Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before March 15, 2018

- **5475:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive integers a, b, c and d such that $\begin{cases} a+b = 14\sqrt{ab-48}, \\ b+c = 14\sqrt{bc-48}, \\ c+d = 14\sqrt{cd-48}, \end{cases}$

  with $a < b < c < d$. Express the values of $b, c,$ and d in terms of a.

- **5476:** *Proposed by Ed Gray, Highland Beach, FL*

  Find all triangles with integer area and perimeter that are numerically equal.

- **5477:** *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Sevrin, Meredinti, Romania*

  Compute:

  $$L = \lim_{n\to\infty}\left(\ln n + \lim_{x\to 0}\frac{1-\sqrt{1+x^2}\,\sqrt[3]{1+x^2}\cdot\ldots\cdot\sqrt[n]{1+x^2}}{x^2}\right).$$

- **5478:** *Proposed by D. M. Btinetu-Giurgiu, "Matei Basarab" National Collge, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzu, Romania*

  Compute:

  $$\int_0^{\pi/2}\cos^2 x\left(\sin x\sin^2\left(\frac{\pi}{2}\cos x\right)+\cos x\sin^2\left(\frac{\pi}{2}\sin x\right)\right)dx.$$

- **5479:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $f:[0,1]\to\Re$ be a continuous convex function. Prove that

  $$\frac{2}{5}\int_0^{1/3}f(t)dt+\frac{3}{10}\int_0^{2/3}f(t)dt \geq \frac{5}{8}\int_0^{8/15}f(t)dt.$$

- **5480:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n \geq 1$ be a nonnegative integer. Prove that in $C[0, 2\pi]$

$$span\{1, \sin x, \sin(2x), \ldots, \sin(nx)\} = span\{1, \sin x, \sin^2 x, \ldots, \sin^n x\}$$

if and only if $n = 1$.

We mention that $span\{v_1, v_2, \ldots, v_k\} = \sum_{j=1}^{k} a_j v_j,\ a_j \in \Re, j = 1, \ldots, k$, denotes the set of all linear combinations with $v_1, v_2, \ldots, v_k$.

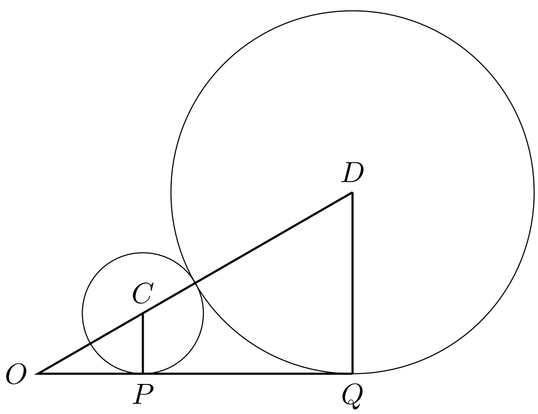
Solutions

- **5457:** *Proposed by Kenneth Korbin, New York, NY*

Given angle A with $\sin A = \dfrac{12}{13}$. A circle with radius 1 and a circle with radius x are each tangent to both sides of the angle. The circles are also tangent to each other. Find x.

Solution by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

The angle bisector passes through the centers C and D of the two circles, and the radii from the centers to the points of tangency P and Q of the circles with a side of the angle make right angles $\angle CPO$ and $\angle DQO$. Thus we have a pair of similar right triangles as follows.



Here $\angle DOQ = A/2$ and $|CD| = 1 + x$.

Suppose $x > 1$. Then $|CP| = 1$ and $|DQ| = x$. We have

$$\sin(A/2) = \frac{|CP|}{|CO|} = \frac{1}{|CO|}$$

so $|CO|\sin(A/2) = 1$. And

$$\sin(A/2) = \frac{|DQ|}{|DO|} = \frac{|DQ|}{|DC| + |CO|} = \frac{x}{1 + x + |CO|}$$

so

$$\sin(A/2) + x\sin(A/2) + |CO|\sin(A/2) = x.$$

Thus

$$x = \frac{1 + \sin(A/2)}{1 - \sin(A/2)}.$$

Now $\sin A = 12/13$ so $\cos A = \pm\sqrt{1 - \sin^2 A} = \pm 5/13$, and thus

$$\sin(A/2) = \sqrt{\frac{1 - \cos(A)}{2}} = 2\sqrt{13}/13 \text{ or } 3\sqrt{13}/13.$$

Therefore

$$x = \frac{13 + 2\sqrt{13}}{13 - 2\sqrt{13}} \approx 3.491 \text{ or } x = \frac{13 + 3\sqrt{13}}{13 - 3\sqrt{13}} \approx 10.908.$$

If $x < 1$ then scale the plane by $1/x$ and appeal to the last paragraph. This gives two more values of x:

$$x = \frac{13 - 2\sqrt{13}}{13 + 2\sqrt{13}} \approx 0.286 \text{ or } x = \frac{13 - 3\sqrt{13}}{13 + 3\sqrt{13}} \approx 0.092.$$

Editor's Comment : **David Stone and John Hawkins of Georgia Southern University, Statesboro, GA** generalized the problem. First, they proved the following lemma:

Let A be an angle, $0 < A < \pi$. Suppose a circle C_1 of radius $r = 1$ is inscribed in A and a *larger* circle C_2 of radius $R = x$ is also inscribed in A, with C_2 tangent to C_1. Then

$$x = \frac{1 + \sin\alpha}{1 - \sin\alpha}, \ \alpha = \frac{1}{2}A.$$

They proved this lemma by showing that it held when angle A is acute and also obtuse. Then they magnified the entire figure by a factor of r, so that the smaller circle C_1 has a radius of r and the larger circle C_2 has a radius of $R = rx$, and this allowed them to generalize the lemma: Let A be an angle, $0 < A < \pi$. Suppose that two circles, circle C_1 of radius r and C_2 of radius R are also inscribed in A, with C_2 tangent to C_1. Then

$$R = \frac{1 + \sin\alpha}{1 - \sin\alpha} r, \ \alpha = \frac{1}{2}A.$$

They went on to say that "with the result stated in this way, we see the co-dependency between r and R – if we know one we know the other." Applying the lemma they went on to solve the problem.

In conclusion they stated the following:

We can apply this result in several interesting ways. For example, as a Corollary, if $A = 60°$, then $\sin\alpha = \sin 30° = \frac{1}{2}$. Let circle C_0 of radius 1 be inscribed in A. Then we have a larger inscribed circle C_1 tangent to C_0 which has radius

$$R_1 = \frac{1 + \sin\alpha}{1 - \sin\alpha} \cdot 1 = \frac{1 + 1/2}{1 - 1/2} = 3.$$

And in continuing on in this manner we have a larger inscribed circle C_2 tangent to C_1 which has radius

$$R_2 = \frac{1+\sin\alpha}{1-\sin\alpha} \cdot 3 = 3^2.$$

There is an infinite sequence of expanding inscribed pairwise tangent circles having radii $R_n = 3^n,\ n \geq 0$.

Not to be outdone, we have a smaller inscribed circle C_{-1}, tangent to C_0 which has radius $R_{-1} = \dfrac{1-\sin\alpha}{1+\sin\alpha} \cdot 1 = \dfrac{1}{3}$

Continuing, there is an infinite sequence of shrinking inscribed pairwise tangent circles of radii $R_{-n} = \dfrac{1}{3^n}, n \geq 0$.

We could carry out this construction for any angle A, (but the numbers won't work out so nicely.)

In summary they stated: Given values for R and r, we can solve the Corollary equation for α and then find A. That is, given tangent circles of radii r and R, with $r < R$, we can compute the angle A which will "circumscribe" the circles: $A = 2\sin^{-1}\left(\dfrac{R-r}{R+r}\right)$.

Also solved by Arkady Alt, San Jose, CA; Charles Burnette, Academia Sinica, Taipei, Taiwan; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; David A. Huckaby, Angelo State University, San Angelo TX; Kee-Wai, Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Vijaya Prasad, Nalluri, India; Trey Smith, Angelo State University, San Angelo, TX; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5458:** *Proposed by Michal Kremzer, Gliwice, Silesia, Poland*

Find two pairs of integers (a,b) from the set $\{1,2,3,4,5,6,7,8,9\}$ such that for all positive integers n, the number

$$c = 537aaa\underbrace{b\ldots b}18403$$

is composite, where there are $2n$ numbers b between a and 1 in the string above.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Note that

$$\begin{aligned} c &= 18403 + b\cdot 10^5 \cdot \underbrace{1...1}_{2n \text{ times}} + a\cdot 10^{2n+5}\cdot 111 + 537\cdot 10^{2n+8} \\ &= 18403 + b00000 \cdot \underbrace{1...1}_{2n \text{ times}} + 10^{2n+5}\cdot 537aaa. \end{aligned}$$

Thus, if $(a,b) \in \{(2,7),(9,7)\}$, since $18403 \equiv_7 0,\ 700000 \equiv_7 0,\ 537222 \equiv_7 0$, and $537999 \equiv_7 0$, where $\equiv_7$ denotes congruence modulo 7, then

$$c \equiv_7 0 + 0\cdot 1\ \underbrace{1...1}_{2n \text{ times}} + 10^{2n+5}\cdot 0 \equiv_7 0,$$

so c is divisible by 7 and, hence composite.

Solution 2 by Ed Gray, Highland Beach, FL

The two pairs which guarantee that $c = 537aaa\underbrace{bbbbbb\ldots bb}_{2n \text{ times}}18403$ is always composite are: $a = 2, b = 7$ and $a = 9, b = 7$. We will show that with these integers, c is always divisible by 7.

A test for divisibility by 7 is as follows: double the last digit and subtract it from the remaining truncated number. If the result is divisible by 7, then so was the original number. As a simple example, consider the number 826. Double the last digit which gives 12, and subtract it from the leading truncated number, which is 82. Then $82 - 12 = 70$, which is divisible by 7, so 826 is divisible by 7.

Now consider our number. It's last digit is 3, and we double it to get 6. Subtracting 6 from the "truncated" number, we have $537aaa\underbrace{bbbbbb\ldots bb}_{2n \text{ times}}1834$.

We note that 1834 is divisible by 7; that if we let $b = 7$, every b will be divisible by 7. It remains to find 2 values for a such that $537aaa$ is divisible by 7. If $a = 2$, we have the number $537222 = 7 \cdot 76746$, and if $a = 9$, we have the number $537999 = 7 \cdot 76857$. This concludes the proof.

Solution 3 by David E. Manes, Oneonta, NY

Two pairs of integers (a, b) that satisfy the problem are $(2, 7)$ and $(9, b)$ where b is any nonnegative integer. For the pair $(2, 7)$, the integer c is always divisible by the prime 7 and for the pair $(9, b)$, c is always divisible by 11.

Given: N is a positive integer and $N = a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0$. Then

$$N \equiv (100a_2 + 10a_1 + a_0) - (100a_5 + 10a_4 + a_3) + (100a_8 + 10a_7 + a_6) - \cdots \pmod 7.$$

For this case, N is divisible by 7 if and only if $N \equiv 0 \pmod 7$. Moreover,

$$N \equiv (-1)^n a_n + (-1)^{n-1} a_{n-1} + \cdots - a_1 + a_0 \pmod{11}$$

and N is divisible by 11 if and only if $N \equiv 0 \pmod{11}$. Let n be a positive integer and define

$$C_n = 537aaab\ldots b18403$$

where the number of digits b is $2n$. If $a = 2$ and $b = 7$, then $C_1 = 5372227718403$, $C_2 = 537222777718403$ and $C_3 = 53722277777718403$. Therefore, modulo 7,

$$\begin{aligned}
C_1 &\equiv 403 - 718 + 227 - 372 + 5 \equiv 4 - 4 + 3 - 1 + 5 \equiv 0 \pmod 7,\\
C_2 &\equiv 403 - 718 + 777 - 222 + 537 \equiv 4 - 4 + 0 - 5 + 5 \equiv 0 \pmod 7,\\
C_3 &\equiv 403 - 718 + 777 - 277 + 722 - 53 \equiv 4 - 4 + 0 - 4 + 1 - 4 \equiv 0 \pmod 7.
\end{aligned}$$

Thus, C_1, C_2 and C_3 are all divisible by 7 and hence, each one is composite. Furthermore, $C_{3n+1} \equiv C_1 \pmod 7$, $C_{3n+2} \equiv C_2 \pmod 7$ and $C_{3n} \equiv C_3 \pmod 7$ for all positive integers n. Hence, if $a = 2$ and $b = 7$, then C_n is always composite since all of these integers are divisible by 7.

If $a = 9$ and b is any nonnegative integer, then the number of digits in C_n is always odd and

$$\begin{aligned} C_n &\equiv 5 - 3 + 7 - a + a - a + b - b + \cdots + b - b + 1 - 8 + 4 - 0 + 3 \\ &\equiv 9 - a \pmod{11}. \end{aligned}$$

Therefore, for all positive integers n, the prime 11 is a divisor of C_n if and only if $a = 9$ and the value of b is superfluous. Hence, C_n is always composite.

Solution 4 Anthony J. Bevelacqu, University of North Dakota, Grand Forks, ND

We have

$$c = 18403 + 10^5 \cdot b \cdot (\overbrace{1 \cdots 1}^{2n}) + 10^{5+2n} \cdot (a \cdot 111 + 10^3 \cdot 537).$$

Note that $c > 18403 = 7 \cdot 11 \cdot 239$.

Since $10 \equiv -1$ mod 11 we have $(\overbrace{1 \cdots 1}^{2n}) \equiv 0$ mod 11 for any n and so $c \equiv -(a+2)$ mod 11. Thus c will be divisible by 11 when $a = 9$ for *any* choice of the digit b and for any non-negative n.

Now if $b = 7$ we have $c \equiv 10^{5+2n}(6a+2)$ mod 7. Thus c will be divisible by 7 when $a = 2$ for any number of digits $b = 7$.

Therefore c will be composite when $(a, b) = (9, b)$ for any choice of the digit b and when $(a, b) = (2, 7)$.

Editor's comments : Most of the other solvers of this problem noticed that an even number of b digits forces the number formed by them alone, to be divisible by 11. Hence, they found the value $a = 9$ makes the number $537aaa18403$ divisible by 11, and so $(9, \text{any digit})$ solves the problem. The solutions listed above pick up another ordered pair. But then **The Honor Students at Ashland University in Ashland, Ohio** upped the ante by finding additional ordered pairs to (9, any other digit). Using MAPLE they found 6 pairs of values for (a, b) that satisfy the problem. They checked these values for all positive integers $n \le 25$. Letting $c = 537aaa\underbrace{b \ldots b}18403$ they found that:

(a, b)	c is divisible by
$(2, 7)$	7
$(4, 1)$	29
$(4, 5)$	13
$(6, 5)$	17
$(6, 9)$	59
$(7, 5)$	89

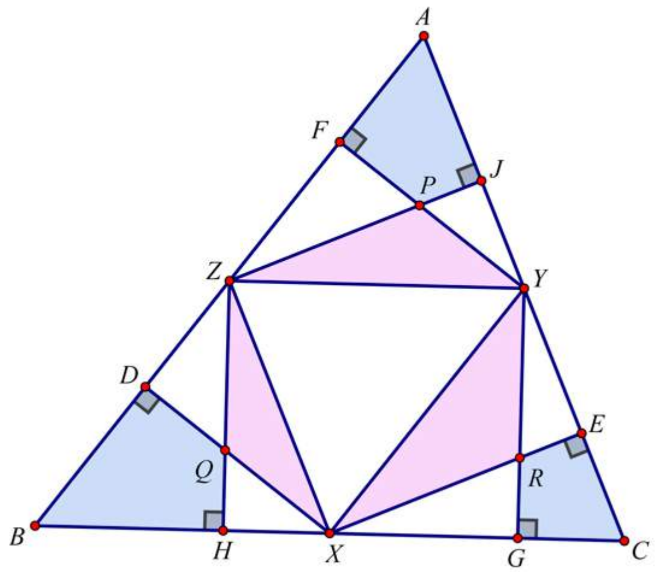
David Stone and John Hawkins of Georgia Southern University, Statesboro, GA found all of the solutions in the above table and an additional one $(4, 8)$, which is divisible by 13. They also found that if $b = 0$ were allowed, then $(2, 0)$ is divisible by 7, for all $n \ge 0$. With respect to the pair $(4, 8)$ they stated that it seemingly has a unique property. If $c_n = 537aaab....b18403$ as defined in the problem, then no single prime divides all c_n, but 7 divides all c_{3k}, 3 divides all c_{3k+1}, and and 13 divides all c_{3k+2}.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Pat Costello, Eastern Kentucky University, Richmond, KY; Kee-Wai Lau, Hong

Kong, China; Zachary Morgan, student at Eastern Kentucky University, Richmond, KY; Nathan Russell, Eastern Kentucky University, Richmond, KY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5459:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

Triangle ABC is an arbitrary acute triangle. Points X, Y, and Z are midpoints of three sides of $\triangle ABC$. Line segments XD and XE are perpendiculars drawn from point X to two of the sides of $\triangle ABC$. Line segments YF and YG are perpendiculars drawn from point Y to two of the sides of $\triangle ABC$. Line segments ZJ and ZH are perpendiculars drawn from point Z to two of the sides of $\triangle ABC$. Moreover, $P = ZJ \cap FY$, $Q = ZH \cap DX$, and $R = YG \cap XE$. Three of the triangles, and three of the quadrilaterals in the figure are shaded. If the sum of the areas of the three shaded triangles is 5, find the sum of the areas of the three shaded quadrilaterals.



Solution 1 by David A. Huckaby, Angelo State University, San Angelo, TX

Let a be the area of triangle ABC. Since Y and Z are the midpoints of AC and AB, respectively, $\triangle AYZ$ is similar to $\triangle ACB$ with a scale factor of 2, so that the area of $\triangle AYZ$ is $\frac{1}{4}a$. Similarly, the areas of $\triangle BXZ$ and $\triangle CXY$ are each $\frac{1}{4}a$, and therefore the area of $\triangle XYZ$ is also $\frac{1}{4}a$.

The area of rectangle $GYZH$ is $\frac{1}{2}a$, since it has the same height and base as $\triangle XYZ$. Similarly, the areas of rectangles $FYXD$ and $EXZJ$ are each $\frac{1}{2}a$.

Consider the sum of the areas of these three rectangles:

$$\begin{aligned}\text{area of three rectangles} &= \text{area of six outer white triangles}\\ &\quad +2(\text{area of three pink triangles}) + 3(\text{area of } \triangle XYZ),\end{aligned}$$

that is,

$$3(\frac{1}{2}a) = \text{area of six outer white triangles} + 2(5) + 3(\frac{1}{4}a),$$

so that the sum of the areas of the six outer white triangles is $\frac{3}{4}a - 10$.

Now consider the sum of the areas of triangles AYZ, BXZ, and CXY:

$$\begin{aligned}\text{area of triangles } AYZ,\ BXZ, \text{ and } CXY &= \text{area of six outer white triangles}\\ &\quad +\text{area of three pink triangles} + \text{area of three blue quadrilaterals},\end{aligned}$$

that is,

$$3(\frac{1}{4}a) = \left[\frac{3}{4}a - 10\right] + 5 + \text{area of three blue quadrilaterals},$$

so that the sum of the areas of the three blue quadrilaterals is 5.

Solution 2 by Andrea Fanchini, Cantú, Italy

We use barycentric coordinates and the usual Conway's notations with reference to the triangle ABC. Then we have $X(0:1:1)$, $\quad Y(1:0:1)$, $\quad Z(1:1:0)$.

• *Coordinates of points* D, E, F, G, H, J.

Line segments XD and XE perpendiculars drawn from point X to two of the sides of $\triangle ABC$ are

$$XAB_{\infty\perp} : (c^2+S_A)x - S_By + S_Bz = 0, \quad XAC_{\infty\perp} : (b^2+S_A)x + S_Cy - S_Cz = 0$$

therefore the points D, E have coordinates

$$D = XAB_{\infty\perp} \cap AB = (S_B : c^2+S_A : 0), \quad E = XAC_{\infty\perp} \cap AC = (S_C : 0 : b^2+S_A)$$

then cyclically

$$G = (0 : S_C : a^2{+}S_B), \quad J = (b^2{+}S_C : 0 : S_A), \quad F = (c^2{+}S_B : S_A : 0), \quad H = (0 : a^2{+}S_C : S_B)$$

• *Coordinates of point* P, Q, R.

Coordinates of point P are

$$P = ZAC_{\infty\perp} \cap YAB_{\infty\perp} = (2S^2 - a^2S_A : S_AS_C : S_AS_B)$$

then cyclically

$$Q = (S_BS_C : 2S^2 - b^2S_B : S_AS_B), \qquad R = (S_BS_C : S_AS_C : 2S^2 - c^2S_C)$$

• *Areas of the three shaded triangles.*

Areas of the three shaded triangles are

$$[PZY] = \frac{S_BS_C}{4S^2}[ABC], \qquad [QZX] = \frac{S_AS_C}{4S^2}[ABC], \qquad [RXY] = \frac{S_AS_B}{4S^2}[ABC]$$

If the sum of the areas of the three shaded triangles is 5, we have

$$[PZY] + [QZX] + [RXY] = \frac{[ABC]}{4}, \quad \Rightarrow \quad [ABC] = 20$$

• *Areas of the three shaded quadrilaterals.*

Area of the quadrilateral $[AFPJ]$ is given from $[AFJ] + [PFJ]$ so

$$[AFJ] = \frac{S_A^2}{4b^2c^2}[ABC], \qquad [PFJ] = \frac{S_A^2S_BS_C}{4b^2c^2S^2}[ABC], \quad \Rightarrow \quad [AFPJ] = \frac{S_A^2(S^2+S_BS_C)}{4b^2c^2S^2}[ABC]$$

then cyclically

$$[BDQH] = \frac{S_B^2(S^2+S_AS_C)}{4a^2c^2S^2}[ABC], \qquad [CERG] = \frac{S_C^2(S^2+S_AS_B)}{4a^2b^2S^2}[ABC]$$

therefore

$$[AFPJ]{+}[BDQH]{+}[CERG] = \frac{a^2S_A^2(S^2+S_BS_C) + b^2S_B^2(S^2+S_AS_C) + c^2S_C^2(S^2+S_AS_B)}{4a^2b^2c^2S^2}[ABC]$$

but $[ABC] = 20$ then

$$[AFPJ]{+}[BDQH]{+}[CERG] = 5\frac{S^2(a^2S_A^2+b^2S_B^2+c^2S_C^2) + S_AS_BS_C(a^2S_A+b^2S_B+c^2S_C)}{a^2b^2c^2S^2}$$

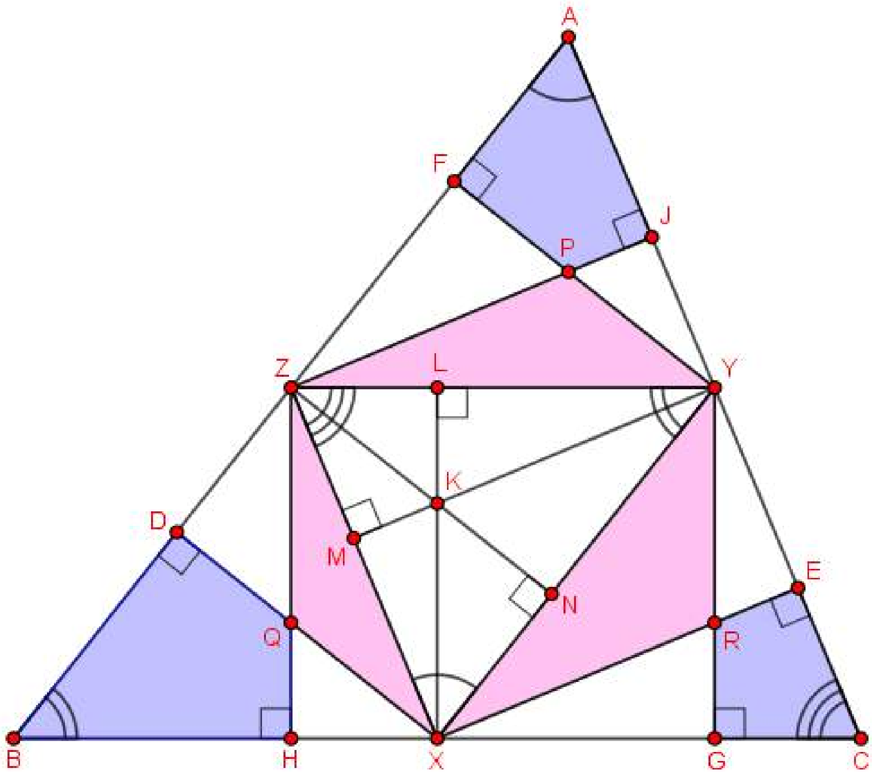
now $a^2S_A + b^2S_B + c^2S_C = 2S^2$ then

$$[AFPJ] + [BDQH] + [CERG] = 5\frac{a^2S_A^2 + b^2S_B^2 + c^2S_C^2 + 2S_AS_BS_C}{a^2b^2c^2}$$

finally $a^2S_A^2 + b^2S_B^2 + c^2S_C^2 + 2S_AS_BS_C = a^2b^2c^2$ so we have that also

$$[AFPJ] + [BDQH] + [CERG] = 5.$$

Solution 3 by Nikos Kalapodis, Patras, Greece



We denote with $[S]$ the area of shape S.
Let XL, YM and ZN be the heights of triangle XYZ and K its orthocenter.
Then the quadrilaterals $PZKY$, $QXKZ$ and $RYKX$ are parallelograms.
It follows that $[PZY] = [KZY]$, $[QZX] = [KZX]$, and $[RXY] = [KXY]$.
Therefore $[PZY] + [QZX] + [RXY] = [KZY] + [KZX] + [KXY] = [XYZ]$ (1).
Furthermore since the triangles AZY, BXZ, CYX and XYZ are congruent with orthocenters P, Q, R and K respectively, it easily follows that $[AFPJ] = [XNKM]$, $[BHQD] = [YLKN]$ and $[CERG] = [ZMKL]$.
Therefore
$[AFPJ] + [BHQD] + [CERG] = [XNKM] + [YLKN] + [ZMKL] = [XYZ]$ (2).
From (1) and (2) we get that
$[AFPZ] + [BHQD] + [CERG] = [PZY] + [QZX] + [RXY] = 5$.

Solution 4 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Triangles ΔAZY, ΔZBX and ΔYXC are equal. Also the sum of the areas of the saded triangles is equal to the area of for example ΔAZY. Also the sum of the areas of the three shaded quadrilaterals is equal to the area of one of the triangles, for example ΔAZY. Therefore, the requested sum is equal to 5.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai, Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Sachit Misra, Nelhi, India; Neculai Stanciu,"George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA and the proposer.

- **5460:** *Proposed by Ángle Plaza,Universidad de Las Palmas de Gran Canaria, Spain*

If $a, b > 0$ and $x, y > 0$ then prove that

$$\frac{a^3}{ax+by} + \frac{b^3}{bx+ay} \geq \frac{a^2+b^2}{x+y}.$$

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo,TX

Since $a, b, x, y > 0$, we have

$$\begin{aligned}
&a^3 (bx+ay)(x+y) + b^3 (ax+by)(x+y) - \left(a^2+b^2\right)(ax+by)(bx+ay) \\
&= a^2 (bx+ay)\left[a(x+y) - (ax+by)\right] + b^2 (ax+by)\left[b(x+y) - (bx+ay)\right] \\
&= a^2 (a-b)\, y\, (bx+ay) + b^2 (b-a)\, y\, (ax+by) \\
&= (a-b)\, y \left[a^2 (bx+ay) - b^2 (ax+by)\right] \\
&= (a-b)\, y \left[ab(a-b)x + \left(a^3-b^3\right) y\right] \\
&= (a-b)^2\, y \left[abx + \left(a^2+ab+b^2\right) y\right] \\
&\geq 0, \qquad (1)
\end{aligned}$$

with equality if and only if $a = b$.

Since $a, b, x, y > 0$, we need only to divide (1) by the positive quantity $(ax+by)(bx+ay)(x+y)$ and to re-arrange terms to obtain the desired inequality. Further, equality is attained if and only if $a = b$.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, NY

Using the Engel form of the Cauchy-Schwarz inequality (or Bergström's inequality) and the AGM inequality, we see that

$$\begin{aligned}
\frac{a^3}{ax+by} + \frac{b^3}{bx+ay} &= \frac{a^4}{a^2x+aby} + \frac{b^4}{b^2x+aby} \\
&\geq \frac{(a^2+b^2)^2}{(a^2+b^2)x+2aby} \\
&\geq \frac{(a^2+b^2)^2}{(a^2+b^2)x+(a^2+b^2)y} = \frac{a^2+b^2}{x+y}.
\end{aligned}$$

Solution 3 by Anna Valkova Tomova, Varna, Bulgaria

We move the expression to the left of the right side of the inequality. Now we have to prove that the new left part is non-negative. Again we will use the capabilities of the mathematical site <http://www.worframalpha.com> to examine the transformed look of this new left-hand side of the inequality.

$$\begin{aligned} & \frac{a^3}{ax+by}+\frac{b^3}{bx+ay}-\frac{a^2+b^2}{x+b} \\ = \quad & \frac{y(a^4y+a^3bx-a^3by-2a^2b^2x+ab^3x-ab^2y+b^4y)}{(x+y)(ay+bx)(ax+by)}. \\ = \quad & \frac{y(a-b)^2(a^2y+abx+aby+b^2y)}{(x+y)ay+bx)(ax+by)}. \end{aligned}$$

Since the numbers involved in the expression are conditionally positive, we have proved the inequality because the equivalent expression is positive too.

Conclusion: The application of information technology enhances the quality of education in mathematics in all of its stages of study. Of course, it should be checked at every stage so as not to allow ridiculous errors. In this sense, "E-Mathematics" does not replace the classic, it continues development with new, more efficient vehicles.

Editor's Comments : **Brian Bradie of Christopher Newport University in Newport News VA** stated that this problem is a generalization of two inequalities that appeared in Problem B-1201 in the February 2017 issue of the Fibonacci Quarterly:

$$\frac{a^3}{aF_n+bF_{n+1}}+\frac{b^3}{bF_n+aF_{n+1}}\geq\frac{a^2+b^2}{F_{n+2}}$$

$$\frac{a^3}{aL_n+bL_{n+1}}+\frac{b^3}{bL_n+aL_{n+1}}\geq\frac{a^2+b^2}{L_{n+2}}$$

Three other generalizations of this problem were made by **D.M.Bătinetu-Giurgiu of the "Matei Basarab" National College in Bucharest, Romania.**

1. A generalization with "two variables:"

If $m\geq 0$ and $a,b,x,y>0$, then $\dfrac{a^{m+2}}{(ax+by)^m}+\dfrac{b^{m+2}}{(bx+ay)^m}\geq\dfrac{a^2+b^2}{(x+y)^m}$.

Proof:

$$\frac{a^{m+2}}{(ax+by)^m}+\frac{b^{m+2}}{(bx+ay)^m}=\frac{a^{2m+2}}{(a^2x+aby)^m}+\frac{b^{2m+2}}{(b^2x+aby)^m}$$

$$=\frac{(a^2)^{m+1}}{(a^2x+aby)^m}+\frac{(b^2)^{m+1}}{(b^2x+aby)^m}\overbrace{\geq}^{J.Radon}\frac{(a^2+b^2)^{m+1}}{(a^2+b^2)x+2aby)^m}\overbrace{\geq}^{AM\geq GM}\frac{(a^2+b^2)^{m+1}}{((a^2+b^2)x+(a^2+b^2)y)^m}$$

$$= \frac{(a^2+b^2)^{m+1}}{(a^2+b^2)^m(x+y)^m} = \frac{a^2+b^2}{(x+y)^m}. \quad \text{Q.E.D.}$$

Corollary 1. If $m = 1$, then we obtain the problem 5460.

2. A generalization with "three variables:"

If $m \geq 0$ and $a, b, c, x, y, z > 0$, then

$$\frac{a^{m+2}}{(ax+by+cz)^m} + \frac{b^{m+2}}{(bx+cy+az)^m} + \frac{c^{m+2}}{(cx+ay+bz)^m} \geq \frac{a^2+b^2+c^2}{(x+y+z)^m}.$$

Proof:

$$\frac{a^{m+2}}{(ax+by+cz)^m} + \frac{b^{m+2}}{(bx+cy+az)^m} + \cdot\frac{c^{m+2}}{(cx+ay+bz)^m}$$

$$= \frac{(a^2)^{m+1}}{(a^2x+aby+acz)^m} + \frac{(b^2)^{m+1}}{(b^2x+bcy+abz)^m} + \frac{(c^2)^{m+1}}{(c^2x+acy+bcz)^m}$$

$$\overbrace{\geq}^{J.Radon} \frac{(a^2+b^2+c^2)^{m+1}}{((a^2+b^2+c^2)x+(ab+bc+ca)y+(bc+ca+ab)z)^m}$$

$$\geq \frac{(a^2+b^2+c^2)^{m+1}}{(a^2+b^2+c^2)^m(x+y+z)^m} = \frac{a^2+b^2+c^2}{(x+y+z)^m}, \quad \text{Q.E.D.}$$

In the last inequality we are utilizing the fact that $a^2+b^2+c^2 \geq ab+bc+ca$ where $a, b, c > 0$.

3. A generalization with "n variables:"

If $t, x, y, a_k > 0, n \in N, n \geq 2, n \in \{1, 2, \ldots, n\}$ such that
$t\sum_{t=1}^{n} a_k^2 \geq \sum_{k=1}^{n} a_k a_{k+1}$, $a_{n+1} = a_1$, then

$$\sum_{k=1}^{n} \frac{a_k^3}{xa_k + ya_{k+1}} \geq \frac{1}{x+ty}\sum_{k=1}^{n} a_k^2.$$

Proof:

$$\sum_{k=1}^{n} \frac{a_k^3}{xa_k+ya_{k+1}} = \sum_{k=1}^{n} \frac{(a_k^2)^2}{xa_k^2+ya_ka_{k+1}} \overbrace{\geq}^{Bergstrom} \frac{\left(\sum_{k=1}^{n} a_k^2\right)^2}{\sum_{k=1}^{n}(xa_k^2+ya_ka_{k+1})}$$

$$\frac{\left(\sum_{k=1}^{n} a_k^2\right)^2}{x\sum_{k=1}^{n} a_k^2 + y\sum_{k=1}^{n} a_k a_{k+1}} = \frac{\left(\sum_{k=1}^{n} a_k^2\right)^2}{x\sum_{k=1}^{n} a_k^2 + ty\sum_{k=1}^{n} a_k^2} = \frac{1}{x+ty}\sum_{k=1}^{n} a_k^2, \quad \text{Q.E.D.}$$

Also solved by Arkady Alt (3 solutions), San Jose, CA; Bruno Salgueiro Fanego Viveiro, Spain; D.M.Bătinetu-Giurgiu of the "Matei Basarab" National College in Bucharest, Romania; D.M.Bătinetu-Giurgiu of the "Matei Basarab" National College in Bucharest, Romania with Neculai Stanciu, "George Emil Palade" School, Buzău, Romania; Brian Bradie, Christopher Newport University, Newport News, VA; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai, Lau, Hong Kong, China; Nikos Kalapodis, Patras, Greece; Moti Levy, Rehovot, Israel; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu, "George Emil Palade" School in Buzau, Romanina with Titu Zvonaru of Comănesti, Romania; and the proposer.

- **5461:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Compute the following sum:

$$\sum_{n=1}^{\infty} \frac{\cos(2n-1)}{(2n-1)^2}.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport, VA

Consider the function $f(x) = \frac{\pi}{2} - x$ on the interval $[0,\pi]$. Because

$$\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right) dx = 0$$

and, for positive integer n,

$$\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)\cos nx\, dx = \begin{cases} \frac{4}{n^2\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

it follows that the Fourier cosine series for f is

$$\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}.$$

The function f is continuous at $x = 1$, so

$$f(1) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)}{(2n-1)^2};$$

therefore,

$$\sum_{n=1}^{\infty}\frac{\cos(2n-1)}{(2n-1)^2}=\frac{\pi}{4}f(1)=\frac{\pi}{4}\left(\frac{\pi}{2}-1\right).$$

Solution 2 by Ed Gray, Highland Beach, FL

Many of these infinite series can be solved by finding a function whose Fourier series expansion results in the given series. The series at hand reprsents an even function so suitable candidates are functions like $f(x)=x^2,\ f(x)=|x|$, etc. A perusal of some functions reveals that the function $f(x)=|x|,\ \pi<x<\pi$, seems just was we need.

The expression is:

1. $f(x)=\dfrac{\pi}{2}-\dfrac{4}{\pi}\displaystyle\sum_{k\geq 1,\text{odd}}^{\infty}\frac{\cos(kx)}{k^2}.$

Since the sum involves odd terms only, we let $k=2n-1$. Further, we eliminate x by letting $x=1$.(Since an even function, $x=-1$ would do just as well.) In either case, $f(1)=f(-1)=|1|=1$ and equation (1) becomes:

2. $1=\dfrac{\pi}{2}-\dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\cos(2n-1)}{(2n-1)^2}$, or

3. $\dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\cos(2n-1)}{(2n-1)^2}=\frac{\pi}{2}-1.$

Multiplying by $\dfrac{\pi}{4}$.

4. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(2n-1)}{(2n-1)^2}=\frac{\pi^2}{8}-\frac{\pi}{4}.$

Solution 3 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\cos(2n-1)}{(2n-1)^2} &= \frac{1}{2}\sum_{n=1}^{\infty}(e^{i(2n-1)}+e^{-i(2n-1)})\int_0^1 t^{2n-2}dt\int_0^1 u^{2n-2}du\\ &= \frac{e^{-i}}{2}\int_0^1\frac{dt}{t^2}\int_0^1\frac{du}{u^2}\sum_{n=1}^{\infty}(tue^{i})^{2k}+\frac{e^{i}}{2}\int_0^1\frac{dt}{t^2}\int_0^1\frac{du}{u^2}\sum_{n=1}^{\infty}(tue^{-i})^{2k}\\ &= \frac{e^{-i}}{2}\int_0^1 dt\int_0^1 du\frac{e^{2i}}{1-(tu)^2e^{2i}}+\frac{e^{i}}{2}\int_0^1 dt\int_0^1 du\frac{e^{-2i}}{1-(tu)^2e^{-2i}}.\end{aligned}$$

The change $x=tu$, $y=u$ yields

$$\frac{e^{i}}{2}\int_0^1\frac{dy}{y}\int_0^y dx\frac{1}{1-x^2e^{2i}}+\frac{e^{-i}}{2}\int_0^1\frac{dy}{y}\int_0^y dx\frac{1}{1-x^2e^{-2i}}$$

$$
\begin{aligned}
&= \frac{e^i}{4}\int_0^1 \frac{dy}{y}\int_0^y dx\left(\frac{1}{1-xe^i}+11+xe^i\right)+\frac{e^{-i}}{4}\int_0^1 \frac{dy}{y}\int_0^y dx\left(\frac{1}{1-xe^{-i}}+\frac{1}{1+xe^{-i}}\right)\\
&= \frac{1}{4}\int_0^1 \frac{dy}{y}\Big[\mathrm{Ln}(1-xe^i)\Big]\Big|_y^0+\frac{1}{4}\int_0^1 \frac{dy}{y}\Big[\mathrm{Ln}(1+xe^i)\Big]\Big|_0^y\\
&= \frac{1}{4}\int_0^1 \frac{dy}{y}\Big[\mathrm{Ln}(1-xe^{-i})\Big]\Big|_y^0+\frac{1}{4}\int_0^1 \frac{dy}{y}\Big[\mathrm{Ln}(1+xe^{-i})\Big]\Big|_0^y\\
&= \frac{-1}{4}\int_0^1 \frac{\mathrm{Ln}(1-ye^i)}{y}dy+\frac{1}{4}\int_0^1 \frac{\mathrm{Ln}(1+ye^i)}{y}dy+\frac{-1}{4}\int_0^1 \frac{\mathrm{Ln}(1-ye^{-i})}{y}dy+\frac{1}{4}\int_0^1 \frac{\mathrm{Ln}(1+ye^{-i})}{y}dy\\
&= \frac{-1}{4}\int_0^{e^i} \frac{\mathrm{Ln}(1-y)}{y}dy+\frac{1}{4}\int_0^{-e^i} \frac{\mathrm{Ln}(1-y)}{y}dy+\frac{-1}{4}\int_0^{e^{-i}} \frac{\mathrm{Ln}(1-y)}{y}dy+\frac{1}{4}\int_0^{-e^{-i}} \mathrm{Ln}(1-y)ydy\\
&= \frac{1}{4}\mathrm{Li}_2(e^i)-\frac{1}{4}\mathrm{Li}_2(-e^i)+\frac{1}{4}\mathrm{Li}_2(e^{-i})-d\frac{1}{4}\mathrm{Li}_2(-e^{-i}).
\end{aligned}
$$

The relation

$$\mathrm{Li}_2(\frac{1}{z})+\mathrm{Li}_2(z)=-\frac{\pi^2}{6}-\frac{(\mathrm{Ln}(-z))^2}{2}$$

gives

$$\frac{1}{4}\left(-\frac{\pi^2}{6}-\frac{(\mathrm{Ln}(-e^i))^2}{2}\right)-\frac{1}{4}\left(-\frac{\pi^2}{6}-\frac{(\mathrm{Ln}(e^i))^2}{2}\right)=-\frac{1}{8}(i(\pi-1))^2+\frac{i^2}{8}=\frac{\pi^2-2\pi}{8}.$$

Solution 4 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

A given 2π-periodic function f can be represented as by the convergent series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos(nx)+b_n\sin(nx)].$$

The convergence of the series means that the sequence $(s_n(x))$ of partial sums, defined by

$$s_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}[a_n\cos(kx)+b_n\sin(kx)],$$

converges at a given point x to $f(x)$, $s_n(x)\to f(x)$. Consider f to be a 2π-periodic function defined by $f(x)=|x|$ for $x\in[-\pi,\pi]$. Note that f is even. Since the product of even functions with the odd function is odd, it follows that $\int_{-n}^{n} f(x)\sin(nx)dx=0$.

Hence $b_n=0$ for all $n\geq 1$. To compute a_n note that the product of an even function with an even function is even, so that

$$a_n=\frac{1}{n}\int_{-n}^{n} f(x)\cos(nx)dx=\frac{2}{n}\int_0^{\pi} f(x)\cos(nx)dx=\frac{2}{n}\int_0^{\pi} x\cos(nx)dx.$$

If $n = 0$ then $a_0 = \dfrac{2}{n}\displaystyle\int_0^{\pi} x dx = \pi$. Hence $a_n = 0$ if n is even and $a_n = -\dfrac{4}{n^2\pi}$ when n is odd, and hence

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}.$$

Since the series $\sum \dfrac{1}{n^2}$ converges, the M−Weierstarss test implies that the above series converges. Furthermore, f is continuous at every point and it is smooth except at points $m\pi$, with m odd. Hence the Fourier series of f converges to f at every point. In particular,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2},$$

for $x \in [-\pi, \pi]$. Substituting $x = 0$, we find that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{n^2}{8},$$

and substituting $x = 1$, we find that

$$\sum_{n=1}^{\infty}\frac{\cos(2n-1)}{(2n-1)^2} = \frac{\pi}{4}\left(\frac{\pi}{2} - 1\right) \approx 0.448302.$$

Also solved by Bruno Salgueiro Fanego Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Henry Ricardo, Westchester Area Math Circle, NY; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova, Varna, Bulgaria, and the proposer.

- **5462:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n \geq 1$ be an integer. Calculate

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1 + \sqrt{\sin(2x)}\right)^n} \mathrm{d}x.$$

Solution 1 by Moti Levy, Rehovot, Israel

First simplification by setting $t = 4x - \pi$,

$$I_n := \int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1 + \sqrt{\sin 2x}\right)^n} dx = \frac{1}{4}\int_{-\pi}^{\pi} \frac{\cos\left(\frac{t}{4} + \frac{\pi}{4}\right)}{\left(1 + \sqrt{\cos\frac{t}{2}}\right)^n} dt = \frac{\sqrt{2}}{4}\int_0^{\pi} \frac{\cos\frac{t}{4}}{\left(1 + \sqrt{\cos\frac{t}{2}}\right)^n} dt$$

Further simplification by the change of variable $w = \frac{1-\sqrt{\cos\frac{t}{2}}}{1+\sqrt{\cos\frac{t}{2}}}$:

$$\cos\frac{t}{2} = \left(\frac{1-w}{1+w}\right)^2, \quad \cos\frac{t}{4} = \frac{\sqrt{2}}{2}\sqrt{1 + \left(\frac{1-w}{1+w}\right)^2}, \; \sin\frac{t}{2} = \sqrt{1 - \left(\frac{1-w}{1+w}\right)^4}$$

$$I_n = \frac{1}{2^n}\int_0^1 (1+w)^{n-2}\left(\frac{1}{\sqrt{w}} - \sqrt{w}\right) dw.$$

By the binomial theorem,

$$(1+w)^{n-2} = \sum_{k=0}^{n-2}\binom{n-2}{k} w^k, \quad n \geq 2$$

after interchanging integration and summation,

$$\begin{aligned}
I_n &= \frac{1}{2^n}\sum_{k=0}^{n-2}\binom{n-2}{k}\int_0^1 \left(w^{k-\frac{1}{2}} - w^{k+\frac{1}{2}}\right) dw \\
&= \frac{1}{2^n}\sum_{k=0}^{n-2}\binom{n-2}{k}\left(\frac{1}{k+\frac{1}{2}} - \frac{1}{k+\frac{3}{2}}\right) \\
&= \frac{1}{2^n}\sum_{k=0}^{n-2}\frac{\binom{n-2}{k}}{\left(k+\frac{1}{2}\right)\left(k+\frac{3}{2}\right)}, \qquad n \geq 2.
\end{aligned}$$

For $n = 1$,

$$\begin{aligned}
I_1 &= \frac{1}{2}\int_0^1 \frac{1-w}{1+w}\frac{1}{\sqrt{w}} dw = \int_0^1 \frac{1-x^2}{1+x^2} dx \\
&= \int_0^1 \left(\frac{2}{1+x^2} - 1\right) dx = \frac{\pi}{2} - 1.
\end{aligned}$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

The transformation $x \to \dfrac{\pi}{2} - x$ yields

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1+\sqrt{\sin(2x)}\right)^n} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\left(1+\sqrt{\sin(2x)}\right)^n} dx.$$

Therefore

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1+\sqrt{\sin(2x)}\right)^n} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\left(1+\sqrt{\sin(2x)}\right)^n} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sqrt{(\cos x + \sin x)^2}}{\left(1+\sqrt{\sin(2x)}\right)^n} dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sqrt{(1+\sin(2x)}}{\left(1+\sqrt{\sin(2x)}\right)^n} dx = \int_0^{\frac{\pi}{4}} \frac{\sqrt{(1+\sin(2x)}}{\left(1+\sqrt{\sin(2x)}\right)^n} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sqrt{1+\sin x}}{\left(1+\sqrt{\sin x}\right)^n} dx$$

$$\underbrace{=}_{y=\sqrt{\sin x}} \int_0^1 \frac{\sqrt{1+y^2}}{(1+y)^n} \cdot \frac{y}{\sqrt{1-y^4}} dy = \int_0^1 \frac{1}{(1+y)^n} \cdot \frac{y}{\sqrt{1-y^2}} dy$$

$$\underbrace{=}_{y=\frac{2z}{1+z^2}} \int_0^1 \frac{1}{1+\frac{2z}{(1+z^2)^n}} \cdot \frac{2z}{1+z^2} \cdot \frac{1+z^2}{1-z^2} \cdot \frac{2(1-z^2)}{(1+z^2)^2} dz = 4\int_0^1 \frac{z(1+z^2)^{n-2}}{(1+z)^{2n}} dz = \int_0^\infty \frac{z(1+z^2)^{n-2}}{(1+z)^{2n}} dz,$$

where we have used that $\displaystyle\int_0^1 \frac{z(1+z^2)^{n-2}}{(1+z)^n} dz = \int_0^\infty \frac{z(1+z^2)^{n-2}}{(1+z)^n} dz,$

as follows by performing the change of variables $z \to 1/z$.

