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In ΔABC the following relationship holds:

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+\frac{w_aw_b+w_bw_c+w_cw_a}{w_a^2+w_b^2+w_c^2}\ge 4$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Tran Hong-Dong Thap-Vietnam; Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Tran Hong-Dong Thap-Vietnam

By Adil Abdullayev:

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\ge\frac{2R-r}{r}=\frac{2R}{r}-1\quad (1)$$

We have: $h_a\le w_a\le m_a$ ***(etc)***

$$\Rightarrow\frac{\sum(w_aw_b)}{\sum w_a^2}\ge\frac{\sum(h_ah_b)}{\sum m_a^2}=\frac{\sum(h_ah_b)}{\frac{3}{4}\sum a^2}\overset{Leibniz}{\ge}\frac{4}{3}\cdot\frac{\sum(h_ah_b)}{9R^2}$$

$$=\frac{4}{3}\cdot\frac{2s^2r}{9R^2}=\frac{8}{27}\cdot\frac{s^2r}{R\cdot R^2}\ge\frac{8}{27}\cdot\frac{\left(3\sqrt{3}\right)^2r}{R^3}=8\left(\frac{r}{R}\right)^3\quad (2)$$

From (1) and (2) we must show that:

$$\frac{2R}{r}-1+8\left(\frac{r}{R}\right)^3\ge 4\Leftrightarrow 2t+\frac{8}{t^3}-5\ge 0\left(t=\frac{R}{r}\ge 2\right)$$

$$\Leftrightarrow 2t^4-5t^3+8\ge 0\Leftrightarrow (t-2)(2t^3-t^2-2t-4)\ge 0$$

It is true because:

$$t \geq 2 \Rightarrow \begin{cases} t-2 \geq 0 \\ 2t^3 - t^2 - 2t - 4 = 2(t-2)^3 + 11t^2 - 26t + 12 > 0 \end{cases}$$

(proved)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\because w_a \geq h_a, \text{etc} \ \& \ w_a^2 \leq s(s-a), \text{etc},$$

$$\therefore \frac{\sum w_a w_b}{\sum w_a^2} \geq \frac{\sum h_a h_b}{\sum s(s-a)} = \sum \frac{\frac{bc \cdot ca}{4R^2}}{s(3s-2s)}$$

$$= \frac{4Rrs}{4R^2S^2}\sum(c) = \frac{4Rrs(2S)}{4R^2S^2} = \frac{2r}{R}$$

$$\Rightarrow \frac{\sum w_a w_b}{\sum w_a^2} \overset{(1)}{\geq} \frac{2r}{R}$$

$$\text{Again, } \sum \frac{r_a}{r_b} = \sum \frac{r_a^2}{r_a r_b} \underset{(2)}{\overset{Bergstrom}{\geq}} \frac{(4R+r)^2}{S^2}$$

$$(1),(2) \Rightarrow LHS \geq \frac{(4R+r)^2}{S^2} + \frac{2r}{R} \geq 4$$

$$\Leftrightarrow \frac{(4R+r)^2}{S^2} \geq \frac{4R-2r}{R} \Leftrightarrow R(4R+r)^2 \overset{(3)}{\geq} S^2(4R-2r)$$

$$\text{Now, RHS of (3)} \overset{Rouche}{\leq} (4R-2r)\left(2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\leq} R(4R+r)^2$$

$$\Leftrightarrow R(4R+r)^2 - (4R-2r)(2R^2+10Rr-r^2) \overset{?}{\geq} 4(2R-r)(R-2r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow (R-2r)(8R^2-12Rr+r^2) \overset{?}{\geq} 4(2R-r)(R-2r)\sqrt{R^2-2Rr}$$

$$\because R - 2r \overset{Euler}{\geq} 0, \therefore \text{it suffices to prove:}$$

$$8R^2 - 12Rr + r^2 \overset{(4)}{\geq} 4(2R-r)\sqrt{R^2-2Rr}$$

$$\because 8R^2 - 12Rr + r^2 = 8R(R-2r) + 4Rr + r^2 > 0 \text{ as } R \overset{Euler}{\geq} 2r$$

$$\therefore (4) \Leftrightarrow (8R^2-12Rr+r^2)^2 - 16(2R-r)^2(R^2-2Rr) > 0$$

$$\Leftrightarrow r^2(4R+r)^2 > 0 \rightarrow \text{true}$$

⇒ (4)⇒(3)⇒ given inequality is true (proved)