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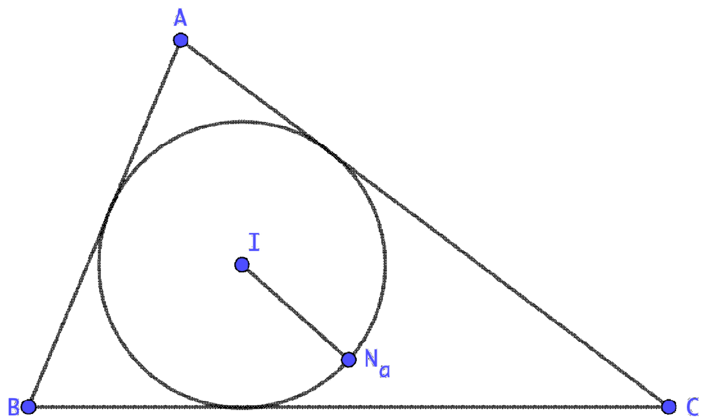

$AB = k, BC = k+3, AC = k+2, k > 0$, I – incenter of ΔABC

N_a – Nagel's point of ΔABC, r – inradius of ABC, $N_a \in (I,r)$

Find: $S(abc)$ (area)

Proposed by Thanasis Gakopoulos-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$a = k+3, b = k+2, c = k, N_a \in (I,r) S(ABC) = ?$$

$$IN_a^2 = r^2 \Rightarrow 9IG^2 = r^2 \Rightarrow \frac{9}{18s}\left(2\sum a^2b + 2\sum ab^2 - \sum a^3 - 9abc\right) = r^2$$

$$\Rightarrow \frac{1}{2s}\left(2\sum ab\,(2s-c) - 2s(s^2-6Rr-3r^2) - 36Rrs\right) = r^2$$

$$\Rightarrow \frac{1}{2s}(4s(s^2+4Rr+r)^2 - 2s(s^2-6Rr-3r^2) - 60Rrs) = r^2$$

$$\Rightarrow \frac{1}{2s}(2s^3+10sr^2-32Rrs) = r^2 \Rightarrow s^2 - 16Rr + 5r^2 = r^2 \Rightarrow s^2 \overset{(1)}{=} 16Rr - 4r^2$$

Now, $abc = 4Rrs \Rightarrow k(k+2)(k+3) = 2Rr(k+k+2+k+3)$

$$\Rightarrow 2Rr \overset{(2)}{=} \frac{k(k+2)(k+3)}{3k+5}$$

Again, $\sum ab = s^2 + 4Rr + r^2 \Rightarrow (k+3)(k+2) + k(k+2) + k(k+3)$

$$= \left(\frac{k+k+2+k+3}{2}\right)^2 + 2\left(\frac{k(k+2)(k+3)}{3k+5}\right) + r^2$$

$$\Rightarrow 4(3k+5)r^2 = 4(3k+5)(3k^2+10k+6) - (3k+5)^3 - 8k(k+2)(k+3)$$

$$\Rightarrow 4(3k+5)r^2 = (k^2-1)(k+5) \Rightarrow r \overset{(3)}{=} \frac{(k^2-1)(k+5)}{4(3k+5)}$$

Plugging (2), (3) in (1): $\frac{(3k+5)^2}{4} = \frac{8k(k+2)(k+3)}{3k+5} - \frac{4(k^2-1)(k+5)}{4(3k+5)}$

$$\Rightarrow \frac{(3k+5)^2}{4} = \frac{7k^3+35k^2+49k+5}{3k+5} \Rightarrow k^3+5k^2-29k-105=0$$

$$\Rightarrow (k+3)(k+7)(k-5)=0 \Rightarrow k=5 \therefore a=8, b=7, c=5 \Rightarrow s=10$$

$$\Rightarrow S(ABC) = \sqrt{(10)(10-8)(10-7)(10-5)} = 10\sqrt{3}$$ ***(Answer)***