Obviously, for $n = 0$, $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1+\sqrt{(2x)}\right)^0} dx = 1.$

For $n = 1$ we have

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1+\sqrt{(2x)}\right)} dx = 2\int_0^{\infty} \frac{z}{(1+z)^2(1+z^2)} dz = \int_0^{\infty} \left(\frac{1}{1+z^2} - \frac{1}{(1+z)^2}\right) dz = \frac{\pi}{2} - 1.$$

For $n \geq 2$ we use the Binomial Theorem to expand the integrand.

$$2\int_0^{\infty} \frac{z(1+z^2)^{n-2}}{(1+z)^{2n}} dz = 2\int_0^{\infty} \frac{z(1+2z+z^2-2z)^{n-2}}{(1+z)^{2n}} dz$$

$$=2\sum_{j=0}^{n-2} \binom{n-2}{j} (-2)^j \int_0^{\infty} \frac{z^{j+1}}{(1+z)^{2j+4}} dz = 2\sum_{j=0}^{n-2} \binom{n-2}{j} (-2)^j \frac{(j+1)!^2}{(2j+3)!}$$

$$= -(n-2)! \sum_{j=0}^{n-2} (-2)^{j+1}(j+1) \frac{(j+1)!}{(n-2-j)!(2j+3)!},$$

where we have used that

$$\int_0^{\infty} \frac{z^{j+1}}{(1+z)^{2j+4}} dz = \frac{j+1}{2j+3} \int_0^{\infty} \frac{z^j}{(1+z)^{2j+3}} dz = \frac{(j+1)j}{(2j+3)(2j+2)} \int_0^{\infty} \frac{z^{(j-1)}}{(1+z)^{(2j+2)}} dz =$$

$\displaystyle \ldots = \frac{(j+1)j}{(2j+3)(2j+2)\cdots(j+3)} \int_0^{\infty} \frac{1}{(1+z)^{(j+3)}} dz = \frac{(j+1)!^2}{(2j+3)!}$, applying repeated integration by parts.

So the integral evaluates to a rational number for all natural numbers n except for $n = 1$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Denote the integral of the problem by I_n. We show that

$$I_n = \begin{cases} \dfrac{\pi - 2}{2}, & \text{for } n = 1 \\[2ex] \dfrac{n\displaystyle\sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{2k+1} - 2^{n-1}}{(n-1)2^{n-1}}, & \text{for } n \geq 2. \end{cases} \tag{1}$$

Let $J_n = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\left(1+\sqrt{\sin(2x)}\right)^n} dx$. By substituting $x = \dfrac{\pi}{2} - y$ into I_n, we see that $I_n = J_n$. Since $1 + \sin(2x) = (\cos x + \sin x)^2$, so

$$I_n = \frac{I_n + J_n}{2} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sqrt{1+\sin(2x)}}{\left(1+\sqrt{\sin(2x)}\right)^n} dx = \frac{1}{4}\int_0^{\pi} \frac{\sqrt{1+\sin x}}{\left(1+\sqrt{\sin x}\right)^n} dx.$$

By putting $x = \pi - y$ we see that

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\sqrt{1+\sin x}}{\left(1+\sqrt{\sin x}\right)^n} dx = \int_0^{\pi/2} \frac{\sqrt{1+\sin y}}{(1+\sqrt{\sin y})^n} dy.$$

Hence, $I_n = \dfrac{1}{2}\displaystyle\int_0^{\pi/2} \dfrac{\sqrt{1+\sin x}}{\left(1+\sqrt{\sin x}\right)^n}dx$. By putting $\sin x = \cos^2\theta$ so that $\cos x dx = -2\sin\theta\cos\theta d\theta$ and so $I_n = \displaystyle\int_0^{\pi/2} \dfrac{\cos\theta d\theta}{(1+\cos\theta)^n}$. We have

$$I_1 = \int_0^{\pi/2} \frac{\cos\theta d\theta}{1+\cos\theta} = \frac{\pi}{2} - \int_0^{\pi/2} \frac{d\theta}{1+\cos\theta} = \frac{\pi}{2} - \frac{1}{2}\int_0^{/2} \sec^2\frac{\theta}{2}d = \frac{\pi-2}{2}.$$

For $n \geq 2$, integrating by parts, we have

$$I_n = \int_0^{\pi/2} \frac{\cos\theta d\theta}{(1+\cos\theta)^n} = \int_0^{\pi/2} \frac{d(\sin\theta)}{(1+\cos\theta)^n} = 1-n\int_0^{\pi/2} \frac{\sin^2\theta d\theta}{(1+\cos\theta)^{n+1}} = 1-n\int_0^{\pi/2} \frac{(1-\cos\theta)d\theta}{(1+\cos\theta)^n}.$$

Hence $(1-n)I_n = 1 - n\displaystyle\int_0^{\pi/2} \dfrac{d\theta}{(1+\cos\theta)^n} = 1 - \dfrac{n}{2^{n-1}}\displaystyle\int_0^{\pi/4} \sec^{2n}\theta d\theta$.

By putting $t = \tan\theta$, we obtain

$$\int_0^{\pi/4} \sec^{2n}\theta\theta = \int_0^1 \left(1+t^2\right)^{n-1} dt + \int_0^1 \left(\sum_{k=0}^{n-1}\binom{n-1}{k}t^{2k}\right) dt = \sum_{k=0}^{n-1}\frac{\binom{n-1}{k}}{2k+1}.$$

Thus (1) holds and this completes the solution.

Also solved by Arkady Alt, San Jose, CA; Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
April 15, 2018

5481: *Proposed by Kenneth Korbin, New York, NY*

A triangle with integer area has integer length sides $(3, x, x+1)$. Find five possible values of x with $x > 4$.

5482: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Prove that if n is a natural number then

$$\frac{(tan5^\circ)^n}{(tan4^\circ)^n+(tan3^\circ)^n}+\frac{(tan4^\circ)^n}{(tan3^\circ)^n+(tan2^\circ)^n}+\frac{(tan3^\circ)^n}{(tan2^\circ)^n+(tan1^\circ)^n}\geq\frac{3}{2}.$$

5483: *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" School Buzău, Romania*

If $a, b > 0$, and $x \in \left(0, \frac{\pi}{2}\right)$ then show that

$$(i) \qquad (a+b)\cdot\frac{\sin x}{x}+\frac{2ab}{a+b}\cdot\frac{\tan x}{x}\geq\frac{6ab}{a+b}.$$

$$(ii) \qquad a\cdot\tan x+\ b\cdot\sin x>2x\sqrt{ab}.$$

5484: *Proposed by Mohsen Soltanifar, Dalla Lana School of Public Health, University of Toronto, Canada*

Let X_1, X_2 be two continuous positive valued random variables on the real line with corresponding mean, median, and mode $\overline{x_1},\ \tilde{x}_1,\ \hat{x}_1$ and $\overline{x_2},\ \tilde{x}_2,\ \hat{x}_2$ respectively. Assume for their associated CDFs, (Cummulative Distribution Functions) we have

$$F_{X_1}(t)\leq F_{X_2}(t) \qquad (t>0).$$

Prove or give a counter example:

$$\text{(i) } \overline{x_2} \leq \overline{x_1}, \quad \text{(ii) } \tilde{x_2} \leq \tilde{x_1}, \quad \text{(iii) } \hat{x}_2 \leq \hat{x}_1.$$

5485: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let x, y, z be three positive real numbers. Show that

$$\prod_{cyclic} (2x + 3y + z + 1) \sum_{cyclic} (4x + 2y + 1)^{-3} \geq 3.$$

5486: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $(x_n)_{n\geq 0}$ be the sequence defined by $x_0 = 0, x_1 = 1, x_2 = 1$ and $x_{n+3} = x_{n+2} + x_{n+1} + x_n + n,\ \forall n \geq 0$. Prove that the series $\sum_{n=1}^{\infty} \frac{x_n}{2^n}$ converges and find its sum.

Solutions

5463: *Proposed by Kenneth Korbin, New York, NY*

Let N be a positive integer. Find triangular numbers x and y such that $x^2 + 14xy + y^2 = \left(72N^2 - 12N - 1\right)^2$.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

The n^{th} triangular number T_n is given by $T_n = \frac{n(n+1)}{2}$. To simplify matters, we will assume that $x \leq y$. Then, by trial and error, we found the following solutions for the first four values of N.

N	$72N^2 - 12N - 1$	x	y
1	59	$T_4 = 10$	$T_6 = 21$
2	263	$T_{10} = 55$	$T_{12} = 78$
3	611	$T_{16} = 136$	$T_{18} = 171$
4	1103	$T_{22} = 253$	$T_{24} = 300$

This leads to the conjecture that one solution consists of

$$x = T_{6N-2} = \frac{(6N-2)(6N-1)}{2} = (3N-1)(6N-1) = 18N^2 - 9N + 1 \tag{1}$$

and

$$y = T_{6N} = \frac{6N(6N+1)}{2} = 3N(6N+1) = 18N^2 + 3N. \tag{2}$$

After some algebraic simplification, we obtain

$$\begin{aligned} x^2+14xy+y^2 &= \left(18N^2-9N+1\right)^2+14\left(18N^2-9N+1\right)\left(18N^2+3N\right) \\ &\quad +\left(18N^2+3N\right)^2 \\ &= 5184N^4-1728N^3+24N+1 \\ &= \left(72N^2-12N-1\right)^2 \end{aligned}$$

and hence, (1) and (2) provide a solution for each $N \geq 1$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We put $x = \dfrac{(aN+b)(aN+b-1)}{2},\ y = \dfrac{(cN+d)(cN+d-1)}{2}$.

$$\frac{(aN+b)^2(aN+b-1)^2}{4}+14\frac{(aN+b)(aN+b-1)(cN+d)^2(cN+d-1)^2}{4}+\frac{(cN+d)^2(cN+d-1)^2}{4}$$
$$=(72N^2-12N-1)^2.$$

By comparing the coefficients of N^4, N^3, N^2, N and the statement of the problem we find the solutions

$$\begin{aligned} (x,y) &= \left(\frac{(6N-1)(6N-2)}{2}, \frac{(6N+1)6N}{2}\right) \text{ and} \\ (x,y) &= \left(\frac{(6N+1)6N}{2}, \frac{(6N-1)(6N-2)}{2}\right). \end{aligned}$$

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony Bevelacqua, University of North Dakota, Grand Forks, ND; Jeremiah Bartz, University of North Dakota, Grand Forks, ND; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Titu Zvonaru, Comănesti and Neculai Stanciu,"George Emil Palade" School Buzău, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5464: *Proposed by Ed Gray, Highland Beach, FL*

Let ABC be an equilateral triangle with side length s that is colored white on the front side and black on the back side. Its orientation is such that vertex A is at lower left, B is its apex, and C is at lower right. We take the paper at B and fold it straight down along the bisector of angle B, thus exposing part of the back side which is black. We continue to fold until the black part becomes 1/2 of the existing figure, the other half being white. The problem is to determine the position of the fold, the distance defined by x (as a function of s) which is the distance from B to the fold.

Solution 1 by David E. Manes, Oneonta, NY

If $x = \dfrac{\sqrt{3}}{2}(2-\sqrt{2})s$, then the area of the resulting black triangle equals the sum of the areas of the two resulting white triangles.

Introduce the coordinates $A(-s/2,0), B(0,\sqrt{3}s/2)$ and $C(s/2,0)$. Then triangle ABC is an equilateral triangle with side length s and altitude $\sqrt{3}s/2)$. In view of the coordinates x and y, let t denote the distance from vertex B to the fold. The equation of the line L containing the points B and C is $y=-\sqrt{3}\left(x-\frac{s}{2}\right)$. Note that for a given value of t, the value of y is given by $y=\frac{\sqrt{3}}{2}s-t$. For example, let $t=\frac{\sqrt{3}s}{4}$. Then $y=\frac{\sqrt{3}s}{4}$ and vertex B has been moved to the origin, thus creating three equilateral triangles two of which are white. Substituting the above value of y in the equation for L yields $x=\frac{s}{4}$ so that the point $P\left(\frac{s}{4},\frac{\sqrt{3}}{4}s\right)$ is a base vertex for the black triangle and an apex for one of the white triangles with side PC. By symmetry, the side length for the black triangle is $2\left(\frac{s}{4}\right)=\frac{s}{2}$ so that its area A_B is given by $A_B=\left(\frac{\sqrt{3}}{4}\right)\left(\frac{s}{2}\right)^2=\frac{\sqrt{3}}{16}s^2$. However, the side length $PC=\sqrt{\left(\frac{s}{4}\right)^2+\left(\frac{\sqrt{3}s}{4}\right)^2}=\frac{s}{2}$, hence the sum of the areas A_W of the two white triangles is $A_W=2\left(\frac{\sqrt{3}}{4}\right)\left(\frac{s}{2}\right)^2=\frac{\sqrt{3}}{8}s^2$. Since $A_W>A_B$, it follows that the value of t is greater than $\frac{\sqrt{3}s}{4}$. For this reason, let $t=\frac{\sqrt{3}}{4}s+k$, where k is a real number such that $0<k<\frac{\sqrt{3}}{4}s$. Then

$$y=\frac{\sqrt{3}}{2}s-t=\frac{\sqrt{3}}{2}-\left(\frac{\sqrt{3}}{4}s+k\right)=\frac{\sqrt{3}}{4}s-k.$$

Substituting this value of y in the equation for L, one obtains $x=\frac{s}{4}+\frac{\sqrt{3}}{3}k$. Hence, the point $P=\left(\frac{s}{4}+\frac{\sqrt{3}}{3}k,\frac{\sqrt{3}}{4}s-k\right)$ is a base vertex for the black triangle and an apex for the white triangle with side PC. Therefore, the side length of the black triangle is $2\left(\frac{s}{4}+\frac{\sqrt{3}}{3}k\right)=\frac{s}{2}+\frac{2\sqrt{3}}{3}k$ so that its area is $A_B=\frac{\sqrt{3}}{4}\left(\frac{s}{2}+\frac{2\sqrt{3}}{3}k\right)^2$. Moreover, for the white triangle

$$PC=\sqrt{\left(\frac{\sqrt{3}}{3}k-\frac{s}{4}\right)^2+\left(\frac{\sqrt{3}}{4}s-k\right)^2}=\frac{s}{2}-\frac{2\sqrt{3}}{3}k.$$

Therefore, the sum of the areas of the two white triangles is $A_W=\frac{\sqrt{3}}{2}\left(\frac{s}{2}-\frac{2\sqrt{3}}{3}k\right)^2$. Setting $A_W=A_B$, we get

$$\frac{\sqrt{3}}{2}\left(\frac{s^2}{4}-\frac{2\sqrt{3}}{3}sk+\frac{4}{3}k^2\right)=\frac{\sqrt{3}}{4}\left(\frac{s^2}{4}+\frac{2\sqrt{3}}{3}sk+\frac{4}{3}k^2\right).$$

This equation simplifies to the following quadratic equation in k:

$$\frac{4}{3}k^2-2\sqrt{3}sk+\frac{s^2}{4}=0$$

with roots $k = \frac{\sqrt{3}}{2}\left(\frac{3}{2} \pm \sqrt{2}\right) s$. The positive square root of 2 yields a value of $k > \frac{\sqrt{3}}{4}s$ and so is inadmissable. Therefore, $k = \frac{\sqrt{3}}{2}\left(\frac{3}{2} - \sqrt{2}\right) s$, whence

$$t = \frac{\sqrt{3}}{4} + k = \frac{\sqrt{3}}{2}(2 - \sqrt{2})s.$$

Observe that using this value of t, the area of the black triangle as well as the sum of the areas of the two white triangles is $\sqrt{3}\left(\frac{3}{2} - \sqrt{2}\right) s^2$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

When B reaches the midpoint of AC, the black part is only 1/3 of the existing figure, which is a trapezium. So we need to push B downwards further. The black part is then an equilateral triangle with base $2x \tan 30^\circ$, height x and hence area $\frac{x^2}{\sqrt{3}}$. The distance between the fold and AC equals $\frac{\sqrt{3}s}{2} - x$. The white part now consists of two congruent equilateral triangles of lengths $\left(\frac{\sqrt{3}s}{2} - x\right) \sec 30^\circ = s - \frac{2x}{\sqrt{3}}$. Since the area of the white part equals the area of the black part, we have $\frac{\sqrt{3}}{2}\left(s - \frac{2x}{\sqrt{3}}\right)^2 = \frac{x^2}{\sqrt{3}}$. Solving, we obtain $x = \frac{\sqrt{3}\left(2 - \sqrt{2}\right) s}{2}$.

Editor's Comment: **David Stone and John Hawkins both Georgia Southern University in Statesboro, GA** generalized the statement as follows: "The problem asked for the configuration in which the black triangle covered half of the final figure. We could just as well determine when the black triangle covers any given portion of the final figure; say one fourth or nine tenths."

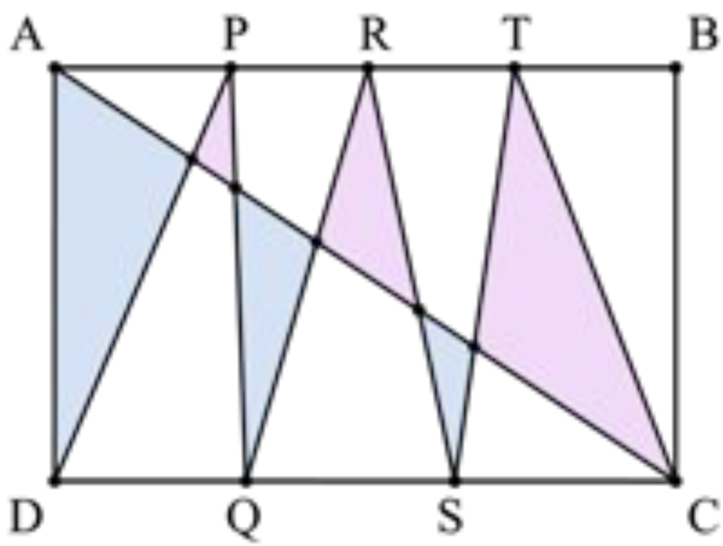
They did this by looking at two cases: 1) $0 < \lambda < \frac{1}{3}$ and 2) $\frac{1}{3} < \lambda < 1$, where the Black Area=$\lambda$Total Area. The constant $\frac{1}{3}$ comes from when the vertex of the Black Triangle lies on the base of the White Triangle. Letting x be the length of the height of the black triangle (measured from its vertex to the fold line) they found that for the first case, where the vertex of the Black Triangle lies in the interior of the White Triangle that: $x = \sqrt{\frac{\lambda}{1+\lambda}} \cdot \frac{\sqrt{3}}{2}s$ and in the second case, where the vertex of the Black Triangle lies in the exterior of the White Triangle that $x = \frac{2\lambda - \sqrt{2\lambda(1-\lambda)}}{3\lambda - 1} \cdot h$, where h is the altitude of the given White Triangle. When $\lambda = \frac{1}{2}$ we obtain the statement of the problem and using their formula reaffirms the above answers.

In concluding their comment they noted, "Another nice example: with $\lambda = \frac{2a^2}{2a^2+1}$, which is very close to 1, we find that $x = \left(1 - \frac{1}{2a+1}\right) h$. That is, in a precisely measurable way, we must fold almost all the way down to get a figure which is almost all black."

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego (two solutions), Viveiro, Spain; David A. Huckaby, Angelo State University San Angelo, TX; and the proposer.

5465: *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

Quadrilateral $ABCD$ is a rectangle with diagonal AC. Points P, R, T, Q and S are on sides AB and DC and they are connected as shown. Three of the triangles inside the rectangle are shaded pink, and three are shaded blue. Which is larger, the sum of the areas of the pink triangles or the sum of the areas of the blue triangles?



Solution by David A. Huckaby, Angelo State University, San Angelo, TX

Let p be the sum of the areas of the pink triangles, b the sum of the areas of the blue triangles, and w the sum of the areas of the three white polygons below the diagonal.

$$\begin{aligned} p + w &= \text{the sum of the areas of triangles } DPQ,\ QRS, \text{ and } STC \\ &= \frac{1}{2}(AD \cdot DQ) + \frac{1}{2}(AD \cdot QS) + \frac{1}{2}(AD \cdot SC) \\ &= \frac{1}{2}\left[AD \cdot (DQ + QS + SC)\right] \\ &= \frac{1}{2}(AD \cdot DC) \\ &= \text{the area of triangle } ADC \\ &= b + w \end{aligned}$$

So $p = b$, that is, the sum of the areas of the pink triangles is equal to the sum of the areas of the blue triangles.

Also solved by Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5466: *Proposed by D.M. Bătinetu– Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" School, Buzău, Romania*

Let $f:(0,+\infty)\to(0,+\infty)$ be a continuous function. Evaluate

$$\lim_{n\to\infty}\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right)dx.$$

Solution 1 by Moti Levy, Rehovot, Israel

The mean value theorem of the integral calculus states:
Let $f(x)$ be continuous function, then

$$\int_a^b f(x)\,dx=(b-a)\,f(\xi),\ \ a\le\xi\le b.$$

Therefore,

$$\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right)dx=\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\right)f\left(\frac{\xi}{n}\right),\quad \frac{n^2}{\sqrt[n]{n!}}\le\xi\le\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}.$$

Taking limits of both sides,

$$\lim_{n\to\infty}\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right)dx=\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\right)\lim_{n\to\infty}f\left(\frac{\xi}{n}\right).$$

Since $f(x)$ is continuous then

$$\lim_{n\to\infty}f\left(\frac{\xi}{n}\right)=f\left(\lim_{n\to\infty}\frac{\xi}{n}\right)=f\left(\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\right)$$

Using Stirling's asymptotic formula, we have

$$\sqrt[n]{n!}\sim\frac{n}{e}.\tag{1}$$

By (1),

$$\frac{n}{\sqrt[n]{n!}}\sim e,\qquad \frac{n^2}{\sqrt[n]{n!}}\sim e\cdot n,$$

which implies that

$$f\left(\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\right)=f(e)$$

and that

$$\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\sim e.$$

We conclude that

$$\lim_{n\to\infty}\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right)dx = ef\left(e\right).$$

Solution 2 by Bruno Salgueiro Fanego,Viveiro, Spain

Let's proceed as in http://www./oei.es/historico/oim/revistaoim/numero 53/261_Bruno.pdf:

Let $n \in N$; since f is continuous on (x_n, x_{n+1}), by the mean value theorem of integral calculus, we have that $\displaystyle\int_{x_n}^{x^{n+1}} f\left(\frac{x}{n}\right)dx = f\left(\frac{\xi_n}{n}\right)(x_{n+1}-x_n)$for some $\xi_n \in (x_n, x_{n+1})$.

Since$\dfrac{x_n}{n} < \dfrac{\xi_n}{n} < \dfrac{x_{n+1}}{n}$,

$$\lim_{n\to\infty}\frac{x_n}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n^{2n}}{n^n n!}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}} = \lim_{n\to\infty}\frac{\dfrac{(n+1)^{n+1}}{(n+1)!}}{\dfrac{n^n}{n!}} = \lim_{n\to\infty}\frac{(n+1)^n(n+1)n!}{(n+1)!n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$$

and $\displaystyle\lim_{n\to\infty}\frac{x_{n+1}}{n} = \lim_{n\to\infty}\frac{x_{n+1}}{n+1}\cdot\frac{n+1}{n} = \lim_{n\to\infty}\frac{x_{n+1}}{n+1}\cdot\lim_{n\to\infty}\frac{n+1}{n} = e\cdot 1 = e$, by the sandwich rule we obtain that $\displaystyle\lim_{n\to\infty}\frac{\xi_n}{n} = e$, and, hence,

$\displaystyle\lim_{n\to\infty} f\left(\frac{\xi_n}{n}\right) = \left(\lim_{n\to\infty}\frac{\xi_n}{n}\right) = f(e)$.

Moreover, from Stolz' rule,

$$\lim_{n\to\infty}(x_{n+1}-x_n) = \lim_{n\to\infty}\frac{(x_{n+1}-x_n)}{(n+1)-n} = \lim_{n\to\infty}\frac{x_n}{n} = e.$$

So, the required limit is equal to

$$\lim_{n\to\infty}\int_{x_n}^{x_{n+1}} f\left(\frac{x}{n}\right)dx = \lim_{n\to\infty} f\left(\frac{\xi_n}{n}\right)\lim_{n\to\infty}(x_{n+1}-x_n) = ef(e).$$

Solution 3 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

This particular problem is similar to Problem 121, which was proposed by D.M. Bătinetu-Giurgiu ("Matei Basarab" National College, Bucharest, Romania) and Neculai Stanciu ("George Emil Palade" School, Buzău, Romania) to the Math Problems Journal, Volume 5, Issue 2 (2015), pp. 420-421. We'll use the following lemma.

Lemma: Let $f : [a,b] \to \Re$ be continuous and $(x_n)_n, (y_n)_n$ two convergent sequences of $[a,b]$ that have the same limit c, then

$$\int_{x_n}^{y_n} f(t)dt = f(c)(y_n - x_n) + O(y_n - x_n).$$

Proof: Let $\epsilon > 0$, then there exists $\delta > 0$ such that $|f(t) - f(c)| < \epsilon$, whenever $|x - c| < \delta$. Since $x_n, y_n \to c$, the there is an $n_0 \in N$ such that $x_n, y_n \in (C - \delta, C + \delta)$, whenever $n > n_0$. Therefore,

$$\left|\int_{x_n}^{y_n} f(t)dt - f(c)(y_n - x_n)\right| \leq \int_{x_n}^{y_n} |f(t) - f(c)|\, dt \leq \epsilon\, |y_n - x_n|\,.$$

Note that the given integral equals

$$I_n = n\int_{\frac{n}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{n\sqrt[n+1]{(n+1)!}}} f(t)dt,$$

this comes directly from the substitution $t = \dfrac{x}{n}$. Let x_n, y_n be the lower, upper bound of the last integral respectively then $x_n, y_n \to e$, since $\dfrac{n}{\sqrt[n]{n!}} \to e$, and thus
$\dfrac{(n+1)^2}{n\sqrt[n+1]{(n+1)!}} = \dfrac{n+1}{n}\dfrac{(n+1)}{\sqrt[n+1]{(n+1)!}} \longrightarrow e,$ as $n \longrightarrow \infty$. Note that by Stolz' theorem

$$n(y_n - x_n) = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}} \longrightarrow e, \text{ as } n \longrightarrow \infty.$$

By the lemma we have

$$I_n = n\left[f(c)(y_n - x_n) + O(y_n - x_n)\right] = ef(e) + O(1),$$

which proves that the limit equals $ef(e)$.

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Soumitra Mandal, Scottish Church College, Chandan -Nagar, West Bengal, India; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova, Varna, Bulgaria, and the proposers.

5467: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

In an arbitrary triangle $\triangle ABC$, let a, b, c denote the lengths of the sides, R its circumradius, and let h_a, h_b, h_c respectively, denote the lengths of the corresponding altitudes. Prove the inequality

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a \cdot h_b \cdot h_c}},$$

and give the conditions under which equality holds.

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

We know that $h_a = (bc)/(2R)$ and cyclic so the inequality actually is

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq \frac{3abc}{2R}\left(\frac{8R^3}{(abc)^2}\right)^{\frac{1}{3}} = 3(abc)^{\frac{1}{3}}.$$

We prove the stronger one

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\geq a+b+c,$$

that is

$$\left(\frac{a^2+bc}{b+c}-a\right)+\left(\frac{b^2+ca}{c+a}-b\right)+\left(\frac{c^2+ab}{a+b}-c\right)\geq 0,$$

or

$$\frac{(a-b)(a-c)}{b+c}+\frac{(b-c)(b-a)}{a+c}+\frac{(c-a)(c-b)}{a+b}\geq 0.$$

We can suppose $a\geq b\geq c$ by symmetry so we come to

$$\frac{(a-b)(a-c)}{b+c}+\frac{(a-c)(b-c)}{a+b}\geq\frac{(a-b)(b-c)}{a+c}.$$

This is implied by

$$\frac{(a-b)(a-c)}{b+c}+\underbrace{\frac{(a-b)(b-c)}{a+b}}_{a-c\geq a-b}\geq\frac{(a-b)(b-c)}{a+c},$$

or

$$\frac{a-c}{b+c}+\frac{b-c}{a+b}\geq\frac{b-c}{a+c}.$$

This is in turn implied by

$$\underbrace{\frac{a-c}{a+c}}_{a\geq b}+\frac{b-c}{a+b}\geq\frac{b-c}{a+c}$$

and this evidently holds true by $a-c\geq b-c\geq 0$. The equality case is $a=b=c$.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

We will prove the following slight improvement:

$$\begin{aligned}\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}&\geq a+b+c\\&\geq 3\sqrt[3]{abc}\\&=\frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a\cdot h_b\cdot h_c}},\end{aligned}\qquad(1)$$

with equality if and only if $a=b=c$.

To begin, we note that

$$\begin{aligned}a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2&=\frac{\left(a^2-b^2\right)^2+\left(b^2-c^2\right)^2+\left(c^2-a^2\right)^2}{2}\\&\geq 0,\end{aligned}\qquad(2)$$

with equality if and only if $a^2 = b^2 = c^2$. Since $a, b, c > 0$, it follows that equality is attained in (2) if and only if $a = b = c$.

Next, we use (2) to obtain

$$\begin{aligned}
&\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \\
&= \frac{\left(a^2-b^2\right)+\left(b^2+bc\right)}{b+c} + \frac{\left(b^2-c^2\right)+\left(c^2+ca\right)}{c+a} + \frac{\left(c^2-a^2\right)+\left(a^2+ab\right)}{a+b} \\
&= \frac{a^2-b^2}{b+c} + \frac{b^2-c^2}{c+a} + \frac{c^2-a^2}{a+b} + a+b+c \\
&= \frac{\left(a^2-c^2\right)+\left(c^2-b^2\right)}{b+c} + \frac{b^2-c^2}{c+a} + \frac{c^2-a^2}{a+b} + a+b+c \\
&= \left(a^2-c^2\right)\left(\frac{1}{b+c} - \frac{1}{a+b}\right) + \left(b^2-c^2\right)\left(\frac{1}{c+a} - \frac{1}{b+c}\right) + a+b+c \\
&= \left(a^2-c^2\right)\frac{a-c}{(a+b)(b+c)} + \left(b^2-c^2\right)\frac{b-a}{(b+c)(c+a)} + a+b+c \\
&= \frac{\left(a^2-c^2\right)^2 + \left(b^2-c^2\right)\left(b^2-a^2\right)}{(a+b)(b+c)(c+a)} + a+b+c \\
&= \frac{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}{(a+b)(b+c)(c+a)} + a+b+c \\
&\geq a+b+c, \qquad (3)
\end{aligned}$$

with equality if and only if $a = b = c$.

Also, the Arithmetic - Geometric Mean Inequality implies that

$$a+b+c \geq 3\sqrt[3]{abc}, \qquad (4)$$

with equality if and only if $a = b = c$.

For the final step, let $K = area\,(\triangle ABC)$. Then,

$$K = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$$

and hence,

$$h_a = \frac{2K}{a}, \quad h_b = \frac{2K}{b}, \quad \text{and} \quad h_c = \frac{2K}{c}.$$

Since $R = \dfrac{abc}{4K}$, we have

$$\begin{aligned}
\frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a \cdot h_b \cdot h_c}} &= \frac{12KR}{2R}\sqrt[3]{\frac{abc}{8K^3}} \\
&= \frac{6K}{2K}\sqrt[3]{abc} \\
&= 3\sqrt[3]{abc}. \qquad (5)
\end{aligned}$$

If we combine (3), (4), and (5), statement (1) follows and equality is attained throughout if and only if $a = b = c$.

Solution 3 by Arkady Alt, San Jose, CA

Let $F = [ABC]$ (area) and let s be its semi-perimeter.
Since $h_a = \dfrac{2F}{a}, h_b = \dfrac{2F}{b}, h_c = \dfrac{2F}{c}$ and $abc = 4RF$ then

$$\sqrt[3]{\frac{1}{h_a h_b h_c}} = \sqrt[3]{\frac{abc}{8F^3}} = \frac{1}{2F}\sqrt[3]{abc} \text{ and}$$

$$\frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a h_b h_c}} = 3\sqrt[3]{abc}.$$

Thus, original inequality becomes

(1) $$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq 3\sqrt[3]{abc}.$$

Since $\dfrac{4a^2}{b+c} \geq 4a - b - c \iff (2a-b-c)^2 \geq 0$ we have

$$\begin{aligned}
\sum_{cyc}\frac{a^2+bc}{b+c} &= \sum_{cyc}\frac{a^2}{b+c} + \sum_{cyc}\frac{bc}{b+c} \geq \sum_{cyc}\frac{4a-b-c}{4} + \sum_{cyc}\frac{bc}{b+c} \\
&= \frac{a+b+c}{2} + \sum_{cyc}\frac{bc}{b+c} = \sum_{cyc}\left(\frac{b+c}{4} + \frac{bc}{b+c}\right) \geq \sum_{cyc} 2\sqrt{\frac{b+c}{4}\cdot\frac{bc}{b+c}} \\
&= \sum_{cyc}\sqrt{bc} \geq 3\sqrt[3]{\sqrt{bc}\cdot\sqrt{ca}\cdot\sqrt{ab}} = 3\sqrt[3]{abc}.
\end{aligned}$$

Solution 4 by Nicusor Zlota,"Traian Vuia" Technical College, Focsani, Romania, and Corneliu-Manescu Avram, Ploiesti, Romania

Assume that $a \geq b \geq c$.

First, we will prove that $\dfrac{a^2+bc}{b+c} + \dfrac{b^2+ca}{c+a} + \dfrac{c^2+ab}{a+b} \geq a+b+c \iff$

$\dfrac{a^2+bc}{b+c} - a + \dfrac{b^2+ca}{c+a} - b + \dfrac{c^2+ab}{a+b} - c \geq 0 \iff$

$\dfrac{(a-b)(a-c)}{b+c} + \dfrac{(b-c)(b-a)}{c+a} + \dfrac{(c-a)(c-b)}{a+b} \geq 0 \iff$

$(a-b)\left(\dfrac{a-c}{b+c} - \dfrac{b-c}{c+a}\right) + (b-a)\left(\dfrac{b-a}{c+a} - \dfrac{c-a}{a+b}\right) + (c-a)\left(\dfrac{a-c}{b+c} - \dfrac{b-c}{c+a}\right) \geq 0$

$(a-b)^2\dfrac{a+b}{(b+c)(c+a)} + (b-c)^2\dfrac{b+c}{(a+b)(c+a)} + (c-a)^2\dfrac{c+a}{(a+b)(b+c)} \geq 0.$

Then, it suffices to prove that

$$a = b = -c \geq \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a h_b h_c}} = \frac{3abc}{2R}\sqrt[3]{\frac{abc}{8S^3}} = \frac{3abc}{2R}\frac{1}{2S}\sqrt[3]{abc} = \sqrt[3]{abc},$$

which is the AM-GM inequality.

Equality holds for $a = b = c$.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy Rehovot, Israel; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro,GA, and the proposer.

$Editor's\ note$: **Hatef I. Arshagi's solution was dedicated to the memory of Mrs. Alieh Ataee**.

5468: *Proposed by Ovidiu Furdui and Alina Sîntămărian, both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all differentiable functions $f : \Re \to \Re$ with $f(0) = 1$ such that $f'(x) = f^2(-x)f(x)$, for all $x \in \Re$.

Solution 1 by Moti Levy, Rehovot, Israel

Let us differentiate both sides of the given differential equation,

$$f''(x) = -2f(-x)f'(-x)\,f(x) + f^2(-x)f'(x). \tag{1}$$

The following two equation are direct consequence of the original equation.

$$f'(-x) = f^2(x)f(-x), \tag{2}$$

$$f^2(-x) = \frac{f'(x)}{f(x)}. \tag{3}$$

After substitution of (2) and (3) in (1), we get differential equation (4) with initial conditions at $x = 0$,

$$f''(x) + 2f^2(x)f'(x) - \frac{\left(f'(x)\right)^2}{f(x)} = 0, \quad f(0) = f'(0) = 1. \tag{4}$$

By the substitution $f(x) = \sqrt{g(x)}$,

$$\begin{aligned} f &= \sqrt{g}, \\ f' &= \frac{1}{2\sqrt{g}}g', \\ f'' &= \frac{1}{2\sqrt{g}}g'' - \frac{1}{4\left(\sqrt{g}\right)^3}\left(g'\right)^2, \end{aligned}$$

we arrive at the equivalent differential equation

$$g^{''}+2gg^{'}-\frac{1}{g}\left(g^{'}\right)^{2}=0,\quad g\left(0\right)=1,g^{'}\left(0\right)=2. \tag{5}$$

Now we want to lower the order of (5) by the substitution $g^{'}=\frac{dg}{dx}=z,\ \ g^{''}=\frac{d^2g}{dx^2}=z\frac{dz}{dg}$,

$$z\frac{dz}{dg}+2gz-\frac{1}{g}z^{2}=0,$$

or

$$g\frac{dz}{dg}-z=-2g^{2}. \tag{6}$$

The solution of (6) is

$$z=cg-2g^{2}.$$

The initial conditions on g dictate that $c=4$, thus we obtain the following differential equation for g,

$$\frac{dg}{dx}=4g\left(x\right)-2g^{2}\left(x\right).$$

or

$$\frac{dx}{dg}=\frac{1}{4g-2g^{2}}.$$

After integration over g, we get

$$x=\frac{1}{4}\ln\frac{g}{2-g}+c$$

or

$$g\left(x\right)=2k\frac{e^{4x}}{ke^{4x}+1}.$$

Again, the initial condition $g\left(0\right)=1$ dictates $k=1$,

$$g\left(x\right)=\frac{2e^{4x}}{e^{4x}+1}.$$

We conclude that the

$$f\left(x\right)=\sqrt{2}\frac{e^{2x}}{\sqrt{e^{4x}+1}}.$$

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

The defining relation

$$f'(x)=f^{2}(-x)f(x), \tag{1}$$

implies that f is continuously differentiable. Setting $-x$ in the relation (1), one gets for every $=x$, $f'(-x)=f^2(x)f(-x)$, Multiplying (1) by $f(x)$ yields for every x

$$f^{'}(x)f(x)=f^{2}(-x)f^{2}(x)=\left[f^{2}(x)f(-x)\right]f(-x)=f^{'}(-x)f(-x),$$

that is $x \to f'(x)f(x)$ is even. Therefore, $\int_{-x}^{x} f'(t)f(t)dt = 2\int_{0}^{x} f'(t)f(t)dt$, and since an antiderivative of $f'f$ is $\frac{f^2}{2}$, this implies that for every x, $f^2(x) + f^2(-x) = 2$. Replacing $f^2(-x)$ in the defining relation one get for every x

$$f'(x) - 2f(x) - f^2(x).$$

This non-linear differential equation seems to have only one solution, namely

$$x \to \frac{\sqrt{2}e^{2x}}{\sqrt{e^{4x}+1}} \tag{2}$$

Conversely, it is easily checked that (2) is indeed a solution to the equation.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that

$$f(x) = \frac{\sqrt{2}e^{2x}}{\sqrt{1+e^{4x}}}. \tag{1}$$

From the given equation , we obtain $f(x)f'(x) = f^2(-x)f^2(x)$, so that

$$f(-x)f'(-x) = f(x)f'(x) \tag{2}$$

Integrating (2) with respect to x, and making use of the fact that $f(0) = 1$, we obtain

$$f^2(-x) = 2 - f^2(x). \tag{3}$$

Substituting (3) into the given equation, we obtain $f'(x) = \left(2 - f^2(x)\right) f(x)$ or $\frac{d(f(x))}{(1-f^2(x))f(x)} = dx$. Integrating boh sides we obtain

$$\frac{\ln\left(f(x)\right)}{2} - \frac{\ln\left(2 - f^2(x)\right)}{4} = x + C,$$

where C is a constant. Since $f(0) = 1$, so $C = 0$. Now (1) follows easily by simple algebra.

Editor's comment: **Anna Tomova of Varna Bulgaria** expressed her solution in terms of a hyperbolic function; $f(x) = \frac{e^x}{\sqrt{\cosh 2x}}$, $f(0) = 1$.

Also solved by Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova, Varna, Bulgaria, and the proposers.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before May 15, 2018

5487: *Proposed by Kenneth Korbin, New York, NY*

Given that $\dfrac{(x+1)^4}{x(x-1)^2} = a$ with $x = \dfrac{b+\sqrt{b-\sqrt{b}}}{b-\sqrt{b-\sqrt{b}}}$. Find positive integers a and b.

5488: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta, Turnu-Severin, Mehedinti, Romania*

Let a, and b be complex numbers. Solve the following equation:

$$x^3 - 3ax^2 + 3(a^2 - b^2)x - a^3 + 3ab^2 - 2b^3 = 0.$$

5489: *Proposed by D.M. Bătinetu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School Buzău, Romania*

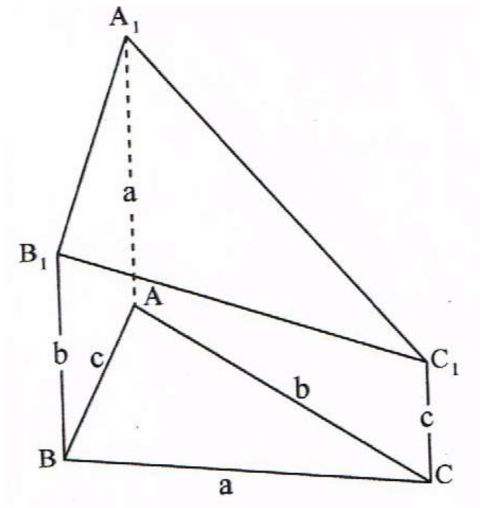
If $a > 0$, compute $\displaystyle\int_0^a \left(x^2 - ax + a^2\right) \arctan(e^x - 1)dx$.

5490: *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

Triangle ABC whose side lengths are a, b, and c lies in plane P. The segmentA_1A, BB_1, CC_1 satisfy:

$$A_1A \perp P,\ B_1B \perp P,\ C_1C \perp P,$$

where $A_1A = a$, $B_1B = b$ and $C_1C = c$, as shown in the figure. Prove that $\triangle A_1B_1C_1$ is acute -angled.



5491: *Proposed by Roger Izard,Dallas, TX*

Let O be the orthocenter of isosceles triangle ABC, $AB = AC$. Let OC meet the line segment AB at point F. If $m = FO$, prove that $c^4 \geq m^4 + 11m^2c^2$.

5492: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let a, b, c, d be four positive numbers such that $ab + ac + ad + bc + bd + cd = 6$. Prove that

$$\sqrt{\frac{abc}{a+b+c+3d}} + \sqrt{\frac{bcd}{b+c+d+3a}} + \sqrt{\frac{cda}{c+d+a+3b}} + \sqrt{\frac{dab}{d+a+b+3c}} \leq 2\sqrt{\frac{2}{3}}.$$

Solutions

5469: *Proposed by Kenneth Korbin, New York, NY*

Let x and y be positive integers that satisfy the equation $3x^2 = 7y^2 + 17$. Find a pair of larger integers that satisfy this equation expressed in terms of x and y.

Solution 1 by Bruno Salugueiro Fanego, Viveiro, Spain

It suffices to find a pair of the type $(ax+by, cx+dy)$ where a, b, c, and d are positive integers and $3(ax+by)^2 - 7(cx+dy)^2 = 3x^2 - 7y^2$.

