RMM Commented Problems Marathon
41 - 60

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ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

PROBLEM TRIANGLE INEQUALITY- 499
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1) In ΔABC

$$\boldsymbol{8\frac{m_a m_b m_c}{h_a h_b h_c} + 1 \ge \frac{(a+b+c)^3}{3abc}}$$

Proposed by Adil Abdullayev - Baku - Azerbaidian

Proof.

We prove the following Lemma

Lemma
2) In ΔABC

$$\boldsymbol{\frac{m_a m_b m_c}{h_a h_b h_c} \ge \frac{R}{2r}.}$$

Proof.

From $m_a \ge \sqrt{p(p-a)}$ *and* $h_a = \dfrac{2S}{a}$ *we have* $m_a m_b m_c \ge Sp$ *and* $h_a h_b h_c = \dfrac{2S^2}{R}$

wherefrom $\dfrac{m_a m_b m_c}{h_a h_b h_c} \ge \dfrac{R}{2r}$.

□

Let's pass to solving the inequality from enunciation.

*Using the **Lemma** and* $a+b+c = 2p, abc = 4Rrp$ *it suffices to prove that:*

$$8 \cdot \frac{R}{2r} + 1 \ge \frac{8p^3}{3 \cdot 4Rrp} \Leftrightarrow \frac{4R+r}{r} \ge \frac{2p^2}{3Rr} \Leftrightarrow 2p^2 \le 3R(4R+r),$$

which follows from Gerretsen's inequality: $p^2 \le 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$$2(4R^2+4Rr+3r^2) \le 3R(4R+r) \Leftrightarrow 4R^2 - 5Rr - 6r^2 \ge 0 \Leftrightarrow (R-2r)(4R+3r) \ge 0$$

obviously from Euler's inequality $R \ge 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark 1.

The inequality can be developed:

3) In ΔABC

$$\lambda \cdot \frac{m_a m_b m_c}{h_a h_b h_c} + 9 - \lambda \geq \frac{(a+b+c)^3}{3abc}, \text{ where } \lambda \geq \frac{16}{3}$$

Proposed by Marin Chirciu - Romania

Proof.

Using ***Lemma*** *and* $a+b+c=2p, abc=4Rrp$ *it suffices to prove that:*

$$\lambda \cdot \frac{R}{2r} + 9 - \lambda \geq \frac{8p^3}{3 \cdot 4Rrp} \Leftrightarrow \frac{\lambda R + (18-2\lambda)r}{2r} \geq \frac{2p^2}{3Rr} \Leftrightarrow 4p^2 \leq 3\lambda R^2 + (54-6\lambda)Rr$$

which follows from Gerretsen's inequality: $p^2 \leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$4(4R^2+4Rr+3r^2) \leq 3\lambda R^2 + (54-6\lambda)Rr \Leftrightarrow (3\lambda-16)R^2 + (38-6\lambda)Rr - 12r^2 \geq 0 \Leftrightarrow$$

$(R-2r)[(3\lambda-16)R+6r] \geq 0,$ *obviously from Euler's inequality* $R \geq 2r$

and the condition $3\lambda - 16 \geq 0$.

Equality holds if and only if the triangle is equilateral.

□

Note

For $\lambda = 8$ *we obtain inequality* ***1***.

Remark 2.

The best inequality having the form of ***3)*** *is:*

4) In ΔABC

$$16\frac{m_a m_b m_c}{h_a h_c h_c} + 11 \geq \frac{(a+b+c)^3}{abc}.$$

Proof.

$$\text{We have } \frac{(a+b+c)^3}{3abc} \leq \frac{16}{3} \cdot \frac{m_a m_b m_c}{h_a h_b h_c} + \frac{11}{3} \overbrace{\leq}^{(1)} \lambda \cdot \frac{m_a m_b m_c}{h_a h_b h_c} + 9 - \lambda,$$

where (1) $\Leftrightarrow \left(\lambda - \frac{16}{3}\right)\frac{m_a m_b m_c}{h_a h_b h_c} \geq \lambda - \frac{16}{3}$, *obviously from* $\lambda \geq \frac{16}{3}$ *is* $\frac{m_a m_b m_c}{h_a h_b h_c} \geq 1$.

Equality holds if and only if the triangle is equilateral.

□

Remark 3.

In the same way we can propose:

5) In ΔABC

$$\lambda \cdot \frac{m_a m_b m_c}{h_a h_b h_c} + 1 - \lambda \geq \frac{a^3+b^3+c^3}{3abc}, \text{ where } \lambda \geq \frac{4}{3}.$$

Proposed by Marin Chirciu - Romania

Proof.

Using the ***Lemma*** *and* $a^3+b^3+c^3=2p(p^2-3r^2-6Rr), abc=4Rrp$ *it suffices to prove that:*

$$\lambda\cdot\frac{R}{2r}+1-\lambda\geq\frac{2p(p^2-3r^2-6Rr)}{3\cdot 4Rrp}\Leftrightarrow\frac{\lambda R+(2-2\lambda)r}{2r}\geq\frac{p^2-3r^2-6Rr}{6Rr}\Leftrightarrow$$

$\Leftrightarrow p^2\leq 3\lambda R^2+(12-6\lambda)Rr+3r^2$, *which follows from Gerretsen's inequality:* $p^2\leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$4R^2+4Rr+3r^2\leq 3\lambda R^2+(12-6\lambda)Rr+3r^2\Leftrightarrow(3\lambda-4)R^2\geq(6\lambda-8)Rr$$

$(3\lambda-4)(R-2r)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$ *and the condition* $3\lambda-4\geq 0$.

Equality holds if and only if the triangle is equilateral.

□

Remark 4.

The best inequality having the form of ***5*** *is:*

6) In ΔABC

$$4\frac{m_a m_b m_c}{h_a h_b h_c}-1\geq\frac{a^3+b^3+c^3}{abc}.$$

Proof.

See solution from ***Remark 2.***

□

7) In ΔABC

$$\lambda\cdot\frac{m_a m_b m_c}{h_a h_b h_c}+1-\lambda\geq\frac{(a+b+c)^2}{3(ab+bc+ca)},\text{ where }\lambda\geq\frac{4}{9}.$$

Proposed by Marin Chirciu - Romania

Proof.

Using ***Lemma*** *and* $a+b+c=2p, ab+bc+ca=p^2+r^2+4Rr$ *it suffices to prove that:*

$$\lambda\cdot\frac{R}{2r}+1-\lambda\geq\frac{4p^2}{3(p^2+r^2+4Rr)}\Leftrightarrow\frac{\lambda R+(2-2\lambda)r}{2r}\geq\frac{4p^2}{3(p^2+r^2+4Rr)}\Leftrightarrow$$

$$\Leftrightarrow 8rp^2\leq(p^2+r^2+4Rr)[3\lambda R+(6-6\lambda)r]\Leftrightarrow$$

$$\Leftrightarrow p^2[4\lambda R-(6\lambda+2)r]+r(4R+r)[3\lambda R+(6-6\lambda)r]\geq 0.$$

We distinguish the cases:

Case 1). If $3\lambda R-(6\lambda+2)r\geq 0$ *the inequality is obvious.*

Case 2). If $3\lambda R-(6\lambda+2)r<0$ *the inequality can be rewritten*

$$p^2[(6\lambda+2)r-3\lambda R]\leq r(4R+r)[3\lambda R+(6-6\lambda)r],$$

which follows from Gerretsen's inequality: $p^2\leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$(4R^2+4Rr+3r^2)[(6\lambda+2)r-3\lambda R]\leq r(4R+r)[3\lambda R+(6-6\lambda)r]$$

$$\Leftrightarrow 3\lambda R^3-2R^2r+(4-9\lambda)Rr^2-6\lambda r^3\geq 0\Leftrightarrow$$

$$\Leftrightarrow(R-2r)[3\lambda R^2+(6\lambda-2)Rr+3\lambda r^2]\geq 0$$

obviously from Euler's inequality $R\geq 2r$ *and the condition* $n\geq\dfrac{4}{3}$.

Equality holds if and only if the triangle is equilateral.

□

Remark 5.

*The best inequality having the form of **7)** is:*

8) In ΔABC

$$4\frac{m_am_bm_c}{h_ah_bh_c}+5\geq\frac{3(a+b+c)^2}{ab+bc+ca}$$

Proof.

*See solution from **Remark 2.***

□

9) In ΔABC

$$\lambda\cdot\frac{m_am_bm_c}{h_ah_bh_c}+1-\lambda\geq\frac{a^2+b^2+c^2}{ab+bc+ca},\ \text{where}\ \lambda\geq\frac{4}{9}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma** and* $a^2+b^2+c^2=2(p^2-r^2-4Rr), ab+bc+ca=p^2+r^2+4Rr$

it suffices to prove that:

$$\lambda\cdot\frac{R}{2r}+1-\lambda\geq\frac{2(p^2-r^2-Rr)}{p^2+r^2+Rr}\Leftrightarrow\frac{\lambda R+(2-2\lambda)r}{2r}\geq\frac{2(p^2-r^2-Rr)}{p^2+r^2+Rr}\Leftrightarrow$$

$$\Leftrightarrow p^2[\lambda R-(2\lambda+2)r]+r[4\lambda R^2+(24-7\lambda)Rr+(6-2\lambda)r^2]\geq 0$$

We distinguish the cases:

Case 1). If $\lambda R-(2\lambda+2)r\geq 0$ *we use Gerretsen's inequality. It remains to prove that:*

$$(16Rr-5r^2)[\lambda R-(2\lambda+2)r]+r[4\lambda R^2+(24-7\lambda)Rr+(6-2\lambda)r^2]\geq 0\Leftrightarrow$$

$$\Leftrightarrow 5\lambda R^2-(11\lambda+2)Rr+(2\lambda+4)r^2\geq 0\Leftrightarrow(R-2r)[5\lambda R-(\lambda+2)r]\geq 0$$

obviously from Euler's inequality $R\geq 2r$ *and the condition* $n\geq\frac{2}{9}$.

Case 2). If $\lambda R-(2\lambda+2)r<0$ *we rewrite the inequality*

$$p^2[(2\lambda+2)r-\lambda R]\leq r[4\lambda R^2+(24-7\lambda)Rr+(6-2\lambda)r^2],$$

which follows from Gerretsen's inequality: $p^2\leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$(4R^2+4Rr+3r^2)[(2\lambda+2)r-\lambda R]\leq r[4\lambda R^2+(24-7\lambda)Rr+(6-2\lambda)r^2]$$

$$\Leftrightarrow\lambda R^3-2R^2r+(4-3\lambda)Rr^2-\lambda r^3\geq 0\Leftrightarrow(R-2r)[\lambda R^2+(2\lambda-2)Rr+\lambda r^2]\geq 0$$

obviously from Euler's inequality $R\geq 2r$ *and the condition* $n\geq\frac{4}{9}$.

Equality holds if and only if the triangle is equilateral.

□

Remark 5.

*The best inequality having the form of **9)** is:*

10) In ΔABC

$$4\frac{m_am_bm_c}{h_ah_bh_c}+5\geq\frac{9(a^2+b^2+c^2)}{ab+bc+ca}.$$

Proof.

*See solution from **Remark 2.***

□

11) In ΔABC

$$\boldsymbol{\lambda \cdot \frac{m_a m_b m_c}{h_a h_b h_c} + 1 - \lambda \geq \frac{3(a^2+b^2+c^2)}{(a+b+c)^2}, \text{ where } \lambda \geq \frac{1}{3}}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma** and* $a^2+b^2+c^2 = 2(p^2-r^2-4Rr), a+b+c = 2p$

it suffices to prove that:

$$\lambda \cdot \frac{R}{2r} + 1 - \lambda \geq \frac{6(p^2-r^2-Rr)}{4p^2} \Leftrightarrow \frac{\lambda R + (2-2\lambda)r}{2r} \geq \frac{3(p^2-r^2-Rr)}{2p^2} \Leftrightarrow$$

$$\Leftrightarrow p^2[\lambda R - (2\lambda+1)r] + 3r^2(4R+r) \geq 0.$$

We distinguish the cases:

Case 1). If $\lambda R - (2\lambda+1)r \geq 0$ *obviously inequality.*

Case 2). If $\lambda R - (\lambda+1)r < 0$ *we rewrite the inequality*

$p^2[(2\lambda+1)r - \lambda R] \leq 3r^2(4R+r)$, *which follows from Gerretsen's inequality:*

$p^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$$(4R^2+4Rr+3r^2)[(2\lambda+1)r - \lambda R] \leq 3r^2(4R+r) \Leftrightarrow$$

$$\Leftrightarrow 4\lambda R^3 - (4\lambda+4)R^2 r + (8-5\lambda)Rr^2 - 6\lambda r^3 \geq 0 \Leftrightarrow (R-2r)[4\lambda R^2 + (4\lambda-4)Rr + 3\lambda r^2] \geq 0$$

obviously from Euler's inequality $R \geq 2r$ *and the condition* $n \geq \frac{1}{3}$.

Equality holds if and only if the triangle is equilateral.

□

Remark 5.

*The best inequality having the form **11)** is:*

12) In ΔABC

$$\boldsymbol{\frac{m_a m_b m_c}{h_a h_b h_c} + 2 \geq \frac{9(a^2+b^2+c^2)}{(a+b+c)^2}.}$$

Proof.

*See the solution from **Remark 2.***

□

13) In ΔABC

$$\boldsymbol{\lambda \cdot \frac{m_a m_b m_c}{h_a h_b h_c} + 2 - \lambda \geq \frac{\sqrt{3}(a+b+c)}{h_a+h_b+h_c}, \text{ where } \lambda \geq \frac{8}{5}.}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the ***Lemma*** *and* $a+b+c=2p, h_a+h_b+h_c=\dfrac{p^2+r^2+4Rr}{2R}$ *it suffices to prove that:*

$$\lambda\cdot\frac{R}{2r}+2-\lambda\geq\frac{\sqrt{3}\cdot 2p\cdot 2R}{p^2+r^2+4Rr}\Leftrightarrow\frac{\lambda R+(4-2\lambda)r}{2r}\geq\frac{4R\cdot p\sqrt{3}}{2p^2}\Leftrightarrow$$
$$\Leftrightarrow p^2[\lambda R-(2\lambda+1)r]+3r^2(4R+r)\geq 0.$$

We distinguish the cases:

Case 1). If $\lambda R-(2\lambda+1)r\geq 0$ *the inequality is obvious.*

Case 2). If $\lambda R-(2\lambda+1)r<0$ *we rewrite the inequality* $p^2[(2\lambda+1)r-\lambda R]\leq 3r^2(4R+r)$, *which follows from Gerretsen's inequality:* $p^2\leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$(4R^2+4Rr+3r^2)[(2\lambda+1)r-\lambda R]\leq 3r^2(4R+r)\Leftrightarrow$$
$$\Leftrightarrow 4\lambda R^3-(4\lambda+4)R^2r+(8-5\lambda)Rr^2-6\lambda r^3\geq 0\Leftrightarrow(R-2r)[4\lambda R^2+(4\lambda-4)Rr+3\lambda r^2]\geq 0$$

obviously from Euler's inequality $R\geq 2r$ *and the condition* $n\geq\dfrac{1}{3}$.

Equality holds if and only if the triangle is equilateral.

□

Remark 5.

The best inequality having the form of ***11)*** *is:*

14) In $\boldsymbol{\Delta ABC}$

$$\frac{m_am_bm_c}{h_ah_bh_c}+2\geq\frac{9(a^2+b^2+c^2)}{(a+b+c)^2}$$

Proof.

See the solution from ***Remark 2.***

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.

E-mail address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

PROBLEM PP 26120
OCTOGON MATHEMATICAL MAGAZINE
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1) In ΔABC

$$\sum r_a(r_b-r_c)^2 \geq \frac{2p^2(p^2-3r^2-12Rr)}{4R+r}$$

Proposed by Mihály Bencze - Romania

Proof.

We prove the following lemma:

Lemma 1.
2) In ΔABC

$$\sum r_a(r_b-r_c)^2 = 4p^2(R-2r).$$

Proof.

We have

$$\sum r_a(r_b-r_c)^2 = \sum r_a(r_b^2+r_c^2-2r_br_c) = \sum r_a(r_b^2+r_c^2) - 6r_ar_br_c =$$

$$= \sum r_a(r_a^2+r_b^2+r_c^2-r_a^2) - 6r_ar_br_c =$$

$$= \sum r_a \sum r_a^2 - \sum r_a^3 - 6r_ar_br_c = (4R+r)\Big[(4R+r)^2-2p^2\Big] - \Big[(4R+r)^3-12Rrp^2\Big] = 4p^2(R-2r)$$

□

Let's solve the inequality in the statement.

*Using **Lemma 1** the inequality can be written:*

$$4p^2(R-2r) \geq \frac{2p^2(p^2-3r^2-12Rr)}{4R+r} \Leftrightarrow p^2 \leq 8R^2-2Rr-r^2$$

which follows from Gerretsen's inequality: $p^2 \leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$4R^2+4Rr+3r^2 \leq 8R^2-2Rr-r^2 \Leftrightarrow 2R^2-3Rr-2r^2 \geq 0 \Leftrightarrow (R-2r)(2R+r) \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark 1.

The inequality can be developed:

3) In ΔABC

$$\sum r_a(r_b-r_c)^2 \geq \frac{np^2(p^2-3r^2-12Rr)}{4R+r}, \text{ where } n \leq 4.$$

Proof.

If $n \leq 0$ the inequality is immediate because $p^2-3r^2-12Rr \geq 0$

true from Gerretsen's inequality: $p^2 \geq 16Rr-5r^2$ and Euler's inequality $R \geq 2r$.

Next we consider $n > 0$.

*Using **Lemma 1** we write the inequality:*

$$4p^2(R-2r) \geq \frac{np^2(p^2-3r^2-12Rr)}{4R+r} \Leftrightarrow np^2 \leq 16R^2+(12n-28)Rr+(3n-8)r^2$$

which follows from Gerretsen's inequality: $p^2 \leq 4R^2+4Rr+3r^2$ and the condition $n > 0$.

It remains to prove that:

$$n(4R^2+4Rr+3r^2) \leq 16R^2+(12n-28)Rr+(3n-8)r^2 \Leftrightarrow (4-n)R^2+(2n-7)Rr-2r^2 \geq 0.$$

$\Leftrightarrow (R-2r)\Big[(4-n)R+r\Big] \geq 0$ *obviously from Euler's inequality $R \geq 2r$*

and the condition $n \leq 4$.

Equality holds if and only if the triangle is equilateral.

□

Note

*For $n = 2$ we obtain inequality **1)**.*

Remark 2.

*The best inequality having the form of **3)** it's obtained for $n = 4$:*

4) In ΔABC

$$\sum r_a(r_b-r_c)^2 \geq \frac{4p^2(p^2-3r^2-12Rr)}{4R+r} \geq \frac{np^2(p^2-3r^2-12Rr)}{4R+r}$$

Proof.

*We use inequality **3)** for $n = 4$ and* $\dfrac{4p^2(p^2-3r^2-12Rr)}{4R+r} \geq \dfrac{np^2(p^2-3r^2-12Rr)}{4R+r}$,

true from $p^2-3r^2-12Rr \geq 0$ and the condition $n \leq 4$.

