overset{(1)}{=}xyz\sqrt[4]{27\sum x^4}\ \&\ x^4-4y^3+6z^2-4x+1\overset{(2)}{=}0$$

Let $x^4=a,y^4=b,z^4=c$***. Then*** $\because x,y,z\neq 0,\therefore\ a,b,c>0$

$$\therefore (1)\Leftrightarrow \frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\overset{(1a)}{=}xyz\sqrt[4]{27\sum a}$$

Now, LHS of (1) $>0\Rightarrow xyz\sqrt[4]{27\sum x^4}>0\Rightarrow xyz>0$

Now, $(xyz)^4=abc\Rightarrow(xyz)^2=\sqrt{abc}\Rightarrow|xyz|=\sqrt[4]{abc}\Rightarrow xyz\overset{(1b)}{=}\sqrt[4]{abc}\ (\because xyz>0)$

(1a), (1b)$\Rightarrow \sum\frac{ab}{c}=\sqrt[4]{abc}\sqrt[4]{27\sum a}\Leftrightarrow\left(\sum\frac{ab}{c}\right)^4\overset{(1c)}{=}abc\left(27\sum a\right)$

Now, $\sum\frac{ab}{c}=\sum\frac{a^2b^2}{abc}=\frac{\sum a^2b^2}{abc}\geq\frac{abc(\sum a)}{abc}=\sum a\Rightarrow\left(\sum\frac{ab}{c}\right)^4\overset{(1d)}{=}\left(\sum a\right)^4$

(1c), (1d)$\Rightarrow abc\left(27\sum a\right)\geq\left(\sum a\right)^4\Rightarrow 27abc\overset{(1e)}{=}\left(\sum a\right)^3$

But $\left(\sum a\right)^3\underset{(i)}{\overset{A-G}{\geq}}27abc$***, equality when*** $a=b=c$

(i), (1e)$\Rightarrow 27abc=\left(\sum a\right)^3\Rightarrow a=b=c\Rightarrow x^4=y^4=z^4\Rightarrow|x|=|y|=|z|\Rightarrow$

$$y=\pm x,z=\pm x$$

Case 1) $y=-x$

$$\textbf{(2)}\Rightarrow x^4+4x^3+6x^2-4x+1\overset{(3)}{=}0$$

Case 1a) $x\leq 0$

Then, $-4x\geq 0\Rightarrow -4x+1\geq 1\overset{(m)}{\geq}0$

& $x^4+4x^3+6x^2=x^2(x^2+4x+6)=x^2(x^2+4x+4+2)=x^2(x+2)^2+2x^2\overset{(n)}{=}0$

(m)+(n)$\Rightarrow x^4+4x^3+6x^2-4x+1>0\Rightarrow$ ***(3) has no roots*** $\Rightarrow$ ***no real solution.***

Case 1b) $x>0$ ***Then,***

$x^4+4x^3+6x^2-4x+1=x^4+4x^3+2x^2+(2x-1)^2>0\Rightarrow$ ***(3) has not roots*** $\Rightarrow$ ***no real solution***

Case 2) $y=x$

(2) $\Rightarrow x^4-4x^3+6x^2-4x+1=0\Rightarrow(x-1)^4=0\Rightarrow x=1\Rightarrow x=y=1$

& $\because xyz>0\therefore z$ ***must be*** >0 ***(as*** $xy=1>0$***)***

$\therefore z^2=x^2\Rightarrow z^2=1\Rightarrow z=1\ (\because z>0)$

$\therefore x=y=z=1$

$\therefore$ ***only possible solution is:*** $x=y=z=1$***.***

Solution 2 by Tran Hong-Vietnam

We have: $xyz>0$***. Must show that:*** $\left(\frac{xy}{z}\right)^4+\left(\frac{yz}{x}\right)^4+\left(\frac{zx}{y}\right)^4\geq xyz\sqrt[4]{27(x^4+y^4+z^4)}$

$$a=\frac{xy}{z};b=\frac{yz}{x};c=\frac{zx}{y}\Rightarrow abc=xyz>0;x^2=ac>0;y^2=ab>0;z^2=bc>0;$$

$$\Rightarrow a,b,c>0$$

$$a^4+b^4+c^4\geq abc\sqrt[4]{27(a^2c^2+a^2b^2+b^2c^2)}$$

$$\Leftrightarrow(a^4+b^4+c^4)^4\geq 27(abc)^4(a^2b^2+b^2c^2+c^2a^2)\quad (*)$$

(*) true because: $\therefore(a^4+b^4+c^4)^3\overset{(Cauchy)}{\geq}\left\{3\sqrt[3]{(abc)^4}\right\}^3=27(abc)^4$

$$\therefore a^4+b^4+c^4\geq a^2b^2+b^2c^2+c^2a^2$$

$\Rightarrow$ ***Equality*** $\Leftrightarrow a=b=c\Rightarrow(x,y,z)\in\{(a,a,a);(-a,-a,a);(a,-a,-a);(-a,a,-a)\}$