Since $(ax+by)^2 - 7(cx+dy)^2 = (3a^2 - 7c^2)x^2 + (3b^2 - 7d^2)y^2 + 2(3ab - 7cd)xy$, it is sufficient that a, b, c, and d satisfy the relations:

$$\begin{cases} 3a^2 - 7b^2 = 3 \\ 3b^2 - 7d^2 = -7 \\ 3ab - 7cd = 0. \end{cases}$$

The pair $(a, c) = (55, 36)$ of positive integers verifies $3a^2 - 7b^2 = 3$, and if it assumed that $d = a = 55$ then it only remains to find a positive integer b such that $3 \cdot b - 7 \cdot 36 = 0$ and $3b^2 - 7 \cdot 55^2 = -7$.

Since $b = 84$ satisfies $3b^2 - 7 \cdot 55^2 = -7$, the pair of larger integers $(55x + 84y, 36x + 55y)$ solves the problem.

Solution 2 by Ed Gray, Highland Beach, FL

Clearly, by inspection, the equation is satisfied by $x = 8, y = 5$. Let the larger integers which satisfy the equation be $x + k = 8 + k$ and $y + m = 5 + m$ then we have:

1. $3(8+k)^2 = 7(5+m)^2 + 17,$ expanding
2. $3(64 + 16k + k^2) = 7(25 + 10m + m^2) + 17$
3. $192 + 48k + 3k^2 = 175 + 70m + 7m^2 + 17$
4. $48k + 3k^2 = 70m + 7m^2$. Let $(k, m) = r$. So,
5. $k = ra, m = rb,\ (a, b) = 1,$ and substituting into 4)
6. $48ra + 3r^2a^2 = 70rb + 7r^2b^2$
7. $48a + 3ra^2 = 70b + 70rb^2$
8. $3a(16 + ra) = 7b(10 + rb)$. Suppose
9. $3a = 10 + rb$ and that
10. $7b = 16 + ra$. Multiply step 9 by a and step10 by b
11. $3a^2 = 10a + arb$
12. $7b^2 = 16b + arb$
13. $3a^2 - 7b^2 = 10a - 16b$
14. $3a^2 - 10a = 7b^2 - 16b,$ and by inspection $b\ = 4, a\ = 6$
15. $3a^2 - 10a = 3(6)^2 - (10)(6) = 108a^2 - 60 = 48$
16. $7b^2 - 16b = 7(4)^2 - (16)(4) = 112 - 64 = 48$. From step 10
17. $7b = 16 + ra$ or $(7)(4) = 28 = 16 + 6r,$ and $r = 2$. From step 5
18. $k = ra = (2)(16) = 12,\ m = rb = (2)(4) = 8$.

Hence the larger integers are $x + k = x + 12 = 20$ and $y + m = y + 8 = 13$.

As a check, substituting $(20, 13)$ into $3x^2 = 7y^2 + 17$, gives equality.

Solution 3 by David E. Manes, Oneonta, NY

We will show that if (x, y) is a solution of $3x^2 = 7y^2 + 17$, then $(55x + 84y, 36x + 55y)$ is another solution of this equation. Furthermore, all positive integer solutions of this equation are given by

$$x_n = \left(4 + \frac{5\epsilon\sqrt{21}}{6}\right)(55 + 12\sqrt{21})^n + \left(4 - \frac{5\epsilon\sqrt{21}}{6}\right)(55 - 12\sqrt{21})^n,$$
$$y_n = \left(\frac{4\sqrt{21}}{7} + \frac{5\epsilon}{2}\right)(55 + 12\sqrt{21})^n + \left(\frac{-4\sqrt{21}}{7} + \frac{5\epsilon}{2}\right)(55 - 12\sqrt{21})^n,$$

where $n \geq 1$ and $\epsilon = \pm 1$.
Re-write the equation as (1) $3x^2 - 7y^2 - 17 = 0$ and note that the two least positive solutions (x, y) are $(8, 5)$ and $(20, 13)$. Construct the recurrent sequences

$$x_{n+1} = \alpha x_n + \beta y_n$$
$$y_{n+1} = \gamma x_n + \delta y_n,$$

where α, β, γ and δ are unknowns and assume that (x_{n+1}, y_{n+1}) is a solution of (1). Then $3(\alpha x_n + \beta y_n)^2 - 7(\gamma x_n + \delta y_n)^2 - 17 = 0$ which expands to

$$(3\alpha^2 - 7\gamma^2)x_n^2 + (3\beta^2 - 7\delta^2)y_n^2 + (6\alpha\beta - 14\gamma\delta)x_n y_n - 17 = 0.$$

Comparing this equation with (1), we get

$$(2)\ 3\alpha^2 - 7\gamma^2 = 3, \quad (3)\ 3\beta^2 - 7\delta^2 = -7, \quad (4)\ 3\alpha\beta = 7\gamma\delta.$$

Squaring equation (4), we get $(3\alpha^2)(3\beta^2) = 49\gamma^2\delta^2$. Using (2) and (3), one obtains

$$(3 + 7\gamma^2)(-7 + 7\delta^2) = 49\gamma^2\delta^2 \quad \text{or} \quad 3\delta^2 - 7\gamma^2 = 3.$$

Subtracting this equation from (2) results in $\alpha = \pm\delta$. Substituting this value in (4), we get $3(\pm\delta)\beta = 7\gamma\delta$ or $\beta = \pm\frac{7}{3}\gamma$. Finally, substituting this value in (3) along with $\delta = \pm\alpha$ yields the equation $3\alpha^2 - 7\gamma^2 = 3$ which is equation (2). The smallest positive integer solution of this equation is $\alpha = 55, \gamma = 36$ so that $\beta = \frac{7}{3}\gamma = 84$. Since we want positive solutions only, we let

$$x_{n+1} = \alpha x_n + \frac{7}{3}\gamma y_n = 55x_n + 84y_n$$
$$y_{n+1} = \gamma x_n + \alpha y_n = 36x_n + 55y_n.$$

Define the 2×2 matrix A over the reals as follows: $A = \begin{pmatrix} 55 & 84 \\ 36 & 55 \end{pmatrix}$. By construction, if $\begin{pmatrix} x \\ y \end{pmatrix}$ is a positive integer solution vector of equation (1), then $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 55x + 84y \\ 36x + 55y \end{pmatrix}$ is also a solution vector with larger integers. Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$, and $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 20 \\ 13 \end{pmatrix}$. Then all positive integer solutions of equation (1) are given by

$$\begin{pmatrix} x'_n \\ y'_n \end{pmatrix} = A^n \begin{pmatrix} 8 \\ 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x''_n \\ y''_n \end{pmatrix} = A^n \begin{pmatrix} 20 \\ 13 \end{pmatrix}$$

for $n \geq 1$. Noting that $A\begin{pmatrix} 8 \\ -5 \end{pmatrix} = \begin{pmatrix} 20 \\ 13 \end{pmatrix}$, we can shorten the above description to

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} 8 \\ \epsilon 5 \end{pmatrix},$$

where $n \geq 1$ and $\epsilon = \pm 1$. To get a closed form expression for the positive integer solutions of equation $3x^2 - 7y^2 - 17 = 0$, we note that A has two distinct eigenvalues $\lambda_1 = 55 + 12\sqrt{21}$ and $\lambda_2 = 55 - 12\sqrt{21}$ with corresponding eigenvectors $\overrightarrow{v_1} = (\sqrt{21}, 3)$ and $\overrightarrow{v_2} = (-\sqrt{21}, 3)$. Therefore, A is diagonalizable; that is, there exists a diagonal matrix $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where λ_1, λ_2 are the eigenvalues of A and an invertible matrix $P = \begin{pmatrix} \sqrt{21} & -\sqrt{21} \\ 3 & 3 \end{pmatrix}$ consisting of the two eigenvectors in columns such that $P^{-1}AP = D$. Therefore, $A = PDP^{-1}$ so that for each positive integer n, $A^n = PD^nP^{-1}$. Hence,

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} 8 \\ \epsilon 5 \end{pmatrix} = \left(PD^nP^{-1}\right) \begin{pmatrix} 8 \\ \epsilon 5 \end{pmatrix},$$

which computationally yields

$$x_n = \left(4 + \frac{5\epsilon\sqrt{21}}{6}\right)(55 + 12\sqrt{21})^n + \left(4 - \frac{5\epsilon\sqrt{21}}{6}\right)(55 - 12\sqrt{21})^n,$$

$$y_n = \left(\frac{4\sqrt{21}}{7} + \frac{5\epsilon}{2}\right)(55 + 12\sqrt{21})^n + \left(\frac{-4\sqrt{21}}{7} + \frac{5\epsilon}{2}\right)(55 - 12\sqrt{21})^n,$$

where $n \geq 1$ and $\epsilon = \pm 1$.
Some examples of solutions (x_n, y_n) to the equation $3x^2 = 7y^2 - 17$ with $\epsilon = 1$ are $(x_1, y_1) = (860, 563)$, $(x_2, y_2) = (94592, 61925)$, $(x_3, y_3) = 10\,404\,260, 6\,811\,187)$ and $(x_4, y_4) = (1\,144\;374\,008, 749\,168\,645)$. Examples of solutions with $\epsilon = -1$ are $(x_1, y_1) = (20, 13)$, $(x_2, y_2) = (2192, 1435)$, $(x_3, y_3) = (241100, 157837)$ and $(x_4, y_4) = (26\,518\,808, 17\,360\,635)$.

Solution 4 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Put $z = 3x$ and rewrite $3x^2 = 7y^2 + 17$ as $z^2 - 21y^2 = 51$. This is a Pell-like equation with two classes of solutions with initial solutions (z, y) of $(24, 5)$ and $(60, 13)$. The fundamental solution of the corresponding Pell equation $z^2 - 21y^2 = 1$ is $55 + 12\sqrt{21}$. From Pell theory, solutions can be generated via $z_{i+1} + y_{i+1}\sqrt{21} = (z_i + y_i\sqrt{21})(55 + 12\sqrt{21})$. This is equivalent to

$$z_{i+1} = 55z_i + 252y_i$$
$$y_{i+1} = 12z_i + 55y_i$$

or back substituting

$$x_{i+1} = 55x_i + 84y_i$$
$$y_{i+1} = 36x_i + 55y_i.$$

Thus (x_{i+1}, y_{i+1}) is a larger pair of positive integer solutions to $3x^2 = 7y^2 + 17$ than the positive integer solutions (x_i, y_i). The initial solutions are listed below and separated by class.

i	x_i	y_i	x_i	y_i
1	8	5	20	13
2	860	563	2192	1435
3	94592	61925	241100	157837
4	10404260	6811187	26518808	17360635
5	1144374008	749168645	2916827780	1909512013

Editor's notes: **Kenneth Korbin, proposer of problem 5469** stated the following:

Given the sequence $t = (\cdots, 33, 7, 2, 3, 13, \cdots)$ with $5t_N = t_{N-1} + t_{N+1}$. Let a and b be a pair of consecutive terms in this sequence with both odd. If $x = \dfrac{a+b}{2}, y = \dfrac{a-b}{2}$ then $3x^2 = 7y^2 + 17$. Example : $x = \dfrac{33+7}{2}, y = \dfrac{33-7}{2}$.

Brian D. Beasley of Presbyterian College in Clinton, SC noted that using Brahmagupta's identity (see [1]), which notes that if $x_1^2 - Ny_1^2 = k_1$ and $x_2^2 - Ny_2^2 = k_2$, then

$$(x_1x_2 + Ny_1y_2)^2 - N(x_1y_2 + x_2y_1)^2 = k_1k_2.$$

Since $55^2 - \left(\frac{7}{3}\right) 36^2 = 1$, if x and y are positive integers with $x^2 - \left(\frac{7}{3}\right) y^2 = \frac{17}{3}$, then

$$(55x + \tfrac{7}{3} \cdot 36y)^2 - \tfrac{7}{3}\,(55y + 36x)^2 = \tfrac{17}{3}.$$

Hence the solution (x, y) produces the larger solution $(55x + 84y, 55y + 36x)$.

Reference:

[1] https://en.wikipedia.org/wiki/Brahmagupta%27s_identity

David Stone and John Hawkins, both at Georgia Southern University in Statesboro, GA approached the problem by looking at the graph of the given statement as a hyperbola, with the question asking us to prove that this graph contains infinitely many lattice points with both coordinates being integers. They did this and then listed a few of the lattice points, two of them being $(94592, 61925)$ and $(241100, 157837)$. They went on to state the following:

The asymptotes of the given hyperbola are $y = \pm\sqrt{\dfrac{3}{7}}x \approx 0.6546536707x$. Our lattice points demonstrate the closeness of the curve to the asymptote. We compute $\dfrac{y}{x}$:

$$\frac{6129}{94592} \approx 0.6546536705$$

$$\frac{157837}{241100} \approx 0.6546536707.$$

Very close! We'd need a piece of graph paper the size of a football field to see that these points are on the hyperbola but not on the asymptote.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony Bevelacqua, University of North Dakota, Grand Forks, ND; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Athens, Greece; Trey Smith, Angelo State University, San Angelo, TX; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Anna V. Tomova, Varna, Bulgaria, and the proposer.

5470: *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

Prove that there are an infinite number of Heronian triangles (triangles whose sides and area are natural numbers), whose side lengths are three consecutive natural numbers.

Solution 1 by Kenneth Korbin, New York, NY

Given a Heronian Triangle with consecutive integer length sides $(b-1, b, b+1)$. Then, a larger such triangle has sides (b^2-3, b^2-2, b^2-1), and another such triangle has sides $\left(-1+2b+\sqrt{3b^2-12}, 2b+\sqrt{3b^2-12}, 1+2b+\sqrt{3b^2-12}\right)$.

Therefore there are infinitely many such triangle. Examples, $(3,4,5), (13,14,15)$, etc.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

Let $(n-1, n, n+1)$ be the sides of the triangle. Then the semiperimeter is $s = \dfrac{3n}{2}$. The area is given by Heron's formula as:

$$\sqrt{s(s-n+1)(s-n)(s-n-1)} = \frac{n}{4}\sqrt{3(n+2)(n-2)}. \qquad (1)$$

Clearly n must be even if (1) represents an integer. We must show that there are infinitely many pairs of integers (m, n) such that $3(2n+2)(2n-2) = 4m^2$ or equivalently $m^2 - 3n^2 = -3$. We see that m must be divisible by 3 and we need therefore to find pairs of integers (m, n) such that $n^2 - 3m^2 = 1$. This is Pell's equation whose general solution is given by $n - \sqrt{3}m = \left(2-\sqrt{3}\right)^k$, where k is an integer. We conclude that (1) is an integer if and one if $n = \left(2-\sqrt{3}\right)^k + \left(2+\sqrt{3}\right)^k,\ k = 0, 1, 2,$ etc.

Solution 3 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Athens, Greece

A triangle whose sides and area are rational numbers is called a *rational triangle.* If the rational triangle is right-angled, it is called a *right-angled rational triangle* or a *rational Pythagorean triangle* or *a numerical right triangle.* If the sides of a rational triangle is of integer length, it is called an *integer triangle.* If further these sides have no common factor greater than unity, the triangle called *primitive integer triangle.* If the integer triangle is right-angled, it is called a Pythagorean triangle. A *Heronian triangle* (named after Heron of Alexandria) is an integer triangle with the additional property that its area is also an integer. A Heronian triangle is called a *primitive Heronian triangle* if sides have no common factor greater than unity. In the 7^{th} century, the Indian mathematician Brahmagupta studied the special case of triangles with consecutive integer sides.

Assume that the consecutive sides of a Brahmagupta triangle are $d-1, d, d+1$, where $d > 4$ is a positive integer. The semiperimeter is $s = \dfrac{3d}{2}$, and by Heron's formula the area A is

$$A = \frac{d}{2}\sqrt{3\left[\left(\frac{d}{2}\right)^2 - 1\right]}. \qquad (1)$$

But A must also be an integer, then the base d of the triangle must be even and the altitude h of the triangle must be an integer multiple of 3 since $16A^2 = 3d^2(d^2-4)$. Since $A = \dfrac{dh}{2}$, this equation reduces to

$$4h^2 = 3(d^2 - 4). \tag{2}$$

If d were odd, then the factors on the right side of (2) would all be odd, a contradiction. Thus d is even and we may write $d = 2y$ (y is a positive integer). The area of the triangle is then $A = hy$, and it follows that h is a rational number. But

$$h^2 = 3(y^2 - 4). \tag{3}$$

is an integer and h itself has to be an integer and hence a multiple of 3. If $h = 3x$, we reduce (3) to the Pell equation $y^2 - 3x^2 = 1$. The Pell equation has an infinity of integer solutions. If $(x, y) = (U, T)$, where $T > 0, U > 0$ the solution with least positive x, all the solutions are given by

$$xy\sqrt{3} = \pm\left(T + U\sqrt{3}\right)^n,$$

where n is an arbitrary integer (Mordell, 1969, p. 53).

[1] Mordell, L.J. (1969). *Diophantine equations.* London Academic Press, Inc.

Solution 4 by Paul M. Harms, North Newton, KS

Let n be a natural number and let $n - 1$ and $n + 1$ be the sides of the triangle. To be a triangle, n must be at least 3. If s is the semi-perimeter, the area of this triangle is

$$\sqrt{s(s-(n-1))(s-n)(s-(n+1))} = \sqrt{\frac{3n}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}+1\right)} = \frac{n}{2^2}\sqrt{3\left(n^2-4\right)}.$$

For area to be a natural number one requirement is that $3\left(n^2 - 4\right)$ be the square of a natural number. Since 3 is a factor inside the square root we need $3\left(n^2 - 4\right) = (3t)^2$ for some natural number t. The last equation is equivalent to $n^2 - 3t^2 = 4$. Letting $n = 2x$ and $t = 2y$ where x and y are natural numbers, the equation becomes $x^2 - 3y^3 = 1$ which is a Pell equation. One solution is $x = x_1 = 2$ and $y = y_1 = 1$. There exist an infinite number of solutions of natural numbers found by equating coefficients of the equation $x_k + y_k\sqrt{3} = \left(2 + \sqrt{3}\right)^k$ for $k = 1, 2, 3, 4$, etc.

For example, when $k = 3, x_3 = 26$ and $y_3 = 15$. In this case, the n for the triangle given above is $n = 2x_3 = 52$. We $n = 2x_k$ we need to make sure that the area is a natural number. Replacing n by $2x_k$ in the area formula, we obtain

$$\frac{x_k}{2}\sqrt{4(3)\left(x_k^2 - 1\right)} = x_k\sqrt{3\left(x_k^2 - 1\right)} = x_k\left(3y_k\right).$$

Thus the area is the product of natural numbers so it is a natural number.

Editor′s comments: **Bruno Salgueiro Fanego of Viveiro, Spain** mentioned in his solution that:
"Discussions of the more general problem of finding all the Heronian triangles whose side lengths are in arithmetic progression can be found, for example, in the articles Heron Triangles with Sides in Arithmetic Progression and Heronian Triangles with Sides in Arithmetic Progression: An Inradius Perspective by J. A. MacDougall and by Herb

Bailey and William Gosnell, respectively."
(http://www.jstor.org/stable/10.4169/math.mag.85.4.290);
Mathematics Magazine Vol. 85, No. 4 (October 2012), pp. 290-294.

This generalization was also mentioned in the solution submitted by **David Stone and John Hawkins both of Georgia Southern University in Statesboro, GA.** They showed in their solution that all primitive Heronian triangles with sides in arithmetic progression had to have one as the difference in the side lengths. Having a common difference greater than 1 produced a Heronian Triangle, but not a primitive one.

Brian D. Beasley of Presbyterian College in Clinton, SC also stated that this problem is well-known, as noted in [1] (below), and that the sequence $\{n_i\}$ (giving the infinitely many Heronian triangles with side lengths $(n_i - 1, n_i, n_i + 1)$, where $\{n_i\}$ is defined by

$$n_1 = 4,\ n_2 = 14, \text{ and } n_{i+2} = 4n_{i+1} - n_i \text{ for } i \geq 1$$

may also be given in the closed form

$$n_i = (2 + \sqrt{3})^i + (2 - \sqrt{3})^i.$$

Reference:

[1] H. W. Gould, A Triangle with Integral Sides and Area, *The Fibonacci Quarterly*, Vol. 11(1) 1973, 27-39.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego of Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Trey Smith, Angelo State University, San Angelo, TX; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposers.

5471: *Proposed by Arkady Alt, San Jose, CA*

For natural numbers p and n where $n \geq 3$ prove that

$$n^{\frac{1}{n^p}} > (n+p)^{\frac{1}{(n+1)(n+2)(n+3)\cdots(n+p)}}.$$

Solution 1 by Moti Levy, Rehovot, Israel

The function $f(x) = x^{\frac{1}{x}}$ is strictly monotone decreasing for $x \geq 3 > e$, since
$f'(x) = x^{\frac{1}{x}} \frac{1}{x^2}(1 - \ln x) < 0, \quad for \quad x > e$. Hence $n + p > n$ implies

$$n^{\frac{1}{n}} > (n+p)^{\frac{1}{(n+p)}}.$$

It follows that

$$\left(n^{\frac{1}{n}}\right)^{\frac{1}{n^{p-1}}} > \left((n+p)^{\frac{1}{(n+p)}}\right)^{\frac{1}{n^{p-1}}},$$

or

$$\left(n^{\frac{1}{n}}\right)^{\frac{1}{n^{p-1}}} = n^{\frac{1}{n^p}} > (n+p)^{\frac{1}{(n+p)n^{p-1}}}.$$

To complete the solution, we note that

$$(n+p)^{\frac{1}{n^{p-1}(n+p)}} > (n+p)^{\frac{1}{(n+1)(n+2)\cdots(n+p)}}.$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

We prove the equivalent inequality

$$\frac{\ln n}{n^p} > \frac{\ln(n+p)}{(n+1)(m+2)\cdots(n+p)}, \qquad (1)$$

by induction on p.

For $x \geq 3$ let $f(x) = \dfrac{\ln x}{x}$. Since $f'(x) = \dfrac{1-\ln x}{x^2} < 0$, so $f(x)$ is strictly decreasing. Hence $f(n) > f(n+1)$ and so (1) is true for $p = 1$. Suppose that (1) is true for $p = k \geq 1$. By the induction assumption, we have

$$\begin{aligned}
\frac{\ln n}{n^{k+1}} &= \frac{1}{n}\left(\frac{\ln n}{n^k}\right) > \frac{\ln(n+k)}{n(n+1)(n+2)\cdots(n+k)} = \\
&= \frac{\ln(n+k+1)}{(n+1)(n+2)\cdots(n+k)(n+k+1)} + \frac{k\ln(n+k)}{n(n+1)(n+2)\cdots(n+k)^2} + \\
&\quad + \frac{1}{(n+1)(n+2)\cdots(n+k)}\left(f(n+k) - f(n+k+1)\right) \\
&> \frac{\ln(n+k+1)}{(n+1)(n+2)\cdots)n+k)(n+k+1)}.
\end{aligned}$$

Thus (1) is true for $p = k+1$ as well and hence true for all positive integers p.

Also solved by Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5472: *Proposed by Francisco Perdomo and Ángel Plaza, both at Universidad Las Palmas de Gran Canaria, Spain*

Let α, β, and γ be the three angles in a non-right triangle. Prove that

$$\frac{1+\sin^2\alpha}{\cos^2\alpha} + \frac{1+\sin^2\beta}{\cos^2\beta} + \frac{1+\sin^2\gamma}{\cos^2\gamma} \geq \frac{1+\sin\alpha\sin\beta}{1-\sin\alpha\sin\beta} + \frac{1+\sin\beta\sin\gamma}{1-\sin\beta\sin\gamma} + \frac{1+\sin\gamma\sin\alpha}{1-\sin\gamma\sin\alpha.}$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We prove more generally that

$$\frac{1+a^2}{1-a^2} + \frac{1+b^2}{1-b^2} + \frac{1+c^2}{1-c^2} \geq \frac{1+ab}{1-ab} + \frac{1+bc}{1-bc} + \frac{1+ca}{1-ca}, \text{ if } 0 \leq a, b, c < 1.$$

The special case follows by putting $a = \sin\alpha, b = \sin\beta, c = \sin\gamma$, with $\alpha + \beta + \gamma = \pi$.

Indeed,

$$\frac{1}{2}\cdot\frac{1+x^2}{1-x^2}+\frac{1}{2}\cdot\frac{1+y^2}{1-y^2}-\frac{1+xy}{1-xy} \;=\; \frac{(x-y)^2(1+xy)}{(1-x^2)(1-y^2)(1-xy)}\geq 0.$$

So

$$\frac{1+a^2}{1-a^2}+\frac{1+b^2}{1-b^2}+\frac{1+c^2}{1-c^2}-\frac{1+ab}{1-ab}-\frac{1+bc}{1-bc}-\frac{1+ca}{1-ca}=\left(\frac{1}{2}\cdot\frac{1+a^2}{1-a^2}+\frac{1}{2}\cdot\frac{1+b^2}{1-b^2}-\frac{1}{2}\cdot\frac{1+ab}{1-ab}\right)+$$
$$\left(\frac{1}{2}\cdot\frac{1+b^2}{1-b^2}+\frac{1}{2}\cdot\frac{1+c^2}{1-c^2}-\frac{1}{2}\cdot\frac{1+bc}{1-bc}\right)+\left(\frac{1}{2}\cdot\frac{1+c^2}{1-c^2}+\frac{1}{2}\cdot\frac{1+a^2}{1-a^2}-\frac{1}{2}\cdot\frac{1+ca}{1-ca}\right)\geq 0.$$

Solution 2 by Moti Levy, Rehovot, Israel

Let $a=\sin\alpha,\ b=\sin\beta$ and $c=\sin\gamma$.
Then the inequality becomes

$$\frac{1+a^2}{1-a^2}+\frac{1+b^2}{1-b^2}+\frac{1+c^2}{1-c^2}\geq\frac{1+ab}{1-ab}+\frac{1+bc}{1-bc}+\frac{1+ca}{1-ca},$$

and since $1+\dfrac{2x}{1-x^2}=\dfrac{1+x^2}{1-x^2}$, then it is equivalent to the following inequality,

$$\frac{a^2}{1-a^2}+\frac{b^2}{1-b^2}+\frac{c^2}{1-c^2}\geq\frac{ab}{1-ab}+\frac{bc}{1-bc}+\frac{ca}{1-ca},\quad 0\leq a,b,c,<1.$$

The function

$$f\left(x\right):=\frac{x^2}{1-x^2}\tag{1}$$

is monotone increasing and convex in $0\leq x<1$, since $f^{'}\left(x\right)=\dfrac{2x}{\left(1-x^2\right)^2}\geq 0$, and $f^{''}\left(x\right)=2\dfrac{3x^2+1}{\left(1-x^2\right)^3}>0$ for $0\leq x<1$.

By definition (1) the right hand side is

$$\frac{ab}{1-ab}+\frac{ca}{1-ca}+\frac{bc}{1-bc}=f\left(\sqrt{ab}\right)+f\left(\sqrt{ca}\right)+f\left(\sqrt{bc}\right),$$

and the left hand side is

$$\frac{a^2}{1-a^2}+\frac{b^2}{1-b^2}+\frac{c^2}{1-c^2}=f\left(a\right)+f\left(b\right)+f\left(c\right).$$

Without loss of generality, we may assume that $a\geq b\geq c$. Then the vector (a,b,c) *majorizes* the vector $\left(\dfrac{a+b}{2},\dfrac{c+a}{2},\dfrac{b+c}{2}\right)$.

By the *Majorizing Inequality*,

$$f\left(a\right)+f\left(b\right)+f\left(c\right)\geq f\left(\frac{a+b}{2}\right)+f\left(\frac{c+a}{2}\right)+f\left(\frac{b+c}{2}\right).\tag{2}$$

By the AM-GM inequality $\dfrac{a+b}{2}\geq\sqrt{ab}$, $\dfrac{c+a}{2}\geq\sqrt{ca}$ and $\dfrac{b+c}{2}\geq\sqrt{bc}$. The function $f\left(x\right)$ is monotone increasing, hence

$$f\left(\frac{a+b}{2}\right)+f\left(\frac{c+a}{2}\right)+f\left(\frac{b+c}{2}\right)\geq f\left(\sqrt{ab}\right)+f\left(\sqrt{ca}\right)+f\left(\sqrt{bc}\right).\tag{3}$$

Inequalities (2) and (3) imply the required result.
In order to make this solution self-contained, the definition of majorizing and the Majorizing Inequality are explained here.
The explanations are excerpted from a nice short article by Murray S. Klamkin (1921-2004) who was one of the greatest problems composer.
M. S. Klamkin, *On a "Problem of the Month"*, Crux Mathematicorum, Volume 28, Number 2, page 86, 2002.
"If A and B are vectors $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)$ where $a_1 \geq a_2 \geq \ldots \geq a_n$, $b_1 \geq b_2 \geq \ldots \geq b_n$, and $a_1 \geq b_1,\ a_1 + a_2 \geq b_1 + b_2$, $a_1 + a_2 + \cdots + a_{n-1} \geq b_1 + b_2 + \cdots + b_{n-1}$, $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$, we say that A majorizes B and write it as $A \succ B$. Then, if F is a convex function,

$$F(a_1) + F(a_2) + \cdots + F(a_n) \geq F(b_1) + F(b_2) + \cdots + F(b_n)."$$

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Soumava Chakraborty, Kolkata, India; Pedro Acosta De Leon, Massachusetts Institute of Technology Cambridge, MA; Bruno Salgueiro Fanego, Viveiro, Spain. Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Athens, Greece, and the proposers.

5473: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $x_1, \cdots, x_n$ be positive real numbers. Prove that for $n \geq 2$, the following inequality holds:

$$\left(\sum_{k=1}^{n} \frac{\sin x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right)\left(\sum_{k=1}^{n} \frac{\cos x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right) \leq \frac{1}{2}\sum_{k=1}^{n}\frac{1}{x_k}.$$

(Here the subscripts are taken modulo n.)

Solution 1 by Moti Levy, Rehovot, Israel

The following three facts will be used in this solution:
1)

$$\left(\sum_{k=1}^{n} a_k \sin x_k\right)\left(\sum_{k=1}^{n} a_k \cos x_k\right) \leq \frac{1}{2}\left(\sum_{k=1}^{n} a_k\right)^2. \tag{4}$$

This can be shown by expanding the left hand side and using the facts that $\sin x_k \cos x_k \leq \frac{1}{2}$ and $\sin x_j \cos x_k + \cos x_j \sin x_k \leq 1$.
2)

$$\left(\sum_{k=1}^{n} \frac{\sqrt{a_k}}{n}\right)^2 \leq \sum_{k=1}^{n} \frac{a_k}{n}. \tag{5}$$

This is implied from $M_{\frac{1}{2}} \leq M_1$ where M_k are power means.
3)

$$\frac{1}{px + qy} \leq \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right), \quad p, q \geq 0 \ \ \textit{and} \ \ p + q = 1. \tag{6}$$

This can be shown by Jensen's inequality.

Now let

$$a_k := \frac{1}{\left((n-1)\,x_k + x_{k+1}\right)^{\frac{1}{2}}}.$$

Then

$$\begin{aligned} LHS &:= \left(\sum_{k=1}^{n} \frac{\sin x_k}{\left((n-1)\,x_k + x_{k+1}\right)^{\frac{1}{2}}}\right)\left(\sum_{k=1}^{n} \frac{\cos x_k}{\left((n-1)\,x_k + x_{k+1}\right)^{\frac{1}{2}}}\right) \\ &= \left(\sum_{k=1}^{n} a_k \sin x_k\right)\left(\sum_{k=1}^{n} a_k \cos x_k\right) \le \frac{1}{2}\left(\sum_{k=1}^{n} a_k\right)^2. \end{aligned}$$

By (5),

$$\begin{aligned} LHS &\le \frac{1}{2}\left(\sum_{k=1}^{n} a_k\right)^2 \le \frac{n}{2}\sum_{k=1}^{n} a_k = \frac{n}{2}\sum_{k=1}^{n} \frac{1}{(n-1)\,x_k + x_{k+1}} \\ &= \frac{1}{2}\sum_{k=1}^{n} \frac{1}{\frac{n-1}{n}x_k + \frac{1}{n}x_{k+1}}. \end{aligned}$$

Set $p = \frac{n-1}{n}$ and $q = \frac{1}{n}$, then by (6)

$$\frac{1}{2}\sum_{k=1}^{n} \frac{1}{\frac{n-1}{n}x_k + \frac{1}{n}x_{k+1}} \le \frac{1}{4}\sum_{k=1}^{n}\left(\frac{1}{x_k} + \frac{1}{x_{k+1}}\right) = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{x_k}.$$

Solution to 2 by Kee-Wai Lau, Hong Kong, China

Since $2ab \le a^2 + b^2$ for any real numbers a and b, so by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & 2\left(\sum_{k=1}^{n} \frac{\sin x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right)\left(\sum_{k=1}^{n} \frac{\cos x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right) \\ \le\ & \left(\sum_{k=1}^{n} \frac{\sin x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right)^2 + \left(\sum_{k=1}^{n} \frac{\cos x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right)^2 \\ \le\ & \left(n\sum_{k=1}^{n} \frac{\sin^2 x_k}{(n-1)x_k + x_{k+1}} + \sum_{k=1}^{n} \frac{\cos^2 x_k}{(n-1)x_k + x_{k+1}}\right) \\ =\ & n\sum_{k=1}^{n} \frac{1}{(n-1)x_k + x_{k+1}}. \end{aligned}$$

Applying Jensen's inequality to the convex function $\frac{1}{x}$ for $x > 0$, we have

$$\frac{n-1}{x_k} + \frac{1}{x_{k+1}} \ge n\left(\frac{1}{\frac{(n-1)x_k + x_{k+1}}{n}}\right) = \frac{n^2}{(n-1)x_k + x_{k+1}}.$$

It follows that $n\sum_{k=1}^{n}\frac{1}{(n-1)x_k+x_{k+1}} \le \frac{1}{n}\left(\sum_{k=1}^{n}\frac{n-1}{x_k}+\sum_{k=1}^{n}\frac{1}{x_{k+1}}\right)=\sum_{k=1}^{n}\frac{1}{x}.$

Thus the inequality of the problem holds.

Solution 3 by Arkady Alt , San Jose, CA

By Cauchy Inequality $\sum_{k=1}^{n}\frac{\sin x_k}{((n-1)\,x_k+x_{k+1})^{1/2}} \le \sqrt{\sum_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}}\cdot\sqrt{\sum_{k=1}^{n}\sin^2 x_k}$

and $\sum_{k=1}^{n}\frac{\cos x_k}{((n-1)\,x_k+x_{k+1})^{1/2}} \le \sqrt{\sum_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}}\cdot\sqrt{\sum_{k=1}^{n}\cos^2 x_k}.$

Also by AM-GM inequality

$$\sqrt{\sum_{k=1}^{n}\sin^2 x_k}\cdot\sqrt{\sum_{k=1}^{n}\cos^2 x_k}\le\frac{1}{2}\left(\sum_{k=1}^{n}\sin^2 x_k+\sum_{k=1}^{n}\cos^2 x_k\right)=\frac{1}{2}\sum_{k=1}^{n}\left(\sin^2 x_k+\cos^2 x_k\right)=\frac{n}{2}.$$

Thus, $\left(\sum_{k=1}^{n}\frac{\sin x_k}{((n-1)\,x_k+x_{k+1})^{1/2}}\right)\left(\sum_{k=1}^{n}\frac{\cos x_k}{((n-1)\,x_k+x_{k+1})^{1/2}}\right)\le\frac{n}{2}\sum_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}$

and it remains to prove the inequality

$$\frac{n}{2}\sum_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}\le\frac{1}{2}\sum_{k=1}^{n}\frac{1}{x_k}\iff\sum_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}\le\frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k}.$$

By the Cauchy Inequality

$$((n-1)\,x_k+x_{k+1})\left(\frac{n-1}{x_k}+\frac{1}{x_{k+1}}\right)\ge n^2\iff\frac{1}{(n-1)\,x_k+x_{k+1}}\le\frac{1}{n^2}\left(\frac{n-1}{x_k}+\frac{1}{x_{k+1}}\right)$$

then $\sum_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}\le\frac{1}{n^2}\sum_{k=1}^{n}\left(\frac{n-1}{x_k}+\frac{1}{x_{k+1}}\right)=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k}.$

Also solved by Ed Gray, Highland Beach, FL; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5474: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b \in \Re, b \neq 0$. Calculate

$$\lim_{n\to\infty}\begin{pmatrix}1-\dfrac{a}{n^2} & \dfrac{b}{n}\\[2ex] \dfrac{b}{n} & 1+\dfrac{a}{n^2}.\end{pmatrix}^n.$$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $M = \begin{pmatrix} 1 - \dfrac{a}{n^2} & \dfrac{b}{n} \\ \dfrac{b}{n} & 1 + \dfrac{a}{n^2} \end{pmatrix}$. The eigenvalues of M are $1 \mp \dfrac{\sqrt{b^2n^2 + a^2}}{n^2}$.

Since they are distinct, M is diagonalizable. It can be obtained by following the diagonalization of M : $\quad M = PDP^{-1}$, where $D = \begin{pmatrix} 1 - \dfrac{\sqrt{b^2n^2 + a^2}}{n^2} & 0 \\ 0 & 1 + \dfrac{\sqrt{b^2n^2 + a^2}}{n^2} \end{pmatrix}$ is the diagonal matrix whose principal diagonal are its eigenvalues and

$P = \begin{pmatrix} \dfrac{-a - \sqrt{b^2n^2 + a^2}}{bn} & \dfrac{-a + \sqrt{b^2n^2 + a^2}}{bn} \\ 1 & 1 \end{pmatrix}$ is the matrix whose columns are the respective eigenvectors which form a basis of R^2. Hence,

$$M^n = P \cdot D^n \cdot P^{-1}$$

$$= \begin{pmatrix} \frac{-a-\sqrt{b^2n^2+a^2}}{bn} & \frac{-a+\sqrt{b^2n^2+a^2}}{bn} \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \left(1 - \frac{\sqrt{b^2n^2+a^2}}{n^2}\right)^n & 0 \\ 0 & \left(1 + \frac{\sqrt{b^2n^2+a^2}}{n^2}\right)^n \end{pmatrix} \cdot$$

$\cdot \begin{pmatrix} \frac{-bn}{2\sqrt{b^2n^2+a^2}} & \frac{-a+\sqrt{b^2n^2+a^2}}{2\sqrt{b^2n^2+a^2}} \\ \frac{bn}{2\sqrt{b^2n^2+a^2}} & \frac{a+\sqrt{b^2n^2+a^2}}{2\sqrt{b^2n^2+a^2}} \end{pmatrix}$, that is $M^n = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ where

$$m_{11} = m_{22} = \frac{an^2}{2\sqrt{b^2n^6 + a^2n^4}} \left(\left(1 - \sqrt{\left(\tfrac{b}{n}\right)^2 + \left(\tfrac{a}{n^2}\right)^2}\right)^n - \left(1 + \sqrt{\left(\tfrac{b}{n}\right)^2 + \left(\tfrac{a}{n^2}\right)^2}\right)^n \right) + \tfrac{1}{2} \left(\left(1 - \sqrt{\left(\tfrac{b}{n}\right)^2 + \left(\tfrac{a}{n^2}\right)^2}\right)^n +$$

and

$$m_{12} = m_{21} = \frac{bn^3}{2\sqrt{b^2n^6 + a^2n^4}} \left(\left(1 + \sqrt{\left(\tfrac{b}{n}\right)^2 + \left(\tfrac{a}{n^2}\right)^2}\right)^n - \left(1 - \sqrt{\left(\tfrac{b}{n}\right)^2 + \left(\tfrac{a}{n^2}\right)^2}\right)^n \right).$$

Thus, as $n \to \infty$,

$$m_{11} = m_{22} \sim \frac{an^2}{2|b|n^3} \left(e^{-n\sqrt{\left(\frac{b}{n}\right)^2 + \left(\frac{a}{n^2}\right)^2}} - e^{n\sqrt{\left(\frac{b}{n}\right)^2 + \left(\frac{a}{n^2}\right)^2}} \right) + \frac{1}{2} \left(e^{-n\sqrt{\left(\frac{b}{n}\right)^2 + \left(\frac{a}{n^2}\right)^2}} + e^{n\sqrt{\left(\frac{b}{n}\right)^2 + \left(\frac{a}{n^2}\right)^2}} \right)$$

$$\underset{n \to \infty}{\sim} \frac{a}{2|b|n} \left(e^{-|b|} - e^{|b|}\right) + \frac{1}{2} \left(e^{-|b|} + e^{|b|}\right) \underset{n \to \infty}{\sim} 0 + \frac{1}{2} \left(e^{-|b|} + e^{|b|}\right) = \cosh |b|$$

and as $n \to \infty$

$$m_{12} = m_{21} \sim \frac{bn^3}{2|b|n^3}\left(e^{n\sqrt{\left(\frac{b}{n}\right)^2+\left(\frac{a}{n^2}\right)^2}} - e^{-n\sqrt{\left(\frac{b}{n}\right)^2)+\left(\frac{a}{n^2}\right)^2}}\right)$$

$$\sim \frac{b}{2|b|n}\left(e^{|b|} - e^{-|b|}\right) = \frac{b}{|b|}\sinh|b|, \text{ so,}$$

$$\lim_{n\to\infty} M^n = \begin{pmatrix} \cosh|b| & \frac{b}{|b|}\sinh|b| \\ \frac{b}{|b|}\sinh|b| & \cosh|b| \end{pmatrix}.$$

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA

Let

$$A = \begin{pmatrix} 1-\dfrac{a}{n^2} & \dfrac{b}{n} \\ \dfrac{b}{n} & 1+\dfrac{a}{n^2} \end{pmatrix}.$$

The eigenvalues of A are

$$\lambda_+ = 1 + \frac{1}{n^2}\sqrt{a^2+n^2b^2} \quad \text{and} \quad \lambda_- = 1 - \frac{1}{n^2}\sqrt{a^2+n^2b^2}.$$

Because $b \neq 0$, these two eigenvalues are distinct, which implies that A is diagonalizable. An eigenvector associated with λ_+ is

$$\begin{pmatrix} b \\ \dfrac{1}{n}(a+\sqrt{a^2+n^2b^2}) \end{pmatrix},$$

and an eigenvector associated with λ_- is

$$\begin{pmatrix} b \\ \dfrac{1}{n}(a-\sqrt{a^2+n^2b^2}) \end{pmatrix}.$$

Thus, if we set

$$P = \begin{pmatrix} b & b \\ \dfrac{1}{n}(a+\sqrt{a^2+n^2b^2}) & \dfrac{1}{n}(a-\sqrt{a^2+n^2b^2}) \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1+\dfrac{1}{n^2}\sqrt{a^2+n^2b^2} & 0 \\ 0 & 1-\dfrac{1}{n^2}\sqrt{a^2+n^2b^2} \end{pmatrix},$$

then $A = PDP^{-1}$ and $A^n = PD^nP^{-1}$. Now,

$$\lim_{n\to\infty} P = \begin{pmatrix} b & b \\ |b| & -|b| \end{pmatrix} \quad \text{and} \quad \lim_{n\to\infty} P^{-1} = -\frac{1}{2b|b|}\begin{pmatrix} -|b| & -b \\ -|b| & b \end{pmatrix}.$$

Moreover,

$$D^n = \begin{pmatrix} \left(1+\dfrac{1}{n^2}\sqrt{a^2+n^2b^2}\right)^n & 0 \\ 0 & \left(1-\dfrac{1}{n^2}\sqrt{a^2+n^2b^2}\right)^n \end{pmatrix},$$

so

$$\lim_{n\to\infty} D^n = \begin{pmatrix} e^{|b|} & 0 \\ 0 & e^{-|b|} \end{pmatrix}.$$

Thus,

$$\begin{aligned} \lim_{n\to\infty} A^n &= -\frac{1}{2b|b|}\begin{pmatrix} b & b \\ |b| & -|b| \end{pmatrix}\begin{pmatrix} e^{|b|} & 0 \\ 0 & e^{-|b|} \end{pmatrix}\begin{pmatrix} -|b| & -b \\ -|b| & b \end{pmatrix} \\ &= \begin{pmatrix} \cosh|b| & \frac{b}{|b|}\sinh|b| \\ \frac{b}{|b|}\sinh|b| & \cosh|b| \end{pmatrix} \\ &= \begin{pmatrix} \cosh b & \sinh b \\ \sinh b & \cosh b \end{pmatrix}. \end{aligned}$$

Remark: This problem is very similar to Problem 1113 from the November 2017 issue of *The College Mathematics Journal.*

Editor′s comment : **David Stone and John Hawkins both at Georgia Southern University in Statesboro, GA** accompanied their solution with the following comment: "At first, we accidentally used $1-\dfrac{a}{n^2}$ in the (2, 2) spot of the matrix and the limit was the same. Perhaps there is a lot of flexibility about this term (since the limit does not depend upon a)."

Also solved by Ulich Abel, Technische Hochschule Mittelhessen, Germany; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Francisco Perdomo and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Anna V. Tomova, Varna, Bulgaria, and the proposer.

Late Solutions

A late solution was received from **Paul M. Harms of Newton, KS to problem 5467.**

Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before June 15, 2018

5493: *Proposed by Kenneth Korbin, New York, NY*

Convex quadrilateral $ABCD$ is inscribed in a circle with diameter $\overline{AC} = 729$. Sides $\overline{AB}$ and $\overline{CD}$ each have positive integer length. Find the perimeter if $\overline{BD}$=715.

5494: *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

If $a \geq b \geq c \geq d$ are the lengths of four segments from which an infinite number of convex quadrilaterals can be constructed, calculate the maximal product of the lengths of the diagonals in such quadrilaterals.

5495: *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" School Buzău, Romania*

Let $\{x_n\}_{n\geq 1},\ x_1 = 1,\ x_n = 1 \cdot \sqrt{3!!} \cdot \sqrt[3]{5!!} \cdot \ldots \sqrt[n]{(2n-1)!!}$.
Find:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}}\right).$$

5496: *Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Let a, b, c be real numbers such that $0 < a < b < c$. Prove that:

$$\sum_{cyclic}\left(e^{a-b} + e^{b-a}\right) \geq 2a - 2c + 3 + \sum_{cyclic}\left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

5497: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

For all integers $n \geq 2$, show that $\displaystyle\prod_{k=1}^{n-1} 2 \sin\left(\frac{k\pi}{n}\right)$ is an integer and determine it.

