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PROBLEM 3647
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1) In ΔABC

$$\sum \frac{(r_a + r_b)(r_a + r_c)}{bc} \geq 9$$

Proposed by Panagiote Ligouras - Italy

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC

$$\sum \frac{(r_a + r_b)(r_a + r_c)}{bc} = \frac{4R + r}{r}$$

Proof.

Using $r_a = \dfrac{S}{s - a}$ *we obtain:*

$$\sum \frac{(r_a + r_b)(r_a + r_c)}{bc} = \sum \frac{\left(\frac{S}{s-a} + \frac{S}{s-b}\right)\left(\frac{S}{s-a} + \frac{S}{s-a}\right)}{bc} = S^2 \sum \frac{\frac{c}{(s-a)(s-b)} \cdot \frac{b}{(s-a)(s-b)}}{bc} =$$

$$= S^2 \sum \frac{1}{(s-a)\prod(s-a)} = \frac{r^2 s^2}{\prod(s-a)} \sum \frac{1}{s-a} = \frac{r^2 s^2}{r^2 s} \cdot \frac{4R + r}{rs} = \frac{4R + r}{r}$$

□

Let's get back to the main problem.

Using the Lemma the inequality can be written: $\dfrac{4R + r}{r} \geq 9 \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having an opposite sense.

3) In ΔABC

$$\sum \frac{(r_a + r_b)(r_a + r_c)}{bc} \leq \frac{9R}{2r}.$$

Proof.

Using the ***Lemma*** *the inequality can be written:* $\dfrac{4R+r}{r} \leq \dfrac{9R}{2r} \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

The double inequality can be written:

4) In ΔABC

$$\mathbf{9 \leq \sum \frac{(r_a + r_b)(r_a + r_c)}{bc} \leq \frac{9R}{2r}}$$

Proof.

See inequalities ***1)*** *and* ***3)****.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Replacing h_a *with* r_a *we propose:*

5) In ΔABC

$$\mathbf{\frac{4r}{R}\left(4 + \frac{r}{R}\right) \leq \sum \frac{(h_a + h_b)(h_a + h_c)}{bc} \leq 2\left(4 + \frac{r}{R}\right)}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.
6) In ΔABC

$$\mathbf{\sum \frac{(h_a + h_b)(h_a + h_c)}{bc} = \frac{5s^2 + r^2 + 4Rr}{4R^2}}$$

Proof.

Using $h_a = \dfrac{2S}{a}$ *we obtain:*

$$\sum \frac{(h_a + h_b)(h_a + h_c)}{bc} = \sum \frac{(\frac{2S}{a} + \frac{2S}{b})(\frac{2S}{a} + \frac{2S}{c})}{bc} = 4S^2 \sum \frac{\frac{a+b}{ab} \cdot \frac{a+c}{ac}}{bc} =$$

$$= \frac{4S^2}{a^2b^2c^2} \sum (a+b)(a+c) = \frac{4r^2s^2}{16R^2r^2s^2} \cdot (5s^2 + r^2 + 4Rr) = \frac{5s^2 + r^2 + 4Rr}{4R^2}$$

which follows from $\sum (a+b)(a+c) = 5s^2 + r^2 + 4Rr.$

□

Let's get back to the main problem.

Left hand inequality.

*Using the **Lemma** we write the inequality:*

$$\frac{5s^2+r^2+4Rr}{4R^2} \geq \frac{4r}{R}\left(4+\frac{r}{R}\right) \Leftrightarrow s^2 \geq 12Rr+3r^2$$

which follows from Gerretsen's inequality: $s^2 \geq 16Rr-5r^2$

It remains to prove that: $16Rr-5r^2 \geq 12Rr+3r^2 \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

The right hand inequality.

*Using the **Lemma** the inequality can be written:*

$$\frac{5s^2+r^2+4Rr}{4R^2} \leq 2\left(4+\frac{r}{R}\right) \Leftrightarrow 5s^2 \leq 32R^2+4Rr-r^2$$

which follows from Gerretsen's inequality: $s^2 \leq 4R^2+4Rr+3r^2$.

It remains to prove that: $5(4R^2+4Rr+3r^2) \leq 32R^2+4Rr-r^2 \Leftrightarrow$

$$\Leftrightarrow 3R^2-4Rr-4r^2 \geq 0 \Leftrightarrow (R-2r)(3R+2r) \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum \frac{(h_a+h_b)(h_a+h_c)}{bc}$ *and* $\sum \frac{(r_a+r_b)(r_a+r_c)}{bc}$ *the following inequality holds:*

7) In ΔABC:

$$\sum \frac{(h_a+h_b)(h_a+h_c)}{bc} \leq \sum \frac{(r_a+r_b)(r_a+r_c)}{bc}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using identities **2)** and **6)** the inequality can be written:*

$$\frac{5s^2+r^2+4Rr}{4R^2} \leq \frac{4R+r}{r} \Leftrightarrow 5s^2r \leq (4R+r)(4R^2-r^2)$$

which follows from Gerretsen's inequality: $s^2 \leq 4R^2+4Rr+3r^2$.

It remains to prove that: $5r(4R^2+4Rr+3r^2) \leq (4R+r)(4R^2-r^2) \Leftrightarrow$

$$\Leftrightarrow 2R^3-2R^2r-3Rr^2-2r^3 \geq 0 \Leftrightarrow (R-2r)(2R^2+2Rr+r^2) \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remarak.

The sequence of inequalities can be written:

8) In ΔABC:

$$\frac{4r}{R}\left(4+\frac{r}{R}\right) \le \sum \frac{(h_a+h_b)(h_a+h_c)}{bc} \le \sum \frac{(r_a+r_b)(r_a+r_c)}{bc} \le \frac{9R}{2r}$$

Proposed by Marin Chirciu - Romania

Proof.

*See inequalities **3)**, **5)** and **7)**.*

Equality holds if and only if the triangle is equilateral.

□

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