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If $a, b, c \in (0, \pi)$ then:

$$\cos^2 a \cdot \cos^2 b \cdot \cos^2 c + (\sin a + \sin b + \sin c)^2 > 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam, Solution 2 by Khaled Abd Imouti-Damascus-Syria, Solution 3 by Amit Dutta-Jamshedpur-India

Solution 1 by Tran Hong-Dong Thap-Vietnam

Let $x = \sin a$; $y = \sin b$; $z = \sin c$. ***Because:*** $0 < a, b, c < \pi \Rightarrow 0 < x, y, z < 1$

$$\prod \cos^2 a + \left(\sum \sin a\right)^2 > 1 \Leftrightarrow \prod (1 - \sin^2 a) + \left(\sum \sin a\right)^2 > 1$$

$$\Leftrightarrow (1 - x^2)(1 - y^2)(1 - z^2) + (x + y + z)^2 > 1$$

$$\Leftrightarrow 1 - (x^2 + y^2 + z^2) + (x^2y^2 + y^2z^2 + z^2x^2) - (xyz)^2 + x^2 + y^2 + z^2 +$$

$$+2(xy + xz + yz) > 1$$

$$\Leftrightarrow (x^2y^2 + y^2z^2 + z^2x^2) + 2(xy + yz + zx) - (xyz)^2 > 0$$

It is true because:

$$(x^2y^2 + y^2z^2 + z^2x^2) + 2xy + 2yz + 2zx \geq 6\sqrt[6]{2^3(xyz)^6} = 6\sqrt{2}xyz$$

and: $6\sqrt{2}xyz > xyz > (xyz)^2$ $(\because 0 < xyz < 1)$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\cos^2 a\cdot\cos^2 b\cdot\cos^2 c+(\sin a+\sin b+\sin c)^2>1, a,b,c\in(0,\pi)$$

$$\left((\sin a+\sin b)+\sin c\right)^2=(\sin a+\sin b)^2+2\sin c\,(\sin a+\sin b)+\sin^2 c$$

$$=\sin^2 a+\sin^2 b+2\sin a\sin b+2\sin c\sin a+2\sin c\sin b+\sin^2 c$$

$$=\sin^2 a+\sin^2 b+\sin^2 c+2\sin a\sin b+2\sin c\sin a+2\sin c\sin b$$

$$\cos^2 a\cdot\cos^2 b\cdot\cos^2 c+\sin^2 a+\sin^2 b+\sin^2 c+2\sin a\sin b+$$

$$+2\sin c\sin a+2\sin c\sin b\overset{?}{>}1$$

But: $(1-\sin^2 a)(1-\sin^2 b)(1-\sin^2 c)=\cos^2 a\cdot\cos^2 b\cdot\cos^2 c$

$$(1-\sin^2 b-\sin^2 a+\sin^2 b\sin^2 a)(1-\sin^2 c)=\cos^2 a\cdot\cos^2 b\cdot\cos^2 c$$

$$1-\sin^2 c-\sin^2 b+\sin^2 b\sin^2 c-\sin^2 a+\sin^2 a\cdot\sin^2 c+\sin^2 b\sin^2 a-$$

$$-\sin^2 a\sin^2 b\sin^2 c+\sin^2 a+\sin^2 b+\sin^2 c+$$

$$+2\sin a\cdot\sin b+2\sin c\sin a+2\sin c\sin b\overset{?}{>}1$$

$$1+\sin^2 b\sin^2 c+\sin^2 a\sin^2 c+\sin^2 b\sin^2 a-\sin^2 a\sin^2 b\sin^2 c+$$

$$+2\sin a\sin b+2\sin c\sin a+2\sin c\sin b\overset{?}{>}1$$

$$\sin^2 b\sin^2 c+\sin^2 a\cdot\sin^2 c+\sin^2 b\cdot\sin^2 a-\sin^2 a\sin^2 b\sin^2 c+$$

$$+2\sin a\sin b+2\sin c\sin a+2\sin c\sin b\overset{?}{>}0$$

$$\sin b\sin c\,(\sin b\sin c+2)+\sin a\sin c\,(\sin a\sin c+2)+$$

$$+\sin b\sin a\,(\sin b\sin a+2)\overset{?}{>}\sin^2 a\sin^2 b\sin^2 c$$

$$\frac{\sin b\sin c+2}{\sin^2 a\sin b\sin c}+\frac{\sin a\sin c+2}{\sin^2 b\sin a\sin c}+\frac{\sin b\sin a+2}{\sin^2 c\sin a\sin b}\overset{?}{>}0$$

because $a,b,c\in(0,\pi)$: $\sin a>0,\sin b>0$ ***and*** $\sin c>0$

So, the inequality is true.

Solution 3 by Amit Dutta-Jamshedpur-India

Let $P=\cos^2 a\cdot\cos^2 b\cdot c+(\sin a+\sin b+\sin c)^2$. ***Put*** $\sin a=p,\sin b=q,\sin c=r$

$$P=(1-p^2)(1-q^2)(1-r^2)+(p+q+r)^2\because a,b,c\in(0,\pi)\Rightarrow p,q,r\in(0,1)$$

$$P=1-p^2-q^2-r^2+(pq)^2+(qr)^2+(pr)^2-(pqr)^2+$$

$$+p^2+q^2+r^2+2pq+2qr+2pr$$

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$$\boldsymbol{P} = \mathbf{1} + (\boldsymbol{pq})^2 + (\boldsymbol{qr})^2 + (\boldsymbol{pr})^2 - (\boldsymbol{pqr})^2 + 2(\boldsymbol{pq} + \boldsymbol{qr} + \boldsymbol{pr})$$

$$\boldsymbol{P} = \mathbf{1} + \boldsymbol{p}^2\boldsymbol{q}^2(\mathbf{1} - \boldsymbol{r}^2) + \boldsymbol{q}^2\boldsymbol{r}^2 + \boldsymbol{p}^2\boldsymbol{r}^2 + 2(\boldsymbol{pq} + \boldsymbol{qr} + \boldsymbol{pr})$$

$$\because \mathbf{0} < r < 1 \Rightarrow 0 < \boldsymbol{r}^2 < 1 \Rightarrow (\mathbf{1} - \boldsymbol{r}^2) > 0$$

$$\therefore \boldsymbol{P} > 1\{\because \boldsymbol{p}^2\boldsymbol{q}^2(\mathbf{1} - \boldsymbol{r}^2) + \boldsymbol{q}^2\boldsymbol{r}^2 + \boldsymbol{p}^2\boldsymbol{q}^2 + 2(\boldsymbol{pq} + \boldsymbol{qr} + \boldsymbol{pr}) > 0\}$$

$\therefore \boldsymbol{P} > 1$***. Proved.***