R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

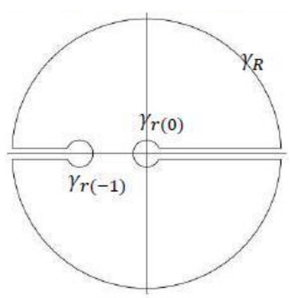
Find:

$$\Omega = \int_0^\infty \frac{\log(x+1)}{x^4+x^2+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Pedro Nagasava-Brazil

1. Let's consider the following complex function and integration path:



$$f(z) = \frac{\log(z)\log(z+1)}{z^4+z^2+1}$$

Therefore:

$$\oint f(z)\,dz = \int_{\gamma_R} f(z)\,dz + \int_{\gamma_{r(0)}} f(z)\,dz + \int_{\gamma r(-1)} f(z)\,dz + \int_{ir}^{R} f(z)\,dz + \int_{R}^{-ir} f(z)\,dz +$$

$$+\int_{-1+ir}^{-R} f(z)\,dz + \int_{-R}^{-1-ir} f(z)\,dz$$

2. Applying the ML inequality it's easy to prove that:

$$\int_{\gamma_R} f(z)\,dz \to 0 \text{ as } R \to \infty;\ \int_{\gamma_{r(0)}} f(z)\,dz \to 0 \text{ as } r \to 0;\ \int_{\gamma(-1)} f(z)\,dz \to 0 \text{ as } r \to 0$$

3. Rewriting and then taking the limits of the first pair of integrals

$$\lim_{R\to\infty}\lim_{r\to 0}\left(\int_{ir}^{R} f(z)\,dz + \int_{R}^{-ir} f(z)\,dz\right) = -2\pi i\int_0^{\infty}\frac{\log(z+1)}{z^4+z^2+1}dz$$

4. Rewriting and then taking the limits of the second pair of integrals:

$$\lim_{R\to\infty}\lim_{r\to 0}\left(\int_{-1+ir}^{-R} f(z)\,dz + \int_{-R}^{-1-ir} f(z)\,dz\right) =$$

$$= \int_{-\infty}^{-1}\frac{\log(z)\left[\log|z+1| + \pi i - (\log|z+1| - \pi i)\right]}{z^4+z^2+1}dz$$

Let $z \to -z$

$$2\pi i\int_1^{\infty}\frac{\log(z)+\pi i}{z^4+z^2+1}dz = 2\pi i\int_1^{\infty}\frac{\log(z)}{z^4+z^2+1}dz + 2\pi\int_1^{\infty}\frac{\pi i}{z^4+z^2+1}dz$$

**** Note: the argument of*** $\log(z)$ ***is given by the contour around*** 0***, hence,*** $\log(z) = \log|z| + \pi i$ ***for both top and bottom sides of the complex plane.***

4.1. Solving the first integral:

$$I_2 = \int_1^{\infty}\frac{\log(z)}{z^4+z^2+1}dz; \text{let } zt \to \frac{1}{z}$$

$$I_2 = -\int_0^1 \frac{z^2\log(z)}{z^4+z^2+1}\left(\frac{1-z^2}{1-z^2}\right)dz = -\int_0^1\left(\frac{z^2\log(z)}{1-z^6} - \frac{z^4\log(z)}{1-z^6}\right)dz$$

$$I_2 = \sum_{k=0}^{\infty}\int_0^1\left(z^{4+6k}\log(z) - z^{2+6k}\log(z)\right)dz = \sum_{k=0}^{\infty}\left[-\frac{1}{(5+6k)^2} + \frac{1}{(3+6k)^2}\right] =$$

$$= -\frac{\psi^{(1)}\left(\frac{5}{6}\right)}{36} + \frac{\zeta(2)}{12}$$

4.2 Solving the second integral:

$$I_3 = \int_1^\infty \frac{dz}{z^4+z^2+1} = \frac{1}{2}\int_1^\infty \frac{1+\frac{1}{z^2}-\left(1-\frac{1}{z^2}\right)}{z^2+\frac{1}{z^2}+1}dz$$

$$I_3 = \frac{1}{2}\Bigg|_1^\infty \left[\frac{1}{\sqrt{3}}\arctan\left(\frac{z-\frac{1}{z}}{\sqrt{3}}\right)-\frac{1}{2}\log\left(\frac{z+\frac{1}{z}-1}{z+\frac{1}{z}+1}\right)\right] = \frac{1}{2}\left(\frac{\pi}{2\sqrt{3}}-\frac{1}{2}\log(3)\right)$$

5. Evaluating the resiudes:

$$\oint f(z)\,dz = 2\pi \lim_{z\to z_k}\sum_{1\le k\le 4}\frac{(z-z_k)\log(z)\log(z+1)}{z^4+z^2+1}$$

$$\oint f(z)\,dz = 2\pi i\left[-\frac{\pi\log(3)}{6\sqrt{3}}+i\pi\left(\frac{\pi}{4\sqrt{3}}-\frac{\log(3)}{4}\right)\right]$$

**** Note: the argument of** $\log(z)\in\,]0;2\pi]$**, thus, the arguments of the residues will be:** $\frac{i\pi}{3},\frac{i2\pi}{3},\frac{i4\pi}{3}$ **and** $\frac{i5\pi}{3}$**; while the argument of** $\log(z+1)\in\,]-\pi,\pi]$**, thus, the arguments of the residues will be:** $\frac{i\pi}{3},\frac{i2\pi}{3},-\frac{i\pi}{3}$ **and** $-\frac{i2\pi}{3}$**.***

6. Gathering all results:

$$-2\pi i\int_0^\infty \frac{\log(z+1)}{z^4+z^2+1}dz + 2\pi i\left(-\frac{\psi^{(1)}\left(\frac{5}{6}\right)}{36}+\frac{\zeta(2)}{12}+\frac{\pi i}{2}\left(\frac{\pi}{2\sqrt{3}}-\frac{1}{2}\log(3)\right)\right) =$$

$$= 2\pi i\left[-\frac{\pi\log(3)}{6\sqrt{3}}+i\pi\left(\frac{\pi}{4\sqrt{3}}-\frac{\log(3)}{4}\right)\right]$$

Hence:

$$\int_0^\infty \frac{\log(x+1)}{x^4+x^2+1}dx = \frac{\pi^2}{72}+\frac{\pi\log(3)}{6\sqrt{3}}-\frac{\psi^{(1)}\left(\frac{5}{6}\right)}{36}$$