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INEQUALITY IN TRIANGLE 949
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1) In ΔABC:

$$\sum (r_b - r_c)^2 \leq \frac{3s^2(R-2r)}{r}$$

Proposed by Adil Abdullayev - Baku - Azerbaijan

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC:

$$\sum (r_b - r_c)^2 = 2(4R+r)^2 - 6s^2.$$

Proof.

Using $r_a = \frac{S}{s-a}$, *we obtain:*

$$\sum (r_b - r_c)^2 = 2\sum r_a^2 - 2\sum r_b r_c = 2\left(\sum r_a\right)^2 - 6\sum r_b r_c = 2(4R+r)^2 - 6s^2$$

which follows from: $\sum r_a = 4R + r$ *and* $\sum r_b r_c = s^2$.

□

Getting back to the main problem:
*Using the **Lemma** we write the inequality:*

$2(4R+r)^2 - 6s^2 \leq \frac{3s^2(R-2r)}{r} \Leftrightarrow 2r(4R+r)^2 \leq 3s^2R$, *which follows from*

Gerretsen's inequality $s^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r}$. *It remains to prove that:*

$2r(4R+r)^2 \leq 3R \cdot \frac{r(4R+r)^2}{R+r} \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasise an inequality having an opposite sense:

3) In ΔABC:

$$\sum (r_b - r_c)^2 \geq \frac{3s^2(R-2r)}{2R-r}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the ***Lemma*** *we write the inequality:*

$$2(4R+r)^2 - 6s^2 \geq \frac{3s^2(R-2r)}{2R-r} \Leftrightarrow 2(2R-r)(4R+r)^2 \geq 3s^2(5R-4r)$$

which follows from Blundon-Gerretsen's inequality $s^2 \leq \dfrac{R(4R+r)^2}{2(2R-r)}$

It remains to prove that:

$$2(2R-r)(4R+r)^2 \geq 3 \cdot \frac{R(4R+r)^2}{2(2R-r)}(5R-4r) \Leftrightarrow (R-2r)^2 \geq 0, \textit{ obvious.}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

4) In ΔABC:

$$\frac{3s^2(R-2r)}{2R-r} \leq \sum (r_b - r_c)^2 \leq \frac{3s^2(R-2r)}{r}$$

Proof.

See inequalities ***1)*** *and* ***3)****.*

Equality holds if and only if the triangle is equilateral.

□

5) In ΔABC:

$$s^2(R-2r)\frac{r^2}{2R^2(R-r)} \leq \sum (h_b - h_c)^2 \leq s^2(R-2r)\frac{4R-3r}{2R^2}.$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.

6) In ΔABC:

$$\sum (h_b - h_c)^2 = \frac{s^4 + s^2(2r^2 - 16Rr) + r^2(4R+r)^2}{2R^2}$$

Proof.

$$\text{Using } h_a = \frac{2S}{a}, \text{ we obtain:}$$

$$\sum(h_b-h_c)^2 = 2\sum h_a^2 - 2\sum h_b h_c = 2\Big(\sum h_a\Big)^2 - 6\sum h_b h_c = 2\Big(\frac{s^2+r^2+4Rr}{2R}\Big)^2 - 6\frac{2s^2r}{R} =$$

$$= \frac{s^4+s^2(2r^2-16Rr)+r^2(4R+r)^2}{2R^2}, \text{ which follows from:}$$

$$\sum h_a = \frac{s^2+r^2+4Rr}{2R} \text{ and } \sum h_b h_c = \frac{2s^2r}{R}$$

□

Let's get back to the main problem:

*Using the **Lemma** we write the inequality:*

$$\frac{2s^2(R-2r)(R-r)}{R^2} \le \sum(h_b-h_c)^2 \le s^2(R-2r)\frac{4R-3r}{2R^2}.$$

