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1) In ΔABC

$$\sum \frac{l_a^2}{h_b h_c} \le \left(\frac{R}{r}\right)^2 - 1$$

Proposed by Mehmet Şahin - Ankara - Turkey

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC

$$\sum \frac{s(s-a)}{h_b h_c} = \frac{s^2 + r^2 - 8Rr}{4r^2}$$

Proof.

Using $h_a = \frac{2S}{a}$ *we obtain:*

$$\sum \frac{s(s-a)}{h_b h_c} = \sum \frac{s(s-a)}{\frac{2S}{b} \cdot \frac{2S}{c}} = \frac{s}{4S^2} \sum bc(s-a) = \frac{s^2 + r^2 - 8Rr}{4r^2},$$ *which follows from:*

$$\sum bc(s-a) = s(s^2 + r^2 - 8Rr)$$

□

Let's get back to the main problem.

Using $l_a^2 \le s(s-a)$ *and* ***Lemma*** *we obtain:*

$$\sum \frac{l_a^2}{h_b h_c} \le \sum \frac{s(s-a)}{h_b h_c} = \frac{s^2 + r^2 - 8Rr}{4r^2} \le \left(\frac{R}{r}\right)^2 - 1$$

where the last inequality is equivalent with: $s^2 \le 4R^2 + 8Rr - 5r^2$

which follows from Gerretsen's inequality: $s^2 \le 4R^2 + 4Rr + 3r^2$

and Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

The inequality can be strengthened:

3) In ΔABC:

$$\sum \frac{l_a^2}{h_b h_c} \le \left(\frac{R}{r}\right)^2 - \frac{R}{r} + 1$$

Proof.

Using $l_a^2 \le s(s-a)$ *and Lemma, we obtain:*

$$\sum \frac{l_a^2}{h_b h_c} \le \sum \frac{s(s-a)}{h_b h_c} = \frac{s^2 + r^2 - 8Rr}{4r^2} \le \left(\frac{R}{r}\right)^2 - \frac{R}{r} + 1$$

where the last inequality is equivalent with:

$s^2 \le 4R^2 + 4Rr + 3r^2$ *(Gerretsen's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **3)** is stronger then inequality **1)**:*

4) In ΔABC:

$$\sum \frac{l_a^2}{h_b h_c} \le \left(\frac{R}{r}\right)^2 - \frac{R}{r} + 1 \le \left(\frac{R}{r}\right)^2 - 1$$

Proof.

*See inequality **3)** and Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having an opposite sense:

Lemma.
5) In ΔABC

$$\sum \frac{l_a^2}{h_b h_c} \ge 3.$$

Proof.

Using $l_a \ge h_a$ *and* $\sum \frac{h_a^2}{h_b h_c} \ge 3$, *which follows from:* $\sum \frac{h_a^2}{h_b h_c} \ge \frac{(\sum h_a)^2}{\sum h_b h_c} \ge 3$.

Above we have used Bergström's inequality and the following inequality:

$(x+y+z)^2 \ge 3(xy+yz+zx)$

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the following sequence of inequalities:

6) In ΔABC:

$$3 \leq \sum \frac{l_a^2}{h_b h_c} \leq \left(\frac{R}{r}\right)^2 - \frac{R}{r} + 1 \leq \left(\frac{R}{r}\right)^2 - 1$$

Proof.

*See inequalities **3)** and **5)**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Regarding the above **Lemma** we propose:*

7) ΔABC

$$\frac{3R}{2r} \leq \sum \frac{s(s-a)}{h_b h_c} \leq \left(\frac{R}{r}\right)^2 - \frac{R}{r} + 1$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma** and Gerretsen's inequality* $16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$

*See inequalities **3)** and **5)**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Replacing h_a with r_a, we propose:

8) In ΔABC:

$$3 \leq \sum \frac{s(s-a)}{r_b r_c} \leq \frac{3R}{2r}$$

Proof.

We use $\sum \frac{s(s-a)}{r_b r_c} = 3$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum \frac{s(s-a)}{r_b r_c}$ *and* $\sum \frac{s(s-a)}{h_b h_c}$, *we can write the following relationship:*

9) In ΔABC

$$\sum \frac{s(s-a)}{r_b r_c} \leq \sum \frac{s(s-a)}{h_b h_c}$$

Proof.

Using the sums $\sum \frac{s(s-a)}{r_b r_c} = 3$ *and* $\sum \frac{s(s-a)}{h_b h_c} = \frac{s^2+r^2-8Rr}{4r^2}$ *we write the following inequality:* $3 \leq \frac{s^2+r^2-8Rr}{4r^2} \Leftrightarrow s^2 \geq 8Rr + 11r^2$*, which follows from Gerretsen's inequality* $s^2 \geq 16Rr - 5r^2$ *and Euler's inequality* $R \geq 2r$.

□

Remark.

We can write the sequence of inequalities:

10) In ΔABC

$$3 \leq \sum \frac{s(s-a)}{r_b r_c} \leq \frac{3R}{2r} \leq \sum \frac{s(s-a)}{h_b h_c} \leq \left(\frac{R}{r}\right)^2 - \frac{R}{r} + 1 \leq \left(\frac{R}{r}\right)^2 - 1$$

Proposed by Marin Chirciu - Romania

Proof.

See inequalities ***7)*** *and* ***8)****.*

Equality holds if and only if the triangle is equilateral.

□

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