Equality holds if and only if the triangle is equilateral.

□

Remark 3.

*Inequality **3)** can also be developed:*

5) In ΔABC

$$\sum r_a(r_b-r_c)^2 \geq \frac{np^2(p^2+(2\lambda-27)r^2-\lambda Rr)}{4R+r}, \text{ where } n \leq 4 \text{ and } \lambda \geq 11.$$

Proposed by Marin Chirciu - Romania

Proof.

If $n \leq 0$ *the inequality is immediate because* $p^2 + (2\lambda - 27)r^2 - \lambda Rr \geq 0$ *true from Gerretsen's inequality:* $p^2 \geq 16Rr - 5r^2$*, Euler's inequality* $R \geq 2r$ *and the condition* $\lambda \geq 11$.

Next we consider $n > 0$.

Using ***Lemma 1*** *we write the inequality:*

$$4p^2(R-2r) \geq \frac{np^2(p^2 + (2\lambda - 27)r^2 - \lambda Rr)}{4R+r} \Leftrightarrow$$

$$\Leftrightarrow 4(R-2r)(4R+r) \geq n(p^2 + (2\lambda - 27)r^2 - \lambda Rr)$$

which follows from Gerretsen's inequality: $p^2 \leq 4R^2 + 4Rr + 3r^2$ *and the condition* $n > 0$.

It remains to prove that:

$$4(R-2r)(4R+r) \geq n(4R^2 + 4Rr + 3r^2 + (2\lambda - 27)r^2 - \lambda Rr) \Leftrightarrow$$

$$\Leftrightarrow (16 - 4n)R^2 + (\lambda n - 4n - 28)Rr + (24 - 2\lambda n - 8)r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)\Big[(16-4n)R + (4+\lambda n - 12n)r\Big] \geq 0$ *obviously from Euler's inequality* $R \geq 2r$ *and the conditions* $n \leq 4, \lambda \geq 11$.

Equality holds if and only if the triangle is equilateral.

□

Note

For $n = 2$ *and* $\lambda = 12$ *we obtain inequality* ***1)****, and for* $\lambda = 16$ *we obtain inequality* ***5).***

Remark 4.

The best inequality having the form of ***5)*** *we obtain for* $n = 4$ *and* $\lambda = 11$*:*

6) In ΔABC

$$\sum r_a(r_b - r_c)^2 \geq \frac{4p^2(p^2 - 5r^2 - 11Rr)}{4R+r} \geq \frac{np^2(p^2 + (2\lambda - 27)r^2 - \lambda Rr)}{4R+r}$$

where $n \leq 4$ and $\lambda \geq 11$.

Proof.

We use inequality ***5)*** *for* $n = 4$ *and* $\lambda = 11$ *and*

$\dfrac{4p^2(p^2 - 5r^2 - 11Rr)}{4R+r} \geq \dfrac{np^2(p^2 + (2\lambda - 27)r^2 - \lambda Rr)}{4R+r}$ *is true from the condition* $n \leq 4$ *and* $p^2 - 5r^2 - 11Rr \geq p^2 + (2\lambda - 27)r^2 - \lambda Rr \Leftrightarrow (\lambda - 11)(R - 2r) \geq 0$, *and the condition* $\lambda \geq 11$.

Equality holds if and only if the triangle is equilateral.

□

Remark 5.

In the same way we can propose:

7) In ΔABC

$$\sum a(b-c)^2 \geq nS(R-2r), \text{ where } n \leq 4.$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 2
8) In ΔABC

$$\sum a(b-c)^2 = 2p(p^2+r^2-14Rr).$$

Proof.

We have

$$\sum a(b-c)^2 = \sum a(b^2+c^2-2bc) = \sum a(b^2+c^2)-6abc = \sum a(a^2+b^2+c^2-a^2)-6abc =$$
$$= \sum a \sum a^2 - \sum a^3 - 6abc = 2p\cdot 2(p^2-r^2-4Rr) - 2p(p^2-3r^2-6Rr) - 6\cdot 4Rrp =$$
$$= 2p(p^2+r^2-14Rr).$$

□

Let's solve the proposed inequality.

*Using **Lemma 2** we write the inequality:*

$2p(p^2+r^2-14Rr) \geq nrp(R-2r)$, *which follows from Gerretsen's inequality:* $p^2 \geq 16Rr-5r^2$

It remains to prove that:

$4r(R-2r) \geq nr(R-2r) \Leftrightarrow (4-n)(R-2r) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$ *and the condition* $n \leq 4$.

Equality holds if and only if the triangle is equilateral.

□

Remark 6.

*The best inequality having the form of **7)** it's obtained for* $n = 4$:

9) In ΔABC

$$\sum a(b-c)^2 \geq 4S(R-2r) \geq nS(R-2r), \text{ where } n \leq 4.$$

Proof.

*See inequality **7)** for* $n = 4$, *and* $4S(R-2r) \geq nS(R-2r) \Leftrightarrow (4-n)(R-2r) \geq 0$,

obviously from $n \leq 4$ *and* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

10) In ΔABC

$$\sum h_a(h_b-h_c)^2 \geq \frac{nS^2(R-2r)}{R^2}, \text{ where } n \leq 2.$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the followin lemma:

Lemma 3.
11) In ΔABC

$$\sum h_a(h_b-h_c)^2=\frac{rp^2(p^2+r^2-14Rr)}{R^2}$$

Proof.

We have:

$$\sum h_a(h_b-h_c)^2=\sum h_a(h_a^2+h_c^2-2h_bh_c)=\sum h_a(h_b^2+h_c^2)-6h_ah_bh_c=$$
$$=\sum h_a(h_a^2+h_b^2+h_c^2-h_a^2)-6abc=$$
$$=\sum h_a\sum h_a^2-\sum h_a^3-6h_ah_bh_c=\frac{rp^2(p^2+r^2-14Rr)}{R^2}, \text{ the last equality follows from:}$$
$$\sum h_a=\frac{p^2+r^2+4Rr}{2R}, \sum h_a^2=\left(\sum h_a\right)^2-2\sum h_bh_c, \sum h_bh_c=\frac{2rp^2}{R}$$
$$\sum h_a^3=\left(\sum h_a\right)^3-3\prod(h_b+h_c), \prod(h_b+h_c)=\frac{rp^2(p^2+r^2+4Rr)}{R^2}$$

□

Let's solve the proposed inequality.

*Using **Lemma 3** we write the inequality:*

$$\frac{rp^2(p^2+r^2-14Rr)}{R^2}\geq\frac{nr^2p^2(R-2r)}{R^2}\Leftrightarrow p^2+r^2-14Rr\geq nr(R-2r)$$

which follows from Gerretsen's inequality: $p^2\geq 16Rr-5r^2$*. It remains to prove that:*
$16Rr-5r^2+r^2-14Rr\geq nr(R-2r)\Leftrightarrow 2r(R-2r)\geq nr(R-2r)\Leftrightarrow(2-n)(R-2r)\geq 0,$
obviously from Euler's inequality $R\geq 2r$ *and the condition* $n\leq 2$.

Equality holds if and only if the triangle is equilateral.

□

Remark 7.

*The best inequality having the form of **10)** it's obtained for* $n=2$:

12. In ΔABC

$$\sum h_a(h_b-h_c)^2\geq\frac{2S^2(R-2r)}{R^2}\geq\frac{nS^2(R-2r)}{R^2}, \text{ where } n\leq 4.$$

Proof.

*See inequality **10)** for* $n=2$, *and* $\frac{2S^2(R-2r)}{R^2}\geq\frac{nS^2(R-2r)}{R^2}\Leftrightarrow(2-n)(R-2r)\geq 0,$

obviously from $n\leq 2$ *and* $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.
E-mail address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

PROBLEMS PP 26038, PP 26039 OCTOGON MATHEMATICAL MAGAZINE ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

Problem PP 26038 Octogon Mathematical Magazine
1) In ΔABC

$$\sum (m_a + m_b)(m_b + m_c) \le \frac{1}{2}(11p^2 - 9r^2 - 36Rr).$$

Proposed by Mihály Bencze - Romania

Proof.

Using $\sum m_a^2 = \frac{3}{4}\sum a^2, 4m_b m_c \le 2a^2 + bc, \sum a^2 = 2(p^2 - r^2 - 4Rr)$ *and*

$\sum bc = p^2 + r^2 + 4Rr$ *we obtain:*

$$\sum (m_a+m_b)(m_b+m_c) = \sum m_a^2 + 3\sum m_b m_c \le \frac{3}{4}\sum a^2 + \frac{3}{4}\sum (2a^2+bc) = \frac{3}{4}(3\sum a^2 + \sum bc) =$$

$$= \frac{3}{4}\Big[6(p^2 - r^2 - 4Rr) + p^2 + r^2 + 4Rr\Big] = \frac{3}{4}(7p^2 - 5r^2 - 20Rr).$$

The inequality we have to prove can be written:

$$\frac{3}{4}(7p^2 - 5r^2 - 20Rr) \le \frac{1}{2}(11p^2 - 9r^2 - 36Rr) \Leftrightarrow p^2 \ge 3r(4R + r)$$

which follows from Gerretsen's inequality: $p^2 \ge 16Rr - 5r^2$

It remains to prove that: $16Rr - 5r^2 \ge 3r(4R + r) \Leftrightarrow R \ge 2r$,

obviously from Euler's inequality.

Equality holds if and only if the triangle is equilateral.

□

Remark 1.

*Inequality **1)** can be developed:*

2) In ΔABC

$$\sum (m_a + \lambda m_b)(m_b + \lambda m_c) \le \frac{\lambda + 1}{4}\Big[(5\lambda + 6)p^2 - (\lambda + 2) \cdot 3r(4R + r)\Big]$$

where $\lambda \in \mathbb{R}$.

Proposed by Marin Chirciu - Romania

Proof.

$$\sum(m_a+\lambda m_b)(m_b+\lambda m_c)=\lambda\sum m_a^2+(\lambda^2+\lambda+1)\sum m_bm_c\leq$$
$$\leq\lambda\cdot\frac{3}{4}\sum a^2+(\lambda^2+\lambda+1)\cdot\frac{1}{4}\sum(2a^2+bc)=$$
$$=\frac{1}{4}\Big[(2\lambda^2+5\lambda+2)\sum a^2+(\lambda^2+\lambda+1)\sum bc\Big]=$$
$$=\frac{1}{4}\Big[(2\lambda^2+5\lambda+2)\cdot 2(p^2-r^2-4Rr)+(\lambda^2+\lambda+1)(p^2+r^2+4Rr)\Big]=$$
$$=\frac{1}{4}\Big[(5\lambda^2+11\lambda+5)p^2-(\lambda^2+3\lambda+1)\cdot 3r(4R+r)\Big].$$

The inequality we have to prove can be written:

$$\frac{1}{4}\Big[(5\lambda^2+11\lambda+5)p^2-(\lambda^2+3\lambda+1)\cdot 3r(4R+r)\Big]\leq$$
$$\leq\frac{\lambda+1}{4}\Big[(5\lambda+6)p^2-(\lambda+2)\cdot 3r(4R+r)\Big]\Leftrightarrow$$

$\Leftrightarrow p^2\geq 3r(4R+r)$, *which follows from* $p^2\geq 16Rr-5r^2$ *(Gerretsen) and* $R\geq 2r$ *(Euler).*

Equality holds if and only if the triangle is equilateral.

□

Note

For $\lambda=1$ *we obtain inequality* ***1)****.*

Problem PP 26039 Octogon Mathematical Magazine

3) In ΔABC

$$\sum\frac{1}{(m_a+m_b)^2}\geq\frac{18}{11p^2-9r^2-36Rr}.$$

Proposed by Mihály Bencze - Romania

Proof.

We use the inequality $x^2+y^2+z^2\geq xy+yz+zx$, *for* $x=\frac{1}{m_a+m_b}, y=\frac{1}{m_b+m_c}$, $z=\frac{1}{m_c+m_a}$ *and inequality* ***1)****. We obtain:*

$$\sum\frac{1}{(m_a+m_b)^2}\geq\sum\frac{1}{(m_a+m_b)(m_b+m_c)}\geq\frac{9}{\sum(m_a+m_b)(m_b+m_c)}\geq$$
$$\geq\frac{9}{\frac{1}{2}(11p^2-9r^2-36Rr)}=\frac{18}{11p^2-9r^2-36Rr}.$$

Equality holds if and only if the triangle is equilateral.

□

Remark 2.

Inequality ***3)*** *can be developed:*

4) In ΔABC

$$\sum\frac{1}{(m_a+\lambda m_b)^2}\geq\frac{36}{(\lambda+1)[(5\lambda+6)p^2-(\lambda+2)\cdot 3r(4R+r)]},\ \text{where } \lambda\geq 0.$$

Proposed by Marin Chirciu - Romania

Proof.

We use the inequality $x^2 + y^2 + z^2 \geq xy + yz + zx$,

for $x = \dfrac{1}{m_a + \lambda m_b}, y = \dfrac{1}{m_b + \lambda m_c}, z = \dfrac{1}{m_c + \lambda m_a}$ *and inequality* ***2)***. *We obtain:*

$$\sum \frac{1}{(m_a + \lambda m_b)^2} \geq \sum \frac{1}{(m_a + \lambda m_b)(m_b + \lambda m_c)} \geq \frac{9}{\sum (m_a + \lambda m_b)(m_b + \lambda m_c)} \geq$$

$$\geq \frac{9}{\frac{\lambda+1}{4}[(5\lambda+6)p^2 - (\lambda+2)\cdot 3r(4R+r)]} = \frac{36}{(\lambda+1)[(5\lambda+6)p^2 - (\lambda+2)\cdot 3r(4R+r)]}.$$

Equality holds if and only if the triangle is equilateral.

□

Note

For $\lambda = 1$ *we obtain inequality* ***3)***.

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.

E-mail address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

PROBLEM TRIANGLE INEQUALITY - 457
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1) In ΔABC

$$\boldsymbol{r_a^3 + r_b^3 + r_c^3 + 24rp^2 \le \left(\frac{9R}{2}\right)^3}$$

Proposed by Daniel Sitaru - Romania

Proof.

Using the known identity in triangle $r_a^3 + r_b^3 + r_c^3 = (4R+r)^3 - 12Rp^2$

the desired inequality can be written: $(4R+r)^3 - 12Rp^2 + 24rp^2 \le \left(\frac{9R}{2}\right)^3 \Leftrightarrow$

$$\Leftrightarrow (4R+r)^3 \le 12p^2(R-2r) + \left(\frac{9R}{2}\right)^3$$

which follows from Gerretsen's inequality: $p^2 \ge 16Rr - 5r^2$ *and the observation that* $R - 2r \ge 0$

It remains to prove that:

$(4R+r)^3 \le 12(16Rr-5r^2)(R-2r)+\left(\frac{9R}{2}\right)^3 \Leftrightarrow 217R^3+1152R^2r-3648Rr^2+952r^3 \ge 0 \Leftrightarrow$

$(R-2r)(217R^2 + 1586Rr - 476r^2) \ge 0$, *obviously from Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

The inequality can be developed:

2) In ΔABC

$$\boldsymbol{r_a^3 + r_b^3 + r_c^3 + nrp^2 \le (n+3)\left(\frac{3R}{2}\right)^3}, \textbf{ where } \boldsymbol{16 \le n \le 24}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the known identity in triangle: $r_a^3 + r_b^3 + r_c^3 = (4R+r)^3 - 12Rp^2$

the requested inequality can be written: $(4R+r)^3 - 12Rp^2 + nrp^2 \le (n+3)\left(\frac{3R}{2}\right)^3 \Leftrightarrow$

$$\Leftrightarrow (4R+r)^3 \le p^2(12R - nr) + (n+3)\left(\frac{3R}{2}\right)^3,$$

which follows from Gerretsen's inequality: $p^2 \ge 16Rr - 5r^2$ *and the observation that* $12R - nr \ge 0$, *true for* $n \le 24$.

It remains to prove that:

$$(4R+r)^3 \le (16Rr - 5r^2)(12R - nr) + (n+3)\left(\frac{3R}{2}\right)^3 \Leftrightarrow$$

$$\Leftrightarrow (27n-431)R^3+1152R^2r-(128n+576)Rr^2+(40n-8)r^3 \geq 0 \Leftrightarrow$$
$$\Leftrightarrow (R-2r)[(27n-431)R^2+(54n+290)Rr+(4-20n)r^2] \geq 0$$

obviously from Euler's inequality $R \geq 2r$ and the condition $27n-431 \geq 0$ checked by $n \geq 6$.

Equality holds if and only if the triangle is equilateral.

□

Note.

*For $n = 24$ we obtain inequality **1)**.*

Remark.

*Taking into account that $r_a r_b r_c = rp^2$ inequality **2)** can be reformulated:*

3) In ΔABC

$$r_a^3+r_b^3+r_c^3+nr_ar_br_c \leq (n+3)\left(\frac{3R}{2}\right)^3, \text{ where } 16 \leq n \leq 24.$$

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.

E-mail address: dansitaru63@yahoo.com

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Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

TRIANGLE INEQUALITY - 532
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1) In ΔABC

$$\frac{\cos^2\frac{A}{2}}{r_a^2}+\frac{\cos^2\frac{B}{2}}{r_b^2}+\frac{\cos^2\frac{C}{2}}{r_c^2}\geq\frac{1}{2Rr}$$

Proposed by Adil Abdullayev - Baku - Azerbaidian

Proof.

We prove the following lemma:

Lemma 1.
2) In ΔABC

$$\frac{\cos^2\frac{A}{2}}{r_a^2}+\frac{\cos^2\frac{B}{2}}{r_b^2}+\frac{\cos^2\frac{C}{2}}{r_c^2}=\frac{1}{r^2}-\frac{1}{2Rr}\left(\frac{4R+r}{p}\right)^2.$$

Proof.

Using the following formulas $\cos^2\frac{A}{2}=\frac{p(p-a)}{bc}$ *and* $r_a=\frac{S}{p-a}$ *we obtain:*

$$\sum\frac{\cos^2\frac{A}{2}}{r_a^2}=\sum\frac{\frac{p(p-a)}{bc}}{\frac{S^2}{(p-a)^2}}=\frac{p}{S^2}\sum\frac{(p-a)^3}{bc}=\frac{p}{r^2p^2}\cdot\frac{\sum a(p-a)^3}{abc}=$$

$$=\frac{1}{r^2p}\cdot\frac{4Rrp^2-2r^2(4R+r)^2}{4Rrp}=\frac{1}{r^2}-\frac{1}{2Rr}\left(\frac{4R+r}{p}\right)^2.$$

*Let's prove inequality **1).***

*Using **Lemma 1** inequality **1)** becomes:*

$\frac{1}{r^2}-\frac{1}{2Rr}\left(\frac{4R+r}{p}\right)^2\geq\frac{1}{2Rr}\Leftrightarrow p^2(2R-r)\geq r(4R+r)^2$, *which is true from*

Gerretsen's inequality $p^2\geq 16Rr-5r^2$. *It remains to prove that*

$(16Rr-5r^2)(2R-r)\geq r(4R+r)^2\Leftrightarrow 8R^2-17Rr+2r^2\geq 0\Leftrightarrow(R-2r)(8R-r)\geq 0$

obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

□

Remark.