$$\therefore a^4-4a^3+6a^2-4a+1=0\ (a>0)\Leftrightarrow(a-1)^4=0\Leftrightarrow a=1$$

$$\Rightarrow(x,y,z)\in\{(1,1,1)\}$$

599. Solve for real numbers:

$$\begin{cases}\dfrac{\tan^{-1}x}{\cot^{-1}x}=e^{\frac{4}{\pi}(\tan^{-1}y-\cot^{-1}y)}\\ \dfrac{\tan^{-1}y}{\cot^{-1}y}=e^{\frac{4}{\pi}(\tan^{-1}x-\cot^{-1}x)}\end{cases}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Solution by Amit Dutta-Jamshedpur-India

$$\frac{\tan^{-1}x}{\cot^{-1}x}=e^{\frac{4}{\pi}(\tan^{-1}y-\cot^{-1}y)}\quad \textbf{(1)}$$

$$\frac{\tan^{-1}y}{\cot^{-1}y}=e^{\frac{4}{\pi}(\tan^{-1}x-\cot^{-1}x)}\quad \textbf{(2)}$$

Equality (1)$\Rightarrow \frac{\tan^{-1}x}{\cot^{-1}x}=e^{\frac{4}{\pi}(\tan^{-1}y-\cot^{-1}y)}$

RHS$>0\Rightarrow LHS=\frac{\tan^{-1}x}{\cot^{-1}x}>0\Rightarrow \tan^{-1}x>0\Rightarrow x>0$

Similarly, from equality (2) $\Rightarrow y>0$*. So,* $x,y>0$

Taking logarithm on both sides of equation (1)

$$\log(\tan^{-1}x)-\log(\cot^{-1}x)=\frac{4}{\pi}(\tan^{-1}y-\cot^{-1}y)\quad \textbf{(3)}$$

Taking log on both sides of equation (2)

$$\log(\tan^{-1}y)-\log(\cot^{-1}y)=\frac{4}{\pi}(\tan^{-1}x-\cot^{-1}x)$$

or $\frac{4}{\pi}(\tan^{-1}x-\cot^{-1}x)=\log(\tan^{-1}y)-\log(\cot^{-1}y)$ ***(4)***

Equality (3)+(4): $\log(\tan^{-1}x)-\log(\cot^{-1}x)+\frac{4}{\pi}(\tan^{-1}x-\cot^{-1}x)=$

$$=\log(\tan^{-1}y)-\log(\cot^{-1}y)+\frac{4}{\pi}(\tan^{-1}y-\cot^{-1}y)$$

Now, let $F(t)=\log(\tan^{-1}t)-\log(\cot^{-1}t)+\frac{4}{\pi}(\tan^{-1}t-\cot^{-1}t)$

$\Rightarrow$ *Equation (3)+(4)*$\Rightarrow F(x)=F(y)$*. Now, differentiation* $F(t)$ *w.r.t* t

$$F'(t)=\frac{1}{\tan^{-1}t\,(1+t^2)}+\frac{1}{\cot^{-1}t\,(1+t^2)}+\frac{4}{\pi}\left(\frac{1}{1+t^2}+\frac{1}{1+t^2}\right)$$

$F'(t)=\frac{1}{(1+t^2)}\left\{\frac{1}{\tan^{-1}t}+\frac{1}{\cot^{-1}t}\right\}+\frac{8}{\pi(1+t^2)}$*. Using power mean inequality*

$$\frac{(\tan^{-1}t)^{-1}+(\cot^{-1}t)^{-1}}{2}\geq\left(\frac{\tan^{-1}t+\cot^{-1}t}{2}\right)^{-1}$$

$$\geq\left(\frac{\pi}{4}\right)^{-1}\quad\left\{\tan^{-1}t+\cot^{-1}t=\frac{\pi}{2}\right\}\Rightarrow\frac{1}{\tan^{-1}t}+\frac{1}{\cot^{-1}t}\geq\frac{8}{\pi}\quad \textbf{(5)}$$

$\Rightarrow F'(t)\geq\frac{8}{\pi(1+t^2)}+\frac{8}{\pi(1+t^2)}\geq\frac{16}{\pi(1+t^2)}>0\Rightarrow F(t)$ *is strictly increasing function*

But $F(x)=F(y)\Rightarrow x=y$*. Equation (1)*$\Rightarrow\frac{\tan^{-1}x}{\cot^{-1}x}=e^{\frac{4}{\pi}(\tan^{-1}x-\cot^{-1}x)}\ \{\because x=y\}$

Taking log on both sides: $\ln(\tan^{-1}x)-\ln(\cot^{-1}x)=\frac{4}{\pi}(\tan^{-1}x-\cot^{-1}x)$