5498: *Proposed by Ovidiu Furdui and Alina Sîntămărian, both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Prove that

$$\sum_{n=1}^{\infty} \frac{\{n!e\}}{n} = \int_0^1 \frac{e^x - 1}{x} dx$$

where $\{a\}$ denotes the fractional part of a.

Solutions

5475: *Proposed by Kenneth Korbin, New York, NY*

Given positive integers a, b, c and d such that $\begin{cases} a+b = 14\sqrt{ab-48}, \\ b+c = 14\sqrt{bc-48}, \\ c+d = 14\sqrt{cd-48}, \end{cases}$ with $a < b < c < d$.

Express the values of b, c, and d in terms of a.

Solution 1 by David E. Manes, Oneonta, NY

If $a = 1 < b = 97 < c = 18817 < d = 3650401$, then it is easily verified that these positive integers satisfy the system of equations. For the first equation $a + b = 14\sqrt{ab-48}$, square both sides, simplify and write the equation as a quadratic in b. Then one obtains $b^2 - (194a)b + (a^2 + 9408) = 0$ with roots $b = 97a \pm 56\sqrt{3(a^2-1)}$. Note that if $a = 1$, then $b = 97$. For the second equation, by symmetry, $c = 97b \pm 56\sqrt{3(b^2-1)}$. If $b = 97$, then $c = 97^2 \pm 56\sqrt{3(97^2-1)} = 18817 \,\text{or}\, 1$. Therefore, $c = 18817$ since $b < c$. Writing c in terms of a, we first note that

$$\begin{aligned} b^2 &= \left(97a \pm 56\sqrt{3(a^2-1)}\right)^2 \\ &= 18817a^2 \pm 10864a\sqrt{3(a^2-1)} - 9408. \end{aligned}$$

Therefore,

$$\begin{aligned} c &= 97b \pm 56\sqrt{3(b^2-1)} \\ &= 97\left(97a \pm 56\sqrt{3(a^2-1)}\right) \pm 56\sqrt{3\left(18817a^2 \pm 10864a\sqrt{3(a^2-1)} - 9409\right)}. \end{aligned}$$

If $a = 1$, then $c = 18817$ using the positive radicals. At this point, let $\alpha = 56\sqrt{3(a^2-1)}$ and $\beta = 56\sqrt{3\left(18817a^2 \pm 10864a\sqrt{3(a^2-1)} - 9409\right.}$. With these substitutions, the equation for c reads $c = 97^2 a \pm 97\alpha \pm \beta$ and if $a = 1$, then $\alpha = 0$ and $\beta = 9408$. For the last equation, $d = 97c \pm 56\sqrt{3(c^2-1)}$ again by symmetry. If $c = 18817$, then $d = 3650401$ or 97 and $c < d$ implies $d \neq 97$. Expressing d in terms of a, observe that

$$\begin{aligned} c^2 &= \left(97^2 a \pm 97\alpha \pm \beta\right)^2 \\ &= 97^4 a^2 + 97^2\alpha^2 + \beta^2 \pm 2 \cdot 97^3 a\alpha \pm 2 \cdot 97^2 a\beta \pm 2 \cdot 97\alpha\beta. \end{aligned}$$

Therefore,

$$\begin{aligned} d &= 97c \pm 56\sqrt{3(c^2-1)} \\ &= 97^3a \pm 97^2\alpha \pm 97\beta \pm 56\sqrt{3\left(97^4a^2+97^2\alpha+\beta^2 \pm 2\cdot 97^3 a\alpha \pm 2\cdot 97^2 a\beta \pm 194\alpha\beta - 1\right)}. \end{aligned}$$

Finally, note that if $a = 1$, then $\alpha = 0, \beta = 9408$ and $d = 3650401$ for the positive signs. This completes the solution.

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

Squaring the equation $x + y = 14\sqrt{xy - 48}$ with x, y positive integers and $x < y$, yields the quadratic equation of $y = y(x)$.

$$y^2 - 194xy + x^2 + 9408 = 0 \qquad (1)$$

The discriminant Δ is $\Delta = 37632(x^2 - 1)$. Since x and y are positive integers, hence $x \geq 1$ and $y_1 = 97x + 56\sqrt{3(x^2-1)}$ or $y_2 = 97x - 56\sqrt{3(x^2-1)}$.

Nevertheless, there is $x < y_2$ which holds for $1 \leq x < 7$. Applying (1) to the given system there is

$b_2 = 97a - 56\sqrt{3(a^2-1)},\ c_2 = 97b_2 - 56\sqrt{3(b_2^2-1)}$ and $d_2 = 97c_2 - 56\sqrt{3(c_2^2-1)}$, with $1 \leq a < 7$.

(1) For $a = 1$, then $b_2 = 97, c_2 = 1$ a contradiction.

(2) For $a = 2$, then $b_2 = 26$, $c_2 = 2$, a contradiction.

(3) For $a = 3, 4, 5$, or 6, b_2 is not an integer. So, the solution (a, b_2, c_2, d_2) is rejected.

Furthermore, there is $x < y_1$, which holds for $x \geq 1$. We may generalize the problem: given positive integers $\{x_i\}_{i=1}^{n}$ such that $x_i + x_{i+1} = 14\sqrt{x_i x_{i+1} - 48}$ with $x_i < x_{i+1}$, then we have the recursive relation

$$x_{i+1} = 97x_i + 56\sqrt{3(x_i^2 - 1)}.$$

Again, applying (1) to the given system there are the following recursive relations:

$b_1 = 97a+56\sqrt{3(a^2-1)}, c_1 = 97b_1+56\sqrt{3(b_1^2-1)}$ and $d_1 = 97c_1+56\sqrt{3(c_1^2-1)}$, with $a \geq 1$.

So, we may list some solutions:

a	b_1	c_1	d_1
1	97	18817	3650401
2	362	70226	13623482
7	1351	262087	50843527
26	5042	978122	189750626
97	18817	3650401	708158977

We can assume that equation (1) is a Diophantine equation. Then, possible solutions are $(x, y) = (1, 97),\ (2, 26),\ (92, 362),\ (26, 5042),\ (97, 18817)$. Equation (1) reduces to a Diophantine equation of the Pell type. We may write $x^2 - 194xy + y^2$ in the matrix form $\begin{pmatrix} x, & y \end{pmatrix} \begin{pmatrix} 1 & -97 \\ -97 & 1 \end{pmatrix} \begin{pmatrix} x, \\ y \end{pmatrix}$. This matrix has eigen vectors $(1, \pm 1)$, which leads us to consider $u_1 = x + y$ and $v_1 = y - x$. Then, equation (1) becomes $49v_1^2 - 48u_1^2 + 9408 = 0$. Since $9408{=}2^6 \cdot 3 \cdot 7^2$, this impies that $u_1 = 7u$ and $v_1 = 12v$ and so, $u^2 - 3v^2 = 4$, a Pell equation. The Pell equation has an infinity of integer solutions in general and the Pell equation implies

$$u = \left(2-\sqrt{3}\right)^n + \left(2+\sqrt{3}\right)^n, \qquad v = \frac{\sqrt{3}}{3}\left[\left(2+\sqrt{3}\right)^n - \left(2-\sqrt{3}\right)^n\right],$$

with $n \in N$, or,

$$x = \frac{u_1 - v_1}{2} = \frac{7u - 12v}{2} = \frac{1}{2}\left[\left(7+4\sqrt{3}\right)\left(2-\sqrt{3}\right)^n + \left(7-4\sqrt{3}\right)\left(2+\sqrt{3}\right)^n\right].$$

So, we may list some solutions:

n	x	y	c_1	d_1
1	2	362	70226	13623482
2	1	97	18817	3650401
3	2	362	70226	13623482
4	7	1351	262087	50843527
5	26	5042	978122	189750626
6	97	18817	3650401	708158977
7	362	70226	13623482	2642885282
8	1351	262087	50843527	9863382151
9	5042	978122	189750626	36810643322
10	18817	3650401	708158977	137379191137

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that either $(a, b, c, d) = (2, 26, 5042, 978122)$ or

$$\left(\frac{(2+\sqrt{3})^k + (2-\sqrt{3})^k}{2},\ 97a + 56\sqrt{3(a^2-1)},\ 18817a + 10864\sqrt{3(a^2-1)},\right.$$
$$\left. 3650401a + 2107560\sqrt{3(a^2-1)}\right),$$

for nonnegative integers k, there are k solutions.

Squaring both sides of the given equations, we obtain respectively,

$$\begin{aligned} b^2 - 194ab + a^2 + 9408 &= 0, \qquad (1) \\ c^2 - 194bc + b^2 + 9408 &= 0, \qquad (2) \\ d^2 - 194cd + c^2 + 9408 &= 0. \qquad (3) \end{aligned}$$

By subtracting (1) from (2), we obtain $(c - a)(c + a - 194b) = 0$. Since $c > a$, so $c = 194b - a$. Similarly by subtracting (2) from (3), we obtain $d = 194c - b = 37635b - 194a$. From (1), we obtain $b = 97a \pm 56\sqrt{3(a^2-1)}$. Since b is an integer, so $3(a^2 - 1)$ is a square, which leads to the Pell Equation

$a^2 - 3t^2 = 1$. It is well known that a must be of the form $\dfrac{(2+\sqrt{3})^k + (2-\sqrt{3})^k}{2}$.
We first suppose that: $b = 97a - 56\sqrt{3(a^2-1)}$, with $a > 1$. Since $b > a$, we deduce with some algebra that $a < 7$. This gives $a = 2$ and hence $b = 26$, $c = 5042,\ d = 978122$.
We next suppose that $b = 97a + 56\sqrt{3(a^2-1)}$, where $a \geq 1$. We the obtain the solutions stated at the beginning and this completes the solution.

Solution 4 by Kenneth Korbin (the proposer) NewYork, NY

Sequence $x = (1, 2, 7, 26, 97, 362, 1351, \ldots, x_N, \ldots)$ with $\mathrm{x}_{N+1} = 4x_N - x_{N-1}$

$$\begin{aligned} x_N + x_{N+4} &= 14x_{N+2} \\ &= 14\sqrt{(x_N)(x_{N+4}) - 48} \end{aligned}$$

$$\begin{aligned} 1 + 97 &= 14(7) = 14\sqrt{(1)(97) - 48} \\ 2 + 362 &= 14(26) = 14\sqrt{(2)(362) - 48} \\ 7 + 1351 &= 14(97) = 14\sqrt{(7)(1351) - 48} \\ \text{etc.} & \end{aligned}$$

If $a = x_N$, then
$b = x_{N+4},\ c = x_{N+8},\ d = x_{N+12}$,

Sequence

$$\begin{aligned} y &= (x_N, x_{N+4}, x_{N+8}, x_{N+12}, x_{N+16}, \cdots) \\ y &= (a, b, c, d, x_{N+16}, x_{N+20}, \cdots) \end{aligned}$$

$c = 194b - a$
$ac - b^2 = 9408$
Therefore, $c = \dfrac{b^2 + 9408}{a}$
$194b - a = \dfrac{b^2 + 9408}{a} = c$
Therefore, $b = 97a + 56\sqrt{3a^2 - 3}$
$c = 194b - a$
Therefore, $c = 18817a + 210864\sqrt{3a^2 - 3}$
$d = 194c - b$
Therefore, $= 36590401a + 2107560\sqrt{3a^2 - 3}$

Editor's Comment : As with previous problems, **David Stone and John Hawkins of Georgia Southern University in Statesboro, GA** attached comments about the problem and their solution that I believe are both informative and instructive to the readership. They are listed below.

Comment 1: The points $(a, b), (b, c)$ and (c, d) all lie on the hyperbola $x^2 - 194xy + y^2 = -9408$.

This is actually the hyperbola $48x^2 - 49y^2 = 4704$, after a 45-degree rotation.
Our hyperbola lies in the first quadrant; it asymptotes are $y = \left(4\sqrt{3}+7\right)^2 x$ and $y = \dfrac{1}{\left(4\sqrt{3}+7\right)^2}x$, and all of the points $(a, b), (b, c)$, and (c, d) lie very close to the steep asymptote whose equation is "almost" $y = 194x$.

Comment 2: Our function f should probably be denoted $f_+(x) = 97x + 56\sqrt{3\,(x^2-1)}$. (See solutions 1, 2 or 3 for the motivation of its derivation.) The companion function $f_-(x) = 97x - 56\sqrt{3\,(x^2-1)}$ actually serves as the inverse of f with suitable restrictions on the domain: $f_+ : [1, \infty) \to [97, \infty)$ is a bijection and $(f_+)^{-1} = f_- : [97, \infty) \to [1, \infty)$. Without the ordering conditions on a, b, c and d, we could use f_+ and f_- randomly to generate solutions based upon an appropriate value for a.

Also solved by Ed Gray, Highland Beach, FL; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.

5476: *Proposed by Ed Gray, Highland Beach, FL*

Find all triangles with integer area and perimeter that are numerically equal.

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY

If we assume that any triangle satisfying our condition must have integer sides, then this is an old problem, that of finding all *equable Heronian triangles* [1], [2].

The solution is that *there exist only five triangles with numerically equal area and perimeter—those with sides* $(6, 8, 10), (5, 12, 13), (6, 25, 29), (7, 15, 20), (9, 10, 17)$. (The first two are the only right triangles.)

Prielipp [3] has proved that a triangle has equal area and perimeter if and only if it can be circumscribed about a circle of radius 2. Kilmer [4] uses Prielipp's result to generate triangles of equal area and perimeter. Markowitz [5] shows that there exists an infinite number of right triangles having rational side lengths for which the area equals the perimeter. Bates [6] has shown that a right triangle ΔABC with $\angle C = 90^0$ has numerically equal area and perimeter if and only if $a + b - c = 4$.

References

[1] "Heronian Triangle," *Wikipedia* article.

[2] "Equable Shape," *Wikipedia* article.

[3] "Area = Perimeter," Robert Prielipp, *Math.Teacher* **78** (February 1985), 90; 127.

[4] "Triangles of Equal Area and Perimeter and Inscribed Circles," Jean Kilmer, *Math.Teacher* **71** (January 1988), 65-70

[5] "Area = Perimeter," Lee Markowitz, *Math.Teacher* **74** (March 1981), 222-223.

[6] "Serendipity on the Area of a Triangle," Madelaine Bates, *Math.Teacher* **72** (April 1979), 273-275.

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

A triangle whose sides and area are rational numbers is called a rational triangle. If the rational triangle is right-angled, it is called a right-angled rational triangle or a numerical right triangle. If the sides of a rational triangle is of integer length, it is called an integer triangle. If further these sides have no common factor greater than unity, the triangle called primitive integer triangle. A Heronian triangle (named after Heron of Alexandria) is an integer triangle with the additional property that its area is also an integer [1].

In 1904, W.A. Whitworth and D. Biddle proved that there are only five Heronian triangles with integer sides whose area equals the perimeter, namely (9,10,17), (7,15,20), (6,25,29), (6,8,10) and (5,12,13) [2]. These Heronian triangles are called perfect triangles. In 1955, R.R. Phelps suggested the problem of finding all Heronian triangles whose integer valued sides add up to twice its area [3]. N. J. Fine solved the last problem and found that there is only one such triangle, namely (3,4,5) [4]. Subbarao [5] proved that the number of integer triangles, whose integer valued-sides add up to λ times their area A is finite, namely $P = \lambda A$, with $\lambda > 0$ and P is perimeter. Furthermore, he showed that the number of integer triangles for $\lambda > \sqrt{8}$ is zero, while he didn't determine the particular values of sides of integer triangles for $\lambda \leq \sqrt{8}$ [specifically, he mentioned other articles about perfect triangles and the right-angled triangle (3,4,5)]. A similar problem concerns the search of the number of Heronian triangles whose integer valued-sides add up to $\frac{1}{n}$ times their area A, namely $A = nP$, with $n \in N$. Markov [6], [7] found an algorithm for the listing of all Heronian triangles with the property $A = nP$. In 1985, Goehl [8] solved that particular problem in the special case of right triangles. In addition, the (3,4,5) right-angled triangle is the integer-sided triangle for which the ratio $\frac{A}{P}$ is a rational number less than 1, it actually has the smallest such ratio [9].

In the usual notation, we have from the hypothesis and the classical area formula of Heron $2s = m$

$$\sqrt{s(s-a)(s-b)(s-c)}. \qquad (1)$$

With $x = s - a, y = s - b, z = s - c$, we have $s = x + y + z$ and (1) becomes

$$4(x + y + z) = xyz. \qquad (2)$$

Suppose (without loss of generality) $x \leq y \leq z$. Then $3 < x$ implies $xyz \geq 16z$ and $x + y + z \leq 3z$ so that

$$4(x + y + z)) \leq 12z < 16z \leq xyz,$$

and (2) would be impossible. Hence, we need try only $x = 1, 2, 3$. (a) For $x = 1$, (2) becomes $(y - 4)(z - 4) = 20$. For $y \geq 9$, then $y - 4 \geq 5$ and $(y - 4)(z - 4) \geq 25$, a contradiction. Furthermore, for $y \leq 4$ and $y = 7$, a contradiction. So, for $y = 5, 6, 8$, we have the following three perfect triangles:
$T_1 = (6, 25, 29), T_2 = (7, 15, 20), T_3 = (9, 10, 17)$.

(b) For $x = 2$, (2) becomes $(y - 2)(z - 2) = 8$. For $y \geq 5$, then $y - 2 \geq 3$ and $(y - 2)(z - 2) \geq 9$, a contradiction. Furthermore, for $y = 1, 2$, a contradiction. So, for $y = 3, 4$ we have the following two perfect triangles: $T_4 = (5, 12, 13)$ and $T_5 = (6, 8, 10)$. Notice that each of the pairs (T_1, T_5) and (T_3, T_5) have a common side. These pairs can be placed along their common sides to form a large triangle in each case [10]. In particular, the perfect triangle T_4 and T_5 are right-angled triangles [11].

(c) For $x = 3$, (2) becomes $(3y - 4)(3z - 4) = 52$. Then, $y \neq 3$, since $3y - 4 = 5$ is not a factor of 52. Furthermore, $y \leq 3$, since $4 \leq y \leq z$ implies $8 \leq 3y - 4$, whence

$(3y-4)(3-4) \geq 64$, a contradiction.

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Comment submitted by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

There are only five examples of triangles with integer sides for which the area and perimeter are equal and integer-valued. These are the triangles for which $(a,b,c) = (5,12,13)$, $(6,8,10)$, $(6,25,29)$, $(7,15,20)$, and $(9,10,17)$. These can be found by using an algorithm described in Reference [1].

However, it's possible to find an infinite number of examples of triangles where at least one side is irrational and yet the area and perimeter are equal and integer-valued. For example, if n is an integer and $n \geq 4$, let $a = 2n - \sqrt{n^2-12}$, $b = 2n$, and $c = 2n + \sqrt{n^2-12}$. When $n = 4$, $(a,b,c) = (6,8,10)$. It is easily shown that when $n > 4$, $n^2 - 12$ cannot be a perfect square and hence, a and c are irrational. We note also that for all $n \geq 4$, $a < b < c$ and $a + b - c = 2\left(n - \sqrt{n^2-12}\right) > 0$. Consequently, we have $a < b < c$ and $a + b > c$, which guarantees that there is a non-degenerate triangle with sides a, b, and c. For this triangle, the perimeter $P = 6n$ and the semi-perimeter $s = 3n$. Then, Heron's Formula for the area A yields

$$\begin{aligned}
A &= \sqrt{s\,(s-a)\,(s-b)\,(s-c)} \\
&= \sqrt{(3n)\left(n+\sqrt{n^2-12}\right)(n)\left(n-\sqrt{n^2-12}\right)} \\
&= \sqrt{3n^2\left[n^2-(n^2-12)\right]} \\
&= \sqrt{36n^2} \\
&= 6n \\
&= P.
\end{aligned}$$

References:

[1] T. Leong, D. Bailey, E. Campbell, C. Diminnie, and P. Swets, *Another Approach to Solving $A = mP$ for Triangles*, Mathematics Magazine **80,** pp. 363 - 368, 2007**.**

Editor's comment : Some readers asked if the side lengths had to be integers, and from the history of the problem we can see that that was the intent originally. But as mentioned in the comment by Bailey, Campbell, Diminnie, and Smith, the side lengths need not be integers and this is the territory where the solution of Stone and Hawkins took them.

Daivd Stone and John Hawkins of Georgia Southern University, seem to have rediscovered a version of the result cited by Bailey, et.al., that the side lengths need not be rational. In their solution they stated and proved the following algorithm:

For any integer $P \geq 21$, there are infinitely many triangles with $A = P$. All such triangles are given by the following prescription;

Let $P \geq 21$ be an integer. Choose b such that $b > 4$ and $2b^3 - Pb^2 + 16P \leq 0$.

Compute $z = b^2 - \dfrac{16P}{P-2b}$.

Let $a = \dfrac{P-b}{2} - \dfrac{1}{2}\sqrt{z}$ and c= $\dfrac{P-b}{2} + \dfrac{1}{2}\sqrt{z}$.

Then a, b, c form a triangle with area and perimeter P.

(After verifying the above algorithm they presented the following examples.)

Example 1: The first example has two irrational sides, but still has $A = P =$ integer.

Let $P = 26$. Then we must choose b such that $2b^3 - 26b^2 + 432 \leq 0$.

That is, $5.146 < b < 11.399$.

Let $b = 8$, then $z = \dfrac{112}{5}$, so $\sqrt{z} = \dfrac{4\sqrt{35}}{5}$.

Thus the other two sides of our triangle are

$a = \dfrac{26-8}{2} - \dfrac{1}{2}\dfrac{4\sqrt{35}}{5} = 9 - \dfrac{2\sqrt{35}}{5} = \dfrac{45-2\sqrt{35}}{5} \approx 6.6336$

and

$c = \dfrac{45+2\sqrt{35}}{5} \approx 11.3664$

Example 2: This example demonstrates that the minimum value for $P, 21$ is actually achieved.

Let $P = 21$. Then we must choose b such that $2b^3 - 21b^2 + 336 \leq 0$.

That is, $6.405 < b < 7.562$.

Let $b=7$. Then $z=1$ so $a=7-\dfrac{1}{2}=\dfrac{13}{2}$ and $=7+\dfrac{1}{2}=\dfrac{15}{2}$.

Here we have the triangle $\left(\dfrac{13}{2},\dfrac{14}{2},\dfrac{15}{2}\right)$ with rational, non-integer sides with $A=P=21$.

Example 3: Note that the previous example used $b=P/3$. This is always a valid value for b. In this case, we have the triangle

$\left(\dfrac{P}{3}-\dfrac{1}{6}\sqrt{P^2-432},\dfrac{P}{3},\dfrac{P}{3}+\dfrac{1}{6}\sqrt{P^2-432}\right)$

which has $A=P$. This triangle has rational sides only for $P=21,24,31,39,56$ and 109.

Also solved by Kee-Wai Lau, Hong Kong, China David E. Manes, Oneonta, NY; Albert Stadler, Herrliberg, Switzerland, and the proposer

5477: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Sevrin, Meredinti, Romania*

Compute:

$$L=\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}\frac{1-\sqrt{1+x^2}\,\sqrt[3]{1+x^2}\cdot\ldots\cdot\sqrt[n]{1+x^2}}{x^2}\right).$$

Solution 1 by Ed Gray, Highland Beach, FL

We rewrite the expression as:

1. $\displaystyle\lim_{x\to 0}\frac{\left[1-(1+x^2)^{\frac{1}{2}+\frac{1}{3}+\frac{1}{n}+\ldots+\frac{1}{n}}\right]}{x^2}$.

2. Let $N=\displaystyle\sum_{k=2}^{k=n}\frac{1}{k}$, i.e., the harmonic series -1

3. Now consider $\displaystyle\lim_{x\to 0}\frac{\left[1-\left(1+x^2\right)^N\right]}{x^2}$.

We expand $(1+x^2)^N$ by the Binomial Theorem:

4. $(1+x^2)^N=1+Nx^2+\dfrac{N(N-1)}{2!}x^4+\ldots$

Then

5. $\displaystyle\lim_{x\to 0}\frac{\left[1-\left(1+Nx^2+\frac{N(N-1)}{2}x^4+\ldots\right)\right]}{x^2}$, or

6. $\displaystyle\lim_{x\to 0}\frac{\left[-Nx^2+\frac{-N(N-1)}{2}x^4+\ldots\right]}{x^2}=\frac{-Nx^2}{x^2}=-N.$

The original limit becomes

7. $\displaystyle\lim_{n\to\infty}(\ln(n)-N)=\lim_{n\to\infty}\left(\ln(n)-\sum_{k=2}^{k=n}\frac{1}{k}\right)=\lim_{n\to\infty}(\ln(n)+1-\text{Harmonic series}).$

The Euler-Mascheroni Constant is defined as $\gamma=\displaystyle\lim_{n\to\infty}\textit{The Harmonic series}-\ln(n)$.

Therefore our expression in step 7 equals $1-\gamma$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Since

$$\lim_{x\to 0}\frac{1-\left(1+x^2\right)^{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}}{x^2}=\lim_{x\to 0}\frac{1-\left(1+x^2\right)^{H_n-1}}{x^2}\left[\frac{0}{0}=Indet.\right]\underset{L'Hospital}{=}$$

$$\lim_{x\to 0}\frac{0-\left(H_n-1\right)\left(1+x\right)^{H_n-2}2x}{2x}=\left(1-H_n\right)\lim_{x\to 0}\left(1+x^2\right)^{H_n-2}=1-H_n,$$

$$L=\lim_{n\to\infty}\left(\ln n+1-H_n\right)=1-\lim_{n\to\infty}\left(H_n-\ln n\right)=1-\gamma,$$

where H_n is the n-th harmonic number and γ is the Euler-Mascheroni constant.

Solution 3 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

$$\sqrt{1+x^2}\,\sqrt[3]{1+x^2}\cdots\sqrt[n]{1+x^2}=(1+x^2)^{\frac{1}{2}+\frac{1}{3}+\frac{1}{n}}=(1+x^2)^{H_n-1}$$

We have

$$\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right)$$

Now

$$\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}=-H_n+1$$

thus

$$L=\lim_{n\to\infty}(\ln-\ln n-\gamma+o(1)+1)=-\gamma+1$$

Solution 4 by Julio Cesar Mohnsam and Mateus Gomes Lucas, both from IFSUL, Campus Pelots-RS, Brazil, and Ricardo Capiberibe Nunes of E.E. Amlio de Caravalho Bas, Campo Grande-MS, Brazil

$$L=\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right)=\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}(1-H_n)(1+x^2)^{H_n-2}\right).$$

because,

$$\lim_{x\to 0}\frac{[1-(1+x^2)^{H_n-1}]}{(x^2)}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{[1-(1+x^2)^{H_n-1}]'}{(x^2)'}=\lim_{x\to 0}(-H_n+1)(1+x^2)^{H_n-2}=-H_n+1$$

Therefore:

$$L=\lim_{n\to\infty}(\ln n-H_n+1)=\lim_{n\to\infty}(\ln n-H_n)+1=1+\lim_{n\to\infty}(\ln n-H_n)=1-\gamma$$

Note: γ is Euler-Mascheroni constant and $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$.

Also solved by Yen Tung Chung, Taichung, Taiwan; Serban George Florin, Romania; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel;

Angel Plaza, University of Las Palmas de Granada Canaria Spain; Ravi Prakash, New Delhi, India; Henry Ricardo, Westchester Area Math Circle, NY; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Shivam Sharma, New Delhi, India; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5478: *Proposed by D. M. Btinetu-Giurgiu, "Matei Basarab" National Collge, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzu, Romania*

Compute:

$$\int_0^{\pi/2} \cos^2 x \left(\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)\right) dx.$$

Solution 1 by Karl Havlak, Angelo State University, San Angelo, TX

Let $u = \cos x$. Then $du = -\sin x dx$ and $\sin(x) = \sqrt{1-u^2}$. We may rewrite the given integral as

$$\int_0^1 \left(u^2 \sin^2\left(\frac{\pi}{2}u\right) + \frac{u^3}{\sqrt{1-u^2}}\sin^2\left(\frac{\pi}{2}\sqrt{1-u^2}\right)\right) du.$$

Considering the second term in the integrand, we let $v = \sqrt{1-u^2}$ so that $dv = -\dfrac{u}{\sqrt{1-u^2}}du$ and $u^2 = 1-v^2$. We may write the integral above as

$$\int_0^1 u^2 \sin^2\left(\frac{\pi}{2}u\right) du + \int_0^1 (1-v^2)\sin^2\left(\frac{\pi}{2}v\right) dv.$$

This reduces to $\displaystyle\int_0^1 \sin^2\left(\frac{\pi}{2}v\right) dv$, which can be easily shown to be $\frac{1}{2}$.

Solution 2 by Moti Levy, Rehovot, Israel)

$$\begin{aligned} I &:= \int_0^{\frac{\pi}{2}} \left(\cos^2 x\right)\left(\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)\right) dx \\ &= \int_0^{\frac{\pi}{2}} \cos^2 x \sin x \sin^2\left(\frac{\pi}{2}\cos x\right) dx + \int_0^{\frac{\pi}{2}} \cos^2 x \cos x \sin^2\left(\frac{\pi}{2}\sin x\right) dx \end{aligned}$$

Change of variables, $u = \cos x$ in the first integral and $v = \sin x$ in the second integral gives

$$\begin{aligned} I &= \int_0^1 u^2 \sin^2\left(\frac{\pi}{2}u\right) du + \int_0^1 \left(1-u^2\right)\sin^2\left(\frac{\pi}{2}u\right) du \\ &= \int_0^1 \sin^2\left(\frac{\pi}{2}u\right) du = \frac{1}{2}. \end{aligned}$$

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$$I = \int_0^{\pi/2} \cos^2 x \left(\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)\right) dx$$

$$= \int_0^{\pi/2} (1-\sin^2 x)\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \int_0^{\pi/2} \cos^3 x \sin^2\left(\frac{\pi}{2}\sin x\right) dx$$

$$= I_1 - I_2 + I_3 \text{ where}$$

$$I_1 = \int_0^{\pi/2} \sin x \sin^2\left(\frac{\pi}{2}\cos x\right), \quad I_2 = \int_0^{\pi/2} \sin^3 x \sin^2\left(\frac{\pi}{2}\cos x\right) dx \text{ and}$$

$$I_3 = \int_0^{\pi/2} \cos^3 x \sin^2\left(\frac{\pi}{2}\sin x\right) dx.$$

Since

$$\begin{aligned} I_1 &= \int_0^{\pi/2} \sin x \frac{1-\cos(\pi\cos x)}{2} dx = \int_0^{\pi/2} \frac{\sin x}{2} - \frac{\sin x(\cos \pi \cos x)}{2} dx \\ &= \left[-\frac{\cos x}{2} - \frac{\sin(\pi\cos x)}{2\pi}\right]_{x=0}^{x=\pi/2} \\ &= \frac{\cos(\pi/2)}{2} - \frac{\sin(\pi\cos(\pi/2))}{2\pi} - \left(-\frac{\cos 0}{2} - \frac{\sin(\pi\cos 0)}{2\pi}\right) \\ &= 0 - 0 + \frac{1}{2} + 0 = \frac{1}{2}. \end{aligned}$$

With the substitution $t = \frac{\pi}{2} - x$, one obtains that

$$\begin{aligned} I_2 &= \int_0^{\pi/2} \sin^3 x \sin^2\left(\frac{\pi}{2}\cos x\right) dx = \int_{\pi/2}^{0} \sin^3\left(\frac{\pi}{2} - t\right) \sin^2\left(\frac{\pi}{2}\cos\left(\frac{\pi}{2} - t\right)\right)(-dt) \\ &= \int_0^{\pi/2} \cos^3 t \sin^2\left(\frac{\pi}{2}\sin t\right) dt = I_3. \end{aligned}$$

The value of the given integral is therefore $I = I_1 = \frac{1}{2}$.

Also solved by Yen Tung Chung, Taichung, Taiwan; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Ravi Prakash, New Delhi, India; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Shivam Sharma, New Delhi, India; Albert Stadler, Herrliberg, Switzerland, and the proposers.

5479: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $f : [0,1] \to \Re$ be a continuous convex function. Prove that

$$\frac{2}{5}\int_0^{1/3} f(t)dt + \frac{3}{10}\int_0^{2/3} f(t)dt \geq \frac{5}{8}\int_0^{8/15} f(t)dt.$$

Solution 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

If $c \in (0,1)$, then for all $t \in [0,1]$, $ct \in [0,c] \subset [0,1]$ and hence, $f(ct)$ is continuous on $[0,1]$. Further, by making the substitution $u = ct$, we get

$$\int_0^1 f(ct)\ dt = \frac{1}{c}\int_0^c f(u)\ du = \frac{1}{c}\int_0^c f(t)\ dt$$

and thus,

$$\int_0^c f(t)\ dt = c\int_0^1 f(ct)\ dt. \tag{1}$$

By (1),

$$\begin{aligned}
&\frac{2}{5}\int_0^{\frac{1}{3}} f(t)\ dt + \frac{3}{10}\int_0^{\frac{2}{3}} f(t)\ dt \\
&= \left(\frac{2}{5}\right)\left(\frac{1}{3}\right)\int_0^1 f\left(\frac{1}{3}t\right)\ dt + \left(\frac{3}{10}\right)\left(\frac{2}{3}\right)\int_0^1 f\left(\frac{2}{3}t\right)\ dt \\
&= \frac{1}{3}\int_0^1 \left[\frac{2}{5}f\left(\frac{1}{3}t\right) + \frac{3}{5}f\left(\frac{2}{3}t\right)\right]\ dt. \qquad (2)
\end{aligned}$$

Since $f(t)$ is convex on $[0,1]$ and $\frac{1}{3}t,\ \frac{2}{3}t \in [0,1]$ for all $t \in [0,1]$, Jensen's Theorem implies that

$$\begin{aligned}
\frac{2}{5}f\left(\frac{1}{3}t\right) + \frac{3}{5}f\left(\frac{2}{3}t\right) &\geq f\left[\left(\frac{2}{5}\right)\left(\frac{1}{3}t\right) + \left(\frac{3}{5}\right)\left(\frac{2}{3}t\right)\right] \\
&= f\left(\frac{8}{15}t\right) \qquad (3)
\end{aligned}$$

for all $t \in [0,1]$. By combining (1), (2), and (3), we obtain

$$\begin{aligned}
\frac{2}{5}\int_0^{\frac{1}{3}} f(t)\ dt + \frac{3}{10}\int_0^{\frac{2}{3}} f(t)\ dt &= \frac{1}{3}\int_0^1 \left[\frac{2}{5}f\left(\frac{1}{3}t\right) + \frac{3}{5}f\left(\frac{2}{3}t\right)\right]\ dt \\
&\geq \frac{1}{3}\int_0^1 f\left(\frac{8}{15}t\right)\ dt \\
&= \left(\frac{1}{3}\right)\left(\frac{15}{8}\right)\int_0^{\frac{8}{15}} f(t)\ dt \\
&= \frac{5}{8}\int_0^{\frac{8}{15}} f(t)\ dt.
\end{aligned}$$

Solution 2 by Michael Brozinsky, Central Islip, NY

We have by the Mean Value Theorem for Integrals that there exists constants C on $\left(0, \frac{1}{3}\right)$, D on $\left(0, \frac{2}{3}\right)$, E on $\left(0, \frac{8}{15}\right)$, S on $\left(\frac{1}{3}, \frac{2}{3}\right)$ such that

$$\int_0^{1/3} f(t)dt = \frac{1}{3}f(C), \int_0^{2/3} f(t)dt = \frac{2}{3}f(D), \int_0^{8/15} f(t)dt = \frac{8}{15}f(E) \text{ and so}$$

$$\frac{1}{5}f(S) = \int_{1/3}^{8/15} f(t)dt = \int_0^{8/15} f(t)dt - \int_0^{1/3} f(t)dt = \frac{8}{15}f(E) - \frac{1}{3}f(C) \text{ and}$$

$$\frac{2}{15}f(T) = \int_{8/15}^{2/3} f(t)dt = \int_0^{2/3} f(t)dt - \int_0^{8/15} f(t)dt = \frac{2}{3}f(D) - \frac{8}{15}f(E)dt.$$

The first of these last two equations gives $f(E) = \frac{3}{8}f(S) + \frac{5}{8}f(C)$ and so the second then gives $f(D) = \frac{1}{5}f(T) + \frac{3}{10}f(S) + \frac{1}{2}f(C)$.

The desired inequality $\frac{2}{5}\int_0^{1/3} f(t)dt + \frac{3}{10}\int_0^{12/3} f(t)dt \geq \frac{5}{8}\int_0^{8/15} f(t)dt$ can be written as

$\frac{2}{15}f(C) + \frac{3}{10}\cdot\frac{2}{3}\cdot f(D) \geq \frac{5}{8}\cdot\frac{8}{15}\cdot f(E)$, or equivalently as

$\frac{2}{15}f(C) + \frac{1}{5}\left(\frac{1}{5}f(T) + \frac{3}{10}f(S) + \frac{1}{2}f(C)\right) \geq \frac{1}{3}\left(\frac{3}{8}f(S) + \frac{5}{8}f(C)\right)$ which is equivalent to

$\left(\frac{2}{15} + \frac{1}{10} - \frac{5}{24}\right)f(C) + \frac{1}{25}f(T) \geq \left(\frac{1}{8} - \frac{3}{50}f(S)\right)$ and then to

$\frac{1}{40}f(C) + \frac{1}{25}f(T) \geq \frac{13}{200}f(S)$ and finally to

$\frac{5}{13}f(C) + \frac{8}{13}f(T) \geq f(S)$ which is true by the convexity since $C < S < T$.

Note: A function is convex on $[a, b]$ means that for all $0 \leq \lambda \leq 1$ whenever $a \leq X < Z \leq b$ we have

$f(X) + \lambda(f(Z) - f(X)) \leq f(X + \lambda(Z - X))$ which can be cast as

$(1-\lambda)f(X) + \lambda f(Z) \geq f(X + \lambda(Z - X))$ and so in the above taking $X = C,\ Z = T,$

and $\lambda = \dfrac{S - C}{T - C}$ we have $X + \lambda(Z - X) = S$.

Solution 3 by Henry Ricardo, Westchester Area Math Circle, NY

If f is convex on $[0, 1]$, then for all $x, y \in [0, 1]$ and for all $\lambda \in [0, 1]$, we have

$$\lambda f(x) + (1-\lambda)f(y) \ \geq \ f(\lambda x + (1-\lambda)y). \qquad (1)$$

Setting $\lambda = 2/5, x = t/3$, and $y = 2t/3$ in (1), $0 \leq t \leq 3/2$, we get

$$\frac{2}{5}f(t/3) \ + \ \frac{3}{5}f(2t/3) \ \geq \ f(8t/15).$$

Integrating from $t = 0$ to $t = 1$ yields

$$\frac{2}{5}\int_0^1 f(t/3)dt \ + \ \frac{3}{5}\int_0^1 f(2t/3)dt \geq \int_0^1 f(8t/15)dt. \quad (2)$$

Setting $u = t/3$ in the first integral of (2), we have

$$\frac{2}{5}\int_0^1 f(t/3)dt \;=\; \frac{6}{5}\int_0^{1/3} f(u)du.$$

Similarly, setting $u = 2t/3$ in the second integral, we get

$$\frac{3}{5}\int_0^1 f(2t/3)dt \;=\; \frac{9}{10}\int_0^{2/3} f(u)du.$$

Finally, setting $u = 8t/15$, we find that

$$\int_0^1 f(8t/15)dt \;=\; \frac{15}{8}\int_0^{8/15} f(u)du.$$

Substituting these into (2) and dividing by 3, we obtain

$$\frac{2}{5}\int_0^{1/3} f(u)du \;+\; \frac{3}{10}\int_0^{2/3} f(u)du \;\geq\; \frac{5}{8}\int_0^{8/15} f(u)du.$$

Solution 4 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

By changing variables it follows that $\alpha \displaystyle\int_0^a f(t)\ dt = \int_0^{\alpha a} f(s/\alpha)\ ds$. Therefore, the left-hand side of the proposed inequality, say LHS, is

$$\begin{aligned} LHS &= \frac{2}{5}\int_0^{1/3} f(t)\ dt + \frac{3}{5}\int_0^{1/3} f(2t)\ dt \\ &\geq \int_0^{1/3} f\left(\frac{2}{5}t + \frac{6}{5}t\right)\ dt \\ &= \int_0^{1/3} f\left(\frac{8}{5}t\right)\ dt \end{aligned}$$

where we have used that f is convex, so $\dfrac{2}{5}f(t) + \dfrac{3}{5}f(2t) \geq f\left(\dfrac{2}{5}t + \dfrac{6}{5}t\right)$. Since the right-hand side is $\dfrac{5}{8}\displaystyle\int_0^{8/15} f(t)\ dt = \int_0^{1/3} f\left(\frac{8}{5}t\right)\ dt$, the conclusion follows.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Albert Sadler, Herrliberg, Switzerland; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece, and the proposer.

5480: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n \geq 1$ be a nonnegative integer. Prove that in $C[0, 2\pi]$

$$span\{1, \sin x, \sin(2x), \ldots, \sin(nx)\} = span\{1, \sin x, \sin^2 x, \ldots, \sin^n x\}$$

if and only if $n = 1$.

We mention that $span\{v_1, v_2, \ldots, v_k\} = \sum_{j=1}^{k} a_j v_j,\ a_j \in \Re, j = 1, \ldots, k$, denotes the set of all linear combinations with $v_1, v_2, \ldots, v_k$.

Solution 1 by Moti Levy, Rehovot, Israel

If $n = 1$ then the spans are trivially equal.

Let $n > 1$. Suppose that $\sin^2 x$ can be expressed as a linear combination of the functions $\{1, sinx, sin(2x), ..., sin(nx)\}$,

$$\sin^2 x = a_0 + \sum_{k=1}^{n} a_k \sin(kx). \tag{1}$$

By setting $x = 0$, we have $a_0 = 0$.

The following definite integral vanish for integer k.

$$\begin{aligned}\int_0^{2\pi} sin^2 x \cdot sin(kx)dx &= \frac{1}{2}\int_0^{2\pi} (1 - \cos(2x)) \cdot sin(kx)dx \\ &= \frac{1}{2}\int_0^{2\pi} sin(kx)dx - \frac{1}{2}\int_0^{2\pi} \cos(2x) \cdot sin(kx)dx = 0.\end{aligned} \tag{2}$$

Now we multiply both sides of (1) by $\sin^2 x$ and integrate from 0 to 2π,

$$\int_0^{2\pi} \sin^4(x)\, dx = \sum_{k=1}^{n} a_k \int_0^{2\pi} sin^2 x \sin(kx)\, dx. \tag{3}$$

The right hand side of (3) is equal to $\frac{3}{4}\pi$ but the left hand side is equal to zero, by (2). This contradiction leads to the conclusion that $\sin^2 x$ cannot be expressed as a linear combination of the functions $\{1, sinx, sin(2x), ..., sin(nx)\}$, hence the spans are not equal for $n > 1$.

Solution 2 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

For $n = 1$ it is certainly true that $span\{1, \sin x\} = span\{1, \sin x\}$.

For $n > 1$,

$$span\{1, \sin x, \sin(2x), \sin(3x), \ldots, \sin(nx)\} \neq span\{1, \sin x, \sin^2 x, \sin^3 x, \ldots, \sin^n x\}$$

because $\sin^2 x$ cannot be written as a linear combination of $1, \sin x, \sin(2x), \sin(3x), \ldots, \sin(nx)$.

To see this, suppose that $\sin^2 x = c_0 \cdot (1) + c_1 \sin(x) + c_2 \sin(2x) + c_3 \sin(3x) + \ldots, +c_n \sin(nx)$.

Then this equation must hold for all values of x in $C\,[0, 2\pi]$.

Letting $x = 0$, shows that $c_0 = 0$.

Letting $x = \dfrac{\pi}{2}$ gives us $1 = 0 + c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot (-1) + c_4 \cdot 0 + c_5 \cdot 1 + \ldots$
or (1) $1 = c_1 - c_3 + c_5 - c_7 + c_9 + \ldots.$

Letting $x = \frac{3\pi}{2}$ gives us $1 = 0 + c_1 \cdot (-1) + c_2 \cdot 0 + c_3 \cdot 1 + c_4 \cdot 0 + c_5 \cdot (-1) + \ldots$
or (2) $1 = -c_1 + c_3 - c_5 + c_7 - c_9 + \ldots$.

The final term in each summation depends upon the of parity of n, but the terms on the right hand sides match up in any case. So, adding (1) + (2) gives us $2 = 0$, which is certainly a contradiction.

Thus, $\sin^2 x$ cannot be written in terms of $1, \sin x, \sin(2x), \sin(3x), \ldots, \sin(nx)$.

Also solved by Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece, and the proposer

Mea Culpa

The names Dionne Bailey, Elsie Campbell, and Charles Diminnie all at Angelo State University in San Angelo, TX were inadvertently omitted from the list of those who had solved problem 5470.

Paolo Perfetti of the Mathematics Department at Tor Vergata University in Rome, Italy should be credited for having solved 5472.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before October 15, 2018

5499: *Proposed by Kenneth Korbin, New York, NY*

Given a triangle with sides $(21, 23, 40)$. The sum of these digits is $2+1+2+3+4+0=12$. Find primitive pythagorean triples in which the sum of the digits is 12 or less.

5500: *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

Without the use of a calculator, show that: $8\sin 20^\circ \cdot \sin 40^\circ \cdot \sin 60^\circ \cdot \sin 80^\circ = \dfrac{3}{2}$.