Let's find an inequality having an opposite sense:

3) In ΔABC

$$\frac{\cos^2\frac{A}{2}}{r_a^2}+\frac{\cos^2\frac{B}{2}}{r_b^2}+\frac{\cos^2\frac{C}{2}}{r_c^2}\le\left(\frac{1}{R}-\frac{1}{r}\right)^2$$

Proposed by Marin Chirciu - Romania

Proof.

*Using **Lemma 1** inequality **3)** can be written:*

$$\frac{1}{r^2}-\frac{1}{2Rr}\left(\frac{4R+r}{p}\right)^2\le\left(\frac{1}{R}-\frac{1}{r}\right)^2\Leftrightarrow p^2\le\frac{R(4R+r)^2}{2(2R-r)}$$

(Blundon - Gerretsen's inequality)

Equality holds if and only if the triangle is equilateral.

□

Remark.

The double inequality can be written:

4. In ΔABC

$$\frac{1}{2Rr}\le\frac{\cos^2\frac{A}{2}}{r_a^2}+\frac{\cos^2\frac{B}{2}}{r_b^2}+\frac{\cos^2\frac{C}{2}}{r_c^2}\le\left(\frac{1}{R}-\frac{1}{r}\right)^2.$$

Proof.

*See inequalities **1)** and **3)**.*

□

Remark.

In the same way we can propose:

5) In ΔABC

$$\frac{1}{R^2p}\le\frac{\sin^2\frac{A}{2}}{r_a^2}+\frac{\sin^2\frac{B}{2}}{r_b^2}+\frac{\sin^2\frac{C}{2}}{r_c^2}\le\frac{1}{4r^2p}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 2.
6) In ΔABC

$$\frac{\sin^2\frac{A}{2}}{r_a^2}+\frac{\sin^2\frac{B}{2}}{r_b^2}+\frac{\sin^2\frac{C}{2}}{r_c^2}=\frac{1}{2Rrp}$$

Proof.

Using the following formulas $\sin^2\dfrac{A}{2}=\dfrac{(p-b)(p-c)}{bc}$ *and* $r_a=\dfrac{S}{p-a}$ *we obtain:*

$$\sum\frac{\sin^2\frac{A}{2}}{r_a^2}=\sum\frac{\frac{(p-b)(p-c)}{bc}}{\frac{S^2}{(p-a)^2}}=\frac{\prod(p-a)}{S^2}\sum\frac{1}{bc}=\frac{r^2p}{r^2p^2}\cdot\frac{\sum a}{abc}=\frac{1}{p}\cdot\frac{2p}{4Rrp}=\frac{1}{2Rrp}.$$

□

Let's prove the double inequality ***5)****.*

Using ***Lemma 2*** *double inequality* ***5)*** *can be written:*

$$\frac{1}{R^2p}\le\frac{1}{2Rrp}\le\frac{1}{4r^2p}\Leftrightarrow 4r^2\le 2Rr\le R^2\Leftrightarrow 2r\le R \text{ (Euler's inequality).}$$

Equality holds if and only if the triangle is equilateral.

□

7) In ΔABC

$$\frac{4}{9R^2}\le\frac{\tan^2\frac{A}{2}}{r_a^2}+\frac{\tan^2\frac{B}{2}}{r_b^2}+\frac{\tan^2\frac{C}{2}}{r_c^2}\le\frac{1}{9r^2}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 3.
8) In ΔABC

$$\frac{\tan^2\frac{A}{2}}{r_a^2}+\frac{\tan^2\frac{B}{2}}{r_b^2}+\frac{\tan^2\frac{C}{2}}{r_c^2}=\frac{3}{p^2}$$

Proof.

Using the following formulas $\tan^2\dfrac{A}{2}=\dfrac{(p-b)(p-c)}{p(p-a)}$ *and* $r_a=\dfrac{S}{p-a}$ *we obtain:*

$$\sum\frac{\tan^2\frac{A}{2}}{r_a^2}=\sum\frac{\frac{(p-b)(p-c)}{p(p-a)}}{\frac{S^2}{(p-a)^2}}=\frac{\prod(p-a)}{S^2p}\sum 1=\frac{r^2p}{r^2p^3}\cdot 3=\frac{3}{p^2}.$$

□

Let's prove the double inequality ***7)****.*

Using Lemma 3 the double inequality ***7)*** *can be written:*

$$\frac{4}{9R^2}\le\frac{3}{p^2}\le\frac{1}{9r^2}\Leftrightarrow 27r^2\le p^2\le\frac{27R^2}{4} \text{ (Mitrinović's inequality).}$$

Equality holds if and only if the triangle is equilateral.

□

9) In ΔABC

$$\frac{1}{r^2} \le \frac{\cot^2\frac{A}{2}}{r_a^2} + \frac{\cot^2\frac{B}{2}}{r_b^2} + \frac{\cot^2\frac{C}{2}}{r_c^2} \le \frac{4R^2 - 10Rr + 5r^2}{r^4}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following Lemma:

10) In ΔABC

$$\frac{\cot^2\frac{A}{2}}{r_a^2} + \frac{\cot^2\frac{B}{2}}{r_b^2} + \frac{\cot^2\frac{C}{2}}{r_c^2} = \frac{p^4 - 16Rrp^2 + 2r^2(4R+r)^2}{r^4p^2}$$

Proof.

Using the following formulas $\cot^2\frac{A}{2} = \frac{p(p-a)}{(p-b)(p-c)}$ *and* $r_a = \frac{S}{p-a}$ *we obtain:*

$$\sum \frac{\cot^2\frac{A}{2}}{r_a^2} = \sum \frac{\frac{p(p-a)}{(p-b)(p-c)}}{\frac{S^2}{(p-a)^2}} = \frac{p}{S^2}\sum \frac{(p-a)^3}{(p-b)(p-c)} = \frac{p}{r^2p^2}\cdot\frac{\sum(p-a)^4}{(p-a)(p-b)(p-c)} =$$

$$\frac{1}{r^2p}\cdot\frac{\sum(p-a)^4}{\prod(p-a)} = \frac{1}{r^2p}\cdot\frac{p^4 - 16Rrp^2 + 2r^2(4R+r)^2}{r^2p} = \frac{p^4 - 16Rrp^2 + 2r^2(4R+r)^2}{r^4p^2}.$$

□

Let's prove the double inequality ***9***.

Using **Lemma 3** *the double inequality* ***9)*** *can be written:*

$$\frac{1}{r^2} \le \frac{p^4 - 16Rrp^2 + 2r^2(4R+r)^2}{r^4p^2} \le \frac{4R^2 - 10Rr + 5r^2}{r^4}.$$

The first inequality can be transformed equivalently:

$$\frac{1}{r^2} \le \frac{p^4 - 16rp^2 + 2r^2(4R+r)^2}{r^4p^2} \Leftrightarrow p^4 - 16Rrp^2 + 2r^2(4R+r)^2 \ge r^2p^2 \Leftrightarrow$$

$$\Leftrightarrow p^2(p^2 - 16Rr - r^2) + 2r^2(4R+r)^2 \ge 0.$$

We distinguish the following cases:

Case 1). If $p^2 - 16Rr - r^2 \ge 0$*, the inequality is equivalent.*

Case 2). If $p^2 - 16Rr - r^2 < 0$*, the inequality can be rewritten:*

$p^2(16Rr + r^2 - p^2) \le 2r^2(4R+r)^2$*, which follows from Gerretsen's inequality:*

$16Rr - 5r^2 \le p^2 \le 4R^2 + 4Rr + 3r^2$*. It remains to prove that:*

$$(4R^2+4Rr+3r^2)(16Rr+r^2-16Rr+5r^2) \le 2r^2(4R+r)^2 \Leftrightarrow R^2 - Rr - 2r^2 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(R+r) \ge 0$*, obviously from Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

Let's prove the second inequality.

We have $\frac{p^4 - 16Rrp^2 + 2r^2(4R+r)^2}{r^4p^2} = \frac{1}{r^4}\left[p^2 - 16Rr + \frac{2r^2(4R+r)^2}{p^2}\right] \le$

$$\le \frac{1}{r^4}\left[4R^2+4Rr+3r^2-16Rr+\frac{2r^2(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right] = \frac{1}{r^4}[4R^2-12Rr+3r^2+2r(R+r)] =$$

$= \frac{4R^2 - 10Rr + 5r^2}{r^4}$, *where, above were used inequalities* $p^2 \leq 4R^2+4Rr+4r^2$ *and*

$p^2 \geq \frac{r(4R+r)^2}{R+r}$, *true from Gerretsen's inequality.*

Equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.

E-mail address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

TRIANGLE INEQUALITY - 524
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1) In ΔABC

$$\frac{bc}{r_br_c}+\frac{ca}{r_cr_a}+\frac{ab}{r_ar_b}\geq 5-\frac{2r}{R}$$

Proposed by Adil Abdullayev - Baku - Azerbaidian

Proof.

We prove the following lemma:

Lemma 1.
2) In ΔABC

$$\frac{bc}{r_br_c}+\frac{ca}{r_cr_a}+\frac{ab}{r_ar_b}=1+\left(\frac{4R+r}{p}\right)^2.$$

Proof.

Using the formula $r_a=\dfrac{S}{p-a}$ *we obtain:*

$$\sum\frac{bc}{r_br_c}=\sum\frac{bc}{\frac{S}{p-b}\cdot\frac{S}{p-c}}=\frac{1}{S^2}\sum bc(p-b)(p-c)=\frac{1}{r^2p^2}\cdot r^2[p^2+(4R+r)^2]=1+\left(\frac{4R+r}{p}\right)^2$$

□

*Let's prove inequality **1)**.*

*Using **Lemma 1** inequality **1)** can be written:*

$1+\left(\dfrac{4R+r}{p}\right)^2\geq 5-\dfrac{2r}{R}\Leftrightarrow p^2\leq\dfrac{R(4R+r)^2}{2(2R-r)}$, *which is Blundon-Gerretsen's inequality.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having on opposite sense:

3) In ΔABC

$$\frac{bc}{r_br_c}+\frac{ca}{r_cr_a}+\frac{ab}{r_ar_b}\leq 2+\frac{R}{r}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using Lemma 1 inequality **3)** can be written:*

$$1+\left(\frac{4R+r}{p}\right)^2 \le 2+\frac{R}{r} \Rightarrow p^2 \ge \frac{r(4R+r)^2}{R+r}$$

which follows from Gerretsen's inequality $p^2 \ge 16Rr - 5r^2$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

The double inequality can be written:

4) In ΔABC

$$\mathbf{5-\frac{2r}{R} \le \frac{bc}{r_br_c}+\frac{ca}{r_cr_a}+\frac{ab}{r_ar_b} \le 2+\frac{R}{r}.}$$

Proof.

*See inequalities **1)** and 3).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

In the same way we can propose:

5) In ΔABC

$$\mathbf{4 \le \frac{bc}{h_bh_c}+\frac{ca}{h_ch_a}+\frac{ab}{h_ah_b} \le 4\left(\frac{R}{r}\right)^2-\frac{3}{4}\cdot\frac{R}{r}+\frac{3}{2}}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 2.
6) In ΔABC

$$\mathbf{\frac{bc}{h_bh_c}+\frac{ca}{h_ch_a}+\frac{ab}{h_ah_b}=\frac{p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2}{4r^2p^2}}$$

Proof.

Using the formula $h_a=\dfrac{2S}{a}$ *we obtain:*

$$\sum\frac{bc}{h_bh_c}=\sum\frac{bc}{\frac{2S}{b}\cdot\frac{2S}{c}}=\frac{1}{4S^2}\sum b^2c^2=\frac{1}{4r^2p^2}[p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2]=$$

$$=\frac{p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2}{4r^2p^2}.$$

□

*Let's prove the double inequality **5)**.*

*Using **Lemma 2** the left inequality from **5)** can be written:*

$$4 \le \frac{p^4 + p^2(2r^2 - 8Rr) + r^2(4R+r)^2}{4r^2p^2} \Leftrightarrow p^4 + p^2(2r^2 - 8Rr) + r^2(4R+r)^2 \ge 16r^2p^2 \Leftrightarrow$$

$$p^4 - p^2(14r^2 + 8Rr) + r^2(4R+r)^2 \ge 0 \Leftrightarrow p^2(p^2 - 14r^2 - 8Rr) + r^2(4R+r)^2 \ge 0.$$

We distinguish the following cases:

Case 1). If $p^2 - 14r^2 - 8Rr \ge 0$, the inequality is obvious.

Case 2). If $p^2 - 14r^2 - 8Rr < 0$, inequality can be rewritten:

$p^2(8Rr + 14r^2 - p^2) \le r^2(4R+r)^2$, which follows from Gerretsen's inequality $16Rr - 5r^2 \le p^2 \le 4R^2 + 4Rr + 3r^2$. It remains to prove that:

$$(4R^2 + 4Rr + 3r^2)(8Rr + 14r^2 - 16Rr + 5r^2) \le r^2(4R+r)^2 \Leftrightarrow$$

$$\Leftrightarrow (4R^2 + 4Rr + 3r^2)(19r - 8R) \le r(4R+r)^2 \Leftrightarrow 8R^3 - 7R^2r - 11Rr^2 - 14r^3 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R - 2r)(8R^2 + 9Rr + 7r^2) \ge 0$, obviously from Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

*Let's prove the right inequality from **5)**:*

We have

$$\frac{bc}{h_bh_c} + \frac{ca}{h_ch_a} + \frac{ab}{h_ah_b} = \frac{p^4 + p^2(2r^2 - 8Rr) + r^2(4R+r)^2}{4r^2p^2} =$$

$$= \frac{1}{4r^2}\left[p^2 + 2r^2 - 8Rr + \frac{r^2(4R+r)^2}{p^2}\right] \le \frac{1}{4r^2}\left[4R^2 + 4Rr + 3r^2 + 2r^2 - 8Rr + \frac{r^2(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right] =$$

$$= \frac{1}{4r^2}(4R^2 - 4Rr + 5r^2 + r(R+r)) = \frac{4R^2 - 3Rr + 6r^2}{4r^2} = 4\left(\frac{R}{r}\right)^2 - \frac{3}{4}\cdot\frac{R}{r} + \frac{3}{2}.$$

In the above inequality we've used $p^2 \le 4R^2 + 4Rr + 3r^2$ and $p^2 \ge \dfrac{r(4R+r)^2}{R+r}$ which follows from Gerretsen's inequality.

Equality holds if and only if the triangle is equilateral.

□

7) In ΔABC

$$2 + \left(\frac{r}{R}\right)^2 \le \frac{h_bh_c}{bc} + \frac{h_ch_a}{ca} + \frac{h_ah_b}{ab} \le \frac{3r}{R}\left(2 - \frac{r}{R}\right).$$

Proposed by Marin Chirciu - Romania

Proof.

Let's prove the following lemma:

Lemma 3.
8) In ΔABC

$$\frac{h_bh_c}{bc} + \frac{h_ch_a}{ca} + \frac{h_ah_b}{ab} = \frac{p^2 - r^2 - 4Rr}{2R^2}.$$

Proof.

Using the formula $h_a = \dfrac{2S}{a}$ *we obtain:*

$$\sum \frac{h_b h_c}{bc} = \sum \frac{\frac{2S}{b} \cdot \frac{2S}{c}}{bc} = 4S^2 \sum \frac{1}{b^2c^2} = 4r^2p^2 \cdot \frac{\sum a^2}{a^2b^2c^2} =$$
$$= 4r^2p^2 \cdot \frac{2(p^2 - r^2 - 4Rr)}{16R^2r^2p^2} = \frac{p^2 - r^2 - 4Rr}{2R^2}$$

□

Let's prove the double inequality ***7)****.*
Using Lemma 3 the double inequality ***7)*** *can be written:*

$$2+\left(\frac{r}{R}\right)^2 \leq \frac{p^2 - r^2 - 4Rr}{2R^2} \leq \frac{3r}{R}\left(2 - \frac{r}{R}\right)$$

which follows from Gerretsen's inequality $16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$.
Equality holds if and only if the triangle is equilateral.

□

9) In ΔABC

$$\boldsymbol{\frac{9r}{2R} \leq \frac{r_b r_c}{bc} + \frac{r_c r_a}{ca} + \frac{r_a r_b}{ab} \leq \frac{9}{4}}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 4.
10) In ΔABC

$$\boldsymbol{\frac{r_b r_c}{bc} + \frac{r_c r_a}{ca} + \frac{r_a r_b}{ab} = 2 + \frac{r}{2R}}$$

Proof.

Using the formula $r_a = \dfrac{S}{p-a}$ *we obtain:*

$$\sum \frac{r_b r_c}{bc} = \sum \frac{\frac{S}{p-b} \cdot \frac{S}{p-c}}{bc} = S^2 \sum \frac{1}{bc(p-b)(p-c)} = r^2p^2 \cdot \frac{4R+r}{2Rr^2p^2} = \frac{4R+r}{2R}.$$

□

Let's prove the double inequality ***9)****.*
Using ***Lemma 4*** *the double inequality* ***9)*** *can be written:*

$$\frac{9r}{2R} \leq 2 + \frac{r}{2R} \leq \frac{9}{4} \Leftrightarrow 2r \leq R \text{ (Euler's inequality).}$$

Equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.
E-mail address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

TRIANGLE INEQUALITY - 548
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1) In ΔABC

$$a^3b^3 + b^3c^3 + c^3a^3 \geq 648R^3r^3.$$

Proposed by Seyram Ibrahimov - Maasilli - Azerbaidian

Proof.

We prove the following lemma:

Lemma 1.
2) In ΔABC

$$a^3b^3 + b^3c^3 + c^3a^3 = p^6 + p^4(3r^2 - 12Rr) + 3p^2r^4 + r^3(4R + r)^3.$$

Proof.

Using the identity $\sum b^2c^2 \sum bc = \sum b^3c^3 + abc\left(\sum a \sum bc - abc\right)$

and the known relationships in triangle: $\sum a = 2p, \sum bc = p^2 + r^2 + 4Rr,$

$\sum b^2c^2 = p^4 + p^2(2r^2 - 8Rr) + r^2(4R + r)^2$ *and* $abc = 4Rrp$ *we obtain*

$$\sum b^3c^3 = p^6 + p^4(3r^2 - 12Rr) + 3p^2r^4 + r^3(4R + r)^3.$$

□

Lemma 2.
3) In ΔABC

$$a^3b^3 + b^3c^3 + c^3a^3 \geq 16r^3(68R^3 - 69R^2r + 30Rr^2 - 4r^3).$$

Proof.

*Using **Lemma 1** we have*

$$p^6+p^4(3r^2-12Rr)+3p^2r^4+r^3(4R+r)^3 = p^4(p^2+3r^2-12Rr)+3p^2r^4+r^3(4R+r)^3 \geq$$

$$\geq (16Rr - 5r^2)^2(16Rr - 5r^2 + 3r^2 - 12Rr) + 3(16Rr - 5r^2)r^4 + r^3(4R + r)^3 =$$

$$= r^3[(16R-5r)^2(4R-2r)+3r^2(16R-5r)+(4R+r)^3] = 16r^3(68R^3-69R^2r+30Rr^2-4r^3).$$

□

*Let's pass to solving inequality **1)**.*

*Using **Lemma 2** it suffices to prove that:*

$16r^3(68R^3-69R^2r+30Rr^2-4r^3) \geq 648R^3r^3 \Leftrightarrow 55R^3-138R^2r+60Rr^2-8r^3 \geq 0 \Leftrightarrow$

$\Leftrightarrow (R-2r)(55R^2-28Rr+4r^2) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.

