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In ΔABC the following relationship holds:

$$\mu(A) < \frac{\pi}{2} \Leftrightarrow r_a < r + 2R$$

$$\mu(A) = \frac{\pi}{2} \Leftrightarrow r_a = r + 2R$$

$$\mu(A) > \frac{\pi}{2} \Leftrightarrow r_a > r + 2R$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Tran Hong-Dong Thap-Vietnam, Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Tran Hong-Dong Thap-Vietnam

We have: $r_a + r_b + r_c = 4R + r; r_b + r_c = 4R\sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{C}{2} + 4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$

$$= 4R\cos\frac{A}{2}\left(\sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{B}{2}\right) = 4R\cos^2\frac{A}{2}$$

$$r_a = r + 2R \Leftrightarrow r_b + r_c = 2R \Leftrightarrow 4R\cos^2\frac{A}{2} = 2R$$

$$\Leftrightarrow \cos^2\frac{A}{2} = \frac{1}{2} \Leftrightarrow \frac{1+\cos A}{2} = \frac{1}{2} \Leftrightarrow \cos A = 0 \overset{0<A<\pi}{\Leftrightarrow} A = \frac{\pi}{2}$$

$$r_a < r + 2R \Leftrightarrow r_b + r_c > 2R \Leftrightarrow 4R\cos^2\frac{A}{2} > 2R$$

$$\Leftrightarrow \cos^2\frac{A}{2} > \frac{1}{2} \Leftrightarrow \frac{1+\cos A}{2} > \frac{1}{2} \Leftrightarrow \cos A > 0 \overset{0<A<\pi}{\Leftrightarrow} 0 < A < \frac{\pi}{2}$$

$$r_a > r + 2R \Leftrightarrow r_b + r_c < 2R \Leftrightarrow 4R\cos^2\frac{A}{2} < 2R$$

$$\Leftrightarrow \cos^2\frac{A}{2} < \frac{1}{2} \Leftrightarrow \frac{1+\cos A}{2} < \frac{1}{2} \Leftrightarrow \cos A < 0 \overset{0<A<\pi}{\Leftrightarrow} \frac{\pi}{2} < A < \pi$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$b + c - a = 4R\cos\frac{A}{2}\cos\frac{B-C}{2} - 4R\sin\frac{A}{2}\cos\frac{A}{2}$$

$$= 4R\cos\frac{A}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right) = 8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Rightarrow s - a \overset{(1)}{=} 4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$r_a - r = \frac{S}{s-a} - \frac{S}{s} = \frac{aS}{s(s-a)} \overset{by\,(1)}{=} \frac{4Rrs\sin^2\frac{A}{2}\cos\frac{A}{2}}{4Rs\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}} = \left(\frac{r}{\frac{r}{4R}}\right)\sin^2\frac{A}{2}$$

$$= 4R\sin^2\frac{A}{2} \Rightarrow \frac{r_a - r}{2R} \overset{(2)}{=} 2\sin^2\frac{A}{2}$$

$$\left.\begin{cases} Now, A < \frac{\pi}{2} \Rightarrow \frac{A}{2} < \frac{\pi}{4} \Rightarrow 2\sin^2\frac{A}{2} < 1 \Rightarrow \frac{r_a - r}{2R} < 1\ (by(2)) \Rightarrow r_a < r + 2R \\ Also, r_a < r + 2R \Rightarrow \frac{r_a - r}{2R} < 1 \Rightarrow 2\sin^2\frac{A}{2} < 1(by\ (2)) \Rightarrow \sin\frac{A}{2} < \frac{1}{\sqrt{2}} \Rightarrow A < \frac{\pi}{2} \end{cases}\right\} \Rightarrow$$

$$A < \frac{\pi}{2} \Leftrightarrow r_a < r + 2R$$

$$\left.\begin{cases} Again, A = \frac{\pi}{2} \Rightarrow \frac{A}{2} = \frac{\pi}{4} \Rightarrow 2\sin^2\frac{A}{2} = 1 \Rightarrow \frac{r_a - r}{2R} = 1(by(2)) \Rightarrow r_a = r + 2R \\ Also, r_a = r + 2R \Rightarrow \frac{r_a - r}{2R} = 1 \Rightarrow 2\sin^2\frac{A}{2} = 1\ (by(2)) \Rightarrow \sin\frac{A}{2} = \frac{1}{\sqrt{2}} \Rightarrow A = \frac{\pi}{2} \end{cases}\right\} \Rightarrow$$

$$A = \frac{\pi}{2} \Leftrightarrow r_a = r + 2R$$

$$\left.\begin{cases} Lastly, A > \frac{\pi}{2} \Rightarrow \frac{A}{2} > \frac{\pi}{4} \Rightarrow 2\sin^2\frac{A}{2} > 1 \Rightarrow \frac{r_a - r}{2R} > 1\ (by(2)) \Rightarrow r_a > r + 2R \\ Also, r_a > r + 2R \Rightarrow \frac{r_a - r}{2R} > 1 \Rightarrow 2\sin^2\frac{A}{2} > 1\ (by(2)) \Rightarrow \sin\frac{A}{2} > \frac{1}{\sqrt{2}} \Rightarrow A > \frac{\pi}{2} \end{cases}\right\} \Rightarrow$$

$$A > \frac{\pi}{2} \Leftrightarrow r_a > r + 2R$$

(Hence proved)