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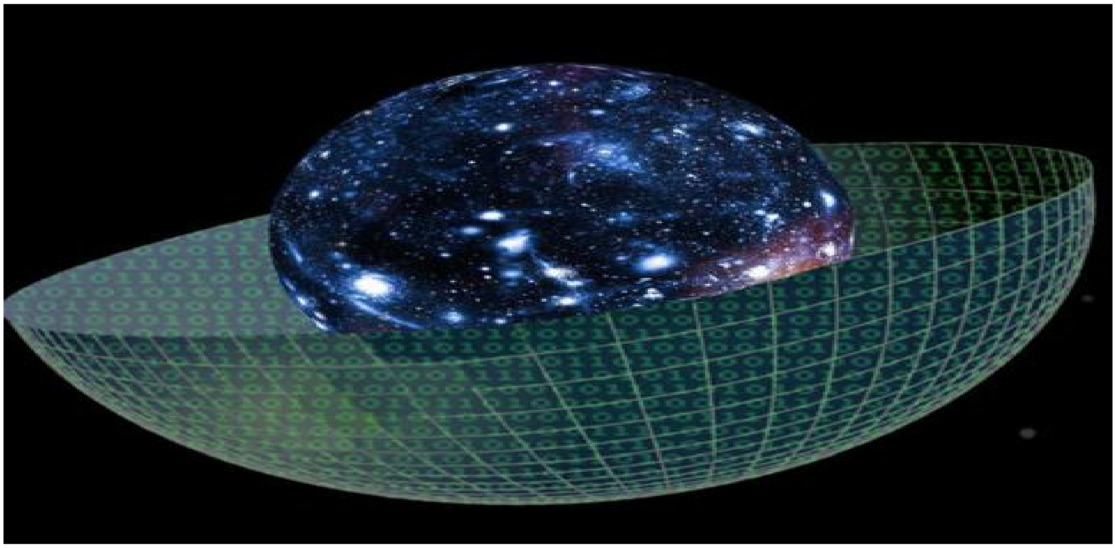

If $0 < a, b, c \leq 16$ then:

$$27\exp\left(\sum_{cyc}\left(\sqrt{\frac{a+2b}{3}}-\sqrt{a}\right)\right) \leq \frac{(a+2b)(b+2c)(c+2a)}{abc}$$

Proposed by Daniel Sitaru – Romania

Solution by Michael Sterghiou-Greece

$$27\exp\left[\sum_{cyc}\left(\sqrt{\frac{a+2b}{3}}-\sqrt{a}\right)\right] \leq \frac{\prod_{cyc}(a+2b)}{abc} \quad \textit{(1)}$$

$$\textit{(1)} \to \sum_{cyc}\left(\sqrt{\frac{a+2b}{3}}-\sqrt{a}\right) \leq \left(\sum_{cyc}\ln\left(\frac{2a+b}{3}\right)\right) - \sum_{cyc}\ln a \ \textit{or}$$

$$\sum_{cyc}(\sqrt{a}-\ln a) \geq \sum_{cyc}\left(\sqrt{\frac{2a+b}{3}}-\ln\left(\frac{2a+b}{3}\right)\right) \quad \textit{(2)}$$

The function $f(t) = \sqrt{t}-\ln t$ on $(0, 16]$ is convex as

$f''(t) = \frac{4-\sqrt{t}}{4t^2} \geq 0$ for $0 < t \leq 16$. Assume WLOG that $a \geq b \geq c$.

Case I: $b \geq \frac{c+a}{2}$. The triad (a, b, c) majorizes the triad $\left(\frac{a+2b}{3} \geq \frac{c+2d}{3} \geq \frac{b+2c}{3}\right)$ as:

$$a \geq \frac{a+2b}{3}$$

$$a+b \geq \frac{a+2b}{3}+\frac{c+2a}{3} \leftrightarrow b \geq c \ \textit{and}\ a+b+c = \sum_{cyc}\frac{a+2b}{3}$$

Case II: $b \leq \frac{c+a}{2}$***. The triad*** $(a \geq b \geq c)$ ***majorizes the triad*** $\left(\frac{a+2a}{3} \geq \frac{a+2b}{3} \geq \frac{b+2c}{3}\right)$ ***as*** $a \geq \frac{c+2a}{3}$ ***and*** $a + b \geq \frac{c+2a}{3} + \frac{a+2b}{3} \leftrightarrow b \geq c$***. Applying Karamata's inequality for the convex function*** $f(t) = \sqrt{t} - \ln t$ ***on*** $(0, 16]$ ***for the above triads for either case I or II we obtain (2). Done!***