E-mail address: dansitaru63@yahoo.com

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Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

TRIANGLE INEQUALITY - 528
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1) In ΔABC

$$\frac{\cot A}{p-a}+\frac{\cot B}{p-b}+\frac{\cot C}{p-c}\le\frac{1}{r}-\frac{R-2r}{2Rr}$$

Proposed by Adil Abdullayev - Baku - Azerbaidian

Proof.

Let's prove the following lemma:

Lemma 1.
2) In ΔABC

$$\frac{\cot A}{p-a}+\frac{\cot B}{p-b}+\frac{\cot C}{p-c}=\frac{5p^2-(4R+r)^2}{2rp^2}.$$

Proof.

We have:

$$\sum\frac{\cot A}{p-a}=\sum\frac{\frac{\cos A}{\sin A}}{p-a}=\sum\frac{\frac{b^2+c^2-a^2}{2bc}\cdot\frac{2R}{a}}{p-a}=\frac{R}{abc}\sum\frac{b^2+c^2-a^2}{p-a}=$$
$$=\frac{R}{4Rrp}\cdot\frac{10p^2-2(4R+r)^2}{p}=\frac{5p^2-(4R+r)^2}{2rp^2}.$$

□

*Let's pass to solving inequality **1)**.*

*Using **Lemma 1** the inequality can be written* $\frac{5p^2-(4R+r)^2}{2rp^2}\le\frac{1}{r}-\frac{R-2r}{2Rr}\Leftrightarrow$

$\Leftrightarrow p^2\le\frac{R(4R+r)^2}{2(2R-r)}$, *which is Blundon's-Gerretsen's inequality.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having an opposite sense:

3) In ΔABC

$$\frac{\cot A}{p-a}+\frac{\cot B}{p-b}+\frac{\cot C}{p-c}\ge\frac{4r-R}{2r^2}.$$

Proof.

Using ***Lemma 1*** *the inequality can be written:*

$$\frac{5p^2-(4R+r)^2}{2rp^2} \geq \frac{4r-R}{2r^2} \Leftrightarrow p^2 \geq \frac{r(4R+r)^2}{R+r}$$

which follows from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

The double inequality can be written:

4) In ΔABC

$$\frac{4r-R}{2r^2} \leq \frac{\cot A}{p-a} + \frac{\cot B}{p-b} + \frac{\cot C}{p-c} \leq \frac{R+2r}{2Rr}.$$

Proof.

See inequalities ***1)*** *and* ***3)***.

Equality holds if and only if the triangle is equilateral.

□

Remark.

In the same way we can propose:

5) In ΔABC

$$\frac{1}{p}\left(15 - \frac{5r}{R} - \frac{4R}{r}\right) \leq \frac{\cos A}{p-a} + \frac{\cos B}{p-b} + \frac{\cos C}{p-c} \leq \frac{1}{p}\left(1 + \frac{r}{R}\right).$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 2.
6) In ΔABC

$$\frac{\cos A}{p-a} + \frac{\cos B}{p-b} + \frac{\cos C}{p-c} = \frac{p^2 - Rr - 4R^2}{Rrp}$$

Proof.

We have $\sum \frac{\cos A}{p-a} = \sum \frac{\frac{b^2+c^2-a^2}{2bc}}{p-a} = \sum \frac{b^2+c^2-a^2}{2(p-a)bc} = \frac{p^2-Rr-4R^2}{Rrp}$

□

*Let's pass to solve the double inequality **5).***

*Using **Lemma 2** the double inequality **5)** can be written*

$$\frac{1}{p}\left(15-\frac{5r}{R}-\frac{4R}{r}\right)\le\frac{p^2-Rr-4r^2}{Rrp}\le\frac{1}{p}\left(1+\frac{r}{R}\right),$$

which follows from Gerretsen's inequality $16Rr-5r^2\le p^2\le 4R^2+4Rr+3r^2$.

Equality holds if and only if the triangle is equilateral.

□

7) In ΔABC

$$\frac{5}{2r}-\frac{1}{R}\le\frac{\csc A}{p-a}+\frac{\csc B}{p-b}+\frac{\csc C}{p-c}\le\frac{1}{2r}\left(2+\frac{R}{r}\right).$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 3.
8) **In ΔABC**

$$\frac{\csc A}{p-a}+\frac{\csc B}{p-b}+\frac{\csc C}{p-c}=\frac{1}{2r}\left[1+\left(\frac{4R+r}{p}\right)^2\right].$$

Proof.

We have:

$$\sum\frac{\csc A}{p-a}=\sum\frac{\frac{1}{\sin A}}{p-a}=\sum\frac{\frac{2R}{a}}{p-a}=2R\sum\frac{1}{a(p-a)}=2R\cdot\frac{p^2+(4R+r)^2}{4Rrp^2}=$$
$$=\frac{1}{2r}\left[1+\left(\frac{4R+r}{p}\right)^2\right]$$

□

*Let's pass to solve the double inequality **7)**.*

*Using **Lemma 3** the double inequality **7)** can be written*

$$\frac{5}{2r}-\frac{1}{R}\le\frac{1}{2r}\left[1+\left(\frac{4R+r}{p}\right)^2\right]\le\frac{1}{2r}\left(2+\frac{R}{r}\right)$$

which follows from Blundon's Gerretsen's inequality $\dfrac{r(4R+r)^2}{R+r}\le p^2\le\dfrac{R(4R+r)^2}{2(2R-r)}$.

Equality holds if and only if the triangle is equilateral.

□

9) In ΔABC

$$\frac{12}{p}\le\frac{\csc^2 A}{p-a}+\frac{\csc^2 B}{p-b}+\frac{\csc^2 C}{p-c}\le\frac{1}{p}\left(\frac{2R^2}{r^2}+\frac{5R}{4r}+\frac{3}{2}\right).$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 4

10) In ΔABC

$$\frac{\csc^2 A}{p-a}+\frac{\csc^2 B}{p-b}+\frac{\csc^2 C}{p-c}=\frac{p^4+p^2(2r^2-4Rr)+r(4R+r)^3}{4r^2p^3}.$$

Proof.

We have

$$\sum\frac{\csc^2 A}{p-a}=\sum\frac{\frac{1}{\sin^2 A}}{p-a}=\sum\frac{\frac{4R^2}{a^2}}{p-a}=4R^2\sum\frac{1}{a^2(p-a)}=$$

$$=4R^2\cdot\frac{p^4+p^2(2r^2-4Rr)+r(4R+r)^3}{16R^2r^2p^3}=\frac{p^4+p^2(2r^2-4Rr)+r(4R+r)^3}{4r^2p^3}.$$

□

Let's pass to solve the double inequality ***9)****.*

Using ***Lemma 4*** *the double inequality* ***7)*** *can be written*

$$\frac{12}{p}\le\frac{p^4+p^2(2r^2-4Rr)+r(4R+r)^3}{4r^2p^3}\le\frac{1}{p}\left(\frac{2R^2}{r^2}+\frac{5R}{4r}+\frac{3}{2}\right).$$

The left inequality is equivalent with:

$$p^4+p^2(2r^2-4Rr)+r(4R+r)^3\ge 48r^2p^2\Leftrightarrow p^2(p^2-46r^2-4Rr)+r(4R+r)^3\ge 0.$$

We distinguish the following cases:

Case 1). If $p^2-46r^2-4Rr\ge 0$, the inequality becomes obviously.

Case 2). If $p^2-46r^2-4Rr<0$, the inequality can be rewritten:

$p^2(46r^2+4Rr-p^2)\le r(4R+r)^3$ *it follows from Blundon-Gerretsen's inequality*

$16Rr-5r^2\le p^2\le\dfrac{R(4R+r)^2}{2(2R-r)}$. *It remains to prove that:*

$$\frac{R(4R+r)^2}{2(2R-r)}\cdot(46r^2+4Rr-16Rr+5r^2)\le r(4R+r)^3\Leftrightarrow 28R^2-55Rr-2r^2\ge 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(28R+r)\ge 0$, *obviously from Euler's inequality $R\ge 2r$.*

Equality holds if and only if the triangle is equilateral.

Let's solve the inequality from the right:

We have $\dfrac{p^4+p^2(2r^2-4Rr)+r(4R+r)^3}{4r^2p^3}=\dfrac{1}{4r^2p}\left[p^2+2r^2-4Rr+\dfrac{r(4R+r)^3}{p^2}\right]\le$

$$\le\frac{1}{4r^2p}\left[4R^2+4Rr+3r^2+2r^2-4Rr+\frac{r(4R+r)^3}{\frac{r(4R+r)}{R+r}}\right]=\frac{1}{4r^2p}[4R^2+5r^2+(4R+r)(R+r)]=$$

$$=\frac{8R^2+5Rr+6r^2}{4r^2p}=\frac{1}{p}\left(\frac{2R^2}{r^2}+\frac{5R}{4r}+\frac{3}{2}\right).$$

In the above inequality we've used $p^2\le 4R^2+4Rr+3r^2$ and $\dfrac{r(4R+r)^2}{R+r}\le p^2$

it follows from Gerretsen's inequality $16Rr-5r^2\le p^2$.

Equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.
E-mail address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

TRIANGLE INEQUALITY - 558
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1) In ΔABC

$$\frac{y+z}{x}\cdot a^2+\frac{z+x}{y}\cdot b^2+\frac{x+y}{z}\cdot c^2\geq 8\sqrt{3}\cdot S$$

where, $x,y,z>0$.

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

Proof.

Using the means inequality we obtain

$$\frac{y+z}{x}\cdot a^2+\frac{z+x}{y}\cdot b^2+\frac{x+y}{z}\cdot c^2=\left(\frac{y}{x}a^2+\frac{x}{y}b^2\right)+\left(\frac{z}{y}b^2+\frac{y}{z}c^2\right)+\left(\frac{x}{z}c^2+\frac{z}{x}a^2\right)\geq$$

$$\geq 2\sqrt{\frac{y}{x}a^2\cdot\frac{x}{y}b^2}+2\sqrt{\frac{z}{y}b^2\cdot\frac{y}{z}c^2}+2\sqrt{\frac{x}{z}c^2\cdot\frac{z}{x}a^2}=2(ab+bc+ca)\geq 8\sqrt{3}\cdot S$$

where the last inequality is true from $ab+bc+ca\geq 4\sqrt{3}S\Leftrightarrow p^2+r^2+4Rr\geq 4\sqrt{3}rp$ *which follows from Gerretsen's inequality* $p^2\geq 16Rr-5r^2$ *and Doucet's inequality* $4R+r\geq p\sqrt{3}$. *It remains to prove that:*

$16Rr-5r^2+r^2+4Rr\geq 4r(4R+r)\Leftrightarrow R\geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral and $x=y=z$.

□

Remark.

The inequality can be developed:

2) In ΔABC

$$\frac{y+z}{x}\cdot a^4+\frac{z+x}{y}\cdot b^4+\frac{x+y}{z}\cdot c^4\geq 32S^2.$$

Proof.

Using the means inequality we obtain:

$$\frac{y+z}{x}\cdot a^4+\frac{z+x}{y}\cdot b^4+\frac{x+y}{z}\cdot c^4=\left(\frac{y}{x}a^4+\frac{x}{y}b^4\right)+\left(\frac{z}{y}b^4+\frac{y}{z}c^4\right)+\left(\frac{x}{z}c^4+\frac{z}{x}a^4\right)\geq$$

$$\geq 2\sqrt{\frac{y}{x}a^4\cdot\frac{x}{y}b^4}+2\sqrt{\frac{z}{y}b^4\cdot\frac{y}{z}c^4}+2\sqrt{\frac{x}{z}c^4\cdot\frac{z}{x}a^4}=2(a^2b^2+b^2c^2+c^2a^2)\geq 2\cdot 16S^2=32S^2$$

where the last inequality is true from $a^2b^2+b^2c^2+c^2a^2\geq 16S^2$ *(F. Goldner's inequality, 1949)*

Proof.

We use the formulas $a^2b^2+b^2c^2+c^2a^2=p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2$ *and* $S^2=r^2p^2$. *We write the inequality:*

$$p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2\ge 16r^2p^2 \Leftrightarrow p^2(p^2-14r^2-8Rr)+r^2(4R+r)^2\ge 0$$

We distinguish the cases:

Case 1). If $p^2-14r^2-8Rr\ge 0$, *the inequality is obvious.*

Case 2). If $p^2-14r^2-8Rr<0$, *the inequality can be rewritten* $p^2(8Rr+14r^2-p^2)\le r^2(4R+r)^2$ *which follows from Gerretsen's inequality* $16Rr-5r^2\le p^2\le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$(4R^2+4Rr+3r^2)(8Rr+14r^2-16Rr+5r^2)\le r^2(4R+r)^2 \Leftrightarrow$$
$$\Leftrightarrow (4R^2+4Rr+3r^2)(19r-8R)\le r(4R+r)^2 \Leftrightarrow 8R^3-7R^2r-11Rr^2-14r^3\ge 0 \Leftrightarrow$$
$$\Leftrightarrow (R-2r)(8R^2+9Rr+7r^2)\ge 0,$$ *obviously from Euler's inequality* $R\ge 2r$.

Equality, for Goldner's inequality holds if and only if the triangle is equilateral.

Equality in ***2)*** *holds if and only if the triangle is equilateral and* $x=y=z$.

□

□

Remark.

The inequality can be generalized:

3) In ΔABC

$$\frac{y+z}{x}\cdot a^{2n}+\frac{z+x}{y}\cdot b^{2n}+\frac{x+y}{z}\cdot c^{2n}\ge 6\left(\frac{4S}{\sqrt{3}}\right)^n$$

where $n\in\mathbb{N}$.

Proposed by Marin Chirciu - Romania

Proof.

Using means inequality we obtain

$$\frac{y+z}{x}\cdot a^{2n}+\frac{z+x}{y}\cdot b^{2n}+\frac{x+y}{z}\cdot c^{2n}=\left(\frac{y}{x}a^{2n}+\frac{x}{y}b^{2n}\right)+\left(\frac{z}{y}b^{2n}+\frac{y}{z}c^{2n}\right)+\left(\frac{x}{z}c^{2n}+\frac{z}{x}a^{2n}\right)\ge$$
$$\ge 2\sqrt{\frac{y}{x}a^{2n}\cdot\frac{x}{y}b^{2n}}+2\sqrt{\frac{z}{y}b^{2n}\cdot\frac{y}{z}c^{2n}}+2\sqrt{\frac{x}{z}c^{2n}\cdot\frac{z}{x}a^{2n}}=$$
$$=2(a^nb^n+b^nc^n+c^na^n)\ge 2\cdot\frac{(ab+bc+ca)^n}{3^{n-1}}\ge 2\cdot\frac{(4\sqrt{3}S)^n}{3^{n-1}}=6\left(\frac{4S}{\sqrt{3}}\right)^n$$

where the penultimate inequality follows from Hölder's inequality,

$$\frac{X^n}{A}+\frac{Y^n}{B}+\frac{Z^n}{C}\ge\frac{(X+Y+Z)^n}{3(A+B+C)}, X,Y,Z,A,B,C>0, n\in\mathbb{N}, n\ge 2$$

and the last inequality is true from $ab+bc+ca\ge 4\sqrt{3}S$

see the solution from inequality ***1)*** *from above.*

Equality holds if and only if the triangle is equilateral and $x=y=z$, *for* $n\ge 1$.

For $n=0$ *we obtain the known inequality* $\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}\ge 6$.

For $n=1$ *we obtain inequality* ***1)***.

For $n=2$ *we obtain inequality* ***2)***.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.

E-mail address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
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SOLUTION
PROBLEM JP104 WINTER 2017
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_c}+\frac{r_c^2}{h_a m_b}\geq\frac{54r^2}{p^2-r^2-4Rr}$$

Proposed by D.M. Bătineţu-Giurgiu - Romania, Martin Lukarevski - Skopje

Proof.

We prove the following lemma:

Lemma 1.
2) ***In*** ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\geq\frac{4(4R+r)^2}{5p^2-3r(4R+r)}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the fact that $h_a\leq m_a$ and Bergström inequality we obtain:

$$\sum\frac{r_a^2}{h_b m_c}\geq\sum\frac{r_a^2}{m_b m_c}\geq\frac{(\sum r_a)^2}{\sum m_b m_c}\geq\frac{(4R+r)^2}{\frac{1}{4}\sum(2a^2+bc)}=\frac{4(4R+r)^2}{2\sum a^2+\sum bc}=$$
$$=\frac{4(4R+r)^2}{2\cdot 2(p^2-r^2-4Rr)+p^2+r^2+4Rr}=\frac{4(4R+r)^2}{5p^2-3r(4R+r)}$$

Equality holds if and only if the triangle is equilateral.

□

Let's pass to solving the inequality from the enunciation.

Using ***Lemma 1*** *it's enough to prove that* $\frac{4(4R+r)^2}{5p^2-3r(4R+r)}\geq\frac{54r^2}{p^2-r^2-4Rr}$.

This inequality can be transformed equivalently:

$$2(4R+r)^2(p^2-r^2-4Rr)\geq 27r^2(5p^2-3r^2-12Rr)\Leftrightarrow$$
$$\Leftrightarrow p^2(32R^2+16Rr-133r^2)\geq 2r(4R+r)^3-81r^3(4R+r)$$

which follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$
and from the observation that $32R^2+16Rr-133r^2>0$ (see Euler's inequality $R\geq 2r$).

It remains to prove that:

$$(16Rr-5r^2)(32R^2+16Rr-133r^2)\geq 2r(4R+r)^3-81r^3(4R+r)\Leftrightarrow$$
$$\Leftrightarrow 32R^3-159Rr^2+62r^3\geq 0\Leftrightarrow (R-2r)(32R^2+64Rr-31r^2)\geq 0$$

obviously from Euler's inequality.

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **1)** can be rewritten:*

1) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\geq\frac{108r^2}{a^2+b^2+c^2}$$

Proof.

*Using **Lemma 1** and the identity $ab+bc+ca=p^2+r^2+4Rr$ it suffices to prove that*

$$\frac{4(4R+r)^2}{5p^2-3r(4R+r)}\geq\frac{108r^2}{p^2+r^2+4Rr}$$

This inequality transformed equivalently:

$$(4R+r)^2(p^2+r^2+4Rr)\geq 27r^2(5p^2-3r^2-12Rr)\Leftrightarrow$$

$$\Leftrightarrow p^2(16R^2+8Rr+r^2-135r^2)+r(4R+r)^3+81r^3(4R+r)\geq 0\Leftrightarrow$$

$$\Leftrightarrow p^2(8R^2+4Rr-67r^2)+32R^3r+24R^2r^2+168Rr^3+41r^4\geq 0$$

We distinguish the following cases:

Case 1). If $8R^2+4Rr-67r^2\geq 0$, the inequality is obvious.