The left side inequality: $\sum(h_b-h_c)^2 \ge s^2(R-2r)\frac{r^2}{2R^2(R-r)}$ *it follows from:*

$$\sum(h_b-h_c)^2 = \frac{s^4+s^2(2r^2-16Rr)+r^2(4R+r)^2}{2R^2} = \frac{s^2(s^2+2r^2-16Rr)+r^2(4R+r)^2}{2R^2} \ge$$

$$\ge \frac{s^2(s^2+2r^2-16Rr)+r^2\cdot 3s^2}{2R^2} = \frac{s^2(s^2+5r^2-16Rr)}{2R^2} \ge$$

$$\ge \frac{s^2(16Rr-5r^2+\frac{r^2(R-2r)}{R-r}+5r^2-16Rr)}{2R^2} = s^2(R-2r)\frac{r^2}{2R^2(R-r)}$$

which follows from Yang Xue Zhi's inequality:

$$s^2 \ge 16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r}$$

□

Note.
Yang Xue Zhi's inequality:

$$16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r} \le s^2 \le 4R^2 + 4Rr + 3r^2 - \frac{r^2(R-2r)}{R-r},$$

strengthen Gerretsen's inequality:

$$16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2.$$

Right hand inequality: $\sum(h_b-h_c)^2 \le s^2(R-2r)\frac{4R-3r}{2R^2}$ *is equivalent with:*

$$\frac{s^4+s^2(2r^2-16Rr)+r^2(4R+r)^2}{2R^2} \le s^2(R-2r)\frac{4R-3r}{2R^2} \Leftrightarrow$$

$\Leftrightarrow r^2(4R+r)^2 \le s^2(4R^2+5Rr+4r^2-s^2)$, *which follows from Gerretsen's inequality:*

$$\frac{r(4R+r)^2}{R+r} \le 16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2.$$

It remains to prove that:

$r^2(4R+r)^2 \le \frac{r(4R+r)^2}{R+r}(4R^2+5Rr+4r^2-4R^2-4Rr-3r^2)$, *obviously, with equality.*

Equality holds if and only if the triangle is equilateral.

Remark.

Between the sums $\sum(h_b - h_c)^2 = \dfrac{s^4 + s^2(2r^2 - 16Rr) + r^2(4R+r)^2}{2R^2}$ *and*

$\sum(r_b - r_c)^2 = 2(4R+r)^2 - 6s^2$ *the following relationship exist:*

7) In ΔABC

$$\boldsymbol{\sum(h_b - h_c)^2 \le \sum(r_b - r_c)^2}$$

Proof.

*Using the identities **2)** and **6)** the inequality can be written:*

$$\frac{s^4 + s^2(2r^2 - 16Rr) + r^2(4R+r)^2}{2R^2} \le 2(4R+r)^2 - 6s^2 \Leftrightarrow$$

$\Leftrightarrow s^2(s^2 + 2r^2 - 16Rr + 12R^2) \le (4R+r)^2(4R^2 - r^2)$, *which follows from*

Blundon-Gerretsen's inequality: $s^2 \le \dfrac{R(4R+r)^2}{2(2R-r)} \le 4R^2 + 4Rr + 3r^2$.

It remains to prove that:

$$\frac{R(4R+r)^2}{2(2R-r)}(4R^2 + 4Rr + 3r^2 + 2r^2 - 16Rr + 12R^2) \le (4R+r)^2(4R^2 - r^2) \Leftrightarrow$$

$\Leftrightarrow 4R^2 - 9Rr + 2r^2 \ge 0 \Leftrightarrow (R-2r)(4R-r) \ge 0$, *obviously, from Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the following sequence of inequalities:

8) In ΔABC:

$$\boldsymbol{s^2(R-2r) \cdot \frac{r^2}{2R^2(R-r)} \le \sum(h_b - h_c)^2 \le \sum(r_b - r_c)^2 \le s^2(R-2r) \cdot \frac{3}{r}}$$

Proof.

*See inequalities **1)**, **5)** and **7)**.*

Equality holds if and only if the triangle is equilateral.

□

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