5501: *Proposed by D.M. Bătinetu-Giurgiu, Bucharest, Romania, Neculai Stanciu, "George Emil Palade" School Buzău, Romania and Titu Zvonaru, Comănesti, Romania*

Determine all real numbers a, b, x, y that simultaneously satisfy the following relations:

$$\begin{cases} (1) \quad ax + by = 5 \\ (2) \quad ax^2 + by^2 = 9 \\ (3) \quad ax^3 + by^3 = 17 \\ (4) \quad ax^4 + by^4 = 33. \end{cases}$$

5502: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Prove that if $a, b, c > 0$ and $a+b+c = e$ then

$$e^{ac^e} \cdot e^{ba^e} \cdot e^{cb^e} > e^e \cdot a^{be^2} \cdot b^{ce^2} \cdot c^{ae^2}.$$

Here, $e = \lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$

5503: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers with $n \geq 2$. Prove that

$$\frac{(a_1^m a_2 + a_2^m a_3 + \cdots + a_n^m a_1)^m}{(a_1^m + a_2^m + \cdots a_n^m)^{m+1}} \leq \frac{1}{n},$$

where m is a positive integer.

5504: *Proposed by Ovidiu Furdui and Alina Sîntămărian both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n \geq 0$ be an integer. Calculate

$$\int_0^1 \frac{x^n}{\left\lfloor \frac{1}{x} \right\rfloor} dx,$$

where $\lfloor x \rfloor$ denotes the integer part of x.

Solutions

5481: *Proposed by Kenneth Korbin, New York, NY*

A triangle with integer area has integer length sides $(3, x, x+1)$. Find five possible values of x with $x > 4$.

Solution 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

For our approach, we will need to find positive integer solutions for the equation

$$m^2 - 8k^2 = 9. \tag{1}$$

One way to do so is to first solve the Pell Equation

$$X^2 - 8Y^2 = 1 \tag{2}$$

and then set $m = 3X$ and $k = 3Y$.

Following the usual process for solving (2), we note that the solution with the smallest X value is $X = 3$, $Y = 1$. Then, all solutions (X_n, Y_n) of (2) can be found by setting

$$X_n + Y_n\sqrt{8} = \left(3 + \sqrt{8}\right)^n$$

for all $n \geq 1$. Then, as described above, we get solutions for (1) by setting $m_n = 3X_n$ and $k_n = 3Y_n$. The first six solutions for (1) and (2) are listed in the following table:

n	X_n	Y_n	m_n	k_n
1	3	1	9	3
2	17	6	51	18
3	99	35	297	105
4	577	204	1,731	612
5	3,363	1,189	10,089	3,567
6	19,601	6,930	58,803	20,790

(3)

For the problem at hand, the semiperimeter s of our triangle is

$$s = \frac{3 + x + (x+1)}{2} = x + 2$$

and Heron's Formula for the area A yields

$$\begin{aligned} A &= \sqrt{s\,(s-3)\,(s-x)\,(s-x-1)} \\ &= \sqrt{(x+2)\,(x-1)\,(2)\,(1)} \\ &= \sqrt{2\,(x^2+x-2)}. \end{aligned}$$

For A to be a positive integer, we must find a positive integer k for which

$$A^2 = 2\left(x^2 + x - 2\right) = 4k^2$$

or

$$x^2 + x - 2 - 2k^2 = 0. \tag{4}$$

By the Quadratic Formula, the positive solution of (4) is

$$\begin{aligned} x &= \frac{-1 + \sqrt{1 + 8\,(k^2+1)}}{2} \\ &= \frac{-1 + \sqrt{8k^2+9}}{2}. \end{aligned}$$

For x to be a positive integer, we will need

$$8k^2 + 9 = m^2$$

or

$$m^2 - 8k^2 = 9$$

for some odd positive integer m. However, table (3) gives us six solutions to use. In each case,

$$x = \frac{-1 + \sqrt{8k^2+9}}{2} = \frac{m-1}{2} \quad \text{and} \quad A = 2k.$$

The solution $m_1 = 9$ and $k_1 = 3$ yields $x = 4$, which is ruled out in the statement of the problem. The other five entries in the table provide five plausible values of x for which A is a positive integer. These values are listed in our final table:

n	m_n	k_n	x	A
2	51	18	25	36
3	297	105	148	210
4	$1,731$	612	865	$1,224$
5	$10,089$	$3,567$	$5,044$	$7,134$
6	$58,803$	$20,790$	$29,401$	$41,580$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Let p and S be the semi perimeter and the area of such a triangle respectively. Then $2p = 3 + x + x + 1 = 2x + 4$ and, by Heron's formula
$S = \sqrt{p(p-3)(p-x)(p-(x+1))} = \sqrt{2x^2+2x-4}$ must be an integer.

It can be easily verified that for each of the five values of $x \in \{25, 148, 865, 5044, 29401\}$ one obtains triangles that have areas of $36, 210, 1224, 7134, 41580$, respectively.

More generally, if $\begin{pmatrix} x_1 \\ S_1 \end{pmatrix} = \begin{pmatrix} 25 \\ 36 \end{pmatrix}$, then the recurrence given by
$\begin{pmatrix} x_{n+1} \\ S_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x_n \\ S_n \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for any integer $n \geq 1$, gives a pair $\begin{pmatrix} x_{n+1} \\ S_{n+1} \end{pmatrix}$ where x_{n+1} is the length of a triangle with integer length sides $(3, x_{n+1}, x_{n+1}+1)$ and S_{n+1} is the integer area of that triangle.

Solution 3 by Julio Cesar Mohnsam and Luiz Lemos Junior, both at IFSUL Campus Pelotas-RS, Brazil

Let p be the semi-perimeter $p = \dfrac{3+x+x+1}{2} = x+2$
The area by Heron is given by:
$A = \sqrt{p(p-3)(p-x)(p-x-1)} = \sqrt{(x+2)(x-1)(2)(1)}$
Then $(x+2)(x-1)(2)$ must be a square, that is, $2x^2+2x-4 = y^2$, follow that:

$$2x^2 + 2x - y^2 - 4 = 0 \tag{1}$$

Multiplying (1) by 8 we have:

$$16x^2 + 16x - 8y^2 - 32 = 0 \tag{2}$$

Adding 4 on both sides of (2) we have:

$$(4x+2)^2 - 8y^2 - 36 = 0 \tag{3}$$

Now make $X = 4x + 2$ and $Y = y$, we have:

$$X^2 - 8Y^2 - 36 = 0 \tag{4}$$

(4) is a diophantine equation of the form $ax^2 - by^2 + c = 0$ in the case of $c = -1$ we have the particular case of the equation of Pell $x^2 - Dy^2 = 1$. If $(a, b)|c$, the equation has a solution. Let's solve (4) using the method Florentin Smarandache [1].
We consider the equation

$$aX^2 - bY^2 + c = 0 \tag{5}$$

and

$$a\alpha^2 - b\beta^2 = a \tag{6}$$

We set the matrix A from (6) as follows:

$$A = \begin{bmatrix} \alpha_0 & \frac{b}{a}\beta_0 \\ \beta_0 & \alpha_0 \end{bmatrix}$$

where (α_0, β_0) are initial solutions of (6).

Now let (X_0, Y_0) are initial solutions of (5), then the general solutions of (5) are given by the following recurrence relation:

$$\begin{bmatrix} X_n \\ Y_n \end{bmatrix} = A^n \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

Thus, by solving (4) we have to first assemble matrix A from $X^2 - 8Y^2 = 1$, note that this Pell equation has initial solution $(3, 1)$, so we have:

$$A = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}$$

But (4) has initial solution $(6, 0)$. So we have to:

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$$

Like $18 = X_1 = 4x_1 + 2 \rightarrow x_1 = 4$ but we have to find $x > 4$.
Thus we calculate A^2, such that:

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}^2 \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 17 & 48 \\ 6 & 17 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 102 \\ 36 \end{bmatrix}$$

As $102 = X_2 = 4x_2 + 2 \rightarrow x_2 = 25$ and the lengths of the first triangle are $(3, 25, 26)$. To find the other values of x we will diagonalize the matrix A. We know that
$A^n = PD^nP^-1$.

$$A = PDP^{-1} = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}$$

Eigenvalues $\lambda_1 = 3 + 2\sqrt{3}$ and $\lambda_2 = 3 - 2\sqrt{3}$
Eigenvectors $\vec{v}_1 = \begin{pmatrix} 2\sqrt{3} \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -2\sqrt{3} \\ 1 \end{pmatrix}$
Therefore $x = \{25, 148, 565, 5044, 29401\}$

[1] Smarandache F. **"Un metodo de resolucion de la ecuacion diofantica**. Gazeta Matematica, Serie 2, Vol. 1, Nr. 2, 1988. Madrid. p. 151-157.

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

Given such a triangle, its semiperimeter is $s = (3 + x + x + 1)/2 = x + 2$. Then by Heron's formula, its area is

$$\Delta = \sqrt{((x+2)(x-1)(2)(1)} = \sqrt{2(x^2 + x - 2)},$$

so we seek integers Δ and x with $x > 4$ such that $\Delta^2 = 2x^2 + 2x - 4$. This equation in turn is equivalent to

$$2\Delta^2 + 9 = (2x+1)^2, \quad \text{or} \quad 2\left(\frac{\Delta}{3}\right)^2 + 1 = \left(\frac{2x+1}{3}\right)^2.$$

We let $a = (2x + 1)/3$ and $b = \Delta/3$ in order to solve the Pellian equation $2b^2 + 1 = a^2$. This equation has infinitely many integer solutions for a and b, which we may describe with the sequences

$$a_0 = 1, a_1 = 3, \text{ and } a_{n+2} = 6a_{n+1} - a_n \text{ for } n \geq 0;$$
$$b_0 = 0, b_1 = 2, \text{ and } b_{n+2} = 6b_{n+1} - b_n \text{ for } n \geq 0.$$

Thus there are infinitely many integers x which satisfy the requirements of the problem, given by the terms x_n (with $n \geq 2$) of the sequence

$$x_0 = 1, x_1 = 4, \text{ and } x_{n+2} = 6x_{n+1} - x_n + 2 \ \text{ for } n \geq 0.$$

In particular, the next five values of x after 4 are 25, 148, 865, 5044, and 29401.

Addenda. (i) We may also describe the above sequences by letting $\gamma = 3 + 2\sqrt{2}$ and $\delta = 3 - 2\sqrt{2}$. Then $a_n = (\gamma^n + \delta^n)/2$ for each $n \geq 0$, which implies that $x_n = (3\gamma^n + 3\delta^n - 2)/4$.

(ii) We further note that the ratios a_n/b_n for $n \geq 1$ occur as every other term in the sequence of converging to the continued fraction representation of $\sqrt{2}$.

Comments by Editor : **Ioannis D. Sfikas of Athens Greece** started his solution off with some nomenclature and bit of history about the problem.
"A triangle whose sides and area are rational numbers is called a *rational triangle.* If the rational triangle is right-angled, it is called a *right-angled rational triangle* or a *rational Pythagorean triangle* or a *numerical right triangle.* If the sides of a rational triangle is of integer length, it is called an *integer triangle.* If further these sides have no common factor greater than unity, the triangle is called a *primitive integer triangle.* If the integer triangle is right-angled, it is called a *Pythagorean triangle. A Heronian triangle* (named after Heron of Alexandria) is an integer triangle with the additional property that its area is also an integer. A Heronian triangle is called *primitive Heronian triangle* if sides have no common factor greater than unity. In the 7th century, the Indian mathematician Brahmagupta studied the special case of triangles with consecutive integer sides."

Kenneth Korbin, the proposer of this problem stated that triangles sides with lengths $(3, x, x+1)$ with $x \geq 4$ have an area of $\sqrt{2x^2 + 2x - 4}$, and are associated with the sequences of $(25, 148, 865, 5044, 29401, \ldots, x_N, \ldots)$ that satisfies the recursion of $x_{N+1} = 6x_N - x_{N-1} + 2$.

David Stone and John Hawkins of Southern Georgia University asked in their solution, why the values of $x \geq 4$? Why not $x > 0$? They then stated that: If

$x = 1$, the triangle $(3, 1, 2)$ is degenerate;
$x = 2$, the triangle $(3, 2, 3)$ has area $\sqrt{8}$ and is not Heronian;
$x = 3$, the triangle $(3, 3, 4)$ has area $\sqrt{20}$ and is not Heronian; and
$x = 4$, the right triangle $(3, 5, 5)$ is too easy.

They then asked: What about those Heronian triangles of the form $(3, x, x+2)$, where x is an integer. Applying Heron's Formula they obtained that $16A^2 = 20x^2 + 40x - 25$ and stated that there are no integer solutions to this for $x \geq 1$ because the left-and side is even while the right hand side is odd.

They then looked at triangles of the form $(3, x, x+3)$ and stated that the only triangle of this form is degenerate. Moreover, no triangle of the form $(3, x, x+d)$ can exist for $d > 3$.

They continued on with the following:

"Thus the problem poser selected the one form that does admit solutions. Still to be investigated: finding the Heronian triangles of the form $(4, x, x+1)$, and those of the form $(4, x, x+2)$, etc."

Their solution ended with the statement: "There are no Heronian triangles of the form $(n, x, x+d)$ for positive integers n and d having opposite parity."

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Jeremiah Bartz, University of North Dakota, Grand Forks, ND; Anthony Bevelacqua University of North Dakota, Grand Forks, ND; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Carl Libis, Columbia Southern University, Orange Beach, AL; David E. Manes, Oneonta, NY; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu,"George Emil Palade" School, Buzău and Tito Zvonaru, Comănesti, Romania; David Stone and John Hawkins of Southern Georgia University, Statesboro, GA, and the proposer.

5482: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Prove that if n is a natural number then

$$\frac{(tan5^\circ)^n}{(tan4^\circ)^n+(tan3^\circ)^n}+\frac{(tan4^\circ)^n}{(tan3^\circ)^n+(tan2^\circ)^n}+\frac{(tan3^\circ)^n}{(tan2^\circ)^n+(tan1^\circ)^n}\geq\frac{3}{2}.$$

Solutions 1 and 2 by Henry Ricardo, Westchester Area Math Circle, NY

Solution 1.

Since, for a fixed natural number n, $(\tan x)^n$ is an increasing positive function for $x\in[0,90^\circ)$, we have

$$\frac{(\tan 5^\circ)^n}{(\tan 4^\circ)^n+(\tan 3^\circ)^n}\geq\frac{(\tan 5^\circ)^n}{(\tan 5^\circ)^n+(\tan 5^\circ)^n}=\frac{1}{2},$$
$$\frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n+(\tan 2^\circ)^n}\geq\frac{(\tan 4^\circ)^n}{(\tan 4^\circ)^n+(\tan 4^\circ)^n}=\frac{1}{2},$$
$$\frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n+(\tan 1^\circ)^n}\geq\frac{(\tan 3^\circ)^n}{(\tan 3^\circ)^n+(\tan 3^\circ)^n}=\frac{1}{2},$$

so that adding these inequalities gives us the desired result. Equality holds if and only if $n=0$ (assuming that 0 is considered a natural number).

Solution 2.

Since, for a fixed natural number n, $(\tan x)^n$ is an increasing positive function for $x\in[0,90^\circ)$, we have

$$\frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n+(\tan 1^\circ)^n}\geq\frac{(\tan 3^\circ)^n}{(\tan 4^\circ)^n+(\tan 5^\circ)^n},$$

and

$$\frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n+(\tan 2^\circ)^n}\geq\frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n+(\tan 5^\circ)^n},$$

so that

$$\sum_{cyclic} \frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n + (\tan 1^\circ)^n} \geq \frac{(\tan 5^\circ)^n}{(\tan 4^\circ)^n + (\tan 3^\circ)^n} + \frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n + (\tan 5^\circ)^n} + \frac{(\tan 3^\circ)^n}{(\tan 4^\circ)^n + (\tan 5^\circ)^n}.$$

Setting $a = (\tan 3^\circ)^n, b = (\tan 4^\circ)^n$, and $c = (\tan 5^\circ)^n$, we see that the right-hand side of the last inequality has the form

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b},$$

for $a, b, c > 0$, which is greater than or equal to 3/2 by Nesbitt's inequality. Equality holds if and only if $n = 0$ (assuming that 0 is considered a natural number).

Solution 3 by Ed Gray, Highland Beach, FL

First we retrieve the required values:
1. $\tan 1^\circ = .017455065$
2. $\tan 2^\circ = .034920769$
3. $\tan 3^\circ = .052407779$
4. $\tan 4^\circ = .069926812$
5. $\tan 5^\circ = .087488664$

We rewrite the problem's equation as:

$$\frac{1}{\dfrac{\tan 4^\circ}{\tan 5^\circ} + \dfrac{\tan 3^\circ}{\tan 5^\circ}} + \frac{1}{\dfrac{\tan 3^\circ}{\tan 4^\circ} + \dfrac{\tan 2^\circ}{\tan 4^\circ}} + \frac{1}{\dfrac{\tan 2^\circ}{\tan 3^\circ} + \dfrac{\tan 1^\circ}{\tan 3^\circ}} \geq \frac{3}{2}$$

Substituting the values from steps 1-5 and performing the indicated divisions we define:

$$f(n) = \frac{1}{(.799267114)^n + (.599023652)^n} + \frac{1}{(.794551256)^n + (.499433116)^n} + \frac{1}{(.66632797)^n + (.333062483)^n}.$$

We note that $f(n)$ is an increasing function of n since the denominators clearly decrease as n increases.

Finally we note that $f(1) = .715158838 + 1.248899272 + 1.000609919 = 2.964668029 > \dfrac{3}{2}$.
Then the equality holds for all n since $f(n)$ is an increasing function.

Solution 4 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

Lemma: For fixed positive reals a, b, c with $a < c,\ b < c$ let $f(x) = \dfrac{c^x}{b^x + a^x}$ for $x \geq 0$.
Then $f(x) \geq \dfrac{1}{2}$, for $x \geq 0$, with equality holding only for $x = 0$.
Proof: We calculate the derivative:

$$f'(x) \quad = \quad \frac{(b^x + a^x)\, c^x \ln c - c^x \,(a^x \ln a + b^x \ln b)}{(b^x + a^x)^2}$$

$$= \quad c^x \frac{(b^x + a^x)\ln c - (a^x \ln a + b^x \ln b)}{(b^x + a^x)^2}$$

$$= \quad c^x \frac{b^x (\ln c - \ln b) + a^x (\ln c - \ln a)}{(b^x + a^x)^2}.$$

The ln function is increasing, so $\ln c > \ln b$ and $\ln c > \ln a$; thus we see that the derivative is positive. Hence the function f is increasing, so $\frac{1}{2} = \text{f}(0) \le f(x)$ for $x \ge 0$. Because the derivative is strictly positive, the function f actually grows: so $f(x) > \frac{1}{2}$ for $x > 0$.

To verify the inequality of the problem, we note that the tangent function is increasing, so in each summand the tangent term in the numerator is larger that each tangent term in the denominator. Hence we can apply the lemma to each of the three summands, forcing the sum $\ge \frac{3}{2}$. Note that equality holds if and only if $n = 0$.

Comment: We can apply the lemma to obtain some ugly inequalities which are clearly true:

$$\frac{3^n}{1^n + 2^n} + \frac{4^n}{2^n + 3^n} + \frac{5^n}{3^n + 4^n} + \cdots + \frac{(n+2)^n}{n^n + (n+1)^n} \quad \ge \quad \frac{n}{2}, \text{ and}$$

$$\frac{[(n+2)!]^n}{[n!]^n + [(n+1)!]^n} \quad \ge \quad \frac{1}{2}.$$

Also solved by Arkady Alt, San Jose, CA (two solutions); Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University of Tor Vergata, Rome, Italy; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposers

5483: *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" School Buzău, Romania*

If $a, b > 0$, and $x \in \left(0, \frac{\pi}{2}\right)$ then show that

$$(i) \qquad (a+b) \cdot \frac{\sin x}{x} + \frac{2ab}{a+b} \cdot \frac{\tan x}{x} \ge \frac{6ab}{a+b}.$$

$$(ii) \qquad a \cdot \tan x + \ b \cdot \sin x > 2x\sqrt{ab}.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Proof of (i).

The AHM yields

$$a+b \geq \frac{4}{\frac{1}{a}+\frac{1}{b}} \quad \Longleftrightarrow \quad (a+b)^2 \geq 4ab$$

and then

$$(a+b)\cdot\frac{\sin x}{x}+\frac{2ab}{a+b}\cdot\frac{\tan x}{x} \geq \frac{4ab}{a+b}\cdot\frac{\sin x}{x}+\frac{2ab}{a+b}\cdot\frac{\tan x}{x}$$

Thus we prove

$$\frac{4ab}{a+b}\cdot\sin x+\frac{2ab}{a+b}\cdot\tan x-\frac{6ab}{a+b}x \geq 0$$

This is equivalent to

$$f(x) \doteq 4\sin x+2\tan x-6x \geq 0$$

$$f'(x)=4\cos x+\frac{2}{\cos^2 x}-6$$

$$f''(x)=-4\sin x+\frac{4\sin x}{\cos^3 x}=4\sin x\left(\frac{4}{\cos^3 x}-1\right)>0$$

via $\cos x \in (0,1)$ for $0<x<\pi/2$. Since $f'(0)=f(0)=0$ we get $f(x)\geq 0$.

Proof of (ii).
Let

$$f(x)=a\cdot\tan x+b\cdot\sin x-2x\sqrt{ab}, \quad f(0)=0$$

$$f'(x)=\frac{a}{\cos^2 x}+b\cos x-2\sqrt{ab} \geq \frac{a}{\cos x}+b\cos x-2\sqrt{ab} \geq 2\sqrt{\frac{ab\cos x}{\cos x}}-2\sqrt{ab}=0$$

and this concludes the proof.

Solution 2 by Arkady Alt, San Jose, CA

(i) First we will prove inequality
$\tan x+2\sin x>3x \iff \dfrac{\tan x}{x}+\dfrac{2\sin x}{x}>3,\ x\in(0,\pi/2)$.
Let $h(x):=\tan x+2\sin x-3x,\ x\in(0,\pi/2)$. Since $h'(x)=\dfrac{1}{\cos^2 x}+2\cos x-3=$
$\dfrac{(2\cos x+1)(1-\cos x)^2}{\cos^2 x}>0,\ x\in(0,\pi/2)$ then $h(x)>h(0)=0$.
Hence, $(a+b)\dfrac{\sin x}{x}+\dfrac{2ab}{a+b}\cdot\dfrac{\tan x}{x}>(a+b)\sin x+\dfrac{2ab}{a+b}\cdot\left(3-\dfrac{2\sin x}{x}\right)=$
$\dfrac{\sin x}{x}\left(a+b-\dfrac{4ab}{a+b}\right)+\dfrac{6ab}{a+b}=\dfrac{\sin x}{x}\cdot\dfrac{(a-b)^2}{a+b}+\dfrac{6ab}{a+b}\geq\dfrac{6ab}{a+b}$.
(ii) Let $h(x):=a\tan x+b\sin x-2x\sqrt{ab}$. Since $h'(x)=\dfrac{a}{\cos^2 x}+b\cos x-2\sqrt{ab}\geq$
$2\sqrt{\dfrac{a}{\cos^2 x}\cdot b\cos x}-2\sqrt{ab}=2\sqrt{ab}\cdot\dfrac{1-\sqrt{\cos x}}{\sqrt{\cos x}}>0$ then $h(x)>h(0)=0 \iff$
$a\tan x+b\sin x>2x\sqrt{ab}$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

It is well known that for $x \in \left(0, \frac{\pi}{2}\right)$, we have $\sin x - \frac{x^3}{6}$ and $\tan x \geq x + \frac{x^3}{3}$. Since $a + b = \frac{4ab + (a-b)^2}{a+b} \geq \frac{4ab}{a+b}$, so the left side of (i) is greater than or equal to

$$\frac{2ab}{a+b}\left(\frac{2\sin x + \tan x}{x}\right) \geq \frac{2ab}{a+b}\left(2\left(1 - \frac{x^2}{6}\right) + \left(1 + \frac{x^2}{3}\right)\right) = \frac{6ab}{a+b},$$

as required.

It is also well known that for $x \in \left(0, \frac{\pi}{2}\right)$, $\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$, so that

$$\sin^2 x = x^2 \cos x \geq \left(x - \frac{x^3}{6}\right)^2 - x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) = \frac{x^4(12 - x^2)}{72} \geq 0.$$

Hence,

$$a \cdot \tan x + b \cdot \sin x \geq 2\sqrt{(a\tan x)(b\sin x)} = 2\sqrt{ab}x\sqrt{\frac{\sin^2 x}{x^2 \cos x}} \geq 2\sqrt{ab}x$$

and (ii) holds.

Also solved by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Southern Georgia University, Statesboro, GA, and the proposers

5484: *Proposed by Mohsen Soltanifar, Dalla Lana School of Public Health, University of Toronto, Canada*

Let X_1, X_2 be two continuous positive valued random variables on the real line with corresponding mean, median, and mode $\overline{x_1}$, $\widetilde{x}_1$, $\widehat{x}_1$ and $\overline{x_2}$, $\widetilde{x}_2$, $\widehat{x}_2$ respectively. Assume for their associated CDFs, (Cumulative Distribution Functions) we have

$$F_{X_1}(t) \leq F_{X_2}(t) \qquad (t > 0).$$

Prove or give a counter example:

$$\text{(i) } \overline{x_2} \leq \overline{x_1}, \quad \text{(ii) } \widetilde{x_2} \leq \widetilde{x_1}, \quad \text{(iii) } \widehat{x}_2 \leq \widehat{x}_1.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

(i) We have $\overline{x_2} = E(X_2) = \int_0^\infty (1 - F_{X2}(t))dt \leq \int_0^\infty (1 - F_{X_1}(t))dt = E\left(X_1\right) = \overline{x_1}$.

(ii) By definition, $F_{X_1}\left(\widetilde{X_1}\right) = F_{X_2}\left(\widetilde{X_2}\right) = \frac{1}{2}$. The functions $t \to F_{X_1}(t)$ and $t \to F_{X_2}(t)$ are monotonically increasing. Therefore $F_{X_2}\left(\widetilde{X_2}\right) \leq F_{X_1}(\widetilde{X_1})$ implies $\widetilde{x_2} \leq \widetilde{x_1}$.

(iii) We construct a counter example as a follows:
Let $p_1(x) = 0$, if $x \leq 1/3$ or $x \geq 1$, $p_1(x) = 9x - 3$, if $1/3 \leq x \leq 2/3$, and $p_1(x) = 9 - 9x$, if $2/3 \leq x \leq 1$.

Let $p_2(x) = 36x$, if $0 \le x \le 1/6$, $p_2(x) = 12 - 36x$, if $1/6 \le x \le 1/3$, $p_2(x) = 0$, if $x \le 0$ or $x \ge 1/3$.

$p_1(x)$ and $p_2(x)$ are probability density functions since $\int_0^1 p_1(x)dx = \int_0^1 p_2(x)dx = 1$.

Obviously, $F_{X_1}(x) = \int_0^x p_1(t)dt \le \int_0^x p_2(t)dt = F_{X_2}(x)$, however $\widetilde{x_1} = 3$, $\widetilde{x_2} = 6$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

We answer (i) and (ii) in the affirmative and (iii) in the negative.

Let X be a continuous random variable on the real line with CDF $F_X(t)$ and mean $\overline{x}$. It is known that $\overline{x} = \int_0^\infty (1 - F_X(t))dt - \int_{-\infty}^0 F_X(t)dt$, provided at least one of the two integrals is finite. Since X_1, X_2 are positive, so $\int_{-\infty}^0 F_{X_1}(t)dt = \int_{-\infty}^0 F_{X_2}(t)dt = 0$ and (i) follows from the fact that

$$\overline{x_2} - \overline{x_1} = \int_0^\infty \left((1 - F_{X_2}(t)) - (1 - F_{X_1}(t))\right) dt = \int_0^\infty \left(F_{X_1}(t) - F_{X_2}(t)\right) dt \le 0.$$

Next we consider the medians, assuming that $\widetilde{x}$ is the least number a satisfying $F_X(a) = \frac{1}{2}$. Suppose, on the contrary, that $\widetilde{x_2} > \widetilde{x_1}$. Since $F_X(t)$ is a non-decreasing function and $x_1 < \frac{\widetilde{x_1} + \widetilde{x_2}}{2} < \widetilde{x_2}$, we have

$$F_{X_2}\left(\widetilde{x_2}\right) = \frac{1}{2} > F_{X_2}\left(\frac{\widetilde{x_1} + \widetilde{x_2}}{2}\right) \ge F_{X_1}\left(\frac{\widetilde{x_1} + \widetilde{x_2}}{2}\right) \ge F_{X_1}\left(\widetilde{x_1}\right) = \frac{1}{2},$$

which is false. This proves (ii).

We now show that (iii) does not necessarily hold. Define the probability density functions $f_{X_1}(t)$ and $f_{X_2}(t)$ of X_1 and X_2 as follows:

$$f_{X_1}(t) = \begin{cases} 0 & x \le 0, \\ \frac{x}{16} & 0 \le x \le 4, \\ \frac{8-x}{16} & 4 \le x \le 8, \\ 0 & x \ge 8 \end{cases} \quad \text{and} \quad f_{X_2}(t) = \begin{cases} 0 & x \le 0, \\ \frac{2x}{25} & 0 \le x \le 5, \\ 0 & x \ge 5. \end{cases}$$

Then

$$F_{X_1}(t)=\begin{cases} 0 & x\le 0,\\ \dfrac{x^2}{32} & 0\le x\le 4,\\ \dfrac{-x^2+16x-32}{32} & 4\le x\le 8,\\ 1 & x\ge 8 \end{cases} \quad \text{and} \quad F_{X_2}(t)=\begin{cases} 0 & x\le 0,\\ \dfrac{x^2}{25} & 0\le x\le 5,\\ 1 & x\ge 5. \end{cases}$$

It is easy to check that $F_{X_1}(t)\le F_{X2}(t)$ for $t>0$, but $\widetilde{x_1}=4<5=\widetilde{x_2}$.
This completes the solution.

Also solved by the proposer.

5485: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let x,y,z be three positive real numbers. Show that

$$\prod_{cyclic}(2x+3y+z+1)\sum_{cyclic}(4x+2y+1)^{-3}\ge 3.$$

Solution 1 by Neculai Stanciu, "George Emil Palade" School, Bazău Romania and Tito Zvonaru, Comănesti, Romania

We denote $4x+2y+1=a,\ 4y+2z+1=b$, and $4z+2x+1=c$. We must prove that

$$\frac{(a+b)(b+c)(c+a)}{8}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\ge 3 \qquad (*)$$

By the AM-GM inequality we have that

$$\frac{(a+b)(b+c)(c+a)}{8} \ge \frac{2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca}}{8}=\frac{8abc}{8}=abc, \qquad (1)$$

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3} \ge 3\cdot\sqrt[3]{\frac{1}{a^3}\cdot\frac{1}{b^3}\cdot\frac{1}{c^3}}=\frac{3}{abc}, \qquad (2)$$

By (1) and (2) we obtain

$$\frac{(a+b)(b+c)(c+a)}{8}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\ge abc\cdot\frac{3}{abc} = 3. \quad \text{I.e. } (*)$$

Solution 2 by Nikos Kalapodis, Patras, Greece

By the AM-GM inequality we have

$$\prod_{cyclic}(2x+3y+z+1)\sum_{cyclic}(4x+2y+1)^{-3}\ge\prod_{cyclic}(2x+3y+z+1)\frac{3}{\prod_{cyclic}(4x+2y+1)}.$$

So, it suffices to prove that $\prod_{cyclic}(2x+3y+z+1)\ge\prod_{cyclic}(4x+2y+1)$.

After expanding the inequality reduces to
$2(x^3+y^3+z^3)+x^2+y^2+z^2+3(xy^2+yz^2+zx^2) \geq 3(x^2y+y^2z+z^2x)+xy+yz+zx+6xyz.$

Since $x^2+y^2+z^2 \geq xy+yz+zx$, it remains to prove that
$2(x^3+y^3+z^3)+3(xy^2+yz^2+zx^2) \geq 3(x^2y+y^2z+z^2x)+6xyz.$

This follows again by using the AM-GM inequality properly:
$2(x^3+y^3+z^3)+3(xy^2+yz^2+zx^2) = 2(x^3+xy^2)+2(y^3+yz^2)+2(z^3+zx^2)+(xy^2+yz^2+zx^2) \geq 4x^2y+4y^2z+4z^2x+(xy^2+yz^2+zx^2) =$
$3(x^2y+y^2z+z^2x)+(x^2y+y^2z+z^2x+xy^2+yz^2+zx^2) \geq 3(x^2y+y^2z+z^2x)+6xyz.$

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Kevin Soto Palacios, Huarmey, Perú; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5486: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $(x_n)_{n\geq 0}$ be the sequence defined by $x_0=0, x_1=1, x_2=1$ and
$x_{n+3}=x_{n+2}+x_{n+1}+x_n+n,\ \forall n \geq 0$. Prove that the series $\displaystyle\sum_{n=1}^{\infty}\frac{x_n}{2^n}$ converges and find its sum.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

The recurrence sequence may be unmasked by generating functions. Let $F(z)$ be the associated generating function. That is, $F(z)=\displaystyle\sum_{n=0}^{\infty}x_n z^n$. Multiplying by z^{n+3} the recurrence relation defining (x_n) and taking into account the initial values it is obtained that

$$F(z)-\left(z+z^2\right)=z\left(F(z)-z\right)+z^2F(z)+z^3F(z)+\frac{z^4}{(1-z)^2}$$

from where $F(z)=\dfrac{z(1-z)^2+x^4}{(z-1)^2(1-z-z^2-z^3)}$.

Since $F(z)$ converges for $|z|<\dfrac{1}{3}\left(\sqrt[3]{17+3\sqrt{33}}-\dfrac{2}{\sqrt[3]{17+3\sqrt{33}}}-1\right)\sim 0.5436...$, then $\displaystyle\sum_{n=1}^{\infty}\frac{x_n}{2^n}=F(1/2)=6.$

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Answer: 6.

Clearly x_n increases and $x_n \geq 1$.

$$\sum_{n=1}^{p} \frac{x_n}{2^n} = \frac{x_1}{2} + \frac{x_2}{4} + \sum_{n=3}^{p} \frac{x_n}{2^n} = \frac{3}{4} + \sum_{n=0}^{p-3} \frac{x_{n+3}}{2^{n+3}} =$$

$$= \frac{3}{4} + \sum_{n=0}^{p-3} \left(\underbrace{\frac{x_{n+2}}{2^{n+3}}}_{I_1} + \underbrace{\frac{x_{n+1}}{2^{n+3}}}_{I_2} + \underbrace{\frac{x_n}{2^{n+3}}}_{I_3} \right) + \sum_{n=0}^{p-3} \frac{n}{2^{n+3}} =$$

$$= \frac{3}{4} + \underbrace{\sum_{n=2}^{p-1} \frac{x_n}{2^{n+1}}}_{I_1} + \underbrace{\sum_{n=1}^{p-2} \frac{x_n}{2^{n+2}}}_{I_2} + \underbrace{\sum_{n=1}^{p-3} \frac{x_n}{2^{n+3}}}_{I_3} + \sum_{n=0}^{p-3} \frac{n}{2^{n+3}} =$$

$$= \frac{3}{4} + \underbrace{-\frac{1}{4} + \sum_{n=1}^{p} \frac{x_n}{2^{n+1}} - \frac{x_p}{2^{p+1}}}_{I_1} + \underbrace{\sum_{n=1}^{p} \frac{x_n}{2^{n+2}} - \frac{x_{p-1}}{2^{p+1}} - \frac{x_p}{2^{p+2}}}_{I_2} +$$

$$+ \underbrace{\sum_{n=1}^{p} \frac{x_n}{2^{n+3}} - \frac{x_{p-2}}{2^{p+1}} - \frac{x_{p-1}}{2^{p+2}} - \frac{x_p}{2^{p+3}}}_{I_3} + \underbrace{\sum_{n=0}^{p-3} \frac{n}{2^{n+3}}}_{\to 1/4 \text{ as } p \to \infty}$$

It follows

$$\frac{1}{8} \sum_{n=1}^{p} \frac{x_n}{2^n} = \frac{1}{2} - \left[\frac{x_p}{2^{p+1}} + \frac{x_{p-1}}{2^{p+1}} + \frac{x_p}{2^{p+2}} + \frac{x_{p-2}}{2^{p+1}} + \frac{x_{p-1}}{2^{p+2}} \right] + \frac{1}{4}$$

Now we prove the

Lemma $x_k/2^k \to 0$.

Proof of the Lemma

First step: the sequence $x_k/2^k$ in monotonic not increasing.

$$\frac{x_{k+3}}{2^{k+3}} = \frac{x_{k+2} + x_{k+1} + x_k + k}{2^{k+3}} \leq \frac{x_{k+2}}{2^{k+2}} \iff \frac{x_{k+1} + x_k + k}{2^{k+3}} \leq \frac{x_{k+2}}{2^{k+3}}$$

that is

$$x_{(k-1)+2} + x_{(k-1)+1} + (k-1) + 1 \leq x_{(k-1)+3}$$

and this is implied by

$$x_{(k-1)+2} + x_{(k-1)+1} + (k-1) + 1 \leq x_{(k-1)+2} + x_{(k-1)+1} + x_{k-1} + (k-1) = x_{(k-1)+3}$$

via $x_{k-1} \geq 1$. The monotonicity of the sequence means that the limit L of $x_k/2^k$ does exist and moreover $0 \leq L < +\infty$. If $L = 0$ the proof is concluded yielding

$$\lim_{p\to\infty} \frac{1}{8} \sum_{n=1}^{p} \frac{x_n}{2^n} = \frac{3}{4} \iff \sum_{n=1}^{\infty} \frac{x_n}{2^n} = 6$$

$L \neq 0$ is impossible as shown by the following argument. We employ the Cesaro–Stolz theorem that states:

$$\lim_{k\to\infty} \frac{x_k}{2^k} = \lim_{k\to\infty} \frac{x_{k+1} - x_k}{2^{k+1} - 2^k}$$

provided that the second limit does exist. We write

$$\frac{x_{k+3}-x_{k+2}}{2^{k+3}-2^{k+2}}=\frac{x_{k+1}+x_k+k}{2^{k+2}}=\frac{1}{2}\frac{x_{k+1}}{2^{k+1}}+\frac{1}{4}\frac{x_k}{2^k}+\frac{k}{2^{k+2}}$$

The existence of the limit $L=\lim_{k\to\infty}\frac{x_k}{2^k}$ would imply

$$L=\frac{1}{2}L+\frac{1}{4}L \implies L=0$$

Solution 3 by Arkady Alt, San Jose, CA

For any sequence $(x_n)_{n\geq 0}$ let $T(x_n):=x_{n+3}-x_{n+2}-x_{n+1}-x_n, n\in N\cup\{0\}$.
Obvious that such defined operator T (we will call it Tribonacci Operator) is linear.
Since $T\left(-\frac{n}{2}\right)=-\frac{n+3}{2}+\frac{n+2}{2}+\frac{n+1}{2}+\frac{n}{2}=n$ then denoting
$u_n:=x_n+\frac{n}{2}, n\in N\cup\{0\}$
we obtain $x_n=u_n-\frac{n}{2}, n\in N\cup\{0\}$ where $T(u_n)=0$ and,
$u_0=0, u_1=1+\frac{1}{2}=\frac{3}{2}, u_2=1+\frac{2}{2}=2.$
Let $(t_n)_{n\geq 0}$ be the sequence defined by $t_0=0, t_1=1, t_2=1$ and
$T(t_n)=0, n\in N\cup\{0\}$.
(Tribonacci Sequence). We have $t_3=2, t_4=4, t_5=7, t_6=13, t_7=24, t_8=44, ...$
Since $\det\begin{pmatrix}0&1&1\\1&1&2\\1&2&4\end{pmatrix}\neq 0$ then for any sequence $(x_n)_{n\geq 0}$ there is triple (c_1,c_2,c_3) of real numbers such that $x_n=c_2t_n+c_2t_{n+1}+c_3t_{n+2}$, that is sequences $(t_n)_{n\geq 0}$, $(t_{n+1})_{n\geq 0}$, $(t_{n+2})_{n\geq 0}$
form a basis of 3-dimesion space $\ker T:=\{(x_n)_{n\geq 0}\mid T(x_n)=0, n\in N\cup\{0\}\}$.
We will find representation u_n as linear combination of t_n, t_{n+1}, t_{n+2},
namely, $u_n=c_1t_n+c_2t_{n+1}+c_3t_{n+2},\ n\in N\cup\{0\}$.
We
have $u_0=c_1t_0+c_2t_1+c_3t_2 \iff c_2+c_3=0, u_1=c_1t_1+c_2t_2+c_3t_3 \iff c_1+c_2+2c_3=\frac{3}{2}$,
$u_2=c_1t_2+c_2t_3+c_3t_4 \iff c_1+2c_2+4c_3=2$. From this system of equations we obtain
$c_3=-c_2, c_1-c_2=\frac{3}{2}, c_1-2c_2=2$.Hence, $c_1=1, c_2=-\frac{1}{2}, c_3=\frac{1}{2}$ and since
$u_n=t_n-\frac{t_{n+1}}{2}+\frac{t_{n+2}}{2}$ we obtain $x_n=t_n-\frac{t_{n+1}}{2}+\frac{t_{n+2}}{2}-\frac{n}{2}=\frac{2t_n-t_{n+1}+t_{n+2}-n}{2}$.
Since radius of convergence of seies $\sum_{n=1}^{\infty}nx^{n-1}$ is 1 and $\sum_{n=1}^{\infty}nx^{n-1}=\frac{1}{(1-x)^2}$
then $\sum_{n=1}^{\infty}\frac{n}{2^n}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{n}{2^{n-1}}=\frac{1}{2}\frac{1}{(1-1/2)^2}=2$ and, therefore, for convergency of $\sum_{n=1}^{\infty}\frac{x_n}{2^n}$ suffice to prove convergency of series $\sum_{n=1}^{\infty}\frac{t_n}{2^n}$.
We can prove that using another basis of $\ker T$ which form sequences $(\alpha^n)_{n\geq 0}, (\beta^n)_{n\geq 0}$, $(\gamma^n)_{n\geq 0}$
where α,β,γ are roots of characteristic equation $x^3-x^2-x-1=0$.
Substitution $x=\frac{4u+1}{3}$ in equation $x^3-x^2-x-1=0$ give us equivalent equation
$4u^3-3u=\frac{19}{8}$

which we solve using substitution $u := \frac{1}{2}\left(t+\frac{1}{t}\right)$. Then equation $4u^3-3u=\frac{19}{8}$ becomes $4\left(\frac{1}{2}\left(t+\frac{1}{t}\right)\right)^3-3\cdot\frac{1}{2}\left(t+\frac{1}{t}\right)=\frac{19}{8} \iff \frac{1}{t^3}+t^3=\frac{19}{4}$.Denoting $z:=t^3$ we obtain

$\frac{1}{z}+z=\frac{19}{4} \iff z=\frac{19-3\sqrt{33}}{8},\frac{19+3\sqrt{33}}{8} \iff t^3=\frac{19-3\sqrt{33}}{8},\frac{19+3\sqrt{33}}{8}$.

Since $\frac{19-3\sqrt{33}}{8}\cdot\frac{19+3\sqrt{33}}{8}=1$ and $u=\frac{1}{2}\left(t+\frac{1}{t}\right)$ then suffices to find $t^3=\frac{19+3\sqrt{33}}{8}$.

We have $t=r(\cos\varphi+i\sin\varphi)$, where $r=\frac{\sqrt[3]{19+3\sqrt{33}}}{2}$ and $\varphi=\frac{2k\pi}{3}, k=1,2,3$.

that is $t_k=\frac{\sqrt[3]{19+3\sqrt{33}}}{2}\omega^k, k=1,2,3$ and $\omega=\ \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},\omega^3=1$.

Thus, denoting $\theta:=\sqrt[3]{19+3\sqrt{33}},\theta^*:=\sqrt[3]{19-3\sqrt{33}}$ we obtain

$\alpha=\frac{1+\theta+\theta^*}{3},\beta=\frac{1+\omega\theta+\omega^2\theta^*}{3}$,

$\gamma=\frac{1+\omega^2\theta+\omega\theta^*}{3}$, the three roots of the equation $x^3-x^2-x-1=0$.

We will prove that $\alpha=\frac{1+\theta+\theta^*}{3}<2$.

First note that by Power Mean–Arithmetic Mean inequality

$p:=\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}\ <2\sqrt[3]{\frac{19+3\sqrt{33}+19-3\sqrt{33}}{2}}=2\sqrt[3]{19}<2\sqrt[3]{27}=6$.

Since $\sqrt[3]{19+3\sqrt{33}}\cdot\sqrt[3]{19-3\sqrt{33}}=\sqrt[3]{19^2-9\cdot 33}=4$ then

$p^3=38+3\sqrt[3]{19+3\sqrt{33}}\cdot\sqrt[3]{19-3\sqrt{33}}\cdot p=38+12p<38+12\cdot 6=110<125=5^3$.

Hence, $\alpha<2$. Also, we obtain $|\beta|,|\gamma|\le\frac{1+\theta+\theta^*}{3}<2$.

Since series $\sum_{n=1}^{\infty}\left(\frac{\alpha}{2}\right)^n,\sum_{n=1}^{\infty}\left(\frac{\beta}{2}\right)^n,\sum_{n=1}^{\infty}\left(\frac{\gamma}{2}\right)^n$ are convergent and t_n is linear combination of

$(\alpha^n)_{n\ge 0},(\beta^n)_{n\ge 0},(\gamma^n)_{n\ge 0}$ then series $\sum_{n=1}^{\infty}\frac{t_n}{2^n}$ convergent as well.

Now we ready to find sum of series $\sum_{n=1}^{\infty}\frac{x_n}{2^n}$.

Let $s_n:=\sum_{k=1}^{n}\frac{t_n}{2^n}$ and $s(x)=\sum_{k=0}^{n}t_{n+1}x^n$. Note also that function

$\frac{1}{1-x-x^2-x^3}$ generates

Tribonacci numbers. Indeed, let $\frac{1}{1-x-x^2-x^3}=\sum_{n=0}^{\infty}a_nx^n$. Then

$\sum_{n=0}^{\infty}a_nx^n\cdot\left(1-x-x^2-x^3\right)=1$

and since

$\sum_{n=0}^{\infty}a_nx^n\cdot\left(1-x-x^2-x^3\right)=\sum_{n=0}^{\infty}a_nx^n-\sum_{n=0}^{\infty}a_nx^{n+1}\ -\sum_{n=0}^{\infty}a_nx^{n+2}-\sum_{n=0}^{\infty}a_nx^{n+3}=$

$a_0+(a_1-a_0)x+(a_2-a_1-a_0)x^2+\sum_{n=3}^{\infty}(a_{n+3}-a_{n+2}-a_{n+1}-a_n)x^{n+3}$ then

$a_0=1,a_1-a_0=a_2-a_1-a_0=0$ implies $a_1=1,a_2=2$ and

$a_{n+3} - a_{n+2} - a_{n+1} - a_n = 0, n \in N \cup \{0\}$. Thus, $a_n = t_{n+1}, n \in N \cup \{0\}$ and, therefore, $\sum_{k=0}^{n} t_{n+1}x^n = s(x) = \dfrac{1}{1-x-x^2-x^3}$. In,
particular, $\sum_{n=1}^{\infty} \dfrac{t_n}{2^n} = \dfrac{1}{2}\sum_{n=1}^{\infty} \dfrac{t_n}{2^{n-1-}} = \dfrac{1}{2}s\left(\dfrac{1}{2}\right) =$
$\dfrac{1}{2} \cdot \dfrac{1}{1-\dfrac{1}{2}-\left(\dfrac{1}{2}\right)^2-\left(\dfrac{1}{2}\right)^3} = 4.$
Then, $\sum_{n=1}^{\infty} \dfrac{x_n}{2^n} = \sum_{n=1}^{\infty} \dfrac{t_n}{2^n} - \sum_{n=1}^{\infty} \dfrac{t_{n+1}}{2^{n+1}} + 2\sum_{n=1}^{\infty} \dfrac{t_{n+2}}{2^{n+2}} - \sum_{n=1}^{\infty} \dfrac{n}{2^{n+1}} =$
$\sum_{n=1}^{\infty} \dfrac{t_n}{2^n} - \sum_{n=2}^{\infty} \dfrac{t_n}{2^n} + 2_{n=3}^{\infty} \dfrac{t_n}{2^n} - \dfrac{1}{2}\sum_{n=1}^{\infty} \dfrac{n}{2^n} =$
$\dfrac{t_1}{2^1} + 2\sum_{n=3}^{\infty} \dfrac{t_n}{2^n} - \dfrac{1}{2} \cdot 2 = -\dfrac{1}{2} + 2\sum_{n=1}^{\infty} \dfrac{t_n}{2^n} - 2\left(\dfrac{t_1}{2^1} + \dfrac{t_2}{2^2}\right) = -\dfrac{1}{2} + 2 \cdot 4 - 2\left(\dfrac{1}{2} + \dfrac{1}{4}\right) = 6.$

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

We show that the given series converges by first using induction to prove that $x_n < 1.95^n$ for each positive integer n. Note that this claim holds for $n \in \{1, 2, 3\}$. Given a positive integer k, if $x_n < 1.95^n$ for $n \in \{k, k+1, k+2\}$, then

$$x_{k+3} < 1.95^{k+2} + 1.95^{k+1} + 1.95^k + k = 1.95^k(6.7525) + k.$$

Thus it suffices to show that $1.95^k(6.7525) + k \le 1.95^{k+3}$, or equivalently $k \le 1.95^k(0.662375)$. This latter inequality holds for each positive integer k (using a separate induction argument). Hence $x_n < 1.95^n$ for $n \ge 1$, so for any positive integer m,

$$\sum_{n=1}^{m} \frac{x_n}{2^n} < \sum_{n=1}^{\infty} \frac{x_n}{2^n} < \sum_{n=1}^{\infty} \frac{1.95^n}{2^n} = \frac{0.975}{1-0.975} = 39.$$

Since its sequence of partial sums is increasing and bounded above, the given series converges.

Next, we let $\sum_{n=1}^{\infty} \dfrac{x_n}{2^n} = L$. Then

$$L = \frac{1}{2} + \frac{1}{4} + \sum_{n=0}^{\infty} \frac{x_{n+2} + x_{n+1} + x_n + n}{2^{n+3}} = \frac{3}{4} + \frac{1}{2}\left(L - \frac{1}{2}\right) + \frac{1}{4}L + \frac{1}{8}L + \sum_{n=0}^{\infty} \frac{n}{2^{n+3}}.$$

Since $\sum_{n=0}^{\infty} \dfrac{n}{2^n} = 2$, we conclude $L = \frac{7}{8}L + \frac{1}{2} + \frac{1}{8}(2)$ and hence $L = 6$.

Also solved by Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler (two solutions), Herrliberg, Switzerland; David Stone and John Hawkins, Southern Georgia University, Statesboro, GA, and the proposer.

$Mea - Culpa$

Arkady Alt of San Jose, CA should have been credited with having solved 5477, and 5478.

Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, all of Angelo State University in San Angelo, TX should have been credited with having solved 5475.

Paul M. Harms, of North Newton, KS should have been credited for having solved 5476.

Anna Valkova Tomova of Varna, Bulgaria should have been credited with having solved 5475 and 5477.

Mea Culpa.

Problems   Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before December 15, 2018

5505: *Proposed by Kenneth Korbin, New York, NY*

Given a Primitive Pythagorean Triple (a, b, c) with $b^2 > 3a^2$. Express in terms of a and b the sides of a Heronian Triangle with area $ab(b^2 - 3a^2)$.

(A Heronian Triangle is a triangle with each side length and area an integer.)

5506: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Find $\Omega = \det\left[\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}^{100}\right]$.

5507: *Proposed by David Benko, University of South Alabama, Mobile, AL*

A car is driving forward on the real axis starting from the origin. Its position at time $0 \leq t$ is $s(t)$. Its speed is a decreasing function: $v(t), 0 \leq t$. Given that the drive has a finite path (that is $\lim\limits_{t \to \infty} s < \infty$), that $v(2t)/v(t)$ has a real limit c as $t \to \infty$, find all possible values of c.

5508: *Proposed by Pedro Pantoja, Natal RN, Brazil*

Let a, b, c be positive real numbers such that $a + b + c = 1$. Find the minimum value of

$$f(a, b, c) = \frac{a}{3ab + 2b} + \frac{b}{3bc + 2c} + \frac{c}{3ca + 2a}.$$

5509: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let x, y, z be positive real numbers that add up to one and such that $0 < \dfrac{x}{y}, \dfrac{y}{z}, \dfrac{z}{x} < \dfrac{\pi}{2}$. Prove that

$$\sqrt{x\cos\left(\frac{y}{z}\right)} + \sqrt{y\cos\left(\frac{z}{x}\right)} + \sqrt{z\cos\left(\frac{x}{y}\right)} < \frac{3}{5}\sqrt{5}.$$

5510: *Proposed by Ovidiu Furdui and Alina Sîntămărian both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\sum_{n=1}^{\infty}\left[4^n\left(\zeta(2n)-1\right)-1\right],$$

where ζ denotes the Riemann zeta function.

Solutions

5487: *Proposed by Kenneth Korbin, New York, NY*

Given that $\dfrac{(x+1)^4}{x(x-1)^2} = a$ with $x = \dfrac{b+\sqrt{b-\sqrt{b}}}{b-\sqrt{b-\sqrt{b}}}$. Find positive integers a and b.

Solution 1 by David E. Manes, Oneonta, NY

If $x = \dfrac{b+\sqrt{b-\sqrt{b}}}{b-\sqrt{b-\sqrt{b}}}$, then $x+1 = \dfrac{2b}{b-\sqrt{b-\sqrt{b}}}$ and $x-1 = \dfrac{2\sqrt{b-\sqrt{b}}}{b-\sqrt{b-\sqrt{b}}}$. Moreover, $(x+1)^4 = \dfrac{16b^4}{(b-\sqrt{b-\sqrt{b}})^4}$ and $(x-1)^2 = \dfrac{4(b-\sqrt{b})}{(b-\sqrt{b-\sqrt{b}})^2}$. Therefore,

$$\begin{aligned} a = \frac{(x+1)^4}{x(x-1)^2} &= \frac{\frac{16b^4}{(b-\sqrt{b-\sqrt{b}})^4}}{\frac{(b+\sqrt{b-\sqrt{b}})(4(b-\sqrt{b})}{(b-\sqrt{b-\sqrt{b}})^3}} \\ &= \frac{16b^4}{4(b-\sqrt{b})(b+\sqrt{b-\sqrt{b}})(b-\sqrt{b-\sqrt{b}})} \\ &= \frac{4b^4}{b^3-b^2-b^2\sqrt{b}+2b\sqrt{b}-b}. \end{aligned}$$

Note that the two terms with $\sqrt{b}$ have opposite signs and cancel off if $b=2$. Let $b=2$. Then $b^3-b^2-b^2\sqrt{b}+2b\sqrt{b}-b=2$ and $a = 2^6/2 = 32$. Hence, $b=2$ and $a=32$ is the unique solution.

Solution 2 by Anthony J. Bevelacqua, University of North Dakota, Great Falls, ND

For notational convenience set $c = \sqrt{b - \sqrt{b}}$. We have $x = \dfrac{b+c}{b-c}$ so $x + 1 = \dfrac{2b}{b-c}$ and $x - 1 = \dfrac{2c}{b-c}$. Thus a is

$$\begin{aligned}\frac{(x+1)^4}{x(x-1)^2} &= \left(\frac{2b}{b-c}\right)^4 \cdot \frac{b-c}{b+c} \cdot \left(\frac{b-c}{2c}\right)^2 \\ &= \frac{4b^4}{(b^2-c^2)c^2}.\end{aligned}$$

and so $a(b^2 - c^2)c^2 = 4b^4$. Now

$$\begin{aligned}(b^2-c^2)c^2 &= (b^2 - b + \sqrt{b})(b - \sqrt{b}) \\ &= (b^3 - b^2 - b) + (2b - b^2)\sqrt{b}\end{aligned}$$

and so

$$a((b^2 - b - 1) + (2-b)\sqrt{b}) = 4b^3.$$

Thus $(2-b)\sqrt{b}$ is a rational number. Therefore either $b = 2$ or $b = d^2$ for some positive integer d.

In the first case our last displayed equation yields $a \cdot 1 = 4 \cdot 2^3$ and so $a = 32$. Thus $a = 32$ and $b = 2$ is a solution to our problem.

In the second case we have

$$(b^2 - b - 1) + (2-b)\sqrt{b} = d^4 - d^3 - d^2 + 2d - 1.$$

Call this n. We have $an = 4b^3$. Since a and b are positive so is n. Since d and n are relatively prime we see that n must be a divisor of 4. If $n = 1$ we have

$$d^4 - d^3 - d^2 + 2d - 1 = 1 \text{ and so } d^4 - d^3 - d^2 + 2d - 2 = 0.$$

By the rational root theorem the only possible positive integer d would be 1 and 2, but neither of these are roots. Similarly $n = 2$ gives $d^4 - d^3 - d^2 + 2d - 3 = 0$ and $n = 4$ gives $d^4 - d^3 - d^2 + 2d - 5 = 0$, but, again, neither of these have positive integer roots. Thus the only solution to our problem is $a = 32$ and $b = 2$.

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

Let $c = b - \sqrt{b - \sqrt{b}}$. Then $x + 1 = 2b/c$ and $x - 1 = 2(b-c)/c$, so

$$a = \frac{(x+1)^4}{x(x-1)^2} = \frac{16b^4}{c^4} \cdot \frac{c^3}{4(b-c)^2(b + \sqrt{b - \sqrt{b}})} = \frac{4b^4}{(b^2 - b + \sqrt{b})(b - \sqrt{b})}.$$

This in turn yields $a = 4b^4/(b^3 - b^2\sqrt{b} - b^2 + 2b\sqrt{b} - b)$. Since a is a positive integer, we must have either $b = n^2$ for some positive integer n or $-b^2 + 2b = 0$. If $b = n^2$, then

$$a = 4n^2 + 4n + 8 + \frac{4(n^3 + n^2 - 3n + 2)}{n^4 - n^3 - n^2 + 2n - 1};$$

the fraction in this latter expression is not an integer for $1 \le n \le 5$ and is strictly between 0 and 1 for $n > 5$, so a is not a positive integer. Thus $-b^2 + 2b = 0$, so $b = 2$ and hence $a = 32$.

Also solved by Michel Bataille, Rouen, France; Ed Gray, Highland Beach, FL; Khanh Hung Vu (Student), Tran Nghia High School, Ho Chi Minh,

Vietnam; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrain University of Athens, Greece; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5488: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta, Turnu-Severin, Mehedinti, Romania*

Let a, and b be complex numbers. Solve the following equation:

$$x^3 - 3ax^2 + 3(a^2 - b^2)x - a^3 + 3ab^2 - 2b^3 = 0.$$

Solution 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

To begin, we note that

$$x^3 - 3ax^2 + 3\left(a^2 - b^2\right)x - a^3 + 3ab^2 - 2b^3$$

can be re-written as

$$\left(x^3 - 3ax^2 + 3a^2x - a^3\right) - 3b^2x + 3ab^2 - 2b^3$$

or

$$(x-a)^3 - 3b^2\,(x-a) - 2b^3.$$

Hence, if we substitute $y = x - a$, the given equation becomes

$$y^3 - 3b^2y - 2b^3 = 0. \tag{1}$$

Next, the left side of equation (1) can be re-grouped to obtain

$$\begin{aligned} y^3 - 3b^2y - 2b^3 &= \left(y^3 + b^3\right) - 3b^2\,(y+b) \\ &= (y+b)\left[\left(y^2 - by + b^2\right) - 3b^2\right] \\ &= (y+b)\left(y^2 - by - 2b^2\right) \\ &= (y+b)^2\,(y-2b)\,. \end{aligned}$$

Therefore, the solutions of (1) are $y = 2b$ and $y = -b$ (double solution).
Finally, since $y = x - a$, the solutions of the original equation are $x = a + 2b$ and $x = a - b$ (double solution).

Solution 2 by Michel Bataille, Rouen, France

Let $p(x)$ denote the polynomial on the left-hand side. Then, a short calculation gives

$$p(X+a) = X^3 - 3b^2X - 2b^3 = (X+b)^2(X-2b)$$

which has $2b$ as a simple root and $-b$ as a double one. It immediately follows that the solution of the given equation are $a-b, a-b, a+2b$.

Solution 3 by Paul M. Harms, North Newton, KS

The equation can be written as $(x-a)^3 - 3ab^2(x-a) - 2b^3 = 0$. If $b = 0$, the solution is $x = a$. If b is not zero, let $x - a = yb$. Then the equation become $b^3(y^3 - 3y - 2) = 0$. We have $y^3 - 3y - 2 = (y-2)(y+1)^2 = 0$. The y solutions are $2, -1$ and -1. The solutions of the equation in the problem are $x = a + 2b$ and $x = a - b$ as a double root.

Solution 4 by G. C. Greubel, Newport News,VA

$$\begin{aligned}
0 &= x^3 - 3\,a\,x^2 + 3(a^2 - b^2)\,x - (a^3 - 3ab^2 + 2b^3) \\
&= x^3 - 3ax^2 + (a-b)(3a+3b)\,x - ((a^2 - 2ab + b^2)(a+2b) \\
&= x^3 - (2(a-b) + (a+2b))\,x^2 + (a-b)((a-b) + 2(a+2b))\,x \\
&\qquad - (a-b)^2(a+2b) \\
&= (x^2 - 2(a-b)\,x + (a-b)^2)(x - (a+2b)) \\
&= (x - (a-b))^2\,(x - (a+2b)).
\end{aligned}$$

From this factorization the solutions of the cubic equation are

$$x \in \{a-b, a-b, a+2b\}.$$

Editor's comment: **David Stone and John Hawkins** made an instructive comment in their solution that merits being repeated. They wrote: “We confess - we did not immediately recognize the factorization. We originally used Cardano’s Formula to find the solutions.

However, there is a line of heuristic reasoning which would lead to the solution. If we consider $a = b$, the equation become $x^3 - 3ax^2 = 0$, which has $x = 0$ as a double root. Hence, the difference $a - b$ could be significant. Trying $x = a - b$ (via synthetic division) then proves to be productive.”

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony J. Bevelacqua, University of North Dakota, Great Falls, ND; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Ioannis D. Sfikas, National and Kapodistrain University of Athens, Greece; Neculai Stanciu “George Emil Palade” School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania (two solutions); David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5489: *Proposed by D.M. Bătinetu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, “George Emil Palade” School Buzău, Romania*

If $a > 0$, compute $\displaystyle\int_0^a \left(x^2 - ax + a^2\right) \arctan(e^x - 1)dx$.

Solution by Soumitra Mandal, Chandar Nagore, India

Let $x = a - y \Rightarrow dx = -dy$, when $x = 0, y = a$; when $x = a; y = 0$.

$$\Omega \;=\; \int_0^a (x^2 - xa + a^2)\tan^{-1}(e^x - 1)dx$$

$$
\begin{aligned}
&= -\int_a^0 \{(a-y)^2 - a(a-y) + a^2\} \tan^{-1}(e^{a-y}-1)dy \\
&= \int_0^a (y^2 - ay + a^2)\tan^{-1}(e^{a-y}-1)dy, \text{therefore,} \\
2\Omega &= \int_0^a (x^2 - ax + a^2)\{\tan^{-1}(e^x - 1) + \tan^{-1}(e^{a-x}-1)\}dx \\
&= \int_0^a (x^2 - xa + a^2)\tan^{-1}\frac{e^x - 1 + e^{a-x} - 1}{1-(e^x-1)(e^{a-x}-1)}dx \\
&= \int_0^a (x^2 - ax + a^2)\tan^{-1}(1)dx = \frac{\pi}{4}\left(\frac{x^3}{3} - a\frac{x^2}{2} + a^2x\Bigg|_{x=0}^{x=a}\right) = \frac{5\pi a^3}{24}. \\
\text{Therefore, } \Omega &= \frac{5\pi a^3}{48}.
\end{aligned}
$$

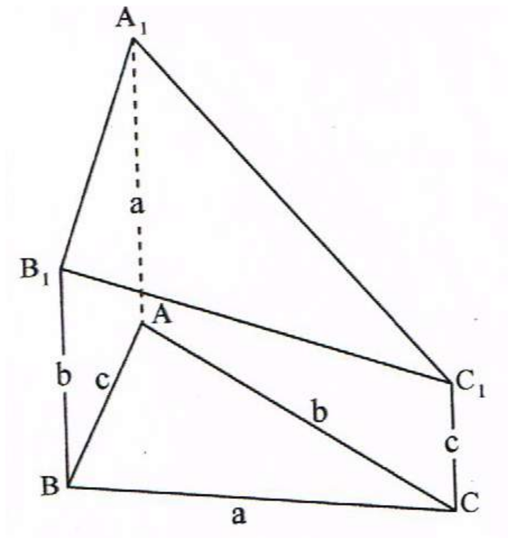
Also solved by Ioannis D. Sfikas, National and Kapodistrain University of Athens, Greece, and the proposers.

5490: *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

Triangle ABC whose side lengths are a, b, and c lies in plane P. The segment$A_1A,\ BB_1,\ CC_1$ satisfy:

$$A_1A \perp P,\ B_1B \perp P,\ C_1C \perp P,$$

where $A_1A = a,\ B_1B = b$ and $C_1C = c$, as shown in the figure. Prove that $\triangle A_1B_1C_1$ is acute -angled.



Solution 1 by Michel Bataille, Rouen, France

We shall use the dot product, recalling that $\overrightarrow{U}\cdot\overrightarrow{V}$ has the same sign as $\cos(\angle(\overrightarrow{U},\overrightarrow{V}))$.
We calculate

$$\begin{aligned}\overrightarrow{A_1B_1}\cdot\overrightarrow{A_1C_1} &= (\overrightarrow{A_1A}+\overrightarrow{AB}+\overrightarrow{BB_1})\cdot(\overrightarrow{A_1A}+\overrightarrow{AC}+\overrightarrow{CC_1})\\ &= a^2+0-ac+0+\overrightarrow{AB}\cdot\overrightarrow{AC}+0-ab+0+bc\\ &= \frac{1}{2}(a^2+b^2+c^2-2ac-2ab+2bc) \quad (\text{since } 2\overrightarrow{AB}\cdot\overrightarrow{AC}=b^2+c^2-a^2)\\ &= \frac{1}{2}(b+c-a)^2.\end{aligned}$$

Thus, $\overrightarrow{A_1B_1}\cdot\overrightarrow{A_1C_1}>0$ and so $\angle B_1A_1C_1$ is acute.
Similarly, we obtain $\overrightarrow{B_1C_1}\cdot\overrightarrow{B_1A_1}=\frac{1}{2}(c+a-b)^2>0$ and
$\overrightarrow{C_1A_1}\cdot\overrightarrow{C_1B_1}=\frac{1}{2}(a+b-c)^2>0$ and therefore $\angle C_1B_1A_1$ and $\angle A_1C_1B_1$ are acute as well.

Solution 2 by Muhammad Alhafi, Al Basel High School, Aleppo, Syria

We will prove that $\overline{B_1C_1}^2<\overline{B_1A_1}^2+\overline{A_1C_1}^2$.

If we draw a line through C_1 *parallel to* $\overline{BC}$ we will see that $a^2+(b-c)^2=\overline{B_1C_1}^2$.

In the same manner we have:

$\overline{A_1B_1}^2 = c^2 + (a-b)^2,\ \overline{A_1C_1}^2 = b^2 + (a-c)^2.$
So the inequality is equivalent to:

$$\begin{aligned} a^2 + (b-c)^2 &< c^2 + (a-b)^2 + b^2 + (a-c)^2 \\ &\iff 2ab + 2ac < a^2 + b^2 + c^2 + 2ab \\ &\iff 2a(b+c) < a^2 + (b+c)^2, \text{ which follows from the AM}-\text{GM inequality.} \end{aligned}$$

Following this line of reasoning we can prove: $\overline{B_1A_1}^2 < \overline{B_1C_1}^2 + \overline{A_1C_1^2}$ and that $\overline{A_1C_1}^2 < \overline{B_1A_1}^2 + \overline{B_1C_1}^2$. Hence, $\triangle A_1B_1C_1$ is acute.

Solution 3 by Michael N. Fried, Ben-Gurion University, Beer Sheva, Israel

Suppose we are given an arbitrary triangle such as ABC with sides $BC = a$, $AC = b$, and $AB = c$. Let the lines AA', BB', CC' with lengths a, b, and c, respectively, be drawn perpendicular to the plane of ABC (see figure 1). Then the triangle $A'B'C'$ with sides $B'C' = a'$, $A'C' = b'$, and $A'B' = c'$ is acute.

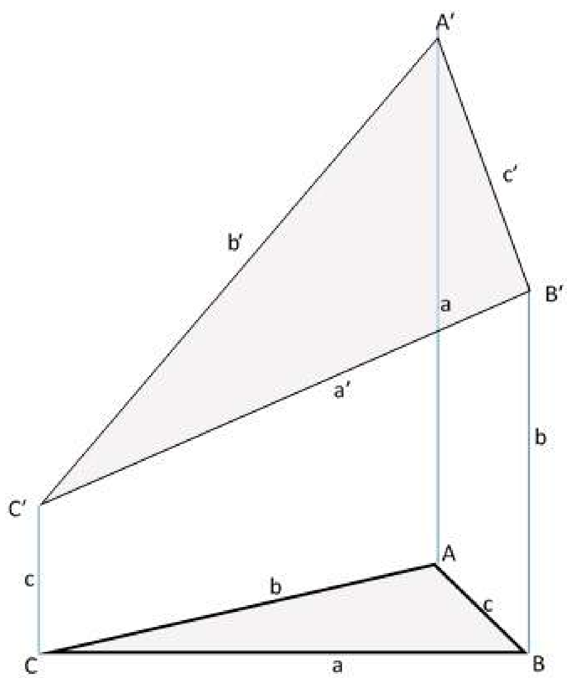


Fig.1

Let us consider first the special case when ABC is an isosceles triangle. First, it is obvious that if ABC is isosceles then also $A'B'C'$ will be isosceles. Moreover, if BC is the base and the angle at A is already acute then the angle at A' will also be acute since $a = a'$ and $c' = b' > b = c$ so that the angle at A' will be less than the angle at A. So we need only consider the case when A is obtuse. In that case, also $a > b = c$.
It makes life easier to consider $A'B'C'$ with respect to the plane UVW drawn through C' (or B') and parallel to ABC so that also $UVW \cong ABC$. In that case, VW coincides with $B'C'$ and $UA' = a - c$ (or $a - b$) (see figure 2).

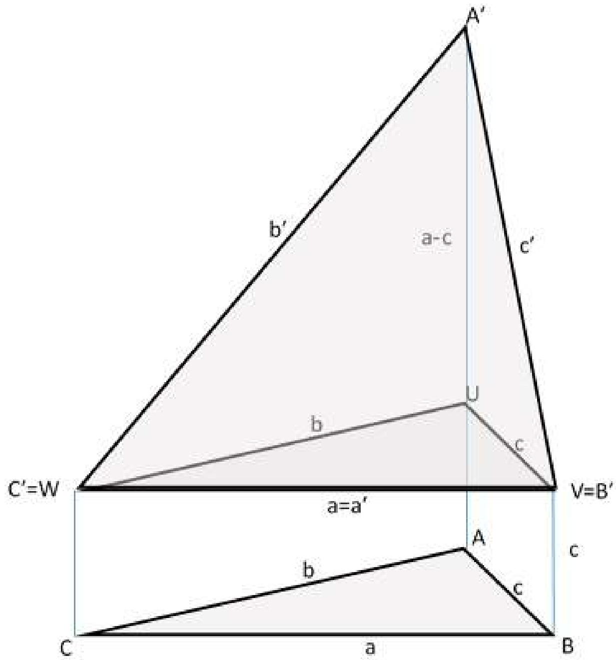


Fig.2

With that out of the way, we need to show that if α is the apex angle at A' then $\alpha < 90°$, or, by the law of cosines, that $2c'^2 \cos\alpha = 2c'^2 - a^2 > 0$. Or since $c'^2 = c^2 + (a-c)^2$:

$$2c^2 + 2(a-c)^2 - a^2 > 0$$

Or, opening parentheses and rearranging:

$$4c^2 - a(4c - a) > 0$$

Note that by the triangle inequality, $2c - a > 0$ so that certainly $4c - a > 0$. By the arithmetic/geometric mean inequality, then, we have (keeping in mind that $a \neq 4c - a$ since otherwise $2c = a$ which is impossible):

$$4c^2 = \left(\frac{a + (4c-a)}{2}\right)^2 > a(4c - a)$$

So, indeed, $4c^2 - a(4c - a) > 0$ and $\alpha < 90°$.
Now, let us consider the case in which ABC is not isosceles. Let us assume that $a > b > c$. As before, consider $A'B'C'$ with respect to the plane UVW drawn through C' and parallel to ABC. Then we have $WB' = b - c$ and $UA' = a - c$ (see figure 3).

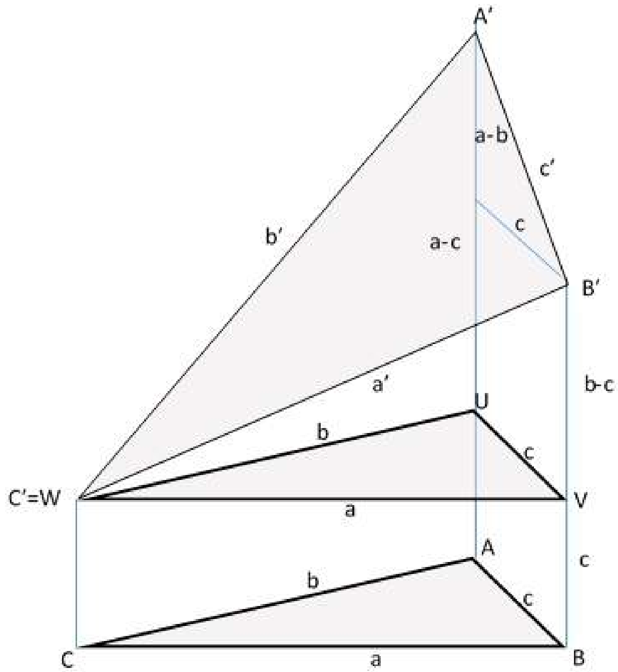


Fig.3

We have then:

$$a'^2 = a^2 + (b-c)^2$$
$$b'^2 = b^2 + (a-c)^2$$
$$c'^2 = c^2 + (a-b)^2$$

Observe that as $a > b > c$, also $a' > b' > c'$, for consider $a'^2 - b'^2$:

$$a'^2 - b'^2 = a^2 + (b-c)^2 - b^2 - (a-c)^2 = (a-b)2c > 0$$

so that $a'^2 > b'^2$. Similarly, we can show that $b'^2 > c'^2$. Since a' is thus the longest side of $A'B'C'$, the angle at A', which we call α', is the largest angle. Therefore, it suffices to show that $\alpha' < 90°$. Again, by the law of cosines this means we must show:

$$2b'c' \cos \alpha' = b'^2 + c'^2 - a'^2 > 0$$

Substituting the expressions above for a', b', and c', we have to show:

$$b^2 + (a-c)^2 + c^2 + (a-b)^2 - a^2 - (b-c)^2 > 0$$

After some algebra, the expression on the left-hand side can be rewritten as follows:

$$c^2 - (a-b)(2c - (a-b))$$

Notice that $a - b > 0$ since we are assuming that a is the longest side of ABC. Also since by the triangle inequality we have $c - (a-b) = b + c - a > 0$, it is certainly true that $2c - (a-b) > 0$. Therefore, again by the arithmetic/geometric-mean inequality, we have:

$$c^2 = \left(\frac{(a-b) + (2c - (a-b))}{2} \right)^2 > (a-b)(2c - (a-b))$$

So, indeed,
$$b'^2 + c'^2 - a'^2 = c^2 - (a-b)(2c-(a-b)) > 0$$
From which we have $\alpha' < 90°$.

Also solved by Yagub N. Aliyev, Problem Solving Group of ADA University, Baku Azerbeaijan; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Ioannis D. Sfikas, National and Kapodistrain University of Athens, Greece; Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposers.

5491: *Proposed by Roger Izard,Dallas, TX*

Let O be the orthocenter of isosceles triangle ABC, $AB = AC$. Let OC meet the line segment AB at point F. If $m = FO$, prove that $c^4 \geq m^4 + 11m^2c^2$.

Solution 1 by Ed Gray, Highland Beach, FL

We assume that c is one of the two equal legs. We re-write the inequality by dividing by c^4, so:

1) $1 \geq \left(\dfrac{m}{c}\right)^4 + 11\left(\dfrac{m}{c}\right)^2$. We attempt to prove the inequality by finding the maximum value of $\dfrac{m}{c}$. We shall use the following notation: vertex A is the apex (top) with angle $2t$. We note that $2t < 90$, otherwise $O = A$, or O is external to the triangle. Vertex B is at lower left, and has value $90 - t$. Vertex C is at lower right, also having a value of $90 - t$. Let P be the mid-point of BC, $y = BF$, $c - y = AF$, $m = OF$, and the base, $BC = s$, so that $BP = PC = \dfrac{s}{2}$. We note that $\triangle FAC$ is a right triangle, so $\angle ACF = 90 - 2t$. Since $\angle ACB = 90 - t$, by subtraction,

2) $\angle FCB = t$. From $\triangle AOF$,

3) $\tan(t) = \dfrac{m}{c-y}$. From $\triangle FCB$,

4) $\sin(t) = \dfrac{y}{s}$, or $y = s \cdot \sin(t)$. From $\triangle ABP$,

5) $\sin(t) = \dfrac{s}{2c}$, or $c = \dfrac{s}{2\sin(t)}$. Substituting (4) and (5) into (3),

6) $m = \tan(t)\dfrac{s}{(2\sin(t)) - s \cdot \sin(t)}$. Dividing (6) by (5),

7) $\dfrac{m}{c} = \dfrac{\sin(t)}{\cos(t)} \cdot \dfrac{s}{2\sin(t)} - s \cdot \sin(t) \cdot 2\sin\dfrac{t}{s}$, or

8) $\dfrac{m}{c} = \dfrac{\sin(t) - 2\sin^3(t)}{\cos(t)}$

9) $\dfrac{d}{dt}\dfrac{m}{c} = \dfrac{\left(\cos(t)\left(\cos(t) - 6\cos(t)\sin^2(t)\right) - \left(\sin(t) - 2\sin^3(t)\right)(-\sin(t)\right)}{\cos^2(t)}$. Simplifying,

10) $8\sin^4(t) - 6\sin^2(t) + 1 = 0$. This is a quadratic equation in $\sin^2(t)$ with roots:

11) $16\sin^2(t) = 6 \pm \sqrt{(36-32)}$, or

12) $\sin^2(t) = \dfrac{1}{2}$, or $\sin^2(t) = \dfrac{1}{4}$. The former is impossible, since $t = 45$, and $2t = 90$, which would put $O = A$. Therefore, $\sin(t) = \frac{1}{2}$, and $t = 30,\ 2t = 60$, and we have an equilateral triangle. Then $c = s, y = \frac{c}{2}$, and from (3)

13) $\tan(30) = \dfrac{m}{\frac{c}{2}}$, and

14) $\dfrac{m}{c} = \dfrac{1}{2}\tan(30) = \dfrac{\sqrt{3}}{6},\ \left(\dfrac{m}{c}\right)^2 = \dfrac{3}{36} = \dfrac{1}{12},\ \left(\dfrac{m}{c}\right)^4 = \dfrac{1}{144}$, so

15) $\left(\dfrac{m}{c}\right)^4 + 11\left(\dfrac{m}{c}\right)^2 = \dfrac{1}{144} + \dfrac{11}{12} < 1$, and the conjecture is proved. Q.E.D.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

The angle α at the vertex A is $\leq \frac{\pi}{2}$, because OC meets the line segment AB. Clearly $AF = AC\cos\alpha$ and $OF = AF\tan\left(\frac{\alpha}{2}\right) = AC\cos\alpha\tan\left(\frac{\alpha}{2}\right)$. Furthermore $\dfrac{OF}{AC} = \dfrac{m}{c}$. Therefore we need to prove that

$$\cos^4\alpha\tan^4\frac{\alpha}{2} + 11\cos^2\alpha\tan^2\frac{\alpha}{2} \leq 1,\ \text{for } 0 \leq \alpha \leq \frac{\alpha}{2}. \qquad (1)$$

We note that

$$y = \cos\alpha\tan\left(\frac{\alpha}{2}\right) = \left(2\cos^2\frac{\alpha}{2} - 1\right)\tan\frac{\alpha}{2} = \left(\frac{2}{1+\tan^2\frac{\alpha}{2}} - 1\right)\tan\frac{\alpha}{2} = \tan\frac{\alpha}{2}\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = x\frac{1-x^2}{1+x^2},$$

where we have put $x = \tan\frac{\alpha}{2}$. Clearly the function $x = \tan\frac{\alpha}{2}$ maps the interval $\left[0, \frac{\alpha}{2}\right]$ to the interval $[0, 1]$. We claim that

$$\max_{0\leq x\leq 1} x\frac{1-x^2}{1+x^2} = \sqrt{\sqrt{5}-2}\frac{\sqrt{5}-1}{2}.$$

Indeed,

$$\frac{d}{dx}x\frac{1-x^2}{1+x^2} = \frac{1-4x^2-x^4}{(1+x^2)^2} = \frac{-\left(x^2+2x+\sqrt{5}\right)\left(x-\sqrt{\sqrt{5}-2}\right)\left(x+\sqrt{\sqrt{5}-2}\right)}{(1+x^2)^2},$$

so the maximum of $x\dfrac{1-x^2}{1+x^2}$ in the interval $[0, 1]$ is assumed at $\sqrt{\sqrt{5}-2}$ and equals

$$\sqrt{\sqrt{5}-2}\frac{3-\sqrt{5}}{\sqrt{5}-1} = \sqrt{\sqrt{5}-2}\frac{\sqrt{5}-1}{2}.$$

Therefore

$$\cos^4\alpha\tan^4\frac{\alpha}{2} + 11\cos^2\alpha\tan^2\frac{\alpha}{2} \leq \left(\sqrt{\sqrt{5}-2}\frac{\sqrt{5}-1}{2}\right)^4 + 11\left(\sqrt{\sqrt{5}-2}\frac{\sqrt{5}-1}{2}\right)^4 = 1,$$

and (1) is proven.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Without loss of generality, let $b = c = 1$. Let $AB = AC$ and AO is perpendicular to BC so AO bisects $\angle BAC$. Let $\angle BAC = 2\theta$, where $0 < \theta \leq \frac{\pi}{4}$.

By considering triangles AOF and ACF, we obtain respectively $m = AF\tan\theta$ and $AF = \cos 2\theta$, so that $m = \tan\theta\cos 2\theta$. Let $t = \tan\theta$, so that $0 < t \le 1$. Then $m = \dfrac{t(1-t^2)}{1+t^2}$. We have $\dfrac{dm}{dt} = \dfrac{1-4t^2-t^4}{(1+t^2)^2}$, which vanishes when $t = \sqrt{\sqrt{5}-2}$, at which m attains its maximum value of $\sqrt{\dfrac{5\sqrt{5}-11}{2}}$. Hence

$$m^4 + 11m^2 \le \frac{123-55\sqrt{5}}{2} + \frac{55\sqrt{5}-121}{2} = 1,$$

and this completes the solution.

Also solved by Ioannis D. Sfikas, National and Kapodistrain University of Athens, Greece, David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5492: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let a, b, c, d be four positive numbers such that $ab + ac + ad + bc + bd + cd = 6$. Prove that

$$\sqrt{\frac{abc}{a+b+c+3d}} + \sqrt{\frac{bcd}{b+c+d+3a}} + \sqrt{\frac{cda}{c+d+a+3b}} + \sqrt{\frac{dab}{d+a+b+3c}} \le 2\sqrt{\frac{2}{3}}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

By the inequality of Cauchy-Schwarz, the left side of the inequality of the problem does not exceed $2\sqrt{\dfrac{abc}{a+b+c+3d} + \dfrac{bcd}{b+c+d+3a} + \dfrac{cda}{c+d+a+3b} + \dfrac{dab}{d+a+b+3c}}$.

From the given relation, we have $d = \dfrac{6-ab-bc-ca}{a+b+c}$, so that

$$\frac{abc}{a+b+c+3d} = \frac{2abc(a+b+c)}{(a-b)^2+(b-c)^2+(c-a)^2+36} \le \frac{abc(a+b+c)}{18}.$$

Similarly,

$$\begin{aligned}
\frac{bcd}{b+c+d+3a} &\le \frac{bcd(b+c+d)}{18} \\
\frac{cda}{c+d+a+3b)} &\le \frac{cda(c+d+a)}{18} \\
\frac{(dab}{d+a+b+3c)} &\le \frac{dab(d+a+b)}{18}.
\end{aligned}$$

Hence the inequality of the problem will follow from

$$abc(a+b+c) + bcd(b+c-d) + cda(c+d+a) + dab(d+a+b) \le 12. \qquad (1)$$

Now it can be checked readily that the left side of (1) equals

$$\frac{2(ab+ac+ad+bc+bd+cd)^2-(a-b)^2(c-d)^2-(b-c)^2(d-a)^2-(c-a)^2(b-d)^2}{6},$$

which does not exceed $\dfrac{(ab+ac+ad+bc+bd+cd)^2}{3}=12.$

This completes the solution.

Solution 2 by Ed Gray, Highland Beach, FL

1) Let $n=a+b+c+d$. Then:

2) $n^2=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd=a^2+b^2+c^2+d^2+12$

3) Suppose that $a=b=c=d=a$. Then (2) becomes:

4) $(4a)^2=4a^2=12$, and $a=1$.

The left side of the inequality becomes:

5) $4\sqrt{1/6)}=2\sqrt{4/6}=2\sqrt{2/3}$, and we see that the inequality becomes an equality. We need show that the expression is a maximum when $a=b=c=d$. We do this by leaving $a=b=1, c=.99, d=1.01$ so that the constant $n=a+b+c+d$ is maintained.

Substituting the new values into the left side,

6) $\sqrt{.99/6.02}+\sqrt{.9999/6}+\sqrt{.9999}+\sqrt{1.01/5.98}=$

7) $.405526605+.408227878+.407549194+.410969976=1.632273698<1.632993162=2\sqrt{2/3}.$

Hence the function is a maximum for $a=b=c=d$, and the inequality is proven.

Solution 3 by Neculai Stanciu "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania

With the Cauchy-Buniakovski-Schwarz inequality we have

$$\frac{abc}{a+b+c+3d}+\frac{bcd}{b+c+d+3a}+\frac{cda}{c+d+a+3b}+\frac{dab}{d+a+b+3c}$$

$$\le\quad 4\left(\frac{abc}{a+b+c+3d}+\frac{bcd}{b+c+d+3a}+\frac{cda}{c+d+a+3b}+\frac{dab}{d+a+b+3c}\right).$$

With the AM−HM inequality we have

$$\frac{abc}{a+b+c+3d}=\frac{abc}{a+d+b+d+c+d}\le\frac{1}{9}\left(\frac{abc}{a+d}+\frac{abc}{b+d}+\frac{abc}{c+d}\right)$$

$$\frac{abc}{a+b+c+3d}+\frac{bcd}{b+c+d+3a}+\frac{cda}{c+d+a+3b}+\frac{dab}{d+a+b+3c}\le$$

$$\le\quad \frac{1}{9}\left(\frac{abc}{a+d}+\frac{bcd}{a+d}+\frac{abc}{b+d}+\frac{acd}{b+d}+\frac{abc}{c+d}+\frac{abd}{c+d}+\frac{bcd}{a+b}+\frac{acd}{a+b}+\frac{bcd}{a+c}+\frac{abd}{a+c}+\frac{abd}{b+c}+\frac{acd}{b+c}\right)$$

$$=\quad \frac{1}{9}\left(bc+ac+ab+cd+bdf+ad\right)=\frac{2}{3}.$$

Hence, by the inequalities from above we obtain the desired inequality!

Solution 4 by Marian Ursărescu, National College "Roman-Voda," Roman, Romania

Cauchy's Inequality implies

$$4\sum \frac{abc}{a+b+c+3d} \geq \left(\sum \sqrt{\frac{abc}{a+b+c+3d}}\right)^2 \Rightarrow$$

$$\sum \sqrt{\frac{abc}{a+b+c+3d}} \leq 2\sqrt{\sum \frac{abc}{a+b+c+3d}} \Rightarrow$$

$$\sum \sqrt{\frac{abc}{a+b+c+3d}} \leq 2\sqrt{\sum \frac{abc}{(a+d)+(b+d)+(c+d)}} \qquad (1)$$

But, $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9 \Rightarrow \frac{1}{x+y+z} \leq \frac{1}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$, which implies

$$\frac{1}{(a+d)+(b+d)+(c+d)} \leq \frac{1}{9}\left(\frac{1}{a+d}+\frac{1}{b+d}+\frac{1}{c+d}\right) \qquad (2)$$

From (1) and(2) we obtain,

$$\sum \sqrt{\frac{abc}{a+b+c+3d}} \leq \frac{2}{3}\sqrt{\sum abc\left(\frac{1}{a+d}+\frac{1}{b+d}+\frac{1}{c+d}\right)}. \qquad (3)$$

But

$$\sum abc\left(\frac{1}{a+d}+\frac{1}{b+d}+\frac{1}{c+d}\right) =$$

$$\frac{abc}{a+d}+\frac{abc}{b+d}+\frac{abc}{c+d}+\frac{bcd}{a+b}+\frac{bcd}{a+c}+\frac{bcd}{a+d}+$$

$$+\frac{cda}{b+a}+\frac{cda}{b+c}+\frac{cda}{b+d}+\frac{dab}{c+a}+\frac{dab}{c+b}+\frac{dab}{c+d} =$$

$$= \frac{bc(a+d)}{a+d}+\frac{ac(b+d)}{b+c}+\frac{ab(c+d)}{c+d}+\frac{bc(a+b)}{a+b}+\frac{4d(a+c)}{a+c}+\frac{ad(4+d)}{4+d} =$$

$$ab + ac + ad + bc + 4d + cd = 6. \qquad (4)$$

Equations (3) and (4) implies that

$$\sum \sqrt{\frac{abc}{a+b+c+3d}} \leq \frac{2}{3}\sqrt{6} = 2\sqrt{\frac{2}{3}}$$

Solutions 5 and 6 by Paolo Perfetti, Department of Mathematics, Tor Vergatta University, Rome, Italy

First Proof The first step uses the concavity of the function $\sqrt{x}$ yielding

$$\sum_{\text{cyc}} \sqrt{\frac{abc}{a+b+c+3d}} \leq 2\sqrt{\sum_{\text{cyc}} \frac{abc}{a+b+c+3d}} \leq 2\sqrt{\frac{2}{3}}$$

that is

$$\sum_{\text{cyc}} \frac{abc}{a+b+c+3d} \leq \frac{2}{3}$$

Cauchy–Schwarz reversed yields

$$\frac{1}{a+d} + \frac{1}{b+d} + \frac{1}{c+d} \geq \frac{9}{a+b+c+3d}$$

so it suffices to prove

$$\frac{1}{9}\Bigg(\frac{abc}{a+d} + \frac{abc}{b+d} + \frac{abc}{c+d} + \frac{bcd}{d+a} + \frac{bcd}{b+a} + \frac{bcd}{c+a} +$$
$$+\frac{cda}{a+b} + \frac{cda}{c+b} + \frac{cda}{d+b} + \frac{dab}{a+c} + \frac{dab}{b+c} + \frac{dab}{d+c}\Bigg) \leq \frac{2}{3}\,\frac{ab+bc+ca+ad+bd+cd}{6}$$

We can rewrite it as

$$\frac{abc}{a+d} + \frac{bcd}{d+a} + \frac{abc}{b+d} + \frac{cda}{d+b} + \frac{abc}{c+d} + \frac{dab}{d+c} + \frac{bcd}{b+a} + \frac{cda}{a+b} + \frac{bcd}{c+a} + \frac{dab}{a+c} +$$
$$+\frac{cda}{c+b} + \frac{dab}{c+b} \leq ab+bc+ca+ad+bd+cd$$

This is actually an equality since

$$\frac{abc}{a+d} + \frac{bcd}{d+a} = bc$$

and so on for the other five cases. This concludes the proof.

Proof 6 (Computer assisted) The first step uses the concavity of the function $\sqrt{x}$ yielding

$$\sum_{\text{cyc}} \sqrt{\frac{abc}{a+b+c+3d}} \leq 2\sqrt{\sum_{\text{cyc}} \frac{abc}{a+b+c+3d}} \leq 2\sqrt{\frac{2}{3}}$$

that is

$$\sum_{\text{cyc}} \frac{abc}{a+b+c+3d} \leq \frac{2}{3}$$

First case $d = 0$. The inequality is

$$\frac{abc}{a+b+c} \leq \frac{2}{3} \tag{1}$$

We know that

$$\begin{aligned} &\left(\sqrt{3}\sqrt{abc(a+b+c)}\right)^2 = 3abc(a+b+c) \leq (ab+bc+ca)^2 \\ \iff \quad & abc(a+b+c) \leq (ab)^2 + (bc)^2 + (ca)^2 \end{aligned}$$

and this holds true by the AGM $(ab)^2 + (ac)^2 \geq a^2bc$ and cyclic. Based on this we can write

$$6 = ab + bc + ca \geq \sqrt{3}\sqrt{abc(a+b+c)} \quad \iff \quad abc(a+b+c) \leq 12$$

which inserted in (1) gives

$$3\frac{12}{a+b+c}\frac{1}{a+b+c} \leq 2 \quad \iff \quad (a+b+c)^2 \geq 18$$

This follows easily by

$$(a+b+c)^2 \geq 3(ab+bc+ca) = 18$$

Second case $d = 1$ which is allowed by the homogeneity of the inequality after writing

$$\sum_{\text{cyc}} \frac{abc}{a+b+c+3d} \leq \frac{2}{3}\frac{ab+bc+cd+da+ac+bd}{6}$$

For $d = 1$ the above inequality becomes

$$\frac{abc}{a+b+c+3} + \frac{bc}{b+c+1+3a} + \frac{ca}{c+1+a+3b} + \frac{ab}{1+a+b+3c} \leq \frac{2}{3} \tag{2}$$

This is a algebraic symmetric inequality in three variables and we employ the so called "UVW" theory. Thus we change variables

$$a+b+c = 3u, \quad ab+bc+ca = 3v^2, \quad abc = w^3$$

By expanding (2) we get

$$\begin{aligned} &A(a,b,c)\sum_{\text{cyc}}\Big(8ab + 3a + 16a^2 + 26a^3 + 16a^4 - 150a^2b^2 + 8a^4bc + 36a^2b^2c + \\ &+42a^3bc + 36a^2bc - 150a^2b^2 + 26a^3b^3 + 3a^5\Big) + \\ &+A(a,b,c)\sum_{\text{sym}}\Big(-11a^3b^2c - 11a^3b + 3a^5b + 16a^4b^2 - 11a^3b^2 + 8a^4b\Big) + \\ &+A(a,b,c)(-150a^2b^2c^2 + 42abc) \doteq A(a,b,c)B(a,b,c) \end{aligned}$$

$$A(a,b,c) = -9(a+b+c+3)(b+c+1+3a)(c+1+a+3b)(1+a+b+3c)$$

Now we prove the **Lemma that:** *The polynomial $B(a,b,c)$ is a concave parabola in the variable w^3.*

Proof of the Lemma We concentrate on the terms of order six, the only terms containing w^6.

$$\sum_{\text{cyc}}(8a^4bc+26a^3b^3)+\sum_{\text{sym}}(-11a^3b^2c+3a^5b+16a^4b^2)-150(abc)^2 \tag{3}$$

and once introduced the new variables (u,v,w), we are interested in those terms containing w^6. We have,

$$\sum_{\text{cyc}}a^4bc=abc\sum_{\text{cyc}}a^3=w^3(3w^3+27u^3-27uv^2)$$

$$\sum_{\text{cyc}}a^3b^3=27v^6-27uv^2w^3+3w^6,\quad \sum_{\text{sym}}a^3b^2c=w^3(9uv^2-3w^3),$$

Moreover,

$$\sum_{\text{sym}}a^5b=\sum_{\text{cyc}}a\sum_{\text{cyc}}a^5-\sum_{\text{cyc}}a^6,\quad \sum_{\text{cyc}}a^5=\sum_{\text{cyc}}a^3\sum_{\text{cyc}}a^2-2\sum_{\text{sym}}a^3b^2$$

Since $a^2+b^2+c^2=9u^2-6v^2$, in $\sum_{\text{sym}}a^5b$ only $\sum_{\text{sym}}a^6$ contains w^6 and precisely

$$\sum_{\text{cyc}}a^6=729u^6-1458u^4v^2+729u^2v^4+162u^3w^3-54v^6-108uv^2w^3+3w^6$$

$$\sum_{\text{sym}}a^4b^2=\sum_{\text{cyc}}a^2\sum_{\text{cyc}}a^4-\sum_{\text{cyc}}a^6$$

The coefficient of the term w^6 of (3) is

$$24+26\cdot 3+11\cdot 3-3\cdot 3-16\cdot 3-150=-72$$

and so the Lemma has been proved.

Since $A(a,b,c)<0$, $-B(a,b,c)$ is a convex parabola whose maximum is attained at one or both the extreme points of variations of w^3. The "UVW" theory states that once fixed the values of u and v, the minimum value of w occurs when $abc=0=w^3$ or when $b=c$ (or cyclic) while the maximum value occurs when $b=c$ (or cyclic). So we need to study two cases.

First case. $c=0$.

$$ab+bc+cd+da+ac+bd=6\iff a=(6-b)/(1+b)$$

Inequality (2) becomes

$$-\frac{(5b^2-16b+14}{3(7+b+b^2)}\le 0$$

which evidently holds true.

Second case. $c=b$.

$$ab + bc + cd + da + ac + bd = 6 \iff a = (6 - b^2 - 2b)/(1 + 2b)$$

Inequality (2) becomes

$$-\frac{(b^2 - 7)(7b^4 - 18b^3 - 27b^2 - 64b - 114)(b - 1)^2}{3(4b + 7b^2 + 7)(-2b + b^2 + 19)(3 + b^2 + 2b)} \leq 0 \tag{4}$$

Clearly $a \geq 0$ so $b \leq \sqrt{7} - 1$ and then $b^2 - 7 \leq 0$. Moreover

$$7b^4 - 18b^3 - 27b^2 \leq 0 \iff b \leq (9 + \sqrt{270})/7$$

and thus $7b^4 - 18b^3 - 27b^2 - 64b - 114 \leq 0$. The conclusion is that (4) holds true and this completes the proof.

Also solved by Michel Bataille, Rouen, France; Ioannis D. Sfikas, National and Kapodistrain University of Athens, Greece, and the proposer.

Mea Culpa

Brian D. Beasley of Presbyterian College in Clinton, SC should have been credited with having solved problem 5510.

Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before January 15, 2019

5511: *Proposed by Kenneth Korbin, New York, NY*

A trapezoid with perimeter $58+14\sqrt{11}$ is inscribed in a circle with diameter $17+7\sqrt{11}$. Find its dimensions if each of its sides is of the form $a+b\sqrt{11}$ where a and b are positive integers.

5512: *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria. Spain*

If $a_k>0$, $(k=1,2,\ldots,n)$ then $\dfrac{n}{\displaystyle\sum_{k=1}^{n}\frac{1}{\frac{1}{k}+a_k}}-\dfrac{n}{\displaystyle\sum_{k=1}^{n}\frac{1}{a_k}}\geq\dfrac{2}{n+1}$.

5513: *Proposed by Michael Brozinsky, Central Islip, NY*

In an $n\times n\times n$ cube partitioned into n^3 congruent cubes by $n-1$ equally spaced planes parallel to each pair of parallel faces, there are 20 times as many non-cubic rectangular parallelepipeds that could be formed as were cubic parallelepipeds. What is n?

5514: *Proposed by D. M. B***a***tinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

If $a\in\left(0,\dfrac{\pi}{2}\right)$ and $b=\arcsin a$, then calculate $\displaystyle\lim_{n\to\infty}\sqrt[n]{n!}\left(\sin\left(\frac{b\cdot\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)-a\right)$.

5515: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let n be a positive integer. Prove that

$$\frac{1}{2^n}\left(\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}+\binom{n-1}{k-1}^2}\right)^2 \geq 1.$$

5516: *Proposed by Ovidiu Furdui and Alina Sîntămărian both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate $\displaystyle\sum_{n=1}^{\infty} n\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}-\frac{1}{2n^2}\right)$.

Solutions

5493: *Proposed by Kenneth Korbin, New York, NY*

Convex quadrilateral $ABCD$ is inscribed in a circle with diameter $\overline{AC}=729$. Sides $\overline{AB}$ and $\overline{CD}$ each have positive integer length. Find the perimeter if $\overline{BD}$=715.

Solution by Bruno Salgueiro Fanego, Viveiro, Spain

Let $a=AB, b=BC, c=CD$ and $d=DA$. Since AC is a diameter of the circumcircle of $ABCD$, $\angle CBA=\dfrac{\pi}{2}=\angle ADC$ and hence the Pythagorean theorem can be applied on $\triangle ABC$ and $\triangle ACD$: $a^2+b^2=7292=c^2+d^2$.

Since $ABCD$ is cyclic, by Ptolemy's theorem $ac+bd=729\times 715$. Thus, $(729^2-a^2)(729^2-c^2)=b^2d^2=(729\cdot 715-ac)^2$, that is, the point with positive integer coordinates (a,c) lies on the ellipse whose equation is $729x^2-1430xy+729y^2-14737464=0$.

From this it follows that

$(a,c)\in\{(279,405),(405,279),(715,729),(729,715)\}$ and since $a<729$ and $c<729$,

$(a,c)\in\{(279,405),(405,279)\}$, so the lengths of the sides of $ABCD$ are

$(a,b,c,d)\in\{(279,180\sqrt{14},\ 405,162\sqrt{14}),\ (279,162\sqrt{14},405,180\sqrt{14})\}$

and hence, the perimeter of $ABCD$ is $a+b+c+d=342(2+\sqrt{14})$.

Editor's comment : **Ioannis D. Sfikas'** solution to this problem started off with comments about its related history. "In Euclidean geometry, Ptolemy's inequality relates the six distances determined by four points in the plane or in a higher-dimensional space. It states for any four points A,B,C,D the following inequality holds:

$$\overline{AB}\cdot\overline{CD}+\overline{BC}\cdot\overline{DA}\geq\overline{AC}\cdot\overline{BD}.$$

As a special case, Ptolemy's theorem states that the inequality becomes an equality exactly when the four points lie in cyclic order on a circle. The inequality does not generalize from Euclidean spaces to arbitrary metric spaces. The spaces where it remains valid are called the Ptolemaic spaces; they include the inner product spaces, Hadamard spaces, and shortest path distances on Ptolemaic graphs.

In other words, Ptolemy's theorem is a relation between the four sides and two diagonals of a cyclic quadrilateral (a quadrilateral whose vertices lie on a common circle). The theorem is named after the Greek astronomer and mathematician Ptolemy. Ptolemy used the theorem as an aid to creating his table of chords, a trigonometric table that he applied to astronomy." Ioannis then went on to solve the problem in the above manner.

Also solved by the Brian D. Beasley, Presbyterian College, Clinton, SC; Cartesian Gains Student Problem Solving Group, Mountain Lakes High School, Mountain Lakes, NJ; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5494: *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

If $a \geq b \geq c \geq d$ are the lengths of four segments from which an infinite number of convex quadrilaterals can be constructed, calculate the maximal product of the lengths of the diagonals in such quadrilaterals.

Solution 1 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

A classical result of Claudius Ptolemy of Alexandria (circa 85AD$-$165AD), known as Ptolemy's theorem, states that for a cyclic quadrilateral with side lengths a, b, c, d (in that order) and diagonals of lengths p and q, the product of the lengths of the diagonals equals the sum of the products of the lengths of the opposite sides, $pq = ac + bd$. For a general convex quadrilateral, we have *Ptolemy's inequality:*

Theorem 1. For a convex quadrilateral with sides of length a, b, c, d (in that order) and diagonals of length p and q, we have $pq \leq ac + bd$.

For the above problem, we have to order α, β, and γ, where $\alpha = ab + cd$, $\beta = ac + bd$ and $\gamma = ad + bc$. Then, we have:

(i) If we have $\alpha \geq \beta$, that means $ab + cd \geq ac + bd$, or $a(b - c) + d(c - b) \geq 0$, which holds.

(ii) If we have $\alpha \geq \gamma$, that means $ab + cd \geq ad + bc$, or $b(a - c) + d(c - a) \geq 0$, or $(b - d)(a - c) \geq 0$, which holds.

So, if $a \geq b \geq c \geq d$ are the lengths of four segments, from which an infinite number of convex quadrilaterals can be constructed, then the maximal product of the lengths of the diagonals in such quadrilaterals is $ab + cd$.

[1] Alsina, Claudi and Nelsen, Roger B. (2009). *When less is more: visualizing basic inequalities, p. 82. Mathematical Association of America.*

Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that the maximal product of the length of the diagonals in such quadrilaterals equals $ab + cd$.

Let $WXYZ$ be a convex quadrilateral such that $\overline{WX} = w,\ \overline{XY} = x,\ \overline{YZ} = y,\ \overline{ZW} = z$. By a result of C.A. Bretschneider, the product of the lengths of the diagonals equals $\sqrt{w^2y^2 + x^2z^2 - 2wxyz\cos(\angle XWZ + \angle XYZ)}$, which does not exceed $\sqrt{w^2y^2 + x^2z^2 + 2wxyz} = wy + xz$. Note that

$$ab + cd = ac + bd + (b - c)(a - d) \geq ac + bd$$

and

$$ab + cd = ad + bc + (a - c)(b - d) \geq ad + bc.$$

Hence in order to obtain the maximum product of the diagonals, we put $w = a,\ x = c,\ y = b$, and $z = d$. It is easy to check that when $WXYZ$ is a cyclic quadrilateral, we have

$$\overline{WY} = \sqrt{\frac{(ab + cd)(bc + ad)}{(ac + bd)}} \text{ and } \overline{XZ} = \sqrt{\frac{(ab + ca)(ac + bd)}{(bc + ad)}},$$

so that the maximum product $ab + cd$ is attained. Hence our claimed maximum.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

The German mathematician Carl Anton Bretschneider derived in 1842 the following generalization of Ptolemy's theorem, regarding the product pq of the diagonals in a convex quadrilateral

$$p^2q^2 = a^2c^2 + b^2d^2 - 2abcd\cos(A + C) \qquad (1)$$

This relation can be considered to be a law of cosines for a quadrilateral. Since $4\cos(A + C) \geq -1$, it also gives a proof of Ptolemy's inequality.

We note that the product p^2q^2 in (1) is maximal if $\cos(A + C) = -1$, i.e., if $A + C = 180^\circ$ which implies that the product pq is maximal if the quadrilateral is a cyclic quadrilateral. In that case we get $pq = ac + bd$.

It remains to determine for which permutation of the sides the term $ac + bd$ is maximal. There are three possibilities, namely

$ab + cd, ac + bd, ad + bc$. Of these three expressions $ab + cd$ is maximal, since

$ab + cd - (ac + bd) = (a - d)(b - c) \geq 0$, and $ab + cd - (ad + bc) = (a - c)(b - d) \geq 0$.

References
[1] Titu Andreescu & Dorian Andrica, Complex Numbers from A to . . . Z, Birkhäuser, 2006, pp. 207-209.

Solution 4 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

The maximal product of the lengths of the diagonals is $ab + cd$. This maximum is achieved when $(AB, BC, CD, DA) = (a, c, b, d)$ is a cyclic quadrilateral.

By considering the vertices as hinges, Thomas proves [1] that any convex quadrilateral can be deformed into a cyclic quadrilateral (having the same side lengths).

In any convex quadrilateral, Ptolemy's Inequality tells us that the product of the diagonals is less than or equal to the sum of the products of the lengths of opposite sides. In a cyclic quadrilateral, Ptolemy's Theorem, tells us that the product of the diagonals equals the sum of the products of the lengths of opposite sides. Given our four appropriate segments, a, b, c, d, there are six ways to arrange them in a convex quadrilateral. By symmetry, only three of these are distinct.

We show these three possibilities with corresponding bound on the product of the diagonals, $AC \cdot BD$:

$(AB, BC, CD, DA) = (a, b, c, d); AC \cdot BD \leq ac + bd$
$(AB, BC, CD, DA) = (a, b, d, c); AC \cdot BD \leq ad + bc$
$(AB, BC, CD, DA) = (a, c, b, d); AC \cdot BD \leq ab + cd.$
The third case gives the largest possible value (because we've placed the two largest sides opposite one another).

Algebraically,
$ac + bd \leq ab + cd \iff 0 \leq (a-d)(b-c)$
which is true by the given ordering, and
$ad + bc \leq ab + cd \iff 0 \leq (a-c)(b-d)$
which is also true by the given ordering.

Summarizing: when the four segments are arranged in a quadrilateral $ABCD$, the product of the diagonals is $\leq AB \cdot CD + BC \cdot AD$; the largest possible value for $AB \cdot CD + BC \cdot AD$ is $ab + cd$, which is achieved when a, b and c, d are opposite sides of a cyclic quadrilateral.

Reference:

1. Peter, Thomas, Maximizing the Area of a Quadrilateral, The College Mathematics Journal, Vol. 34, No. 4 (September 2003), pp. 315-316.

Also solved by Kenneth Korbin, New York, NY; David E. Manes, Oneonta, NY, and the proposers.

5495: *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" School Buzău, Romania*

Let $\{x_n\}_{n\geq 1},\ x_1 = 1,\ x_n = 1 \cdot \sqrt{3!!} \cdot \sqrt[3]{5!!} \cdot \ldots \sqrt[n]{(2n-1)!!}$.
Find:

$$\lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}} \right).$$

Solution 1 by Moti Levy, Rehovot, Israel

$$(2n-1)!! = \frac{(2n)!}{2^n n!}. \qquad (1)$$

Using Stirling's asymptotic formula, we have

$$n! \sim \frac{n^n}{e^n}. \tag{2}$$

Applying (2) to (1) yields

$$\sqrt[n]{(2n-1)!!} \sim \left(\frac{(2n)^{2n}}{e^{2n}2^n n!}\right)^{\frac{1}{n}} = \frac{2n}{e} \tag{3}$$

Now we use (3) to approximate x_n,

$$x_n \sim \prod_{k=1}^{n} 2ke = \frac{2^n n!}{e^n} \sim \frac{2^n \frac{n^n}{e^n}}{e^n} = \frac{2^n n^n}{e^{2n}},$$

or

$$\sqrt[n]{x_n} \sim \frac{2n}{e^2}.$$

Hence,

$$\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}} \sim \frac{e^2}{2}(n+1) - \frac{e^2}{2}n = \frac{e^2}{2},$$

and we conclude that $\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}}\right) = \frac{e^2}{2} \cong 3.694\,5.$

Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that the limit of the problem equals $\frac{e^2}{2}$.

We need the following knows results for positive integers n.

$$\begin{aligned}
\ln(n!) &= \left(n+\frac{1}{2}\right)\ln n - n + A + O\left(\frac{1}{n}\right), && (1)\\
\sum_{k=1}^{n}\frac{1}{k} &= \ln n + B + O\left(\frac{1}{n}\right), && (2)\\
\ln\left(1+\frac{1}{n}\right) &= \frac{1}{n} + O\left(\frac{1}{n^2}\right). && (3)
\end{aligned}$$

where A and B are constants.

By (1) we have

$$\ln((2k)!) - \ln(k!) = \ln(k!) + (2\ln 2 - 1)k + \frac{\ln 2}{2} + O\left(\frac{1}{k}\right). \tag{4}$$

Next we show that

$$_n = n\ln n + (\ln 2 - 2)n + \frac{(1+\ln 2)\ln n}{2} + O(1) \tag{5}$$

In fact by (4) we have

$$\ln x_n = \sum_{k=1}^{n} \frac{\ln((2k-1)!!)}{k}. \quad = \quad \sum_{k=1}^{n} \frac{\ln((2k)!) - \ln(k!) - (\ln 2)k}{k}$$

$$= \sum_{k=1}^{n} \left(\ln k + \ln 2 - 1 + \frac{\ln 2}{2k} + O\left(\frac{1}{k^2}\right) \right)$$

$$= \ln(n!) + (\ln 2 - 1)n + \frac{\ln 2}{2} \sum_{k=1}^{n} \frac{1}{n} + O(1),$$

and (5) follows from (1) and (2).

Let $f(n) = 2\ln n - \dfrac{\ln x_n}{n}$. By (5), we obtain

$$f(n) = \ln n + 2 - 2 + O\left(\frac{\ln n}{n}\right). \qquad (6)$$

We next show that

$$f(n+1) - f(n) = \frac{1}{n} + O\left(\frac{\ln n}{n^2}\right). \qquad (7)$$

In fact

$$f(n+1) - f(n) = 2(\ln(n+1) - \ln n) - \left(\frac{\ln x_{n+1}}{n+1} - \frac{\ln x_n}{n} \right)$$

$$= 2\ln\left(1 + \frac{1}{n}\right) + \frac{\ln x_n}{n(n+1)} - \frac{\ln(2n+2)! - \ln(n+1)! - (\ln 2)(n+1)}{(n+1)^2},$$

and (7) follows readily from (3),(5) and (4). By the mean value theorem, we have

$$e^{f(n+1)} - e^{f(n)} = (f(n+1) - f(n))\, e^t, \qquad (8)$$

where t is a number lying between $f(n)$ and $f(n+1)$. By (6), both $e^{f(n+1)}$ and $e^{f(n)}$ equal $\dfrac{e^2 n}{2}\left(1 + O\left(\dfrac{\ln n}{n}\right)\right)$. Hence, by (7) and (8),

$$\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}} = e^{f(n+1)} - e^{f(n)} = \frac{e^2}{2}\left(1 + O\left(\frac{\ln n}{n}\right)\right),$$

and our claim for the limit follows.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

We will use the lemma from Solution 3 to Problem 5398 that appeared in this Column (see Nov. 2016 issue) that stated: *"If the positive sequence* (p_n) *is such that* $\lim_{n\to\infty} \dfrac{p_{n+1}}{np_n} = p > 0$, *then* $\lim_{n\to\infty} \left(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{p_n} \right) = \dfrac{p}{e}$*."*

We let $\{p_n\}_{n\geq 1},\ p_n = \dfrac{n^{2n}}{x_n}$. Then

$$\lim_{n\to\infty}\frac{p_{n+1}}{np_n} = \lim_{n\to\infty}\frac{\dfrac{(n+1)^{2n+2}}{x_n\ \sqrt[n+1]{(2n+1)!!}}}{n\dfrac{n^{2n}}{x_n}}$$

$$= \lim_{n\to\infty}\frac{(n+1)^{2n+2}}{n^{2n+1}\ \sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty}\frac{(n+1)^{2n}(n+1)^2}{n^{2n}n\ \sqrt[n+1]{(2n+1)!!}}$$

$$= \lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^n\right)^2\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)^{2(n+1)}}}{\sqrt[n+1]{n^{n+1}(2n+1)!!}} = e^2\lim_{n\to\infty}\sqrt[n]{\frac{n^{2n}}{(n-1)^n(2n-1)!!}}$$

$$\underset{root\ crition}{=} e^2\lim_{n\to\infty}\frac{\dfrac{(n+1)^{2(n+1)}}{n^{n+1}(2n+1)!!}}{\dfrac{n^{2n}}{(n-1)^n(2n-1)!!}} = e^2\lim_{n\to\infty}\frac{(n+1)^{2n}(n+1)^2(n-1)^n}{n^n n^{2n}(2n+1)}$$

$$= e^2\lim_{n\to\infty}\frac{(n+1)^{2n}(n+1)^2(n-1)^n}{n^{2n}n(2n+1)n^n}$$

$$= e^2\lim_{n\to\infty}\frac{(n+1)^{2n}}{n^{2n}}\lim_{n\to\infty}\frac{(n+1)^2}{n^{2n+1}}\lim_{n\to\infty}\frac{(n-1)^n}{n^n}$$

$$= e^2\lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^n\right)^2\frac{1}{2}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^2e^2\frac{1}{2}e^{-1}$$

$$= \frac{e^3}{2} =: p > 0, \quad \text{which implies by the lemma mentioned above,}$$

that the required limit is $\lim\limits_{n\to\infty}\left(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{p_n}\right) = \dfrac{p}{e} = \dfrac{e^2}{2}$.

Editor′s comment : In addition to the above solution **Bruno Salgueiro Fanego** stated that a more general form of the problem was published by the authors′ of 5495 in the journal *La Gaceta de la Real Sociedad Matemática Espanãola* vol. 17 (3), 2014, pp. 523-524. (available at http://gaceta.rsme.es/abrir.phd?id=1218.) Therein they showed: If $\{a_n\}_{n\geq 1}$ is a sequence of real positive numbers such that $\lim\limits_{n\to\infty}(a_{n+1}-a_n) = a \neq 0$, then

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{\prod_{k=1}^{n+1} f(a_k)}} - \frac{(n)^2}{\sqrt[n]{\prod_{k=1}^{n} f(a_k)}}\right) = \frac{e}{ca}.$$

Letting $a_n = n$ and $f(n) = \sqrt[n]{(2n-1)!!}$ gives the desired result.

Solution 4 by Michel Bataille, Rouen, France

Let $u_n = \frac{n^2}{\sqrt[n]{x_n}} = \left(\frac{n^{2n}}{x_n}\right)^{1/n}$. We show that $\lim\limits_{n\to\infty}(u_{n+1}-u_n) = \frac{e^2}{2}$.

To this end, we first recall that $(2n-1)!! = \dfrac{(2n)!}{2^n(n!)}$ and the following asymptotic expansion as $n \to \infty$:

$$\ln(n!) = n\ln(n) - n + \frac{\ln(n)}{2} + \ln(\sqrt{2\pi}) + \frac{1}{12n} + o(1/n),$$

from which we readily deduce

$$\ln\left(\frac{(2n)!}{n!}\right) = \ln((2n)!) - \ln(n!) = n\ln(n) + n(2\ln(2)-1) + \frac{\ln(2)}{2} - \frac{1}{24n} + o(1/n).$$

Now, we have

$$\ln(u_n) = 2\ln(n) - \frac{\ln(x_n)}{n} = 2\ln(n) - \frac{1}{n}\ln\left(\prod_{k=1}^{n}\frac{1}{2}\cdot\left(\frac{(2k)!}{k!}\right)^{1/k}\right) = 2\ln(n) + \ln(2) - \frac{s_n}{n} \quad (1)$$

where $s_n = \sum\limits_{k=1}^{n}\frac{1}{k}\cdot\ln\left(\frac{(2k)!}{k!}\right)$.

Consider the sequence $\{y_n\}_{n\geq 2}$ defined by

$$y_n = s_n - n\ln(n) - (2\ln(2)-2)n - \frac{1+\ln(2)}{2}\cdot\ln(n).$$

For $n \to \infty$, we calculate

$$\begin{aligned}
y_n - y_{n-1} &= s_n - s_{n-1} - n\ln(n) + (n-1)\ln(n-1) - (2\ln(2)-2) + \frac{1+\ln(2)}{2}\cdot\ln\left(1-\frac{1}{n}\right)\\
&= \frac{1}{n}\left(\ln\left(\frac{(2n)!}{n!}\right)\right) + n\ln\left(1-\frac{1}{n}\right) - \ln(n) - \frac{1-\ln(2)}{2}\ln\left(1-\frac{1}{n}\right) + (2-\ln(2))\\
&= 2\ln(2) - 1 + \frac{\ln(2)}{2n} - \frac{1}{24n^2} + o(1/n^2) + n\left(-\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} + o(1/n^3)\right)\\
&\qquad -\frac{1-\ln(2)}{2}\left(-\frac{1}{n} - \frac{1}{2n^2} + o(1/n^2)\right) + 2 - 2\ln(2)\\
&= \frac{a}{n^2} + o(1/n^2)
\end{aligned}$$

where we set $a = -\frac{1+2\ln(2)}{8}$. Thus, the series $\sum_{n=2}^{\infty}(y_n - y_{n-1})$ is convergent. Let S denotes its sum. Then, we may write $\sum_{k=2}^{n}(y_k - y_{k-1}) = S + o(1)$ and so $y_n = b + o(1)$ as $n \to \infty$ (where $b = S + y_1$).
It follows that

$$s_n = n\ln(n) + (2\ln(2)-2)n + \frac{1+\ln(2)}{2}\cdot\ln(n) + b + o(1)$$

as $n \to \infty$.
From (1), we now obtain

$$\ln(u_n) = \ln(n) + 2 - \ln(2) - \frac{1+\ln(2)}{2}\cdot\frac{\ln(n)}{n} - \frac{b}{n} + o(1/n).$$

First, we deduce that $\ln(u_n) = \ln(n) + 2 - \ln(2) + o(1)$, hence $u_n = e^{\ln(n)+2-\ln(2)} \cdot e^{o(1)}$ and so $u_n \sim\ n \cdot \frac{e^2}{2}$. Second, the calculation of $\ln(u_{n+1}) - \ln(u_n)$ easily leads to

$$\ln(u_{n+1}) - \ln(u_n) = \ln\left(1 + \frac{1}{n}\right) + o(1/n) = \frac{1}{n} + o(1/n).$$

(Note that
$\frac{\ln(n)}{n} - \frac{\ln(n+1)}{n+1} = \frac{1}{n}\left((\ln n)\left(1 - (1+1/n)^{-1}\right) + o(1)\right) = \frac{1}{n}\left(-\frac{\ln(n)}{n} + o(1)\right) = o(1/n)$ as $n \to \infty$.)
Since

$$u_{n+1} - u_n = u_n\left(\frac{u_{n+1}}{u_n} - 1\right) = u_n\left(e^{\ln(u_{n+1})-\ln(u_n)} - 1\right)$$

we finally arrive at

$$u_{n+1} - u_n\ \sim\ u_n(\ln(u_{n+1}) - \ln(u_n))\ \sim\ n \cdot \frac{e^2}{2} \cdot \left(\frac{1}{n}\right)\ \sim\ \frac{e^2}{2}$$

and the result follows.

Also solved by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Marian Ursarescu - Romania, and the proposers.

5496: *Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Let a, b, c be real numbers such that $0 < a < b < c$. Prove that:

$$\sum_{cyclic}\left(e^{a-b} + e^{b-a}\right) \geq 2a - 2c + 3 + \sum_{cyclic}\left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY

For $x > 0$ we apply the known inequality $e^x > x + 1$ to $x = a - b, b - c$, and $a - c$ to get

$$e^{a-b} > a - b + 1,\quad e^{b-c} > b - c + 1,\quad e^{a-c} > a - c + 1,$$

respectively. Adding these inequalities yields

$$e^{a-b}\ +\ e^{b-c}\ +\ e^{a-c}\ >\ 2a\ -\ 2c\ +\ 3. \tag{1}$$

For $x > y$, we see that
$e^{x-y} > (x/y)^{\sqrt{xy}} \iff x - y > \sqrt{xy}\ln(x/y) \iff \sqrt{xy} < (x-y)/(\ln x - \ln y)$, which is the left-hand member of the *logarithmic mean inequality.* Thus we have, since $0 < a < b < c$,

$$e^{b-a} > \left(\frac{b}{a}\right)^{\sqrt{ab}},\quad e^{c-b} > \left(\frac{c}{b}\right)^{\sqrt{bc}},\quad e^{c-a} > \left(\frac{c}{a}\right)^{\sqrt{ac}} > \left(\frac{a}{c}\right)^{\sqrt{ac}}. \tag{2}$$

Adding (1) and (2) , we find that

$$\sum_{cyclic}\left(e^{a-b}+e^{b-a}\right) > 2a-2c+3+\sum_{cyclic}\left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We will prove the slightly stronger inequality

$$\sum_{cyclic}\left(e^{a-b}+e^{b-a}\right) \geq a-c+3+\sum_{cyclic}\left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

We will use the inequalities

$e^x \geq 1+x,\ x$ real, (1)

$1 \geq \left(\frac{y}{x}\right)^{\sqrt{xy}}, 0 \leq y \leq x,$ (2)

$e^{y-x} \geq \left(\frac{y}{x}\right)^{\sqrt{xy}},\ y \geq x,$ (3)

(1) and (2) are clear, while (3) is equivalent to each of the following lines:

$y-x \geq \sqrt{xy}\log\left(\frac{y}{x}\right),$

$\sqrt{\frac{y}{x}}-\sqrt{\frac{x}{y}} \geq \log\left(\frac{y}{x}\right),$

$x-\frac{1}{x}-\log x = \int_1^x\left(1+\frac{1}{t^2}-\frac{1}{t}\right)dt \geq 0,\ x \geq 1$ which holds true.

Thus

$$\sum_{cyclic}\left(e^{a-b}+e^{b-a}\right) \geq 1+a-b+\left(\frac{b}{a}\right)^{\sqrt{ab}}+1+b-c+\left(\frac{b}{c}\right)^{\sqrt{bc}}+1+c-a+a^{a-c}$$

$$\begin{aligned}
&= 3+\left(\frac{b}{a}\right)^{\sqrt{ab}}+\left(\frac{c}{b}\right)^{\sqrt{bc}}+e^{a-c}\\
&\geq 3+\left(\frac{b}{a}\right)^{\sqrt{ab}}+\left(\frac{c}{b}\right)^{\sqrt{bc}}+1+a-c\\
&\geq 3+\left(\frac{b}{a}\right)^{\sqrt{ab}}+\left(\frac{c}{b}\right)^{\sqrt{bc}}+\left(\frac{a}{c}\right)^{\sqrt{bc}}+a-c.
\end{aligned}$$

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China, and the proposer.

5497: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

For all integers $n \geq 2$, show that $\prod_{k=1}^{n-1} 2\,\sin\left(\frac{k\pi}{n}\right)$ is an integer and determine it.

Solution 1 by Kee-Wai Lau, Hong Kong, China

It is proved as formula 12 on p. 227 Chapter XII of [1] that

$$\sin n\theta = \sin\theta \prod_{k=1}^{n-1} 2\sin\left(\theta + \frac{k\pi}{n}\right).$$

Since $\lim_{\theta\to\infty} \frac{\sin n\theta}{\sin\theta} = n$, so $\prod_{k=1}^{n-1} 2\sin\left(\theta + \frac{k\pi}{n}\right) = n.$

Reference:

1. D.V. Durell and A. Robinson: *Advanced Trigonometry,* Dover Publication, Inc., New York 2003.

Solution 2 by David E. Manes, Oneonta, NY

Subtracting the complex equation $e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i\sin x$ from the equation $e^{ix} = \cos x + i\sin x$, one obtains the formula

$$2\sin x = \frac{1}{i}\left(e^{ix} - e^{-ix}\right).$$

Therefore,

$$\begin{aligned}\prod_{k=1}^{n-1} 2\sin\left(\frac{k\pi}{n}\right) &= \prod_{k=1}^{n-1} \frac{1}{i}\left(e^{i\pi k/n} - e^{-i\pi k/n}\right)\\ &= \left(\frac{1}{i}\right)^{n-1}\left(\prod_{k=1}^{n-1} e^{i\pi k/n}\right)\left(\prod_{k=1}^{n-1}\left(1 - e^{-2i\pi k/n}\right)\right).\end{aligned}$$

Note that

$$\begin{aligned}\prod_{k=1}^{n-1} e^{i\pi k/n} &= e^{\left(\sum_{k=1}^{n-1} i\pi k/n\right)} = e^{(i\pi/n)\sum_{k=1}^{n-1} k} = e^{(i\pi/n)((n-1)(n)/2)}\\ &= e^{(i\pi/2)(n-1)} = \left(e^{i\pi/2}\right)^{n-1} = \left(\cos(\pi/2) + i\sin(\pi/2)\right)^{n-1} = i^{n-1}.\end{aligned}$$

Hence,

$$\prod_{k=1}^{n-1} 2\sin\left(\frac{k\pi}{n}\right) = \prod_{k=1}^{n-1}\left(1 - e^{-2i\pi k/n}\right) = f(1),$$

where $f(x) = \prod_{k=1}^{n-1}\left(x - e^{-2i\pi k/n}\right)$. The zeros of the polynomial $f(x)$ are the non-trivial n^{th} roots of unity so that

$$f(x) = \frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1}.$$

Therefore, $f(1) = n$. Hence, if $n \geq 0$, then

$$\prod_{k=1}^{n-1} 2\sin\left(\frac{k\pi}{n}\right) = n.$$

Solution 3 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Let $\zeta = e^{\pi i/n}$. Then $\zeta^2 = e^{2\pi i/n}$ is a primitive n-th root of unity. So $\zeta^2, \ldots, \zeta^{2(n-1)}$ are the roots of

$$\frac{x^n - 1}{x - 1} = x^{n-1} + \cdots + 1$$

and therefore

$$x^{n-1} + \cdots + 1 = \prod_{k=1}^{n-1}(x - \zeta^{2k}).$$

Let $x = 1$ to find

$$\begin{aligned} n &= \prod_{k=1}^{n-1}(1 - \zeta^{2k}) \\ &= \prod_{k=1}^{n-1} -\zeta^k(\zeta^k - \zeta^{-k}). \end{aligned}$$

Since $\zeta^k = e^{k\pi i/n} = \cos(k\pi/n) + i\sin(k\pi/n)$ we have $\zeta^k - \zeta^{-k} = 2i\sin(k\pi/n)$. Thus

$$n = \prod_{k=1}^{n-1} -2i\zeta^k \sin\left(\frac{k\pi}{n}\right).$$

Finally, since each $\sin(k\pi/n) > 0$ and each $|-2i\zeta^k| = 2$ for $k = 1, \ldots, n-1$ we have

$$n = \prod_{k=1}^{n-1} 2\sin\left(\frac{k\pi}{n}\right)$$

by taking the absolute value of the last expression.

Editor's comment : **Paul M. Harms of North Newton KS** mentioned in his solution to 5497 that Wikipedia's "List of Trigonometric Identities" includes $\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$, and from this the value of n immediately follows.

Also solved by Michel Bataille, Rouen, France; Bruno Salgueiro Fanego (three solutions), Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton KS; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ángle Plaza, University of Las Palmas de Gran Canaria, Spain; Henry Ricardo, Westchester Area Math Circle, NY; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Marian Ursărescu, Romania, and the proposer.

5498: *Proposed by Ovidiu Furdui and Alina Sîntămărian, both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Prove that

$$\sum_{n=1}^{\infty} \frac{\{n!e\}}{n} = \int_0^1 \frac{e^x - 1}{x} dx$$

where $\{a\}$ denotes the fractional part of a.

Solution 1 by Pedro H. O. Pantoja, Natal/RN, Brazil

By Taylor's formula,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e^{\alpha}}{(n+1)!}, \quad \alpha \in (0,1),$$

and this implies that

$$n!e = n!\left(\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) + \frac{e^{\alpha}}{n+1}, \quad \alpha \in (0,1).$$

Therefore,

$$\{n!e\} = n!e - \lfloor n!e \rfloor = n!\left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}\right) \Rightarrow$$

$$\{n!e\} = n!\left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\right) = \sum_{k=1}^{\infty} \frac{1}{(n+1)(n+2)\cdots(n+k)}.$$

We have,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\{n!e\}}{n} &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{n(n+1)(n+2)\cdots(n+k)} \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{k!}\int_0^1 (1-x)^k x^{n-1} dx \\
&= \int_0^1 \sum_{k=1}^{\infty} \frac{(1-x)^k}{k!} \sum_{n=1}^{\infty} x^{n-1} dx \\
&= \int_0^1 (e^{1-x} - 1) \cdot \frac{1}{1-x} dx \\
&= \int_0^1 \frac{e^{1-x} - 1}{1-x} dx \\
&= \int_0^1 \frac{e^y - 1}{y} dy,
\end{aligned}$$

where in the last integral, we used the substitution $y = 1 - x$.

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Both the sides of the equality are equal to

$$\sum_{k=1}^{\infty} \frac{1}{k \cdot k!}$$

$$\{n!e\} = n!\left\{2 + \frac{1}{2} + \frac{1}{3!} + \ldots + \frac{1}{m!} + \frac{1}{(m+1)!}\ldots\right\} = \left\{\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots\right\}$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{(n+1)(n+2)\ldots(n+k)} < \sum_{k=1}^{\infty} 2^{-k} = 1$$

it follows that

$$\left\{\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots\right\} = \sum_{k=1}^{\infty} \frac{1}{(n+1)(n+2)\ldots(n+k)}$$

and

$$\sum_{n=1}^{\infty} \frac{\{n!e\}}{n} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{n(n+1)(n+2)\ldots(n+k)} =$$

$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)\ldots(n+k)} = \sum_{k=1}^{\infty} \frac{1}{k \cdot k!} \qquad (1)$$

and finally

$$\int_0^1 \frac{e^x - 1}{x} dx = \int_0^1 \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} dx = \sum_{k=1}^{\infty} \int_0^1 \frac{x^{k-1}}{k!} dx = \sum_{k=1}^{\infty} \frac{1}{k \cdot k!}$$

For proving (1) let's write $a_n = 1/(n(n+1)\cdots(n+k))$.

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+k+1} \iff a_{n+1}(n+1) - na_n = -a_{n+1}k$$

Telescoping

$$\sum_{n=1}^{N} a_{n+1}(n+1) - na_n = \underbrace{a_{N+1}(N+1)}_{\to 0} - a_1 = -k\sum_{n=1}^{N} a_{n+1}$$

and

$$\sum_{k=1}^{\infty} a_k = \frac{1+k}{k(k+1)!} = \frac{1}{k \cdot k!}$$

Solution 3 by Michel Bataille, Rouen, France

From $\frac{e^x-1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$ for $x \in (0,1]$ and

$$\sum_{n=1}^{\infty} \int_0^1 \left|\frac{x^{n-1}}{n!}\right| dx = \sum_{n=1}^{\infty} \frac{1}{n \cdot (n!)} < \infty,$$

we deduce that

$$\int_0^1 \frac{e^x - 1}{x}\, dx = \int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \right) dx = \sum_{n=1}^{\infty} \left(\int_0^1 \frac{x^{n-1}}{n!}\, dx \right) = \sum_{n=1}^{\infty} \frac{1}{n \cdot (n!)}. \qquad (1)$$

On the other hand, for $n \geq 1$ we have

$$(n!)e = (n!) \sum_{j=0}^{\infty} \frac{1}{j!} = a_n + \sum_{k=1}^{\infty} \frac{1}{(n+1)\cdots(n+k)}$$

where $a_n = n! + (n-1)!\binom{n}{1} + (n-2)!\binom{n}{2} + \cdots + 1!\binom{n}{n-1} + 1$ is a positive integer and

$$0 < \sum_{k=1}^{\infty} \frac{1}{(n+1)\cdots(n+k)} < \sum_{k=1}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{n} \leq 1.$$

It follows that

$$\{n!e\} = \sum_{k=1}^{\infty} \frac{1}{(n+1)\cdots(n+k)}$$

and so

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\{n!e\}}{n} &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{n(n+1)\cdots(n+k)} \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n(n+1)\cdots(n+k)} \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)\cdots(n+k-1)} - \frac{1}{(n+1)\cdots(n+k)} \right) \right) \\
&= \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{1}{1 \cdot 2 \cdots k}.
\end{aligned}$$

Finally we obtain $\sum\limits_{n=1}^{\infty} \frac{\{n!e\}}{n} = \sum\limits_{k=1}^{\infty} \frac{1}{k \cdot (k!)}$, and comparing with (1) gives the required result.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposers.

Mea Culpa

Brian D. Beasley of Presbyterian College in Clinton, SC should have been credited with having solved 5482, and **Albert Stadler of Herrliberg, Switzerland** should have been credited with having solved 5488.

Titu Zvonaru of Comănesti, Romania noted that proof number 2 (of the 6 shown) for problem 5492 is incomplete. The question asked us to prove that a certain inequality held that was subject to a constraint on the variables. The author of the solution found values of the variables that produced equality, and then by taking other values in small epsilon neighborhoods around this point that produced equality, showed that the resulting values of the expression were smaller than the value that gave equality. Up to here, everything is fine. But it was then concluded that the point giving equality was a local maximum. The method used was very similar to the one that is often used in obtaining saddle and extrema points vis-a-vis Lagrange Multipliers. Admittedly there is some hand-waving in using this approach, and this is what Titu noticed. The approach used in this problem can tell us when the inequality goes awry, but it cannot be used to prove with absolute certainty that the inequality holds. For that, derivative tests within the theory of Lagrange Multipliers, must be used.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before February 15, 2019

5517: *Proposed by Kenneth Korbin, New York, NY*

Find positive integers (a, b, c) such that
$\arccos\left(\dfrac{a}{1331}\right) = \arccos\left(\dfrac{b}{1331}\right) + \arccos\left(\dfrac{c}{1331}\right)$ with $a < b < c$.