Case 2). If $8R^2+4Rr-67r^2<0$, the inequality can be rewritten:

$$32R^3r+24R^2r^2+168Rr^3+41r^4\geq p^2(67r^2-4Rr-8r^2)$$

which follows from Gerretsen's inequality $p^2\leq 4R^2+4Rr+3r^2$. It remains to prove that:

$$32R^3r+24R^2r^2+168Rr^3+41r^4\geq(4R^2+4Rr+3r^2)(67r^2-4Rr-8R^2)\Leftrightarrow$$

$$\Leftrightarrow 8R^4+20R^3r-51R^2r^2-22Rr^3-40r^4\geq 0\Leftrightarrow(R-2r)(8R^3+36R^2r+21Rr^2+20r^3)\geq 0,$$

obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

5. In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\geq\frac{108r^2}{ab+bc+ca}\geq\frac{108r^2}{a^2+b^2+c^2}.$$

Proof.

*We use inequality **4)** and inequality $a^2+b^2+c^2\geq ab+bc+ca$.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **4)** can also be strengthened:*

6) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\geq\frac{9r\sqrt{3}}{p}$$

Proof.

*Using **Lemma 1** it suffices to prove that* $\frac{4(4R+r)^2}{5p^2-3r(4R+r)}\geq\frac{9r\sqrt{3}}{p}$.

This inequality can be transformed equivalently:

$4p(4R+r)^2\geq 9r\sqrt{3}(5p^2-3r^2-12Rr)$, *which follows from Mitrinović's inequality* $p\geq 3r\sqrt{3}$.

It suffices to prove that

$4\cdot 3r\sqrt{3}(4R+r)^2\geq 9r\sqrt{3}(5p^2-3r^2-12Rr)\Leftrightarrow 4(4R+r)^2\geq 15p^2-9r(4R+r)\Leftrightarrow$

$\Leftrightarrow 4(4R+r)^2+9r(4R+r)\geq 15p^2$, *true from Gerretsen's inequality*

$p^2\leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$4(4R+r)^2+9r(4R+r)\geq 15(4R^2+4Rr+3r^2)\Leftrightarrow R^2+2Rr-8r^2\geq 0\Leftrightarrow (R-2r)(R+4r)\geq 0$

obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **6)** is stronger than inequality **4)**.*

7) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\geq\frac{9r\sqrt{3}}{p}\geq\frac{108r^2}{ab+bc+ca}$$

Proof.

*We use inequality **6)** and the known inequality in triangle* $ab+bc+ca\geq 4\sqrt{3}S$

□

Remark.

*Inequality **6)** can also be strengthened:*

8) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\geq\frac{2p\sqrt{3}}{3R}$$

Proof.

Using ***Lemma 1*** *it suffices to prove that* $\frac{4(4R+r)^2}{5p^2-3r(4R+r)} \geq \frac{2p\sqrt{3}}{3R}$.

This inequality can be transformed equivalently:

$6R(4R+r)^2 \geq p\sqrt{3}(5p^2-3r^2-12Rr)$, *which follows from Doucet's inequality* $4R+r \geq p\sqrt{3}$. *It remains to prove that*

$6R(4R+r)^2 \geq (4R+r)(5p^2-3r^2-12Rr) \Leftrightarrow 6R(4R+r) \geq 5p^2-3r^2-12Rr$

true from Gerretsen's inequality $p^2 \leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$6R(4R+r) \geq 5(4R^2+4Rr+3r^2)-3r^2-12Rr \Leftrightarrow 2R^2-Rr-6r^2 \geq 0 \Leftrightarrow (R-r)(2R+3r) \geq 0$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality ***8)*** *is stronger than inequality* ***6)***.

9) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b} \geq \frac{2p\sqrt{3}}{3R} \geq \frac{9r\sqrt{3}}{p}.$$

Remark.

We use inequality ***8)*** *and the known inequality in triangle* $2p^2 \geq 27Rr$

(true from Gerretsen's inequality $p^2 \geq 16Rr-5r^2$ *and Euler's inequality* $R \geq 2r$*).*

Remark.

We can write the following inequalities:

10) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b} \geq \frac{4(4R+r)^2}{5p^2-3r(4R+r)} \geq \frac{9r\sqrt{3}}{p} \geq \frac{108r^2}{p^2+r^2+4Rr} \geq \frac{54r^2}{p^2-r^2-4Rr}$$

Proof.

We use ***Lemma 1*** *and the above inequalities.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having an apposite sense:

11) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b} \leq \left(\frac{R}{r}\right)^2-\frac{3}{4}\cdot\frac{R}{r}+\frac{1}{2}.$$

Proof.

Let's prove the following lemma:

Lemma 2.
12) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\le\frac{p^2(r-8R)+(4R+r)^3}{4rp^2}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the fact that $h_a \le m_a$ we obtain:

$$\sum\frac{r_a^2}{h_b m_c}\le\sum\frac{r_a^2}{h_b h_c}=\sum\frac{\frac{S^2}{(p-a)^2}}{\frac{2S}{b}\cdot\frac{2S}{c}}=\frac{1}{4}\sum\frac{bc}{(p-a)^2}=\frac{1}{4}\det\frac{p^2(r-8R)+(4R+r)^3}{rp^2}=$$
$$=\frac{p^2(r-8R)+(4R+r)^3}{4rp^2}$$

The equality holds if and only if the triangle is equilateral.

□

*Let's pass to solving inequality **11)**.*

*Using **Lemma 2** it suffices to prove that* $\frac{p^2(r-8R)+(4R+r)^3}{4rp^2}\le\left(\frac{R}{r}\right)^2-\frac{3}{4}\cdot\frac{R}{r}+\frac{1}{2}$

This inequality can be transformed equivalently:

$$p^2(r-8R)+(4R+r)^3\le p^2(4R^2-3Rr+2r^2)\Leftrightarrow p^2(4R^2+5Rr+r^2)\ge r(4R+r)^3$$

which follows from inequality $p^2\ge\frac{r(4R+r)^2}{R+r}$

(true from Gerretsen's inequality $p^2\ge 16Rr-5r^2$ and Euler's inequality $R\ge 2r$).

The equality holds if and only if the triangle is equilateral.

□

Remark.

The double inequality can be written:

1) In ΔABC

$$\frac{4(4R+r)^2}{5p^2-3r(4R+r)}\le\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\le\frac{p^2(r-8R)+(4R+r)^3}{4rp^2}$$

Proof.

*See **Lemma 1 and Lemma 2***

The equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.
E-mail address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

PROBLEM 584 - INEQUALITY IN TRIANGLE ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1) In ΔABC

$$\frac{m_a^4}{h_b h_c} + \frac{m_b^4}{h_c h_a} + \frac{m_c^4}{h_a h_b} \geq \frac{3}{4}(p^2 + r^2 + 4Rr).$$

Proposed by Boris Colakovic - Belgrade - Serbia

Proof.

Using Bergström inequality we obtain:

$$\sum \frac{m_a^4}{h_b h_c} \geq \frac{(\sum m_a^2)^2}{\sum h_b h_c} = \frac{(\frac{3}{4}\sum a^2)^2}{\frac{2rp^2}{R}} = \frac{\frac{9}{16}(\sum a^2)^2}{\frac{2rp^2}{R}} \geq \frac{9R(\sum bc)^2}{32rp^2} =$$

$$= \frac{9R(p^2 + r^2 + 4Rr)^2}{32rp^2} \geq \frac{3}{4}(p^2 + r^2 + 4Rr)$$

where the last inequality is equivalent with:

$$3R(p^2 + r^2 + 4Rr) \geq 8rp^2 \Leftrightarrow p^2(3R - 8r) + 3Rr(4R + r) \geq 0.$$

We distinguish the cases:

Case 1). If $3R - 8r \geq 0$*, the inequality is obvious.*

Case 2). If $3R - 8r < 0$*, the inequality can be rewritten* $3Rr(4R+r) \geq p^2(8r - 3R)$ *which is true from Gerretsen's inequality* $p^2 \geq 16Rr - 5r^2$*. It remains to prove that:*

$$3Rr(4R + r) \geq (16Rr - 5r^2)(8r - 3R) \Leftrightarrow 3R^2 - 2R^2 r - 5Rr^2 - 6r^3 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R - 2r)(3R^2 + 4Rr + 3r^2) \geq 0 \text{ obviously from Euler's inequality } R \geq 2r.$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **1)** can be written:*

2) In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{\frac{m_a^4}{h_b h_c} + \frac{m_b^4}{h_c h_a} + \frac{m_c^4}{h_a h_b} \geq \frac{3}{4}(ab + bc + ca).}$$

Proof.

We use the identity $ab + bc + ca = p^2 + r^2 + 4Rr$.

□

Remark.

Inequality 2) can be strengthened:

3) In ΔABC

$$\frac{m_a^4}{h_b h_c}+\frac{m_b^4}{h_c h_a}+\frac{m_c^4}{h_a h_b}\geq\frac{3}{4}(a^2+b^2+c^2)$$

Proof.

Using Bergström's inequality, we obtain:

$$\sum\frac{m_a^4}{h_b h_c}\geq\frac{(\sum m_a^2)^2}{\sum h_b h_c}=\frac{(\frac{3}{4}\sum a^2)^2}{\frac{2rp^2}{R}}=\frac{\frac{9}{16}(\sum a^2)^2}{\frac{2rp^2}{R}}\geq\frac{9R(\sum a^2)^2}{32rp^2}\geq\frac{3}{4\sum a^2}$$

where the last inequality is equivalent with:

$$3R\sum a^2\geq 8rp^2\Leftrightarrow 3R\cdot 2(p^2-r^2-4Rr)\geq 8rp^2\Leftrightarrow p^2(3R-4r)\geq 3Rr(4Rr+r)$$

which is true from Gerretsen's inequality $p^2\geq 16Rr-5r^2$. It remains to prove that:

$$(16Rr-5r^2)(3R-4r)\geq 3Rr(4R+r)\Leftrightarrow 18R^2-41Rr+10r^2\geq 0\Leftrightarrow(R-2r)(18R-5r)\geq 0$$

obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality 3) is stronger than inequality 2):

4) In ΔABC

$$\frac{m_a^4}{h_b h_c}+\frac{m_b^4}{h_c h_a}+\frac{m_c^4}{h_a h_b}\geq\frac{3}{4}(a^2+b^2+c^2)\geq\frac{3}{4}(ab+bc+ca).$$

Proof.

See inequality 3) and $a^2+b^2+c^2\geq ab+bc+ca$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality 3) can be also strengthened:

5) In ΔABC

$$\frac{m_a^4}{h_b h_c}+\frac{m_b^4}{h_c h_a}+\frac{m_c^4}{h_a h_b}\geq\frac{9}{4}\cdot\frac{a^3+b^3+c^3}{a+b+c}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemmas:

Lemma 1.

6) In ΔABC

$$\frac{m_a^4}{h_b h_c}+\frac{m_b^4}{h_c h_a}+\frac{m_c^4}{h_a h_b}=\frac{2p^6-p^4(23Rr+2r^2)+p^2(10R^2r^2-19Rr^3-2r^4)+2r^3(4R+r)^3}{8r^2p^2}$$

Proof.

$$\sum \frac{m_a^4}{h_b h_c} = \sum \frac{(m_a^2)^2}{\frac{2S}{b}\cdot\frac{2S}{c}} = \frac{1}{4S^2}\sum bc\left(\frac{2b^2+2c^2-a^2}{4}\right)^2 = \frac{1}{64S^2}\sum bc(E-3a^2)^2 =$$

$$= \frac{2p^6-p^4(23Rr+2r^2)+p^2(10R^2r^2-19Rr^3-2r^4)+2r^3(4R+r)^3}{8r^2p^2}, \text{ where } E = 2\sum a^2.$$

□

Lemma 2.
7) In ΔABC

$$\frac{m_a^4}{h_b h_c} + \frac{m_b^4}{h_c h_a} + \frac{m_c^4}{h_a h_b} \geq \frac{77R^3-112R^2r+25Rr^2-2r^3}{4R}.$$

Proof.

*Using **Lemma 1** we obtain:*

$$\frac{m_a^4}{h_b h_c} + \frac{m_b^4}{h_c h_a} + \frac{m_c^4}{h_a h_b} = \frac{2p^6-p^4(23Rr+2r^2)+p^2(10R^2r^2-19Rr^3-2r^4)+2r^3(4Rr+r)^3}{8r^2p^2} =$$

$$= \frac{1}{8r^2}\left[2p^4-p^2(23Rr+2r^2)+10R^2r^2-19Rr^3-2r^4+\frac{2r^3(4R+r)^3}{p^2}\right] =$$

$$= \frac{1}{8r^2}\left[p^2(2p^2-23Rr-2r^2)+10R^2r^2-19Rr^3-2r^4+\frac{2r^3(4R+r)^3}{p}\right] \geq$$

$$\geq \frac{1}{8r^2}\left[(16Rr-5r^2)\Big(2(16Rr-5r^2)-23Rr-2r^2\Big)+10R^2r^2-19Rr^3-2r^4+\frac{2r^3(4R+r)^3}{\frac{R(4R+r)^2}{2(2R-r)}}\right] =$$

$$= \frac{77R^3-112R^2r+25Rr^2-2r^3}{4R}, \text{ where the last inequality follows from}$$

Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$ *and Blundon's inequality* $p^2 \leq \dfrac{R(4R+r)^2}{2(2R-r)}$.

□

*Let's pass to solving inequality **5)**.*

*Using **Lemma 2** and the identities* $a^3+b^3+c^3 = 2p(p^2-3r^2-6Rr)$ *and* $a+b+c = 2p$

It suffices to prove that $\dfrac{77R^3-112R^2r+25Rr^2-2r^3}{4R} \geq \dfrac{9}{4}\cdot\dfrac{2p(p^2-3r^2-6Rr)}{2p} \Leftrightarrow$

$$77R^3-112R^2r+25Rr^2-2r^3 \geq 9R(p^2-3r^2-6Rr)$$

which follows from Gerretsen's inequality $p^2 \leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$77R^3-112R^2r+25Rr^2-2r^3 \geq 9R(4R^2+4Rr+3r^2-3r^2-6Rr) \Leftrightarrow$$

$$41R^3-94R^2r+25Rr^2-2r^3 \geq 0 \Leftrightarrow (R-2r)(41R^2-12Rr+r^2) \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **5)** is stronger than inequality **3)**:*

8) In ΔABC

$$\frac{m_a^4}{h_b h_c}+\frac{m_b^4}{h_c h_a}+\frac{m_c^4}{h_a h_b}\geq\frac{9}{4}\cdot\frac{a^3+b^3+c^3}{a+b+c}\geq\frac{3}{4}(a^2+b^2+c^2).$$

Proof.

*See inequality **5)** and*

$$\frac{9}{4}\cdot\frac{a^3+b^3+c^3}{a+b+c}\geq\frac{3}{4}(a^2+b^2+c^2)\Leftrightarrow a^3+b^3+c^3\geq\frac{1}{3}(a+b+c)(a^2+b^2+c^2)$$

true from Chebysev's inequality.

Equality holds if and only if the triangle is equilateral.

□

Remark.

The following inequalities can be written:

9. In ΔABC

$$\frac{m_a^4}{h_b h_c}+\frac{m_b^4}{h_c h_a}+\frac{m_c^4}{h_a h_b}\geq\frac{77R^3-112R^2r+25Rr^2-2r^3}{4R}\geq\frac{9}{4}\cdot\frac{a^3+b^3+c^3}{a+b+c}\geq$$

$$\geq\frac{3}{4}(a^2+b^2+c^2)\geq\frac{3}{4}(ab+bc+ca)$$

Proof.

*See inequalities **7)**, **8)**, and **4)**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having an apposite sense.

10) In ΔABC

$$\frac{m_a^4}{h_b h_c}+\frac{m_b^4}{h_c h_a}+\frac{m_c^4}{h_b h_a}\leq\frac{4R^4-37r^4}{r^2}.$$

Proof.

*Using **Lemma 1** we obtain:*

$$\frac{m_a^4}{h_b h_c}+\frac{m_b^4}{h_c h_a}+\frac{m_c^4}{h_a h_b}=\frac{2p^6-p^4(23Rr+2r^2)+p^2(10R^2r^2-19Rr^3-2r^4)+2r^3(4R+r)^3}{8r^2p^2}=$$

$$=\frac{1}{8r^2}\left[2p^4-p^2(23Rr+2r^2)+10R^2r^2-19Rr^3-2r^4+\frac{2r^3(4R+r)^3}{p^2}\right]=$$

$$=\frac{1}{8r^2}\left[p^2(2p^2-23Rr-2r^2)+10R^2r^2-19Rr^3-2r^4+\frac{2r^3(4R+r)^3}{p^2}\right]\leq$$

$$\leq\frac{1}{8r^2}\left[(4R^2+4Rr+3r^2)\Big(2(4R^2+4Rr+3r^2)-23Rr-2r^2\Big)+10R^2r^2-19Rr^3-2r^4+\frac{2r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$= \frac{16R^4 - 14R^3r - R^2r^2 - 19Rr^3 + 6r^4}{4r^2} \leq \frac{4R^4 - 37r^4}{r^2}$ *where the last inequality follows from Euler's inequality* $R \geq 2r$ *and the penultimate from Gerretsen's inequality*

$$p^2 \leq 4R^2 + 4Rr + 3r^2 \text{ and } p^2 \geq \frac{r(4R+r)^2}{R+r}$$

true from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

The double inequality can be written:

11) In ΔABC

$$\boldsymbol{\frac{21R^3 + 48r^3}{4R} \leq \frac{m_a^4}{h_b h_c} + \frac{m_b^4}{h_c h_a} + \frac{m_c^4}{h_a h_b} \leq \frac{4R^4 - 37r^4}{r^2}}.$$

Proposed by Marin Chirciu - Romania

Proof.

*See inequalities **10), 7)** and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.

E-mail address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

PROBLEM 573
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1) In ΔABC

$$\sum\left(\frac{1}{b^2}+\frac{1}{c^2}\right)\geq\frac{27}{2}\cdot\frac{1}{r_a^2+r_b^2+r_c^2}$$

Proposed by Seyran Ibrahimov - Maasilli - Azerbaidian

Remark.

Inequality can be strengthened:

2) In ΔABC

$$\sum\left(\frac{1}{b^2}+\frac{1}{c^2}\right)\geq\frac{27}{2}\cdot\frac{1}{r_ar_b+r_br_c+r_cr_a}$$

Proof.

We prove the following lemma:

Lemma.
3) In ΔABC

$$\sum\left(\frac{1}{b^2}+\frac{1}{c^2}\right)=\frac{p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2}{8p^2R^2r^2}.$$

Proof.

