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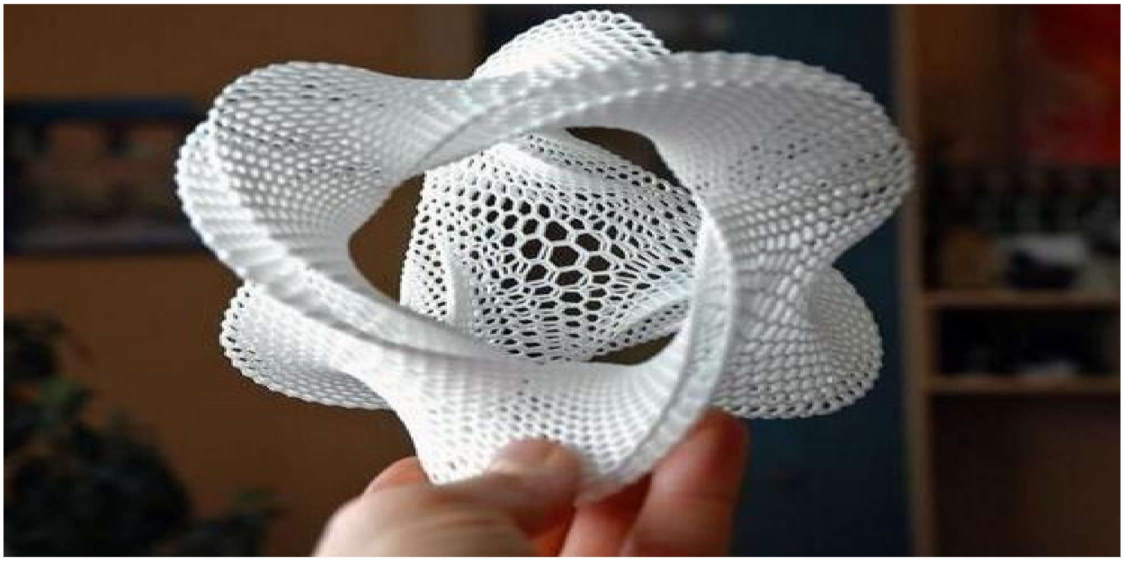

x, y, z – real numbers different in pairs $x+y+z=0$. Find $\min\Omega$

$$\Omega=(x^2+y^2+z^2)\left(\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}\right)$$

Proposed by Le Ngo Duc-Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam; Solution 2 by Michael Sterghiou-Greece

Solution 1 by Tran Hong-Dong Thap-Vietnam

$x+y+z=0\Leftrightarrow z=-x-y;(x\neq y;x\neq z;y\neq z)$. *We must show that:* $\Omega\geq\frac{9}{2}$

$$\Leftrightarrow[x^2+y^2+(x+y)^2]\left[\frac{1}{(x-y)^2}+\frac{1}{(2y+x)^2}+\frac{1}{(x+2y)^2}\right]\geq\frac{9}{2}$$

$$\Leftrightarrow 2[2x^2+2y^2+2xy][(2y+x)^2(x+2y)^2+(x-y)^2(x+2y)^2+(x-y)^2(y+2x)^2]\geq$$

$$\geq 9\{(x-y)(x+2y)(y+2x)\}^2\Leftrightarrow 243x^4y^2+486x^3y^3+243x^2y^4\geq 0$$

$$\Leftrightarrow 243(x^2y^2)(x^2+2xy+y^2)\geq 0\Leftrightarrow 243(xy)^2(x+y)^2\geq 0$$ *(true for x, y)*

Equality:

$$x=0\Rightarrow z=-y\neq 0;\ y=0\Leftrightarrow z=-x\neq 0;\ x=-y(x,y\neq 0)\Rightarrow z=-(-y)-y=0$$

Hence, $\Omega_{\min}=\frac{9}{2}\Leftrightarrow(x;y;z)=(0;t;-t)$ *or* $(x;y;z)=(t;-t;0)$

or $(t;0;-t)$ *(with $t\neq 0$)*

Solution 2 by Michael Sterghiou-Greece

If $n\in\mathbb{N}^*, a,b,c\in\mathbb{R}$: $\prod_{cyc}(a-b)^2\neq 0\wedge\sum_{cyc}a=0$ *find the min of:*

$$A=\left(\sum_{cyc}a^{2n}\right)\left(\sum_{cyc}\frac{1}{(a-b)^{2n}}\right)\quad (T)$$

WLOG assume $a>b>0>c$. *We have* $c=-(a+b)$. $A\cdot b^{2n}\cdot\frac{1}{b^{2n}}$ *gives* $A=[x^{2n}+1+(x+1)^{2n}]\cdot\left[\frac{1}{(x-1)^{2n}}+\frac{1}{(x+2)^{2n}}+\frac{1}{(2x+1)^{2n}}\right]$ *with* $x=\frac{a}{b}>1$ *[If* $a>0>b>c$ *we eliminate* a *and with* $x=\frac{c}{b}>1$ *we are at the same situation]. (T) expands to:*

$$\left(\frac{x}{x-1}\right)^{2n}+\left(\frac{1}{x-1}\right)^{2n}+\left(\frac{x+1}{x-1}\right)^{2n}+\left(\frac{x}{x+2}\right)^{2n}+\left(\frac{1}{x+2}\right)^{2n}+\left(\frac{x+1}{x+2}\right)^{2n}+$$

$$+\left(\frac{1}{2x+1}\right)^{2n}+\left(\frac{x}{2x+1}\right)^{2n}+\left(\frac{x+1}{2x+1}\right)^{2n}\quad (1)$$

The terms $\left(\frac{x}{x-1}\right)^{2n}+\left(\frac{x+1}{x+2}\right)^{2n}\underset{AM-GM}{\geq}2\left[\frac{x^2+x}{x^2+x-2}\right]^n\geq 2^*$ *and also the terms* $\left(\frac{x+1}{x-1}\right)^{2n}+\left(\frac{x}{x+2}\right)^{2n}\geq 2^*$. *In addition:* $\left(\frac{1}{x-1}\right)^{2n}+\left(\frac{1}{x+2}\right)^{2n}+\left(\frac{1}{2x+1}\right)^{2n}\geq 0^*$. *Now, the function,* $f(x)=\left(\frac{x}{2x+1}\right)^{2n}+\left(\frac{x+1}{2x+1}\right)^{2n}$ *is decreasing because:*

$$f'(x)=2n\cdot\frac{1}{(2x+1)^2}\left[\left(\frac{x}{2x+1}\right)^{2n}-\left(\frac{x+1}{2x+1}\right)^{2n}\right]<0,\text{ therefore}$$

$$f(x)\geq\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\left[\left(\frac{x}{2x+1}\right)^{2n}+\left(\frac{x+1}{2x+1}\right)^{2n}\right]=2\left(\frac{1}{2}\right)^{2n}=\left(\frac{1}{2}\right)^{2n-1}$$

Summing up the terms of (1) we get $A\geq 2+2+0+\left(\frac{1}{2}\right)^{2n-1}$

or A *minimum* $=4+\left(\frac{1}{2}\right)^{2n-1}$

** We observe that these* $\min$ *are achieved when* $x\to+\infty$ *therefore it is legitimate to write at the limit* $x\to\infty$ *that* $A\geq 4+\left(\frac{1}{2}\right)^{2n-1}$