5518: *Proposed by Roger Izard, Dallas, TX*

Let triangle PQR be equilateral and let it intersect another triangle ABC at points U, U', W, W', V, V' such that WU', UV', VW' are equal in length, and triangles $AU'W, BV'U, CW'V$ are equal in area (see Figure 1). Show that triangle ABC must then also be equilateral

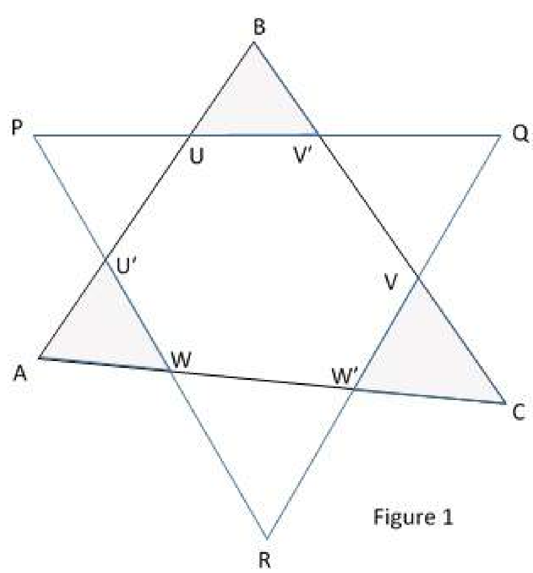

Figure 1

5519: *Proposed by Titu Zvonaru, Comănesti, Romania*

Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + \frac{2abc}{a^3 + b^3 + c^3} \geq \frac{11}{3}.$$

5520: *Proposed by Raquel León (student) and Angel Plaza, University of Las Palmas de Gran Canaria, Spain*

Let n be a positive integer. Prove that

$$\sum_{k=0}^{2n} \binom{2n+k}{k}\binom{2n}{k}\frac{(-1)^k}{2^k}\frac{1}{k+1} = 0.$$

5521: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $a > 0$ be a real number. If f is an odd non-constant real function having second derivative in the interval $[-a, a]$ and $f'(-a) = f'(a) = 0$, then prove that there exists a point $c \in (-a, a)$ such that

$$\frac{1}{2} f''(c) \geq \frac{|f(a)|}{a^2}$$

5522: *Proposed by Ovidiu Furdui and Cornel Vălean from Technical University of Cluj-Napoca, Cluj-Napoca, Romania and Timiş, Romania, respectively*

Calculate

$$\int_0^1 \int_0^1 \frac{\log(1-x) - \log(1-y)}{x-y} dxdy.$$

Solutions

5499: *Proposed by Kenneth Korbin, New York, NY*

Given a triangle with sides $(21, 23, 40)$. The sum of these digits is $2 + 1 + 2 + 3 + 4 + 0 = 12$. Find primitive pythagorean triples in which the sum of the digits is 12 or less.

Solution 1 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

The two best-known primitive pythagorean triples $(3, 4, 5)$ and $(5, 12, 13)$ have this property. We will give two infinite families of such triples.

Recall that $a = s^2 - t^2$, $b = 2st$, $c = s^2 + t^2$ is a primitive pythagorean triple for any $s > t \geq 1$ with $\gcd(s, t) = 1$ and s and t of opposite parity.

1. For any $n \geq 1$ let $s = 10^n + 1$ and $t = 10^n$. Then

$$\begin{array}{rcccl} a & = & s^2 - t^2 & = & 2 \cdot 10^n + 1 \\ b & = & 2st & = & 2 \cdot 10^{2n} + 2 \cdot 10^n \\ c & = & s^2 + t^2 & = & 2 \cdot 10^{2n} + 2 \cdot 10^n + 1 \end{array}$$

is a primitive pythagorean triple. The sum of the digits in (a, b, c) is

$$2 + 1 + 2 + 2 + 2 + 2 + 1 = 12.$$

2. For any $n \geq 1$ let $s = 10^{2n} + 1$ and $t = 10^n$. Then

$$\begin{array}{rcccl} a & = & s^2 - t^2 & = & 10^{4n} + 10^{2n} + 1 \\ b & = & 2st & = & 2 \cdot 10^{3n} + 2 \cdot 10^n \\ c & = & s^2 + t^2 & = & 10^{4n} + 3 \cdot 10^{2n} + 1 \end{array}$$

is a primitive pythagorean triple. The sum of the digits in (a, b, c) is

$$1 + 1 + 1 + 2 + 2 + 1 + 3 + 1 = 12.$$

Solution 2 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

To begin, we note the well-known result that (a, b, c) is a primitive Pythagorean triple, with $a^2 + b^2 = c^2$ and a odd if and only if $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$ for some positive integers m and n such that $m > n$, $\gcd(m, n) = 1$, and m and n have opposite parity.

We can provide an infinite family of primitive Pythagorean triples for which the sum of the digits is 12 by choosing $m = 10^k + 1$ and $n = 10^k$ with $k \geq 0$. Then,

$$a_k = \left(10^k + 1\right)^2 - 10^{2k} = 2 \times 10^k + 1,$$

$$b_k = 2\left(10^k + 1\right)\left(10^k\right) = 2 \times 10^{2k} + 2 \times 10^k,$$

and

$$c_k = \left(10^k + 1\right)^2 + 10^{2k} = 2 \times 10^{2k} + 2 \times 10^k + 1$$

for $k \geq 0$. As noted above, $(a_k . b_k, c_k)$ is a primitive Pythagorean triple for each $k \geq 0$. Further, in each case, the sum of the non-zero digits for a_k, b_k, and c_k is $(2 + 1) + (2 + 2) + (2 + 2 + 1) = 12$. In particular, when $k = 0$, we have $(a_0, b_0, c_0) = (3, 4, 5)$, the best known primitive Pythagorean triple.

Another example is $(a, b, c) = (5, 12, 13)$. However, we haven't been able to generalize this in a manner similar to that shown above. Also, we haven't found any other examples of primitive Pythagorean triples (a, b, c) for which the sum of the digits of a, b, and c is 12 or less.

Editor's comment : **David Stone and John Hawkins, both of Georgia Southern University in Stateboro, GA** stated that a computer search revealed no triples with total digit sum < 12. In each triple with total digit sum 12, the 12 was achieved as

3+4+5. They also found no triples with total digit sum 13 or 14. They went on to find the above mentioned infinite class with digit sum15 and ended their solution with the comment: "We do not know whether there are other triples with total digit sum 12. Note that $x+y+z = 2ab + \left(b^2-a^2\right) + \left(b^2+a^2\right) = 2b(ab)$. For any integer w we know that $w = \text{Digitsum}(w) \mod 3$. Thus Digitsum (x)+ Digitsum (y)+ Digitsum $(z) = x+y+z = 2b(a+b) \mod 3$. So if the total digit sum is 12, then $b = 0 \mod 3$ or $a+b = 0 \mod 3$. That is, there are restrictions on the generators a and b."

Also solved by Ed Gray, Highland Beach, FL; David E. Manes, Oneonta, NY; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5500: *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

Without the use of a calculator, show that: $8\sin 20^\circ \cdot \sin 40^\circ \cdot \sin 60^\circ \cdot \sin 80^\circ = \dfrac{3}{2}$.

Solution 1 by David A. Huckaby, Angelo State University, San Angelo, TX

$$\begin{aligned}
&8\sin 20^\circ \cdot \sin 40^\circ \cdot \sin 60^\circ \cdot \sin 80^\circ\\
&\quad = 8\sin(30^\circ - 10^\circ)\cdot \sin(30^\circ + 10^\circ)\cdot \sin 60^\circ \cdot \sin(90^\circ - 10^\circ)\\
&\quad = 8[\sin 30^\circ \cdot \cos 10^\circ - \cos 30^\circ \cdot \sin 10^\circ]\cdot[\sin 30^\circ \cdot \cos 10^\circ + \cos 30^\circ \cdot \sin 10^\circ]\\
&\qquad\quad \cdot \frac{\sqrt{3}}{2}\cdot[\sin 90^\circ \cdot \cos 10^\circ - \cos 90^\circ \cdot \sin 10^\circ]\\
&\quad = 8\left[\frac{1}{2}\cdot\cos 10^\circ - \frac{\sqrt{3}}{2}\cdot \sin 10^\circ\right]\cdot\left[\frac{1}{2}\cdot\cos 10^\circ + \frac{\sqrt{3}}{2}\cdot \sin 10^\circ\right]\cdot\frac{\sqrt{3}}{2}\cdot[\cos 10^\circ]\\
&\quad = \sqrt{3}\cos 10^\circ \cdot \left[\cos 10^\circ - \sqrt{3}\sin 10^\circ\right]\cdot\left[\cos 10^\circ + \sqrt{3}\sin 10^\circ\right]\\
&\quad = \sqrt{3}\cos 10^\circ \cdot \left[\cos^2 10^\circ - 3\sin^2 10^\circ\right]\\
&\quad = \sqrt{3}\left[\cos^3 10^\circ - 3\sin^2 10^\circ \cdot \cos 10^\circ\right]\\
&\quad = \sqrt{3}\cos 30^\circ\\
&\quad = \sqrt{3}\cdot\frac{\sqrt{3}}{2}\\
&\quad = \frac{3}{2}
\end{aligned}$$

Solution 2 by Cartesian Gains Student Problem Solving Group, Mountain Lakes High School, Mountain Lakes, NJ

We use the well known formula: $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

Converting to radians, we rewrite the left side of the equation as

$$8\cdot\left(\frac{1}{2i}\right)^4\cdot\left(e^{i\pi/9} - e^{-i\pi/9}\right)\cdot\left(e^{i2\pi/9} - e^{-i2\pi/9}\right)\cdot\left(e^{i3\pi/9} - e^{-i3\pi/9}\right)\cdot\left(e^{i4\pi/9} - e^{-i4\pi/9}\right).$$

Expanding gives:

$$\frac{1}{2} \cdot \left(e^{i(10\pi/9)} + e^{-i(10\pi/9)} - e^{i(8\pi/9)} - e^{-i(8\pi/9)} - e^{i(6\pi/9)} - e^{-i(6\pi/9)} + 2 \right) \tag{1}$$

We use the fact that on the unit circle $e^{i(10\pi/9)}$ represents the same complex number as $e^{-i(8\pi/9)}$. Similarly, $e^{i(8\pi/9)} = e^{-i(10\pi/9)}$. These terms cancel out in our equation.

Additionally,

$$-\left(e^{i(6\pi/9)} + e^{-i(6\pi/9)}\right) = -\left(\cos\frac{6\pi}{9} + i\sin\frac{6\pi}{9} + \cos\frac{-6\pi}{9} + i\sin\frac{-6\pi}{9}\right) = -2\cos(6\pi/9) = 1.$$

Therefore, equation (1) reduces to: $\frac{1}{2}(1+2) = \frac{3}{2}$.

Editor′s comments: **Albert Stadler of Herrliberg, Switzerland** and several other solvers, noticed that this problem is a special case of problem 5497, which asked us to find a closed form of

$$\prod_{k=1}^{n-1} 2\sin\left(\frac{k\pi}{n}\right).$$

He showed that

$$\prod_{k=1}^{n-1} 2\sin\left(\frac{k\pi}{n}\right) = \lim_{x} 1 \frac{x^n - 1}{x - 1} = n.$$

By symmetry $\sin\left(\frac{k\pi}{n}\right) = \sin\left(\frac{n-k)\pi}{n}\right)$, so if n is odd then

$$\prod_{k=1}^{\frac{n-1}{2}} 2\sin\left(\frac{k\pi}{n}\right) = \sqrt{n}.$$

So problem is 5500 is the special case with $n = 9$.

Yagub Alyiev of ADA University in Baku, Azerbaijan, mentor to the two students listed below from his university who solved the problem, sent two web addresses wherein animated solutions can be found. See:

https://www.youtube.com/watch?v=Tc58b2AGFf4 (and)

https://www.youtube.com/watch?v=zAiXPhPvWpct=187s.

Also solved by Arkady Alt; San Jose, CA; Michel Bataille, Rouen, France; Brian D. Beasley (two solutions), Presbyterian College, Clinton, SC; Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND; Scott H. Brown, Montgomery, AL; Michael Brozinsky, Central Islip, NY; Elsie Campbell, Dionne T. Bailey, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX; : Michael C. Faleski, Delta College, University Center, MI; Bruno Salgueiro Fango, Viveiro, Spain; Ed Gray, Highland Beach, FL; Vagif Hamzayev(student), ADA University, Baku, Azerbaijan; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Carl Libis, Columbia Southern University, Orange Beach, AL;

David E. Manes, Oneonta, NY; Kamal Mustafayev (student), ADA University, Baku, Azerbaijan; Pedro H.O. Pantoja, Natal/RN, Brazil; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ioannis D. Sfikas (two solutions), National and Kapodistrian University of Athens, Greece; Digby Smith, Mount Royal University, Calgary, Canada; Albert Stadler of Herrliberg, Switzerland; Neculai Stanciu, "George Emil Palade" School Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins of Georgia Southern University, Statesboro, GA; Daniel Văcaru, Pitesti, Romania, and the proposers.

5501: *Proposed by D.M. Bătinetu-Giurgiu, Bucharest, Romania, Neculai Stanciu, "George Emil Palade" School Buzău, Romania and Titu Zvonaru, Comănesti, Romania*

Determine all real numbers a, b, x, y that simultaneously satisfy the following relations:

$$\begin{cases} (1) \quad ax+by=5 \\ (2) \quad ax^2+by^2=9 \\ (3) \quad ax^3+by^3=17 \\ (4) \quad ax^4+by^4=33. \end{cases}$$

Solution 1 by Stanley Rabinowitz, Chelmsford, MA

(1) $ax+by=5$.
(2) $ax^2+by^2=9$.
(3) $ax^3+by^3=17$.
(4) $ax^4+by^4=33$.
(5) $ax=5-by$, by (1).
(6) $(5-by)xy+by^3=9y$, by (2) and (5).
(7) $ax^2=9-by^2$, by (2).
(8) $(9-by^2)x+by^3=17$, by (3) and (7).
Subtracting (8) from (6) gives
(9) $5xy-9x=9y-17$.
(10) $ax^3=17by^3$, by (3).
(11) $(17-by^3)x+by^4=33$, by (4) and (10).
Multiplying (8) by y and subtracting (11) yields
(12) $9xy-17x=17y-33$.
Subtracting 5 times (12) from 9 times (9) and dividing the result by 4 gives (13) $x=-y+3$.
Substituting this value of x into (9) and simplifying yields: $-5(y-1)(y-2)=0$.
Therefore, $y=1$ or 2.

Suppose $y=1$. Then, $x=2$, by (13). Thus, $2a+b=5$ and $4a+b=9$, by (1) and (2). Hence, $a=2$ and $b=1$. That is, $(x,y,a,b)=(2,1,2,1)$.

Similarly, if $y=2$, then $(x,y,a,b)=(1,2,1,2)$.

Note that this result holds in any commutative ring with unity, which has no zero divisors and $5 \neq 0$.

Solution 2 by David E. Manes, Oneonta, NY

Writing $5 = 2^2 + 1$, $9 = 2^3 + 1$, $17 = 2^4 + 1$ and $33 = 2^5 + 1$, one notes that two of the solutions (a, b, x, y) for the system of equations are $(1, 2, 1, 2)$ and $(2, 1, 2, 1)$. We will show that these are the only solutions. Let A be the augmented 4×3 matrix for the system of equations where a and b are regarded as the unknowns and the powers of x and y are regarded as the coefficients. Then

$$A = \begin{bmatrix} x & y & 5 \\ x^2 & y^2 & 9 \\ x^3 & y^3 & 17 \\ x^4 & y^4 & 33 \end{bmatrix}.$$

Row-reducing A, we find that it is row-equivalent to the matrix R given by

$$R = \begin{bmatrix} 1 & 0 & \frac{5y-9}{x(y-x)} \\ 0 & 1 & \frac{9-5x}{y(y-x)} \\ 0 & 0 & 17 - 9y - 9x + 5xy \\ 0 & 0 & 33 - 9(x^2 + y^2 + xy) + 5xy(y + x) \end{bmatrix}.$$

If $x = 1$ and $y = 2$, then the two expressions $17 - 9y - 9x + 5xy$ and $33 - 9(x^2 + y^2 + xy) + 5xy(y + x)$ both equal 0. Therefore, $a = 1$ and $b = 2$ since $\frac{5y-9}{x(y-x)} = 1$ and $\frac{9-5x}{y(y-x)} = 2$ when $x = 1$ and $y = 2$. If $x = 2$ and $y = 1$, then $17 - 9x - 9y + 5xy = 33 - 9(x^2 + y^2 + xy) + 5xy(y + x) = 0$ so that $a = 2$ and $b = 1$. Working with residues modulo 3, one finds that the equation $17 - 9y - 9x + 5xy \equiv 0 \pmod 3$ if and only if $x \equiv 1 \pmod 3$ and $y \equiv 2 \pmod 3$ or $x \equiv 2 \pmod 3$ and $y \equiv 1 \pmod 3$. Furthermore, these residues have to be least residues since otherwise, the residues can be made to satisfy the first equation in the system, but not the second.

Also solved by Arkady Alt; San Jose, CA; Hatef Arshagi, Guilford Technical Community College, Jamestown, NC; Michel Bataille, Rouen, France; Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony Bevelacqua, University of North Dakota, Grand Forks, ND; Cartesian Gains Student Problem Solving Group, Mountain Lakes High School, Mountain Lakes, NJ; Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX; Ed Gray, Highland Beach,FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Perfetti Paolo, Department of Mathematics, Tor Vergata University, Rome, Italy; Ioannis D. Sfikas (two solutions), National and Kapodistrian University of Athens, Greece; Digby Smith, Mount Royal University, Calgary, Canada; Albert Stadler of Herrliberg, Switzerland; David Stone and John Hawkins,

Georgia Southern University, Statesboro, GA; Daniel Văcaru, Pitesti, Romania, and the proposers.

5502: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Prove that if $a, b, c > 0$ and $a + b + c = e$ then

$$e^{ac^e} \cdot e^{ba^e} \cdot e^{cb^e} > e^e \cdot a^{be^2} \cdot b^{ce^2} \cdot c^{ae^2}.$$

Here, $e = \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

The inequality is equivalent to

$$ac^e + ba^e + cb^e > e + be^2 \ln a + ce^2 \ln b + ae^2 \ln c$$

that is

$$a(c^e - e^2 \ln c) + b(a^e - e^2 \ln a) + c(b^e - e^2 \ln b) > e$$

Let $f(x) = x^e - e^2 \ln x$.

$$f''(x) = e(e-1)x^{e-2} + \frac{e^2}{x^2} > 0$$

Thus by Jensens's inequality

$$e \sum_{\text{cyc}} \frac{a}{e}(c^e - e^2 \ln c) \geq e\left[\left(\frac{a+b+c}{e}\right)^e - a^2 \ln \frac{a+b+c}{e}\right] = e$$

Solution 2 by Moti Levy, Rehovot, Israel

The function $\ln x$ is monotone increasing, then by applying log function on both sides of the inequality, we get

$$ac^e + ba^e + cb^e > e + be^2 \ln a + ce^2 \ln b + ae^2 \ln c, \tag{1}$$

or

$$\frac{a}{e}c^e + \frac{b}{e}a^e + \frac{c}{e}b^e > 1 + e^2\left(\frac{b}{e}\ln a + \frac{c}{e}\ln b + \frac{a}{e}\ln c\right). \tag{2}$$

The function $\ln x$ is concave, hence

$$\ln\left(\frac{ab+bc+ca}{e}\right) \geq \frac{b}{e}\ln a + \frac{c}{e}\ln b + \frac{a}{e}\ln c. \tag{3}$$

Thus we get for the right hand side of inequality (2) :

$$1 - e^2 + e^2 \ln(ab+bc+ca) \geq 1 + e^2\left(\frac{b}{e}\ln a + \frac{c}{e}\ln b + \frac{a}{e}\ln c\right). \tag{4}$$

The function x^e is convex, hence we get for the left hand side of inequality (2):

$$\frac{a}{e}c^e + \frac{b}{e}a^e + \frac{c}{e}b^e \geq \left(\frac{ab+bc+ca}{e}\right)^e. \tag{5}$$

By (4) and (5), to finish the solution, we have to show that

$$\left(\frac{ab+bc+ca}{e}\right)^e > 1 - e^2 + e^2 \ln(ab+bc+ca). \qquad (6)$$

Let us denote

$$x := (ab+bc+ca)^e. \qquad (7)$$

Since $ab+bc+ca \le \frac{e^2}{3}$, then

$$0 < x \le \left(\frac{e^2}{3}\right)^e. \qquad (8)$$

Setting (7) in (6), we need to show that

$$\frac{x}{e^e} > 1 - e^2 + e\ln x, \text{ for } 0 < x \le \left(\frac{e^2}{3}\right)^e,$$

or that

$$f(x) := x - e^{1+e}\ln x + e^e\left(e^2-1\right) > 0, \text{ for } 0 < x \le \left(\frac{e^2}{3}\right)^e. \qquad (9)$$

One can easily check that $f'(x) = 1 - \frac{e^{1+e}}{x} < 0$ for $0 < x \le \left(\frac{e^2}{3}\right)^e$. Hence, $f(x)$ is monotone decreasing function for $0 < x \le \left(\frac{e^2}{3}\right)^e$. Moreover, $\lim_{x\to 0} f(x) = +\infty$ and $f\left(\left(\frac{e^2}{3}\right)^e\right) = \left(\frac{e^2}{3}\right)^e - e^{1+e}\left(\ln\left(\frac{e^2}{3}\right)^e\right) + e^e\left(e^2-1\right) \cong 7.4789 > 0$. These and the monotonicity of $f(x)$ imply that $x - e^{1+e}\ln x + e^e\left(e^2-1\right) > 0$, for $0 < x \le \left(\frac{e^2}{3}\right)^e$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

For $0 < x < 1$, let $f(x)$ be the convex function $x^e - e^2\ln x$. By taking logarithms, we see that the inequality of the problem is equivalent to

$$af(c) + bf(a) + cf(b) > e. \qquad (1)$$

Let $\gamma_1 = \dfrac{a}{e}, \gamma_2 = \dfrac{b}{e}$ and $\gamma_3 = \dfrac{c}{e}$. By Jensen's inequality, the left side of (1) is greater than or equal to $ef(\gamma_1 c + \gamma_2 a + \gamma_3 b) = ef\left(\dfrac{ab+bc+ca}{e}\right)$.

Since $f'(x) = \dfrac{e(x^e - e)}{x} < 0$ and

$ab+bc+ca = \dfrac{2(a+b+c)^2 - (a-b)^2 - (b-c)^2 - (c-a)^2}{6} \le \dfrac{e^3}{3}$, so

$$f\left(\frac{ab+bc+ca}{e}\right) \ge f\left(\frac{e}{3}\right) = 1.49\cdots > 1.$$

Thus (1) holds and this completes the solution.

Solution 4 by Michel Bataille, Rouen, France

Taking logarithms and arranging, we see that the inequality is equivalent to

$$\frac{a}{e}\cdot c^e + \frac{b}{e}\cdot a^e + \frac{c}{e}\cdot b^e > 1 + e^2\left(\frac{b}{e}\cdot\ln a + \frac{c}{e}\cdot\ln b + \frac{a}{e}\cdot\ln c\right).$$

Since the functions $x \mapsto x^e$ and $x \mapsto \ln x$ are respectively convex and concave on $(0, \infty)$, Jensen's inequality yields

$$\frac{a}{e} \cdot c^e + \frac{b}{e} \cdot a^e + \frac{c}{e} \cdot b^e \geq \left(\frac{ab+bc+ca}{e}\right)^e$$

and

$$\frac{b}{e} \cdot \ln a + \frac{c}{e} \cdot \ln b + \frac{a}{e} \cdot \ln c \leq \ln\left(\frac{ab+bc+ca}{e}\right).$$

Therefore, it is sufficient to prove that

$$U^e - e^2 \ln U - 1 > 0 \tag{1}$$

where $U = \frac{ab+bc+ca}{e}$.
Since $e^2 = (a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \geq 3(ab+bc+ca)$, we have $U \leq \frac{e}{3}$, hence $U \in (0,1)$.
Now, let $f(x) = x^e - e^2 \ln x - 1$. The function f satisfies $f(1) = 0$ and $f'(x) = \frac{e(x^e - e)}{x}$. It follows that f is strictly decreasing on the interval $(0,1]$ and so $f(U) > f(1)$, which is the desired inequality (1).

Also solved by Ed Gray, Highland Beach, FL; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler of Herrliberg, Switzerland, and the proposer.

5503: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers with $n \geq 2$. Prove that

$$\frac{(a_1^m a_2 + a_2^m a_3 + \cdots + a_n^m a_1)^m}{(a_1^m + a_2^m + \cdots a_n^m)^{m+1}} \leq \frac{1}{n},$$

where m is a positive integer.

Solution 1 by Michel Bataille, Rouen, France

Let $I_m(a_1, a_2, \ldots, a_n)$ denote the proposed inequality.
First we suppose that $I_1(a_1, a_2, \ldots, a_n)$ holds for all $a_1, \ldots, a_n > 0$, in other words that

$$n(a_1a_2 + a_2a_3 + \cdots + a_na_1) \leq (a_1 + a_2 + \cdots + a_n)^2 \tag{1}$$

for all positive $a_1, \ldots, a_n$ and we show that if m is an integer with $m \geq 2$, then $I_m(a_1, \ldots, a_n)$ also holds for all positive $a_1, \ldots, a_n$.
Let m be an integer with $m \geq 2$ and let $a_1, \ldots, a_n > 0$. Applying (1) with $a_1^m, \ldots, a_n^m$ instead of $a_1, \ldots, a_n$, respectively, we obtain $(a_1^m + \cdots + a_n^m)^2 \geq n(a_1^m a_2^m + \cdots + a_n^m a_1^m)$ so that

$$(a_1^m + \cdots + a_n^m)^{m+1} = (a_1^m + \cdots + a_n^m)^2 (a_1^m + \cdots + a_n^m)^{m-1} \geq n(a_1^m a_2^m + \cdots + a_n^m a_1^m)(a_1^m + \cdots + a_n^m)^{m-1}.$$

But from Holder's inequality, we have

$$(a_1^m a_2^m + \cdots + a_n^m a_1^m)(a_1^m + \cdots + a_n^m)^{m-1} \geq (a_1^m a_2 + a_2^m a_3 + \cdots a_n^m a_1)^m$$

and it follows that $(a_1^m + \cdots + a_n^m)^{m+1} \geq n(a_1^m a_2 + a_2^m a_3 + \cdots a_n^m a_1)^m$, which is the desired inequality $I_m(a_1, \ldots, a_n)$.

Now, we show that (1) holds for all positive $a_1, \ldots, a_n$ if and only if $n \leq 4$.

Suppose that $n \geq 5$. If (1) holds for all $a_1, a_2, \ldots, a_n > 0$, then in particular it holds if we take $a_1 = a_2 = \cdots = a_{n-2} = \varepsilon$ and $a_{n-1} = a_n = 1$ where ε is an arbitrary positive number. This provides the inequality $n((n-3)\varepsilon^2 + 2\varepsilon + 1) \leq ((n-2)\varepsilon + 2)^2$. Letting $\varepsilon \to 0^+$, we obtain $n \leq 4$, a contradiction. Thus, we must have $n \leq 4$.

Conversely, if $n = 2$ (resp. $n = 3$, resp, $n = 4$), it is easily checked that (1) is equivalent to $(a_1 - a_2)^2 \geq 0$ (resp. $(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_1)^2 \geq 0$, resp. $(a_1 - a_2 + a_3 - a_4)^2 \geq 0$) and so $I_1(a_1, \ldots, a_n)$ holds for all $a_1, \ldots, a_n > 0$ when $n = 2, 3$ or 4.

In conclusion, $I_m(a_1, a_2, \ldots, a_n)$ holds for any positive integer m and all positive real numbers $a_1, \ldots, a_n$ if and only if $n \leq 4$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

The statement is wrong in general. It is true for $n = 2$ and for any real $m \geq 1$, since by Hölder's inequality,

$$(x^m y + xy^m)^m = x^m y^m (x^{m-1} + y^{m-1})^m \leq x^m y^m \left((1^m + 1^m)^{\frac{1}{m}} (x^m + y^m)^{\frac{m-1}{m}} \right)^m =$$

$$= x^m y^m 2 (x^m + y^m)^{m-1} \leq \frac{1}{2}(x^m + y^m)^{m+1},$$

and the last inequality is equivalent to $x^m y^m \leq \frac{1}{4}(x^m + y^m)^2$, which is clearly true.

The statement is true as well for $n = 3$ and any real $m \geq 1$, since by Hölder's inequality,

$$(x^m y + y^m z + z^m x)^m = x^m y^m z^m \left(\frac{x^{m-1}}{z} + \frac{y^{m-1}}{x} + \frac{x^{m-1}}{z} + \frac{z^{m-1}}{y} \right)^m \leq$$

$$\leq x^m y^m z^m \left(\left(\frac{1}{z^m} + \frac{1}{x^m} + \frac{1}{y^m} \right)^{\frac{1}{m}} (x^m + y^m + z^m)^{\frac{m-1}{m}} \right)^m =$$

$$= (x^m y^m + y^m z^m + z^m x^m)(x^m + y^m + z^m)^{m-1} \leq \frac{1}{3}(x^m + y^m + z^m)^{m+1},$$

and the last inequality is equivalent to $ab + bc + ca \leq \frac{1}{3}(a + b + c)^2$, with $a = x^m, b = y^m, c = z^m$, which is clearly true, since it is equivalent to $ab + bc + ca \leq a^2 + b^2 + c^2$ (which is true because of Cauchy-Schwarz).

The problem statement is not true in general, We construct counterexamples as follows: Let $a = 1$ for $1 \leq i \leq k, a_i = 0$ for $k + 1 \leq i \leq n$. Then $(a_1^m a_2 + a_2^m a)_3 + \cdots + a_n^m a_1 = k - 1$ and $a_1^{m+1} + a_2^{m+1} + \cdots + a_n^{m+1} = k$. The stated inequality then reads as

$$(k-1)^m \leq \frac{1}{n} k^{m+1}, \tag{2}$$

which fails for an infinity of triples (k, m, n). For instance (2) is wrong for $(n-2, 1, n)$, if $n \geq 5$, it is wrong for $(n-3, 2, n)$, if $n \geq 8$ and it is wrong for $(n-4, 3, n)$ if $n \geq 12$.

Purists may argue that these are not real counter-examples, since $a_i = 0$, for $k + 1 \leq i \leq n$, so that not all a_i are strictly positive. However we may replace 0 by $\epsilon > 0$ and make ϵ sufficiently small to reach the same conclusion.

Also solved by Arkady Alt; San Jose, CA; Ed Gray, Highland Beach, FL; Perfetti Paolo, Department of Mathematics, Tor Vergata University Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Ioannis D. Sfikas (two solutions), National and Kapodistrain University of Athens, Greece, and the proposer.

5504: *Proposed by Ovidiu Furdui and Alina Sîntămărian bothat the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n \geq 0$ be an integer. Calculate

$$\int_0^1 \frac{x^n}{\left\lfloor \frac{1}{x} \right\rfloor} \mathrm{d}x,$$

where $\lfloor x \rfloor$ denotes the integer part of x.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We have

$$\int_0^1 \frac{x^n}{\left\lfloor \frac{1}{x} \right\rfloor} dx = \lim_{K\to\infty} \sum_{k=1}^{K} k \int_{\frac{1}{k+1}}^{\frac{1}{k}} x^n dx = \lim_{K\to\infty} \sum_{k=1}^{K} k \int_{\frac{1}{k+1}}^{\frac{1}{k}} x^n dx = \lim_{K\to\infty} \sum_{k=1}^{K} \frac{k}{n+1} \left(\frac{1}{k^{n+1}} - \frac{1}{(k+1)^{n+1}} \right)$$

$$= \frac{1}{n+1} \lim_{K\to\infty} \sum_{k=1}^{K} \left(\frac{1}{k^n} - \frac{k+1-1}{(k+1)^{n+1}} \right) = \frac{1}{n+1} \lim_{K\to\infty} \sum_{k=1}^{K} \left(\frac{1}{k^n} - \frac{1}{(k+1)^n} + \frac{1}{(k+1)^{n+1}} \right)$$

$$= \frac{1}{n+1} \lim_{K\to\infty} \left(1 - \frac{1}{(K+1)^n} + \sum_{k=1}^{K} \frac{1}{(k+1)^{n+1}} \right) = \frac{\zeta(n+1)}{n+1}.$$

Solution 2 by Stanley Rabinowitz, Chelmsford, MA

We start by breaking the interval $(0,1)$ up into subintervals over which the function $\left\lfloor \frac{1}{x} \right\rfloor$ is constant.

$$\begin{aligned}
\int_0^1 \frac{x^n}{\left\lfloor \frac{1}{x} \right\rfloor} dx &= \sum_{k=1}^{\infty} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{x^n}{\left\lfloor \frac{1}{x} \right\rfloor} dx \\
&= \sum_{k=1}^{\infty} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{x^n}{k} dx \\
&= \sum_{k=1}^{\infty} \left[\frac{x^{n+1}}{k(n+1)} \right]_{\frac{1}{k+1}}^{\frac{1}{k}} \\
&= \frac{1}{n+1} \sum_{k=1}^{\infty} \left[\frac{\left(\frac{1}{k}\right)^{n+1}}{k} - \frac{\left(\frac{1}{k+1}\right)^{n+1}}{k} \right] \\
&= \frac{1}{n+1} \sum_{k=1}^{\infty} \left[\frac{1}{k^{n+2}} - \frac{1}{k(k+1)^{n+1}} \right]
\end{aligned}$$

$$= \frac{1}{n+1}\sum_{k=1}^{\infty}[A-B]$$

where

$$A=\sum_{k=1}^{\infty}\frac{1}{k^{n+2}}=\zeta(n+2) \quad \text{and} \quad B=\sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{1}{(k+1)^{n+1}}\right]$$

and where $\zeta(n)=\sum_{k=1}^{\infty}\frac{1}{k^n}$ is the Riemann zeta function.

Now note that by the formula for the sum of a geometric progression,

$$\sum_{i=2}^{n+1}\frac{1}{(k+1)^i}=\frac{1}{k(k+1)}-\frac{1}{k}\left[\frac{1}{(k+1)^{n+1}}\right].$$

So

$$\begin{aligned}
B &= \sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{1}{(k+1)^{n+1}}\right] \\
&= \sum_{k=1}^{\infty}\left[\frac{1}{k(k+1)}-\sum_{i=2}^{n+1}\frac{1}{(k+1)^i}\right] \\
&= \sum_{k=1}^{\infty}\left[\frac{1}{k(k+1)}\right]-\sum_{k=1}^{\infty}\left[\sum_{i=2}^{n+1}\frac{1}{(k+1)^i}\right] \\
&= \sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}\right]-\sum_{i=2}^{n+1}\left[\sum_{k=1}^{\infty}\frac{1}{(k+1)^i}\right] \\
&= 1-\sum_{i=2}^{n+1}(\zeta(i)-1) \\
&= n+1-\sum_{i=2}^{n+1}\zeta(i)
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^1\frac{x^n}{\left\lfloor\frac{1}{x}\right\rfloor}\,dx &= \frac{1}{n+1}[A-B] \\
&= \frac{1}{n+1}\left[\zeta(n+2)-\left(n+1-\sum_{i=2}^{n+1}\zeta(i)\right)\right] \\
&= \frac{1}{n+1}\left[\sum_{i=2}^{n+2}\zeta(i)-(n+1)\right] \\
&= \frac{1}{n+1}\left[\sum_{i=2}^{n+2}\zeta(i)\right]-1.
\end{aligned}$$

Solution 3 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

For $x \in (0,1]$ if $\dfrac{1}{k+1} < x \leq \dfrac{1}{k}$, then $k \leq \left\lfloor \dfrac{1}{x} \right\rfloor < k+1$. That is for x, $\left\lfloor \dfrac{1}{x} \right\rfloor = k$. Therefore,

$$\begin{aligned}
\int_0^1 \frac{x^n}{\lfloor \frac{1}{x} \rfloor} &= \sum_{k=1}^{\infty} \int_{1/k+1}^{1/k} \frac{x^n}{k} dx \\
&= \sum_{k=1}^{\infty} \frac{1}{k(n+1)} \left(\frac{1}{k^{n+1}} - \frac{1}{(k+1)^{n+1}} \right) \\
&= \frac{1}{n+1} \sum_{k=1}^{\infty} \left(\frac{1}{k^{n+2}} - \frac{1}{k(k+1)^{n+1}} \right) \\
&= \frac{1}{n+1} \sum_{k=1}^{\infty} \left(\frac{1}{k^{n+2}} + \frac{1}{(k+1)^{n+1}} + \cdots + \frac{1}{k+1} - \frac{1}{k} \right),
\end{aligned}$$

from where

$$\int_0^1 \frac{x^n}{\left\lfloor \dfrac{1}{x} \right\rfloor} dx = -1 + \frac{\sum_{j=2}^{n+2} \zeta(j)}{n+1}.$$

Solution 4 by Moti Levy, Rehovot, Israel

The first step is to substitute $y = \frac{1}{x}$ and then to split the integration range into intervals $[k, k+1]$, $k \geq 1$.

$$\begin{aligned}
\int_0^1 \frac{x^n}{\lfloor \frac{1}{x} \rfloor} dx = \int_1^{\infty} \frac{y^{-n-2}}{\lfloor y \rfloor} dy &= \sum_{k=1}^{\infty} \int_k^{k+1} \frac{y^{-n-2}}{k} dy \\
&= \frac{1}{(n+1)} \sum_{k=1}^{\infty} \left(\frac{1}{k^{n+2}} - \frac{1}{k\,(k+1)^{n+1}} \right) \\
&= \frac{1}{(n+1)} \left(\zeta\,(n+2) - \sum_{k=1}^{\infty} \frac{1}{k\,(k+1)^{n+1}} \right).
\end{aligned}$$

Let $S_n := \sum_{k=1}^{\infty} \frac{1}{k(k+1)^n}$.

$$\begin{aligned}
S_{n+1} - S_n &= \sum_{k=1}^{\infty} \left(\frac{1}{k\,(k+1)^{n+1}} - \frac{1}{k\,(k+1)^n} \right) \\
&= \sum_{k=1}^{\infty} \frac{1}{(k+1)^n} \left(\frac{1}{k\,(k+1)} - \frac{1}{k} \right) \\
&= -\sum_{k=1}^{\infty} \frac{1}{(k+1)^{n+1}} = 1 - \zeta\,(n+1)\,. \qquad (10)
\end{aligned}$$

It is easy to see that

$$S_1 := \sum_{k=1}^{\infty} \frac{1}{k\,(k+1)} = 1. \tag{11}$$

It follows from (10) and (11) that

$$S_n = n - \sum_{k=1}^{n-1} \zeta\,(k+1)\,.$$

$$\begin{aligned}
\int_0^1 \frac{x^n}{\left\lfloor \frac{1}{x} \right\rfloor} dx &= \frac{1}{n+1}\left(\zeta\,(n+2) - S_{n+1}\right) \\
&= \frac{1}{n+1}\left(\zeta\,(n+2) - (n+1) + \sum_{k=1}^{n} \zeta\,(k+1)\right) \\
&= \frac{1}{n+1}\left(\sum_{k=2}^{n+2} \zeta\,(k)\right) - 1.
\end{aligned}$$

Solution 5 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We shall show that the value of the integral is (the average of the“first” $n+1$ values of the Riemann zeta function) minus 1; that is, $\dfrac{\zeta(2)+\zeta(3)+\ldots+\zeta(n+2)}{n+1} - 1$.

We shall need the following result, which seems interesting in its own right.
Lemma 1: For
$m \geq 1, 1 + x + x(x+1) + x(x+1)^2 + x(x+1)^3 + \ldots + x(x+1)^{m-1} = (x+1)^m$.

Proof by induction. The identity is clearly true for $m = 1$. Upon the induction hypothesis,

$$\begin{aligned}
& 1 + x + x(x+1) + x(x+1)^2 + x(x+1)^3 + \ldots + x(x+1)^{m-1} = x(x+1)^m \\
= \ & (x+1)^m + x(x+1)^m \\
= \ & (x+1)^m(1+x) \\
= \ & (x+1)^{m+1}.
\end{aligned}$$

This leads to the following result about a partial fractions decomposition.

Lemma 2:
$$\frac{1}{k(k+1)^m} = \frac{1}{k} - \frac{1}{(k+1)^m} - \frac{1}{(k+1)^{m-1}} - \frac{1}{(k+1)^{m-2}} - \cdots - \frac{1}{(k+1)^2} - \frac{1}{k+1}.$$

Proof: After clearing fractions, we see that this identity is equivalent to
$1 = (k+1)^n - k - k(k+1) - k(k+1)^2 - \ldots - k(k+1)^{n-2} - k(k+1)^{n-1}$, which is true by Lemma 1.

Now we are in position to calculate the given integral. Note that
$\left\lfloor \dfrac{1}{x} \right\rfloor = k \iff k \leq \dfrac{1}{x} < k+1 \iff \dfrac{1}{k+1} x \leq \dfrac{1}{k}$.

Thus

$$\int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{x^n}{\left\lfloor \frac{1}{x} \right\rfloor} dx = \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{x^n}{k} dx = \frac{1}{k}\frac{1}{n+1}\left[\left(\frac{1}{k}\right)^{n+1} - \left(\frac{1}{k+1}\right)^{n+1}\right] = \frac{1}{k}\frac{1}{n+1}\left[\frac{1}{k^{n+1}} - \frac{1}{(k+1)^{n+1}}\right].$$

Therefore,

$$\int_0^1 \frac{x^n}{\left\lfloor \frac{1}{x} \right\rfloor} dx = \sum_{k=1}^{\infty} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{x^n}{\left\lfloor \frac{1}{x} \right\rfloor} dx = \sum_{k=1}^{\infty} \frac{1}{k}\frac{1}{n+1}\left[\frac{1}{k^{n+1}} - \frac{1}{(k+1)^{n+1}}\right]$$

$$= \quad \frac{1}{n+1}\sum_{k=1}^{\infty}\left[\frac{1}{k^{n+2}} - \frac{1}{k(k+1)^{n+1}}\right]$$

$$= \quad \frac{1}{n+1}\sum_{k=1}^{\infty}\left[\frac{1}{k^{n+2}} - \left\{\frac{1}{k} - \frac{1}{(k+1)^{n+1}} - \frac{1}{(k+1)^{n}} - \frac{1}{(k+1)^{n-1}} - \cdots - \frac{1}{(k+1)^{2}} - \frac{1}{k+1}\right\}\right]$$

$$\underset{\text{by Lemma 2}}{=} \quad \frac{1}{n+1}\sum_{k=1}^{\infty}\left[\frac{1}{k^{n+2}} - \frac{1}{k} + \frac{1}{(k+1)^{n+1}} + \frac{1}{(k+1)^{n}} + \frac{1}{(k+1)^{n-1}} + \cdots + \frac{1}{(k+1)^{2}} + \frac{1}{k+1}\right]$$

$$= \quad \frac{1}{n}\left\{\sum_{k=1}^{\infty}\frac{1}{k^{n+2}} + \sum_{k=1}^{\infty}\frac{1}{(k+1)^{n+1}} + \sum_{k=1}^{\infty}\frac{1}{(k+1)^{n}} + \cdots + \sum_{k=1}^{\infty}\frac{1}{(k+1)^{2}} - \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)\right\}.$$

The final sum in this expression telescopes, and its sum is 1.

Each of the other sums is a shifted version of the zeta function:

$$\sum_{i=1}^{\infty}\frac{1}{(k+1)^m} = \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \cdots = \left(-1 + \frac{1}{1^m}\right) + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \cdots = -1 = \zeta(m).$$

Therefore,

$$\int_0^1 \frac{x^n}{\left\lfloor \frac{1}{x} \right\rfloor} dx = \frac{1}{n+1}\left\{\sum_{k=1}^{\infty}\frac{1}{k^{n+2}} + \sum_{k=1}^{\infty}\frac{1}{(k+1)^{n+1}} + \sum_{k=1}^{\infty}\frac{1}{(k+1)^{n}} + \cdots + \sum_{k=1}^{\infty}\frac{1}{(k+1)^{2}} - \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)\right\}$$

$$= \quad \frac{1}{n+1}\left\{[-1+\zeta(n+2)] + [-1+\zeta(n+1)] + [-1+\zeta(n)] + [-1+\zeta(n-1)] + \cdots + [-1+\zeta(2)] - 1\right\}$$

$$= \quad \frac{1}{n+1}\left\{\zeta(2) + \zeta(3) + \cdots + \zeta(n) + \zeta(n+1) + \zeta(n+2) - n \cdot 1 - 1\right\}$$

$$= \quad \frac{1}{n+1}\left\{\zeta(2) + \zeta(3) + \cdots + \zeta(n) + \zeta(n+1) + \zeta(n+2)\right\} - 1.$$

There are $n+1$ terms inside the braces, so we have our promised result:

$$\int_0^1 \frac{x^n}{\left\lfloor \frac{1}{x} \right\rfloor} dx =$$

(the average of the first n + 1 values of the Riemann zeta function) minus 1.

Comment: There are other variants of this answer, because the values of the zeta function for even n can be expressed in terms of the Bernoulli numbers. That would not make the answer any nicer though.

Also solved by Michel Bataille, Rouen, France; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas (two solutions), National and Kapodistrian University of Athens, Greece; Perfetti Paolo, Department of Mathematics, Tor Vergata University Rome, Italy; Daniel Văcaru, Pitesti, Romania, and the proposers.

Mea Culpa

Ioannis D. Sfikas of National and Kapodistrian in University of Athens, Greece should have been credited with having solved 5495 and 5496.

Carl Libis of Columbia Southern University in Orange Beach, AL should have been credited with having solved 5497.

Correction: Problem 5514 in the November 2018 issue of this column should have been stated as:

If $a \in (0,1)$ and $b = \arcsin a$, then calculate $\lim\limits_{n\to\infty} \sqrt[n]{n!}\left(\sin\left(\frac{b \cdot \sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right) - a\right)$.