We have $\sum\left(\frac{1}{b^2}+\frac{1}{c^2}\right)=2\sum\frac{1}{a^2}=\frac{2\sum b^2c^2}{a^2b^2c^2}=\frac{2[p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2}{16p^2R^2r^2}=$

$$=\frac{p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2}{8p^2R^2r^2}.$$

□

*Let's pass to solving inequality **2)**.*

*Using **Lemma** and the known identity in triangle* $r_ar_b+r_br_c+r_cr_a=p^2$

we write the inequality $\frac{p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2}{8p^2R^2r^2}\geq\frac{27}{2p^2}\Leftrightarrow$

$\Leftrightarrow p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2\geq 108R^2r^2\Leftrightarrow$

$\Leftrightarrow p^2(p^2+2r^2-8Rr)+r^2(4R+r)^2\leq 108R^2r^2$, *which follows from Gerretsen's inequality* $p^2\geq 16Rr-5r^2$ *and from the observation that* $p^2+2r^2-8Rr>0$.

It suffices to prove that: $(16Rr-5r^2)(16Rr-5r^2+2r^2-8Rr)+r^2(4R+r)^2\geq 108R^2r^2\Leftrightarrow$

$\Leftrightarrow 9R^2-20Rr+4r^2\geq 0\Leftrightarrow(R-2r)(9R-2r)\geq 0$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **2)** is stronger than inequality **1)**:*

4) In ΔABC

$$\sum\left(\frac{1}{b^2}+\frac{1}{c^2}\right) \geq \frac{27}{2}\cdot\frac{1}{r_ar_b+r_br_c+r_cr_a} \geq \frac{27}{2}\cdot\frac{1}{r_a^2+r_b^2+r_c^2}.$$

Proof.

*See inequality **2)** and $r_a^2+r_b^2+r_c^2 \geq r_ar_b+r_br_c+r_cr_a$.*

Equality holds if and only if the triangle is equilateral.

□

*Inequality **2)** can be strengthened:*

5) In ΔABC

$$\sum\left(\frac{1}{b^2}+\frac{1}{c^2}\right) \geq \frac{8R^2+Rr-2r^2}{8R^3r}$$

Proof.

*Using **Lemma** the inequality can be written*

$$\frac{p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2}{8p^2R^2r^2} \geq \frac{8R^2+Rr-2r^2}{8R^3r} \text{ which follows from}$$

$$\frac{p^4+p^2(2r^2-8Rr)+r^2(4Rr+r)^2}{8p^2R^2r^2} = \frac{1}{8R^2r^2}\left[p^2+2r^2-8Rr+\frac{r^2(4R+r)^2}{p^2}\right] \geq \frac{8R^2+Rr-2r^2}{8R^3r}.$$

where the last inequality follows from Gerretsen's inequality $p^2 \geq 16Rr-5r^2$

and Blundon's inequality $p^2 \leq \dfrac{R(4R+r)^2}{2(2R-r)}$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **5)** is stronger than inequality **2)**:*

6) In ΔABC

$$\sum\left(\frac{1}{b^2}+\frac{1}{c^2}\right) \geq \frac{8R^2+Rr-2r^2}{8R^3r} \geq \frac{27}{2}\cdot\frac{1}{r_ar_b+r_br_c+r_cr_a}$$

Proof.

*See inequality **5)**, identity* $r_ar_b + r_br_c + r_cr_a = p^2$ *and* $\dfrac{8R^2+Rr-2r^2}{8R^3r} \geq \dfrac{27}{2p^2}$ *which follows from Gerretsen's inequality* $p^2 \geq 16Rr - 5r^2$. *It remains to prove that:*

$$74R^3 - 14R^2r - 37Rr^2 + 10r^3 \geq 0 \Leftrightarrow (R-2r)(20R^2+16Rr-5r^2) \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

We can write the following inequalities:

7) In ΔABC

$$\sum\left(\frac{1}{b^2}+\frac{1}{c^2}\right) \geq \frac{8R^2+Rr-2r^2}{8R^3r} \geq \frac{1}{2Rr} \geq \frac{27}{2}\cdot\frac{1}{r_ar_b+r_br_c+r_cr_a} \geq$$

$$\geq \frac{17R-2r}{8R^3} \geq \frac{2}{R^2} \geq \frac{27}{2}\cdot\frac{1}{r_a^2+r_b^2+r_c^2}$$

Proof.

*See inequalities **5)**, Euler's inequality* $2p^2 \geq 27Rr$ *and Gerretsen's inequality* $p^2 \geq 16Rr - 5r^2$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having an opposite sense:

8) In ΔABC

$$\sum\left(\frac{1}{b^2}+\frac{1}{c^2}\right) \leq \frac{4R^2-3Rr+6r^2}{8R^2r^2}.$$

Proof.

*Using **Lemma** the inequality can be be written:*

$\dfrac{p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2}{8p^2R^2r^2} \leq \dfrac{4R^2-3Rr+6r^2}{8R^2r^2}$, *which follows from writing:*

$$\frac{p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2}{8p^2R^2r^2} = \frac{1}{8R^2r^2}\left[p^2+2r^2-8Rr+\frac{r^2(4R+r)^2}{p^2}\right]$$

and the Gerretsen's inequality: $\dfrac{r(4R+r)^2}{R+r} \leq 16Rr-5r^2 \leq p^2 \leq 4R^2+4Rr+3r^2$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

9) In ΔABC

$$\frac{8R^2+Rr-2r^2}{8R^3r} \leq \sum\left(\frac{1}{b^2}+\frac{1}{c^2}\right) \leq \frac{4R^2-3Rr+6r^2}{8R^2r^2}.$$

Proposed by Marin Chirciu - Romania

Proof.

*See inequalities **5)** and **8)**.*

Equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.

E-mail address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

PROBLEM UP.147.
ROMANIAN MATHEMATICAL MAGAZINE
AUTUMN EDITION 2018

MARIN CHIRCIU

1) In ΔABC

$$\frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{r_b^2}{\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{r_c^2}{\tan\frac{A}{2}\tan\frac{B}{2}}\geq\frac{9(a^2+b^2+c^2)}{4}$$

Proposed by Hoang Le Nhat Tung - Hanoi -Vietnam

Proof.
We prove the following lemma:
Lemma:
2) In ΔABC

$$\frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{r_b^2}{\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{r_c^2}{\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{(4R+r)^3-12Rp^2}{r}$$

Proof.

Using $r_a=\dfrac{S}{s-a}$ *and* $\dfrac{B}{2}\tan\dfrac{C}{2}=\dfrac{s-a}{s}$ *we obtain:*

$$\sum\frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}}=\sum\frac{\frac{S^2}{(s-a)^2}}{\frac{s-a}{s}}=S^2s\sum\frac{1}{(s-a)^3}=S^2\cdot s\cdot\frac{(4R+r)^3-12Rs^2}{S^3}=$$
$$=\frac{(4R+r)^3-12Rs^2}{r}$$

□

Back to the main problem:

Using the Lemma and $a^2+b^2+c^2=2(s^2-r^2-4Rr)$ *we write the inequality:*

$$\frac{(4R+r)^3-12Rs^2}{r}\geq\frac{9}{4}\cdot2(s^2-r^2-4Rr)\Leftrightarrow 2(4R+r)^3+9r^2(4R+r)\geq 3s^2(8R+3r)$$

which follows from Gerretsen's inequality: $s^2\leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$2(4R+r)^3+9r^2(4R+r)\geq 3(4R^2+4Rr+3r^2)(8R+3r)\Leftrightarrow 8R^3-9R^2r-12Rr^2-4r^3\geq 0\Leftrightarrow$$
$$\Leftrightarrow (R-2r)(8R^2+7Rr+2r^2)\geq 0$$ *obviously from Euler's inequality* $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality can be strengthened:

3) In ΔABC

$$\frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{r_b^2}{\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{r_c^2}{\tan\frac{A}{2}\tan\frac{B}{2}}\geq\frac{27}{4}\cdot\frac{a^3+b^3+c^3}{a+b+c}$$

Proposed by Marin Chirciu - Romania

Proof.

Using Lemma and $a^3+b^3+c^3=2s(s^2-3r^2-6Rr)$ *the inequality can be written:*

$$\frac{(4R+r)^3-12Rs^2}{r}\geq\frac{27}{4}\cdot\frac{2s(s^2-3r^2-6Rr)}{2s}\Leftrightarrow 4(4R+r)^3+27r^2(6R+3r)\geq$$

$\geq 3s^2(16R+9r)$ *which follows from Gerrentsen's inequality:* $s^2\leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$4(4R+r)^3+27r^2(6R+3r)\geq 3(4R^2+4Rr+3r^2)(16R+9r)\Leftrightarrow$$

$$\Leftrightarrow 32R^3-54R^2r-21Rr^2+2r^3\geq 0\Leftrightarrow(R-2r)(32R^2+10Rr-r^2)\geq 0$$

obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality 3) is stronger then inequality 1):

4) In ΔABC:

$$\frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{r_b^2}{\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{r_c^2}{\tan\frac{A}{2}\tan\frac{B}{2}}\geq\frac{27}{4}\cdot\frac{a^3+b^3+c^3}{a+b+c}\geq\frac{9}{4}(a^2+b^2+c^2).$$

Proof.

We use inequality 3) and:

$$\frac{27}{4}\cdot\frac{a^3+b^3+c^3}{a+b+c}\geq\frac{9}{4}(a^2+b^2+c^2)\Leftrightarrow 3(a^3+b^3+c^3)\geq(a+b+c)(a^2+b^2+c^2)\Leftrightarrow$$

$\Leftrightarrow 2\sum a^3\geq\sum ab(a+b)$, *which follows from* $a^3+b^3\geq ab(a+b)\Leftrightarrow(a+b)(a-b)^2\geq 0$ *and the analogs. The equality holds if and only if the triangle is equilateral.*

□

Remark.

Let's obtain an inequality of opposite sense:

5) In ΔABC:

$$\frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{r_b^2}{\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{r_c^2}{\tan\frac{A}{2}\tan\frac{B}{2}}\leq\frac{81R}{8r}(9r^2-32r^2)$$

Proposed by Marin Chirciu - Romania

Proof.

Using the Lemma we write the inequality:

$\frac{(4R+r)^3-12Rs^2}{r} \leq \frac{81R}{8r}(9R^2-32r^2)$, *which follows from Euler's inequality* $r \leq \frac{R}{2}$ *and Mitrinovič's inequality:* $s^2 \geq 27r^2$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

6) In ΔABC:

$$\frac{27}{2}(5Rr-4r^2) \leq \frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{r_b^2}{\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{r_c^2}{\tan\frac{A}{2}\tan\frac{B}{2}} \leq \frac{81R}{8r}(9R^2-32r^2)$$

Proposed by Marin Chirciu - Romania

Proof.

We use 3), 5) and Gerretsen's inequality $s^2 \geq 16Rr-5r^2$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

In the same way we can propose:

7) In ΔABC:

$$\frac{r_a^2}{\cot\frac{B}{2}\cot\frac{C}{2}}+\frac{r_b^2}{\cot\frac{C}{2}\cot\frac{A}{2}}+\frac{r_c^2}{\cot\frac{A}{2}\cot\frac{B}{2}} = r(4Rr+r)$$

Proof.

Using $r_a=\frac{S}{s-a}$ *and* $\cot\frac{B}{2}\cot\frac{C}{2}=\frac{s}{s-a}$ *we obtain:*

$$\sum\frac{r_a^2}{\cot\frac{B}{2}\cot\frac{C}{2}}=\sum\frac{\frac{S^2}{(s-a)^2}}{\frac{s}{s-a}}=\frac{S^2}{s}\sum\frac{1}{s-a}=\frac{S^2}{s}\cdot\frac{4R+r}{S}=r(4R+r).$$

□

8) In ΔABC:

$$9r^2 \leq \frac{r_a^2}{\cot\frac{B}{2}\cot\frac{C}{2}}+\frac{r_b^2}{\cot\frac{C}{2}\cot\frac{A}{2}}+\frac{r_c^2}{\cot\frac{A}{2}\cot\frac{B}{2}} \leq \frac{9Rr}{2}.$$

Proposed by Marin Chirciu - Romania

Proof.

Using the inequality $\sum\frac{r_a^2}{\cot\frac{B}{2}\cot\frac{C}{2}}=r(4R+r)$ *and Euler's inequality* $R \geq 2$.

Equality holds if and only if the triangle is equilateral.

□

9) In ΔABC:

$$\frac{r_a^2}{\tan^2\frac{A}{2}}+\frac{r_b^2}{\tan^2\frac{B}{2}}+\frac{r_c^2}{\tan^2\frac{C}{2}}=3s^2$$

Proof.

$$\text{Using } r_a=\frac{S}{s-a} \text{ and } \tan^2\frac{A}{2}=\frac{(s-b)(s-c)}{s(s-a)} \text{ we obtain:}$$

$$\sum\frac{r_a^2}{\tan^2\frac{A}{2}}=\sum\frac{\frac{S^2}{(s-a)^2}}{\frac{(s-b)(s-c)}{s(s-a)}}=S^2s\sum\frac{1}{(s-a)(s-b)(s-c)}=r^2s^3\cdot\frac{3}{r^2s}=3s^2.$$

□

10) In ΔABC:

$$81r^2\le\frac{r_a^2}{\tan^2\frac{A}{2}}+\frac{r_b^2}{\tan^2\frac{B}{2}}+\frac{r_c^2}{\tan^2\frac{C}{2}}\le\frac{81R^2}{4}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the identity $\sum\frac{r_a^2}{\tan^2\frac{A}{2}}=3p^2$ *and Mitrinovič's inequality* $27r^2\le s^2\le\frac{27R^2}{4}$.

Equality holds if and only if the triangle is equilateral.

□

11) In ΔABC:

$$\frac{r_a^2}{\cot^2\frac{A}{2}}+\frac{r_b^2}{\cot^2\frac{B}{2}}+\frac{r_c^2}{\cot^2\frac{C}{2}}=\frac{2s^4-16s^2R(4R+r)+(4R+r)^4}{s^2}$$

Proof.

$$\text{Using } r_a=\frac{S}{s-a} \text{ and } \cot^2\frac{A}{2}=\frac{s(s-a)}{(s-b)(s-c)} \text{ we obtain:}$$

$$\sum\frac{r_a^2}{\cot^2\frac{A}{2}}=\sum\frac{\frac{S^2}{(s-a)^2}}{\frac{s(s-a)}{(s-b)(s-c)}}=\frac{S^2}{s}\sum\frac{(s-b)(s-c)}{(s-a)^3}=$$

$$=\frac{r^2s^2}{s}\cdot\frac{2s^4-16s^2R(4R+r)+(4R+r)^4}{r^2s^3}=\frac{2s^4-16s^2R(4R+r)+(4R+r)^4}{s^2}.$$

□

12) In ΔABC:

$$\frac{r_a^2}{\cot^2\frac{A}{2}}+\frac{r_b^2}{\cot^2\frac{B}{2}}+\frac{r_c^2}{\cot^2\frac{C}{2}}\ge\frac{9Rr}{2}.$$

Proposed by Marin Chirciu - Romania

Proof.

Using the identity $\frac{r_a^2}{\cot^2\frac{A}{2}}+\frac{r_b^2}{\cot^2\frac{B}{2}}+\frac{r_c^2}{\cot^2\frac{C}{2}}=\frac{2s^4-16s^2R(4R+r)+(4R+r)^4}{s^2}$

we write the inequality:

$$\frac{2s^4-16s^2R(4R+r)+(4R+r)^4}{s^2}\geq\frac{9Rr}{2}\Leftrightarrow 2(4R+r)^4\geq s^2(128R^2+41Rr-4s^2)$$

which follows from Blundon-Gerretsen inequality $16Rr-5r^2\leq s^2\leq\frac{R(4R+r)^2}{2(2R-r)}$.

It remains to prove that:

$$2(4R+r)^4\geq\frac{R(4R+r)^2}{2(2R-r)}[128R^2+41Rr-4(16Rr-5r^2)]\Leftrightarrow 23R^2-44Rr-4r^2\geq 0\Leftrightarrow$$

$\Leftrightarrow(R-2R)(23R+2r)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.

E-mail address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

PROBLEM X.31
ROMANIAN MATHEMATICAL MAGAZINE
NO. 21/2018

MARIN CHIRCIU

1. In ΔABC the following relationship holds:

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}+\frac{2r}{R}\geq 4$$

Proposed by Marian Ursărescu - Romania

Proof.

We have $\sum\frac{a^2}{bc}=\frac{\sum a^3}{abc}=\frac{2s(s^2-3r^2-6Rr)}{4Rrs}=\frac{s^2-3r^2-6Rr}{2Rr}$

The inequality can be written: $\frac{s^2-3r^2-6Rr}{2Rr}+\frac{2r}{R}\geq 4\Leftrightarrow s^2\geq 14Rr-r^2$

which follows from Gerretsen's inequality: $s^2\geq 16Rr-5r^2$

It remains to prove that: $16Rr-5r^2\geq 14Rr-r^2\Leftrightarrow R\geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

The inequality can be extended:

2) In ΔABC:

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}+n\cdot\frac{r}{R}\geq 3+\frac{n}{2},\text{ where } n\leq 4.$$

Proposed by Marin Chirciu - Romania

Proof.

If $n<0$*, the inequality is banal, because* $\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\geq 3$*, from means inequality,*

and $n\cdot\frac{r}{R}\geq\frac{n}{2}\Leftrightarrow R\geq 2r$ *(Euler's inequality). Next, we use* $n\geq 0$*.*

Using the following identity: $\sum\frac{a^2}{bc}=\frac{s^2-3r^2-6Rr}{2Rr}$*, we write the inequality:*

$\frac{s^2-3r^2-6Rr}{2Rr}+\frac{nr}{R}\geq\frac{n+6}{3}\Leftrightarrow s^2\geq Rr(n+12)+r^2(3-2n)$ *which follows from Gerretsen's inequality:* $s^2\geq 16Rr-5r^2$*. It remains to prove that:*

$$16Rr-5r^2\geq Rr(n+12)+r^2(3-2n)\Leftrightarrow$$
$$\Leftrightarrow 16Rr-5r^2\geq Rr(n+12)+r^2(3-2n)\Leftrightarrow R(4-n)\geq 2r(4-n),$$

true from Euler's inequality $R \geq 2r$ and the condition from hypothesis $n \leq 4$

Equality holds if and only if the triangle is equilateral.

□

Remark.

For $n = 2$ we obtain Problem X.31 from RMM 21/2018.

Remark.

In the same way we can propose:

3) In ΔABC:

$$\frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2} + n \cdot \frac{r}{R} \geq 3 + \frac{n}{2}, \text{ where } n \leq \frac{8}{5}.$$

Proposed by Marin Chirciu - Romania

Proof.

If $n < 0$, the inequality is banal, because $\frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2} \geq 3$, from means inequality, and $n \cdot \frac{r}{R} \geq \frac{n}{2} \Leftrightarrow R \geq 2r$ (Euler's inequality). Next we use $n \geq 0$.