Let $G(x)=\ln(\tan^{-1}x)-\ln(\cot^{-1}x)-\frac{4}{\pi}(\tan^{-1}x-\cot^{-1}x)$

$$G'(x)=\frac{1}{\tan^{-1}x\,(1+x^2)}+\frac{1}{\cot^{-1}x\,(1+x^2)}-\frac{4}{\pi}\left\{\frac{1}{1+x^2}+\frac{1}{1+x^2}\right\}$$

$$G'(x)=\frac{1}{(1+x^2)}\left\{\frac{1}{\tan^{-1}x}+\frac{1}{\cot^{-1}x}-\frac{8}{\pi}\right\}$$

From (V) $\Rightarrow\frac{1}{\tan^{-1}x}+\frac{1}{\cot^{-1}x}\geq\frac{8}{\pi}\Rightarrow G'(x)\geq 0\Rightarrow G(x)$ *is an increasing function*

So, $G(x)=0$ *can have only one real root. Also, we can see that real root exists only when* $\tan^{-1}x=\cot^{-1}x\Rightarrow x=1\Rightarrow x=1$ *is the only possible real root.*

600. Solve the following system:

$$\begin{cases}x^e+y^e+z^e+u^e=\dfrac{56}{12\pi}\\(xy)^e+(xz)^e+(xu)^e+(yz)^e+(yu)^e+(zt)^e=\dfrac{89}{12\pi^2}\\(xyz)^e+(xyu)^e+(xzu)^e+(yzu)^e=\dfrac{56}{12\pi^3}\\(xyzu)^e=\dfrac{1}{\pi^4}\end{cases}$$

Proposed by Jhoaw Carlos-La Paz-Bolivia

Solution by Amit Dutta-Jamshedpur-India

Let $\pi x^e=a,\pi y^e=b,\pi z^e=c,\pi u^e=d$. *This system of equations reduces to*

$\begin{cases}\sum a=\frac{56}{12}\\\sum ab=\frac{89}{12}\\\sum abc=\frac{56}{12}\\abcd=1\end{cases}$. *Let us create a biquadratic equation in which* a,b,c,d *are its roots.*

$$t^4-\left(\frac{56}{12}\right)t^3+\left(\frac{89}{12}\right)t^2-\left(\frac{56}{12}\right)t+1=0;\ 12t^4-56t^3+89t^2-56t+12=0$$

Dividing throughout by t^2 $(\because t\neq 0)\Rightarrow 12t^2-56t+89-\frac{56}{t}+\frac{12}{t^2}=0$

$$\Rightarrow 12\left(t^2+\frac{1}{t^2}\right)-56\left(t+\frac{1}{t}\right)+89=0\Rightarrow 12\left[\left(t+\frac{1}{t}\right)^2-2\right]-56\left(t+\frac{1}{t}\right)+89=0$$

$$\Rightarrow 12\left(t+\frac{1}{t}\right)^2 - 56\left(t+\frac{1}{t}\right) + 65 = 0 \text{ Let } k = \left(t+\frac{1}{t}\right) \Rightarrow 12k^2 - 56k + 65 = 0$$

$$k = \frac{5}{2}, \frac{13}{6};\ t + \frac{1}{t} = \frac{5}{2},\ \ 2t^2 - 5t + 2 = 0;\ t = 2, \frac{1}{2} \text{ Or } t + \frac{1}{t} = \frac{13}{6};\ 6t^2 - 13t + 6 = 0$$

$$t = \frac{3}{2}, \frac{2}{3} \Rightarrow t = 2, \frac{1}{2}, \frac{3}{2}, \frac{2}{3} \Bigg| \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}$$

$$a = \pi x^e = 2 \Rightarrow x = \left(\frac{2}{\pi}\right)^{\frac{1}{e}};\ b = \pi y^2 = \frac{1}{2} \Rightarrow y = \left(\frac{1}{2\pi}\right)^{\frac{1}{e}};\ c = \pi z^e = \frac{3}{2} \Rightarrow z = \left(\frac{3}{2\pi}\right)^{\frac{1}{e}}$$

$$d = \pi u^e = \frac{2}{3} \Rightarrow u = \left(\frac{2}{3\pi}\right)^{\frac{1}{e}}$$

$$(a, b, c, d) \equiv (\pi x^e, \pi y^e, \pi z^e, \pi u^e) \equiv \left(2, \frac{1}{2}, \frac{3}{2}, \frac{2}{3}\right)$$

$$\begin{cases} x = \left(\frac{2}{\pi}\right)^{\frac{1}{e}} \\ y = \left(\frac{1}{2\pi}\right)^{\frac{1}{e}} \\ z = \left(\frac{3}{2\pi}\right)^{\frac{1}{e}} \\ u = \left(\frac{2}{3\pi}\right)^{\frac{1}{e}} \end{cases}$$

Note: Since (x, y, z, u) are symmetric in the given problem, so any combination of the above set is possible for (x, y, z, u).

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru