R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

In ΔABC the following relationship holds:

$$\frac{r_a}{\cos^4\frac{A}{2}}+\frac{r_b}{\cos^4\frac{B}{2}}+\frac{r_c}{\cos^4\frac{C}{4}}\ge\frac{36R^2}{4R+r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Tran Hong-Dong Thap-Vietnam, Solution 2 by Marian Ursărescu-Romania, Solution 3 by Soumava Chakraborty-Kolkata-India

Solution 1 by Tran Hong-Dong Thap-Vietnam

Suppose: $A\le B\le C\Rightarrow\cos^4\frac{A}{2}\ge\cos^4\frac{B}{2}\ge\cos^4\frac{C}{2}\Rightarrow\frac{1}{\cos^4\frac{A}{2}}\le\frac{1}{\cos^4\frac{B}{2}}\le\frac{1}{\cos^4\frac{C}{2}}$ *and*

$$a\le b\le c\Rightarrow s-a\ge s-b\ge s-c\Rightarrow\frac{S}{s-a}\le\frac{S}{s-b}\le\frac{S}{s-c}\Rightarrow r_a\le r_b\le r_c$$

$$\Rightarrow LHS=\sum\frac{r_a}{\cos^4\frac{A}{2}}\overset{Chebyshev}{\ge}\frac{1}{3}(r_a+r_b+r_c)\left(\sum\frac{1}{\cos^4\frac{A}{2}}\right)$$

$$\overset{(\sum x^2\ge\sum xy)}{\ge}\frac{1}{3}(4R+r)\cdot\left[\sum\frac{1}{\left(\cos\frac{A}{2}\cos\frac{B}{2}\right)^2}\right]=\frac{1}{3}(4R+r)\cdot\frac{8R(4R+r)}{s^2}$$

We must show that: $\frac{2R(4R+r)^2}{3s^2}\ge\frac{9R^2}{4R+r}\Leftrightarrow 2(4R+r)^3\ge 27s^2R$ ***(*)***

$$s^2 \le 4R^2 + 4Rr + 3r^2 \Rightarrow 27R(4R^2 + 4Rr + 3r^2) \overset{(1)}{\le} 2(4R + r)^3$$

$$\textit{(1)} \Leftrightarrow (R - 2r)(20R^2 + 28Rr - r^2) \ge 0$$

It is true because:

$$R \ge 2r \Rightarrow \begin{cases} R - 2r \ge 0 \\ 20R^2 + 28Rr - r^2 \ge 20 \cdot 4r^2 + 28 \cdot 2r^2 - r^2 = 135r^2 > 0 \end{cases} \Rightarrow \textit{(*) true.}$$

Proved.

Solution 2 by Marian Ursărescu-Romania

We must show: $\left(\frac{r_a}{\cos^4\frac{A}{2}} + \frac{r_b}{\cos^4\frac{B}{2}} + \frac{r_c}{\cos^4\frac{C}{2}}\right)(4R + r) \ge 36R^2$ ***(1)***

But $4R + r = r_a + r_b + r_c$ ***(2)***

From (1)+(2) we must show that: $\left(\frac{r_a}{\cos^4\frac{A}{2}} + \frac{r_b}{\cos^4\frac{B}{2}} + \frac{r_c}{\cos^4\frac{C}{2}}\right)(r_a + r_b + r_c) \ge 36R^2$ ***(3)***

From Cauchy's inequality:

$$\left(\frac{r_a}{\cos^4\frac{A}{2}} + \frac{r_b}{\cos^4\frac{B}{2}} + \frac{r_c}{\cos^4\frac{C}{2}}\right)(r_a + r_b + r_c) \ge \left(\frac{r_a}{\cos^2\frac{A}{2}} + \frac{r_b}{\cos^2\frac{B}{2}} + \frac{r_c}{\cos^2\frac{C}{2}}\right)^2$$

From (3)+ (4) we must show: $\frac{r_a}{\cos^2\frac{A}{2}} + \frac{r_b}{\cos^2\frac{B}{2}} + \frac{r_c}{\cos^2\frac{C}{2}} \ge 6R$ ***(5). But***

$$\left.\begin{array}{c} \left(\frac{r_a}{\cos^2\frac{A}{2}} + \frac{r_b}{\cos^2\frac{B}{2}} + \frac{r_c}{\cos^2\frac{C}{2}}\right)\left(r_a\cos^2\frac{A}{2} + r_b\cos^2\frac{B}{2} + r_c\cos^2\frac{C}{2}\right) \overset{Cauchy}{\ge} (r_a + r_b + r_c)^2 = (4R + r)^2 \\ \sum r_a\cos^2\frac{A}{2} = \frac{s^2}{2R} \end{array}\right\} \Rightarrow$$

$$\left(\frac{r_a}{\cos^2\frac{A}{2}} + \frac{r_b}{\cos^2\frac{B}{2}} + \frac{r_c}{\cos^2\frac{C}{2}}\right) \cdot \frac{s^2}{2R} \ge (4R + r)^2 \Rightarrow$$

$$\frac{r_a}{\cos^2\frac{A}{2}} + \frac{r_b}{\cos^2\frac{B}{2}} + \frac{r_c}{\cos^2\frac{C}{2}} \ge \frac{2R(4R+r)^2}{s^2} \quad \textit{(6)}$$

From (5)+(6) we must show:

$\frac{2R(4R+r)^2}{s^2} \ge 6R \Leftrightarrow (4R + r)^2 \ge 3s^2$***, true because it's Doucet's inequality.***

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\Delta ABC, \sum \frac{r_a}{\cos^4\frac{A}{2}} \ge \frac{36R^2}{4R+r}$$

$$\sum \frac{r_a}{\cos^4\frac{A}{2}} = \sum \frac{r_a^3}{r_a^2\cos^4\frac{A}{2}} = \sum \frac{r_a^3}{\frac{r^2s^2}{(s-a)^2}\cdot\frac{s^2(s-a)^2}{b^2c^2}} = \sum \frac{r_a^3(a^2b^2c^2)}{a^2r^2s^4}$$

$$= \frac{16R^2r^2s^2}{r^2s^4}\sum \frac{r_a^3}{a^2} \overset{Radon}{\ge} \left(\frac{16R^2}{s^2}\right)\left(\frac{(\sum r_a)^3}{(\sum a)^2}\right) = \frac{16R^2(4R+r)^3}{4s^4} = \frac{16R^2(4R+r)^4}{4s^4(4R+r)}$$

$$\overset{Trucht}{\ge} \frac{4R^2(3s^2)^2}{s^4(4R+r)} = \frac{36R^2}{4R+r}$$ ***(Proved)***