We have $\sum \frac{bc}{a^2} = \frac{\sum (bc)^3}{(abc)^2} = \frac{s^6 + s^4(3r^2 - 12Rr) + 3s^2r^4 + r^3(4R+r)^3}{(4Rrs)^2}$

Using the identity $\sum \frac{bc}{a^2} = \frac{s^6 + s^4(3r^2 - 12Rr) + 3s^2r^4 + r^3(4Rr+r)^3}{(4Rrs)^2}$

we write the inequality: $\frac{s^6 + s^4(3r^2 - 12Rr) + 3s^2r^4 + r^3(4R+r)^3}{(4Rrs)^2} + \frac{nr}{R} \geq \frac{n+6}{3} \Leftrightarrow$

$$s^2[s^4 + s^2(3r^2 - 12Rr) + 3r^4 + 16nRr^3 - (8n+48)R^2r^2] + r^3(4R+r)^3 \geq 0$$

We distinguish the following cases:

Case 1). If $[s^4 + s^2(3r^2 - 12Rr) + 3r^4 + 16nRr^3 - (8n+48)R^2r^2] \geq 0$,

the inequality is obvious.

Case 2). If $[s^4 + s^2(3r^2 - 12Rr) + 3r^4 + 16nRr^3 - (8n+48)R^2r^2] < 0$,

The inequality can be rewritten:

$$r^3(4R+r)^3 \geq s^2[(8n+48)R^2r^2 - 16nRr^3 - 3r^4 + s^2(12Rr - 3r^2 - s^2)],$$

which follows from Blundon-Gerretsen's inequality: $16Rr - 5r^2 \leq s^2 \leq \frac{R(4R+r)^2}{2(2Rr-r)}$ *and the observation that:* $12Rr - 3r^2 - s^2 < 0$. *It remains to prove that:* $r^3(4R+r)^3 \geq$

$$\geq \frac{R(4R+r)^2}{2(2R-r)}[(8n+48)R^2r^2 - 16nRr^3 - 3r^4 + (16Rr - 5r^2)(12Rr - 3r^2 - (16Rr - 5r^2))]$$

$$\Leftrightarrow 2r(4R+r)(2R-r) \geq R[R^2(8n-16) + Rr(52-16n) - 13r^2] \Leftrightarrow$$

$$\Leftrightarrow R^3(16-8n) + R^2r(16n-36) + 9Rr^2 - 2r^3 \geq 0 \Leftrightarrow (R-2r)[R^2(16-8n) - 4Rr + r^2] \geq 0,$$

true from Euler's inequality $R \geq 2r$ and the condition from hypothesis $n \leq \frac{8}{5}$.

Equality holds if and only if the triangle is equilateral.

□

4) In ΔABC:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+n\cdot\frac{r}{R}\geq 3+\frac{n}{2},\text{ where } n\leq\frac{2}{5}$$

Proposed by Marin Chirciu - Romania

Proof.

If $n<0$, *the inequality is banal, because* $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$, *from means inequality and* $n\cdot\frac{r}{R}\geq\frac{n}{2}\Leftrightarrow R\geq 2r$*(Euler's inequality). Next, we use* $n\geq 0$.

Using Bergström's inequality, we obtain:

$$\sum\frac{a}{b}=\sum\frac{a^2}{ab}\geq\frac{(a+b+c)^2}{ab+bc+ca}=\frac{4s^2}{s^2+r^2+4Rr}.$$

It suffices to prove that:

$$\frac{4s^2}{s^2+r^2+4Rr}+\frac{nr}{R}\geq\frac{n+6}{3}\Leftrightarrow s^2[(2-n)R+2nr]\geq R^2r(4n+24)+Rr(6-7n)-2r^3,$$

which follows from Gerretsen's inequality: $s^2\geq 16Rr-5r^2$ *and the remark that* $2-n>0$. *It remains to prove that:*

$$(16Rr-5r^2)[(2-n)R+2nr]\geq R^2r(4n+24)+Rr(6-7n)-2r^3\Leftrightarrow$$
$$R^2(2-5n)+Rr(11n-4)-2nr^2\geq 0\Leftrightarrow(R-2r)[R(2-5n)+nr]\geq 0,$$

true from Euler's inequality $R\geq 2r$ *and the condition from hypothesis* $n\leq\frac{2}{5}$.

Equality holds if and only if the triangle is equilateral.

□

5) In ΔABC:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+n\cdot\frac{r}{R}\geq\frac{3}{2}+\frac{n}{2},\text{ where } n\leq\frac{1}{3}.$$

Proposed by Marin Chirciu - Romania

Proof.

If $n<0$, *the inequality is banal, because* $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$, *(Nesbitt's inequality), and* $n\cdot\frac{r}{R}\geq\frac{n}{2}\Leftrightarrow R\geq 2r$ *(Euler's inequality). Next, we use* $n\geq 0$.

Using the identity $\sum\frac{a}{b+c}=\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}$, *we write the inequality:*

$$\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}+\frac{nr}{R}\geq\frac{n+3}{2}\Leftrightarrow s^2[R(1-n)+2nr]\geq r[(2n+10)R^2+(7-3n)Rr-2nr^2],$$

which follows from Gerretsen's inequality: $s^2\geq 16Rr-5r^2$. *It remains to prove that:*

$$(16Rr-5r^2)[R(1-n)+2nr]\geq r[(2n+10)R^2+(7-3n)Rr-2nr^2]\Leftrightarrow$$
$$(3-9n)R^2+(20n-6)Rr-4nr^2\geq 0\Leftrightarrow(R-2r)[R(3-9n)+2nr]\geq 0,$$

true, from Euler's inequality $R\geq 2r$ *and the condition from hypothesis* $n\leq\frac{1}{3}$.

Equality holds if and only if the triangle is equilateral.

□

6) Prove that in any triangle ABC the following inequality holds:

$$\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2+n\frac{r}{R}\geq\frac{3}{4}+\frac{n}{2},\ \text{where } n\leq\frac{9}{10}.$$

Proposed by Marin Chirciu - Romania

Proof.

If $n<0$, the inequality is banal, because $\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2\geq\frac{3}{4}$, from the inequality $x^2+y^2+z^2\geq\frac{(x+y+z)^2}{3}$, where $x=\frac{a}{b+c}, y=\frac{b}{c+a}, z=\frac{c}{a+b}$ and Nesbitt's inequality $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$, and $n\cdot\frac{r}{R}\geq\frac{n}{2}\Leftrightarrow R\geq 2r$ (Euler's inequality). Next, we use $n\geq 0$. We have:

$$\sum\left(\frac{a}{b+c}\right)^2=\frac{\sum a^2(a+b)^2(a+c)^2}{\prod(b+c)^2}.$$

Using $\sum a^2(a+b)^2(a+c)^2=8s^2[s^4-s^2(4Rr+6r^2)+r^2(6R^2+4Rr+r^2)]$ *and* $\prod(b+c)=2s(s^2+r^2+2Rr)$, *we have:*

$$\sum\left(\frac{a}{b+c}\right)^2=\frac{2[s^4-s^2(4Rr+6r^2)+r^2(6R^2+4Rr+r^2)]}{(s^2+r^2+2Rr)^2}.$$

The inequality is written:

$$\frac{2[s^4-s^2(4Rr+6r^2)+r^2(6R^2+4Rr+r^2)]}{(s^2+r^2+2Rr)^2}+n\frac{r}{R}\geq\frac{3}{4}+\frac{n}{2}\Leftrightarrow$$

$$\Leftrightarrow 8R[s^4-s^2(4Rr+6r^2)+r^2(6R^2+4Rr+r^2)]\geq((2n+3)R-4nr)(s^2+r^2+2Rr)^2\Leftrightarrow$$

$$\Leftrightarrow s^2[((5-2n)R+4nr)s^2-(8n+44)R^2r+(12n-54)Rr^2+8nr^3]+$$

$$+r^2[(36-8n)R^3+(8n+20)R^2r+(14n+5)Rr^2+4nr^3]\geq 0$$

We distinguish the following cases:

Case 1). If $((5-2n)R+4nr)s^2-(8n+44)R^2r+(12n-54)Rr^2+8nr^3\geq 0$, the inequality is obvious.

Case 2). If $((5-2n)R+4nr)s^2-(8n+44)R^2r+(12n-54)Rr^2+8nr^3<0$, the inequality is written:

$$r^2[(36-8n)R^3+(8n+20)R^2r+(14n+5)Rr^2+4nr^3]\geq$$

$$\geq s^2[(8n+44)R^2r+(54-12n)Rr^2-8nr^3-((5-2n)R+4nr)s^2]$$

resulting from Gerretsen's inequality $16Rr-5r^2\leq s^2\leq 4R^2+4Rr+3r^2$.

It remains to show that:

$$r^2[(36-8n)R^3+(8n+20)R^2r+(14n+5)Rr^2+4nr^3]\geq$$

$$\geq(4R^2+4Rr+3r^2)[(8n+44)R^2r+(54-12n)Rr^2-8nr^3-((5-2n)R+4nr)(16Rr-5r^2)]$$

$$\Leftrightarrow(144-160n)R^4+(176n-136)R^3r+(184n-188)R^2r^2+(224n-232)Rr^3-32nr^4\geq 0$$

$$\Leftrightarrow(36-40n)R^4+(44-34)R^3r+(46n-47)R^2r^2+(56n-58)Rr^3-8nr^4\geq 0\Leftrightarrow$$

$$(R-2r)[(36-40n)R^3+(38-36n)R^2r+(29-26n)Rr^2+4nr^3]\geq 0$$

true from Euler's inequality. Equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, ROMANIA.
Email address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

PROBLEM JP.152
ROMANIAN MATHEMATICAL MAGAZINE
NO. 11, WINTER EDITION 2018

MARIN CHIRCIU

1. Let ABC be a triangle, h_a, h_b, h_c denote the lengths of altitudes, l_a, l_b, l_c denote the lengths of inner bisectors, and r_a, r_b, r_c be its exradii. Prove that:

$$\frac{h_a r_a}{l_a^2} + \frac{h_b r_b}{l_b^2} + \frac{h_c r_c}{l_c^2} \geq 3$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC:

$$\frac{h_a r_a}{l_a^2} + \frac{h_b r_b}{l_b^2} + \frac{h_c r_c}{l_c^2} = \frac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr}$$

Proof.

Using $h_a = \frac{2S}{a}, r_a = \frac{S}{s-a}, l_a = \frac{2bc}{b+c}\cos\frac{A}{2}, \cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$, *we obtain:*

$$\sum \frac{h_a r_a}{l_a^2} = \sum \frac{\frac{2S}{a} \cdot \frac{S}{s-a}}{(\frac{2bc}{b+c}\cos\frac{A}{2})^2} = \frac{2S^2}{4abcs} \sum \frac{(b+c)^2}{(s-a)^2} =$$

$$= \frac{r}{8R} \cdot \frac{2(8R^2 + 8Rr + 3r^2 - s^2)}{r^2} = \frac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr}$$

□

Let's return to the main problem:

The inequality we have to prove: $\frac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr} \geq 3 \Leftrightarrow s^2 \leq 8R^2 - 4Rr + 3r^2$

which follows from Gerretsen's inequality: $s^2 \leq 4R^2 + 4Rr + 3r^2$

It remains to prove that:

$4R^2 + 4Rr + 3r^2 \leq 8R^2 - 4Rr + 3r^2 \Leftrightarrow 4R^2 \geq 8Rr \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

□

Remark.

The inequality can be strengthened.

3) In ΔABC :

$$\frac{h_a r_a}{l_a^2}+\frac{h_b r_b}{l_b^2}+\frac{h_c r_c}{l_c^2}\geq\frac{R}{r}+1$$

Proposed by Marin Chirciu - Romania

Proof.

Using ***Lemma*** *and Gerretsen's inequality:* $s^2\leq 4R^2+4Rr+3r^2$ *we obtain:*

$$\sum\frac{h_a r_a}{l_a^2}=\frac{8R^2+8Rr+3r^2-s^2}{4Rr}\geq\frac{8R^2+8Rr+3r^2-4R^2-4Rr-3r^2}{4Rr}=$$

$$=\frac{4R^2+4Rr}{4Rr}=\frac{R}{r}+1$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality 3) is stronger than inequality 1):

4) In ΔABC:

$$\frac{h_a r_a}{l_a^2}+\frac{h_b r_b}{l_b^2}+\frac{h_c r_c}{l_c^2}\geq\frac{R}{r}+1\geq 3.$$

Proof.

See inequality 3) is $\frac{R}{r}+1\geq 3\Leftrightarrow R\geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasises an inequality having an opposite sense:

5) In ΔABC :

$$\frac{h_a r_a}{l_a^2}+\frac{h_b r_b}{l_b^2}+\frac{h_c r_c}{l_c^2}\leq 2\Big(\frac{R}{2}-\frac{r}{R}\Big)$$

Proof.

Using ***Lemma*** *and Gerretsen's inequality:* $s^2\geq 16Rr-5r^2$ *we obtain:*

$$\sum\frac{h_a r_a}{l_a^2}=\frac{8R^2+8Rr+3r^2-s^2}{4Rr}\leq\frac{8R^2+8Rr+3r^2-16Rr+5r^2}{4Rr}=$$

$$=\frac{8R^2-8Rr+8r^2}{4Rr}=\frac{2(R^2-Rr+r^2)}{Rr}\leq\frac{2(R^2-r^2)}{Rr}=2\Big(\frac{R}{r}-\frac{r}{R}\Big).$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

6) In ΔABC:

$$\frac{R}{r}+1\le\frac{h_ar_a}{l_a^2}+\frac{h_br_b}{l_b^2}+\frac{h_cr_c}{l_c^2}\le 2\left(\frac{R}{r}-\frac{r}{R}\right)$$

Proposed by Marin Chirciu - Romania

Proof.

See inequalities 3) and 5).

Equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, ROMANIA.

Email address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

INEQUALITY IN TRIANGLE 867
ROMANIAN MATHEMATICAL MAGAZINE

MARIN CHIRCIU

1. Let ABC be a triangle. Prove that:

$$\sum r_a(h_b+h_c)^2 \geq 12sS.$$

Proposed by Mehmet Şahin - Ankara - Turkey

Proof.

We prove the following lemma:

Lemma 1.
2) In $\Delta ABC : \sum r_a(h_b+h_c)^2 = \frac{s^2(s^2-3r^2)}{R}$.

Proof.

Using $r_a = \dfrac{S}{s-a}$ *and* $h_a = \dfrac{2S}{a}$ *we obtain:*

$$\sum r_a(h_b+h_c)^2 = \sum \frac{S}{s-a}\left(\frac{2S}{b}+\frac{2S}{c}\right)^2 = 4S^3\sum\frac{(b+c)^2}{b^2c^2(s-a)} = 4r^3s^3\cdot\frac{s^2-3r^2}{4sRr^3} = $$

$$= \frac{s^2(s^2-3r^2)}{R}.$$

In the above equality we've used: $\sum \dfrac{(b+c)^2}{b^2c^2(s-a)} = \dfrac{s^2-3r^2}{4sRr^3}$, *which follows from:*

$$\sum a^2(b+c)^2(s-b)(s-c) = 4s^2Rr(s^2-3r^2), abc = 4Rrs \text{ and } \prod(s-a) = r^2s.$$

□

Let's get back to the main problem:

*Using **Lemma 1** the inequality can be written:*

$\dfrac{s^2(s^2-3r^2)}{R} \geq 12rs^2 \Leftrightarrow s^2 \geq 12Rr+3r^2$, *which follows from Gerretsen's inequality*

$s^2 \geq 16Rr-5r^2$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasises an inequality having an opposite sense.

3) In $\Delta ABC : \sum r_a(h_b+h_c)^2 \leq 6Rs^2$

Proposed by Marin Chirciu - Romania

Proof.

*Using **Lemma 1** we write the inequality:*

$\frac{s^2(s^2-3r^2)}{R} \leq 6Rs^2 \Leftrightarrow s^2 \leq 6R^2+3r^2$, *which follows from Gerretsen's inequality* $s^2 \leq 4R^2+4Rr+3r^2$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

4) In $\Delta ABC : 12rs^2 \leq \sum r_a(h_b+h_c)^2 \leq 6Rs^2$.

Proof.

*See inequalities **1)** and **3)**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Changing r_a *with* h_a *we can build inequalities similar to those above.*

5) In $\Delta ABC : \sum h_a(r_a+r_c)^2 \geq 12sS$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 2.

6) In $\Delta ABC : \sum h_a(r_b+r_c)^2 = 4s^2(2R-r)$.

Proof.

Using $r_a = \frac{S}{s-a}$ *and* $h_a = \frac{2S}{a}$ *we obtain:*

$$\sum h_a(r_b+r_c)^2 = \sum \frac{2S}{a}\left(\frac{S}{s-b}+\frac{S}{s-c}\right)^2 = 2S^3 \sum \frac{a}{(s-b)^2(s-c)^2} =$$

$$= 2r^3s^3 \cdot \frac{2(2R-r)}{sr^3} = 4s^2(2R-r)$$

In the above inequality we've used: $\sum \frac{a}{(s-b)^2(s-c)^2} = \frac{2(2R-r)}{sr^3}$

which follows from: $\sum a(s-a)^2 = 2sr(2R-r)$ *and* $\prod(s-a) = r^2s$.

□

Let's get back to the main problem:

*Using **Lemma 2** we write the inequality:*

$4s^2(2R-r) \geq 12rs^2 \Leftrightarrow R \geq 2r$ *(Euler's inequality* $R \geq 2r$*).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasises an inequality having an opposite sense.

7) In $\Delta ABC : \sum h_a(r_b + r_c)^2 \le 2R(4R + r)^2$.

Proposed by Marin Chirciu - Romania

Proof.

*Using **Lemma 2** we write the inequality:*

$4s^2(2R-r) \le 2R(4R+r)^2 \Leftrightarrow s^2 \le \frac{R(4R+r)^2}{2(2R-r)}$, *which is Blundon-Gerretsen's inequality.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

8) In $\Delta ABC : 12rs^2 \le \sum h_a(r_b + r_c)^2 \le 2R(4R + r)^2$.

Proof.

*See inequalities **5)** and **7)**.*

Equality holds if and only if the triangle is equilateral.

□

9) In $\Delta ABC : 324r^3 \le \sum r_a(r_b + r_c)^2 \le \frac{81R^3}{2}$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 3.

10) In $\Delta ABC : \sum r_a(r_b + r_c)^2 = 4s^2(R + r)$.

Proof.

Using $r_a = \frac{S}{s-a}$ *we obtain:*

$$\sum r_a(r_b + r_c)^2 = \sum \frac{S}{s-a}\left(\frac{S}{s-b} + \frac{S}{s-c}\right)^2 = S^3 \sum \frac{1}{s-a} \cdot \frac{a^2}{(s-b)^2(s-c)^2} =$$

$$= \frac{S^3}{\prod(s-a)} \sum \frac{a^2}{(s-b)(s-c)} = \frac{r^3s^3}{r^2s} \cdot \frac{4(R+r)}{r} = 4s^2(R+r).$$

In the above inequality we've used: $\sum \frac{a^2}{(s-b)(s-c)} = \frac{4(R+r)}{r}$

which follows from: $\sum a^2(s-a) = 2sr(R+r)$ *and* $\prod(s-a) = r^2s$.

□

Let's get back to the main problem:

Using **Lemma 3** *the double inequality can be written:*

$$324r^3 \leq 4s^2(R+r) \leq \frac{81R^3}{2},$$ *which follows from Gerretsen's inequality:*

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

11) In $\Delta ABC : 48s^2 \cdot \frac{r^3}{R^2} \leq \sum h_a(h_b+h_c)^2 \leq 12s^2 r$.

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 4.
12) In $\Delta ABC : \sum h_a(h_b+h_c)^2 = \frac{r}{R^2} \cdot s^2(s^2+r^2+10Rr)$.

Proof.

Using $h_a = \frac{2S}{a}$ *we obtain:*

$$\sum h_a(h_b+h_c)^2 = \sum \frac{2S}{a}\left(\frac{2S}{b}+\frac{2S}{c}\right)^2 = 8S^3 \sum \frac{1}{a} \cdot \frac{(b+c)^2}{b^2c^2} = \frac{8S^3}{abc} \sum \frac{(b+c)^2}{bc} =$$

$$= \frac{8r^3s^3}{4Rrs} \cdot \frac{s^2+r^2+10Rr}{2Rr} = \frac{r}{R^2} \cdot s^2(s^2+r^2+10Rr).$$

In the above equality we've used: $\sum \frac{(b+c)^2}{bc} = \frac{s^2+r^2+10Rr}{2Rr}$

which follows from: $\sum a(b+c)^2 = 2s(s^2+r^2+10Rr)$ *and* $abc = 4Rrs$.

□

Let's get back to the main problem:

Using **Lemma 4** *the double inequality can be written:*

$$48s^2 \cdot \frac{r^3}{R^2} \leq \frac{r}{R^2} \cdot s^2(s^2+r^2+10Rr) \leq 12s^2 r,$$ *which follows from Gerretsen's inequality*

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

13) In $\Delta ABC : \sum r_a^2(h_b+h_c)^2 \geq 36S^2$

Proposed by Marin Chirciu - Romania

Proof.

With means inequality we have:

$$(1)\quad \sum r_a^2(h_b+h_c)^2 \geq \sum r_a^2\cdot 4h_bh_c = 4\sum r_a^2h_bh_c = \frac{8r}{R}\cdot s^2(8R^2+2Rr-s^2)$$

which follows from: $\sum r_a^2h_bh_c = \frac{2r}{R}\cdot s^2(8R^2+2Rr-s^2)$, *because:*

$$\sum r_a^2h_bh_c = \sum\left(\frac{S}{s-a}\right)^2\cdot\frac{2S}{b}\cdot\frac{2S}{c} = 4S^4\sum\frac{1}{bc(s-a)^2},$$

$$\sum\frac{1}{bc(s-a)^2} = \frac{\sum a(s-b)^2(s-c)^2}{abc\prod(s-a)},$$

$$\sum a(s-b)^2(s-c)^2 = 2sr^2(8R^2+2Rr-s^2), abc = 4Rrs, \prod(s-a) = sr^2.$$

In order to prove $\sum r_a^2(h_b+h_c)^2 \geq 36S^2$ *using* (1) *it suffices to prove that:*

$\frac{8r}{R}\cdot s^2(8R^2+2Rr-s^2) \geq 36S^2 \Leftrightarrow 2s^2 \leq 16R^2-5Rr$, *true from*

Gerretsen's inequality $s^2 \leq 4R^2+4Rr+3r^2$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

14) In $\Delta ABC : \sum h_a^2(r_b+r_c)^2 \geq 36S^2$

Proposed by Marin Chirciu - Romania

Proof.

With means inequality we have:

(1)

$$\sum h_a^2(r_b+r_c)^2 \geq \sum h_a^2\cdot 4r_br_c = 4\sum h_a^2r_br_c = \frac{s^2}{R^2}\cdot[s^4+s^2(2r^2-12Rr)+r^3(4Rr+r)]$$

which follows from: $\sum h_a^2r_br_c = \frac{s^2}{4R^2}\cdot[s^4+s^2(2r^2-12Rr)+r^3(4R+r)]$, *because*

$$\sum h_a^2r_br_c = \sum\left(\frac{2S}{a}\right)^2\cdot\frac{S}{s-b}\cdot\frac{S}{s-c} = 4S^4\sum\frac{1}{a^2(s-b)(s-c)},$$

$$\sum\frac{1}{a^2(s-b)(s-c)} = \frac{\sum b^2c^2(s-a)}{(abc)^2\prod(s-a)},$$

$$\sum b^2c^2(s-a) = s[s^4+s^2(2r^2-12Rr)+r^3(4Rr+r)], abc = 4Rrs, \prod(s-a) = sr^2.$$

In order to prove $\sum h_a^2(r_b+r_c)^2 \geq 36S^2$ *using* (1) *it suffices to prove that:*

$$\frac{s^2}{R^2}\cdot[s^4+s^2(2r^2-12Rr)+r^3(4R+r)] \geq 36S^2 \Leftrightarrow$$

$s^4+s^2(2R^2-12Rr)+r^3(4R+r) \geq 36R^2r^2$, *true from Gerretsen's inequality*

$s^2 \geq 16Rr-5r^2$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

15) In $\Delta ABC : \sum r_a^2(r_b+r_c)^2 \geq 36Sr^2$.

Proof.

With means inequality we have:

$$\sum r_a^2(r_b+r_c)^2 \geq \sum r_a^2 \cdot 4r_br_c = 4r_ar_br_c \sum r_a = 4 \cdot s^2r(4R+r) \geq 4 \cdot s^2r \cdot 9r = 36Sr^2.$$

Equality holds if and only if the triangle is equilateral.

□

16) In $\Delta ABC : \sum h_a^2(h_b + h_c)^2 \geq \left(\frac{12Sr}{R}\right)^2$

Proof.

With means inequality we have:

$$\sum h_a^2(h_b + h_c)^2 \geq \sum h_a^2 \cdot 4h_bh_c = 4h_ah_bh_c \sum h_a = 4 \cdot \frac{s^2r^2}{R} \cdot \frac{s^2 + r^2 + 4Rr}{2r} \geq$$

$$\geq 4 \cdot \frac{s^2r^2}{R} \cdot \frac{36r^2}{2R} = \frac{144S^2r^2}{R^2} = \left(\frac{12Sr}{R}\right)^2.$$

Equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, ROMANIA.

Email address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

INEQUALITY IN TRIANGLE 873
ROMANIAN MATHEMATICAL MAGAZINE

MARIN CHIRCIU

1. In ΔABC the following relationship holds:

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}\ge 2+\frac{r}{2R}$$

Proposed by Adil Abdullayev - Baku - Azerbaijan

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC:

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}=\frac{s^2+5r^2+2Rr}{8Rr}$$

Proof.

Using $m_a^2=\dfrac{2b^2+2c^2-a^2}{4}$ *we obtain:*

$$\sum\frac{m_a^2}{bc}=\sum\frac{\frac{2b^2+2c^2-a^2}{4}}{bc}=\frac{1}{4}\sum\frac{2b^2+2c^2-a^2}{bc}=\frac{s^2+5r^2+2Rr}{8Rr},\ because$$

$$\sum\frac{2b^2+2c^2-a^2}{bc}=\frac{\sum a(2b^2+2c^2-a^2)}{abc},\ and\ \sum a(2b^2+2c^2-a^2)=$$

$$=2\sum a^2\sum a-3\sum a^3,\sum a=2s,\sum a^2=2(s^2-r^2-4Rr),$$

$$\sum a^3=2s(s^2-3r^2-6Rr),abc=4Rrs.$$

□

Let's return to the main problem:

Using ***Lemma*** *the inequality that we have to prove can be written:*

$\dfrac{s^2+5r^2+2Rr}{8Rr}\ge 2+\dfrac{r}{2R}\Leftrightarrow s^2\ge 14Rr-r^2$, *which follows from Gerretsen's inequality:* $s^2\ge 16Rr-5r^2$. *It remains to prove that:* $16Rr-5r^2\ge 14Rr-r^2\Leftrightarrow$

$\Leftrightarrow R\ge 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

The inequality can be strengthened:

3) In ΔABC:

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}\geq\frac{9}{4}$$

Proof.

*Using **Lemma** we write the inequality:*

$$\frac{s^2+5r^2+2Rr}{8Rr}\geq\frac{9}{4}\Leftrightarrow s^2\geq 16Rr-5r^2 \text{ (Gerretsen's inequality)}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **3)** is stronger than inequality **1)**:*

4) In ΔABC:

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}\geq\frac{9}{4}\geq 2+\frac{r}{2R}$$

Proof.

*See **3)** and Euler's inequality $R\geq 2r$.*

Equality holds if and only if the triangle is equilateral.

□

Let's emphasises an inequality having an opposite sense:

5) In ΔABC:

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}\leq\frac{9R}{8r}$$

Proposed by Marin Chirciu - Romania

*Using **Lemma** the inequality we have to prove can be written:*

$\frac{s^2+5r^2+2Rr}{8Rr}\leq\frac{9R}{8r}\Leftrightarrow s^2\leq 9R^2-2Rr-5r^2$, *which follows from Gerretsen's inequality:* $s^2\leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$4R^2+4Rr+3r^2\leq 9R^2-2Rr-5r^2\Leftrightarrow 5R^2-6Rr-8r^2\geq 0\Leftrightarrow (R-2r)(5R+4r)\geq 0,$$

obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

Remark.

6) In ΔABC:

$$\frac{9}{4}\leq\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}\leq\frac{9R}{8r}.$$

Proof.

*See inequalities **3)** and **5)**.*

Equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, ROMANIA.
Email address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

INEQUALITY IN TRIANGLE 881
ROMANIAN MATHEMATICAL MAGAZINE

MARIN CHIRCIU

1. In ΔABC:

$$\frac{bc}{al_a}+\frac{ca}{bl_b}+\frac{ab}{cl_c}\le\frac{9R^2}{2S}$$

Proposed by Mehmet Şahin - Ankara - Turkey

Proof.

Using $l_a=\frac{2bc}{b+c}\cos\frac{A}{2}$ *we obtain:*

$$\frac{1}{al_a}=\frac{1}{a\cdot\frac{2bc}{b+c}\cos\frac{A}{2}}=\frac{b+c}{2abc\cdot\cos\frac{A}{2}}=\frac{2R(\sin B+\sin C)}{2abc\cdot\cos\frac{A}{2}}=\frac{R\cdot 2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{4RS\cdot\frac{A}{2}}=$$

$$=\frac{R\cdot 2\cos\frac{A}{2}\cos\frac{B-C}{2}}{4RS\cdot\cos\frac{A}{2}}=\frac{\cos\frac{B-C}{2}}{2S},\ \textit{wherefrom}\ \frac{bc}{al_a}=\frac{bc\cdot\cos\frac{B-C}{2}}{2S}.$$

Because $\cos\frac{B-C}{2}\le 1$ *and* $\sum bc\le\sum a^2\le 9R^2$ *(Leibniz's inequality), it follows:*

$$\sum\frac{bc}{al_a}=\sum\frac{bc\cdot\cos\frac{B-C}{2}}{2S}\le\sum\frac{bc}{2S}\le\frac{9R^2}{2S}.$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasises an inequality having an opposite sense.

2) In ΔABC:

$$\frac{bc}{al_a}+\frac{ca}{bl_b}+\frac{ab}{cl_c}\ge\frac{18r}{s}$$

Proposed by Marin Chirciu - Romania

Proof.

Using $\frac{bc}{al_a}=\frac{bc\cdot\cos\frac{B-C}{2}}{2S}$ *we obtain*

$$\sum\frac{bc}{al_a}=\sum\frac{bc\cdot\cos\frac{B-C}{2}}{2S}=\frac{1}{2S}\sum bc\cdot\cos\frac{B-C}{2}\tag{1}$$

With means inequality and $abc=4RS,\prod\cos\frac{B-C}{2}=\frac{s^2+r^2+2Rr}{8R^2}$ *we obtain:*

$$\sum bc\cdot\cos\frac{B-C}{2}\ge 3\sqrt[3]{\prod bc\cdot\cos\frac{B-C}{2}}=3\sqrt[3]{(abc)^2\prod\cos\frac{B-C}{2}}=$$

(2)

$$= 3\sqrt[3]{(4RS)^2 \cdot \frac{s^2+r^2+2Rr}{8R^2}} = 3\sqrt[3]{2s^2r^2(s^2+r^2+2Rr)} \geq 3\sqrt[3]{(12r^2)^3} = 3\cdot 12r^2 = 36r^2$$

We've used above $s^2 \geq 16Rr - 5r^2$ (Gerretsen) $s \geq 3r\sqrt{3}$ (Mitrinovic) and $R \geq 2r$ (Euler). From (1) and (2) it follows the conclusion. Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

3) In ΔABC:

$$\frac{18r}{s} \leq \frac{bc}{al_a} + \frac{ca}{bl_b} + \frac{ab}{cl_c} \leq \frac{9R^2}{2S}$$

Proof.

*See inequalities **1)** and **2)**.*

Equality holds if and only if the triangle is equilateral.

Remark.

The double inequality can be strengthened:

4) In ΔABC:

$$2\sqrt{3} \leq \frac{bc}{al_a} + \frac{ca}{bl_b} + \frac{ab}{cl_c} \leq \frac{2(R+r)^2}{S}$$

Proposed by Marin Chirciu - Romania

Proof.

Inequality from the left side: $\frac{bc}{al_a} + \frac{ca}{bl_b} + \frac{ab}{cl_c} \geq 2\sqrt{3}$ it follows from:

(1) *The proof of **2)** implies* $\sum \frac{bc}{al_a} \geq \frac{3}{2S}\sqrt[3]{2s^2r^2(s^2+r^2+2Rr)}$

(2) *Then* $\frac{3}{2S}\sqrt[3]{2s^2r^2(s^2+r^2+2Rr)} \geq 2\sqrt{3}$

$$\Leftrightarrow 3\sqrt[3]{2s^2r^2(s^2+r^2+2Rr)} \geq 4rs\sqrt{3} \Leftrightarrow$$

$$\Leftrightarrow 27 \cdot 2s^2r^2(s^2+r^2+2Rr) \geq 64r^3s^3 \cdot 3\sqrt{3} \Leftrightarrow 9(s^2+r^2+2Rr) \geq 32rs\sqrt{3}$$

which follows from Doucet's inequality $4R + r \geq s\sqrt{3}$. It remains to prove that:

$$9(s^2+r^2+2Rr) \geq 32r(4R+r) \Leftrightarrow 9s^2 \geq 110Rr + 23r^2,$$

true from Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2$ and Euler's inequality $R \geq 2r$.

It suffices to prove that:

$$9(16Rr - 5r^2) \geq 110Rr + 23r^2 \Leftrightarrow R \geq 2r.$$

From (1) and (2) we obtain $\frac{bc}{al_a} + \frac{ca}{bl_b} + \frac{ab}{cl_c} \geq 2\sqrt{3}$.

Inequality from the right side: $\frac{bc}{al_a} + \frac{ca}{bl_b} + \frac{ab}{cl_c} \leq \frac{2(R+r)^2}{S}$.

*The proof of **1)** implies:*

$$(1) \qquad \sum \frac{bc}{al_a} = \sum \frac{bc \cdot \cos \frac{B-C}{2}}{2S} \leq \sum \frac{bc}{2S}$$

With identity $\sum bc = s^2+r^2+4Rr$ *and Gerretsen's inequality* $s^2 \leq 4R^2+4Rr+3r^2$ *we have:*

(2)

$$\sum bc = s^2+r^2+4Rr \leq 4R^2+4Rr+3r^2+r^2+4Rr = 4R^2+8Rr+4r^2 = 4(R+r)^2$$

From (1) *and* (2) *it follows* $\frac{bc}{al_a} + \frac{ca}{bl_b} + \frac{ab}{cl_c} \leq \frac{1}{2S} \cdot 4(R+r)^2 = \frac{2(R+r)^2}{S}$.

□

Equality holds if and only if the triangle is equilateral. □

Remark.

The double inequality ***4)*** *is stronger than* ***3)***.

5) In ΔABC:

$$\frac{18r}{s} \leq 2\sqrt{3} \leq \frac{bc}{al_a} + \frac{ca}{bl_b} + \frac{ab}{cl_c} \leq \frac{2(R+r)^2}{S} \leq \frac{9R^2}{2S}$$

Proposed by Mehmet Şahin - Turkey, Marin Chirciu - Romania

Proof.

See ***4)****, Euler's inequality* $R \geq 2r$ *and Mitrinovic's inequality* $s \geq 3r\sqrt{3}$.

Equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, ROMANIA.

Email address: dansitaru63@yahoo.com

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

INEQUALITY IN TRIANGLE 847
ROMANIAN MATHEMATICAL MAGAZINE

MARIN CHIRCIU

1. In ΔABC

$$4(m_a+m_b+m_c)\geq\sum\frac{r_a+r}{r_a-r}(h_b+h_c).$$

Proposed by Bogdan Fustei - Romania

Proof.

We prove the following lemma:

Lemma.
1) In ΔABC

$$\sum\frac{r_a+r}{r_a-r}(h_b+h_c)=\frac{3s^2-r^2-4Rr}{R}$$

Using $r_a=\frac{S}{s-a}, r=\frac{S}{s}$ *and* $h_a=\frac{2S}{a}$ *we obtain:*

$$\sum\frac{r_a+r}{r_a-r}(h_b+h_c)=\sum\frac{\frac{S}{s-a}+\frac{S}{s}}{\frac{S}{s-a}-\frac{S}{s}}\left(\frac{2S}{b}+\frac{2S}{c}\right)=\frac{2S}{abc}\sum(b+c)^2=$$

$$=\frac{1}{2R}\cdot 2(3s^2-r^2-4Rr)=\frac{3s^2-r^2-4Rr}{R}$$

Let's get back to the main problem:

With Tereshin's inequality $m_a\geq\frac{b^2+c^2}{4R}$ *we obtain:*

$$\sum m_a\geq\sum\frac{b^2+c^2}{4R}=\frac{2\sum a^2}{4R}=\frac{2(s^2-r^2-4Rr)}{2R}=\frac{s^2-r^2-4Rr}{R}.$$

*Using **Lemma** and* $\sum m_a\geq\frac{s^2-r^2-4Rr}{R}$

It suffices to prove that:

$$4\cdot\frac{s^2-r^2-4Rr}{R}\geq\frac{3s^2-r^2-4Rr}{R}\Leftrightarrow s^2\geq 12Rr+3r^2,$$

which follows from Gerretsen's inequality: $s^2\geq 16Rr-5r^2$ *and Euler's inequality* $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasises an inequality having an opposite sense.

2) In ΔABC:

$$4(h_a+h_b+h_c)\leq\sum\frac{r_a+r}{r_a-r}(h_a+h_c).$$

Proof.

Using ***Lemma*** *and* $\sum h_a = \dfrac{s^2 + r^2 + 4Rr}{2R}$ *we write the inequality:*

$$4 \cdot \frac{s^2 + r^2 + 4Rr}{2R} \leq \frac{3s^2 - r^2 - 4Rr}{R} \Leftrightarrow s^2 \geq 12Rr + 3r^2,$$

which follows from Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

3) In ΔABC**:**

$$4(h_a + h_b + h_c) \leq \sum \frac{r_a + r}{r_a - r}(h_b + h_c) \leq 4(m_a + m_b + m_c).$$

Proof.

See inequalities ***1)*** *and* ***2)****.*

Equality holds if and only if the triangle is equilateral.

□

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, ROMANIA.

Email address: dansitaru63@yahoo.com