RMM Commented Problems Marathon
21 - 40

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ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

INEQUALITY IN TRIANGLE 305
ROMANIAN MATHEMATICAL MAGAZINE
2017

MARIN CHIRCIU

1. In ΔABC

$$\boldsymbol{\frac{R}{2r}+\frac{3p^2}{(4R+r)^2}\geq 2}$$

Proposed by Adil Abdullayev - Baku - Azerbaidian

Proof.

Using Gerretsen's inequality $p^2\geq 16Rr-5r^2$, it's enough to prove that:

$$\frac{R}{2r}+\frac{3(16Rr-5r^2)}{(4R+r)^2}\geq\Leftrightarrow 16R^3-56R^2r+65Rr^2-34Rr^2\geq 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(16R^2-24Rr+17r^2)\geq 0,$ *obviously from Euler's inequality* $R\geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark

The inequality can be developed:

2. In ΔABC

$$\boldsymbol{n\cdot\frac{R}{r}+k\cdot\frac{p^2}{(4R+r)^2}\geq 2n+\frac{k}{3},\ \textit{where}\ 15n\geq 2k\geq 0.}$$

Proposed by Marin Chirciu - Romania

Proof.

Using Gerretsen's inequality $p^2\geq 16Rr-5r^2$ and the conditions $n\geq 0, k\geq 0$ it's enough to prove that:

$$n\cdot\frac{R}{r}+k\cdot\frac{16Rr-5r^2}{(4R+r)^2}\geq 2n+\frac{k}{3}\Leftrightarrow$$

$$48nR^3-(72n+16k)R^2r+(40k-45n)Rr^2-(6n+18k)r^3\geq 0$$

$\Leftrightarrow (R-2r)(48nR^2+(24n-16k)Rr+(3n+8k)r^2)\geq 0,$ *obviously from Euler's inequality* $R\geq 2r$ *and the observation that* $48nR^2+(24n-16k)Rr+(3n+8k)r^2\geq 0$ *for* $15n\geq 2k\geq 0$.

The equality holds if and only if the triangle is equilateral or $n=k=0$.

□

Remark

The inequality can be reformulated:

3. In ΔABC

$$\boldsymbol{\frac{R}{r}+\lambda\cdot\frac{p^2}{(4R+r)^2}\geq 2+\frac{\lambda}{3}}, \textbf{ where } \boldsymbol{0\leq\lambda\leq\frac{15}{2}}.$$

Proof.

*In **2.** we divide with n and we denote $\frac{k}{n}=\lambda$.*

The equality holds if and only if the triangle is equilateral.

*For $\lambda=6$ we obtain inequality **1.***

$\square$

Remark

The inequality can be reformulated:

4. In ΔABC

$$\boldsymbol{\lambda\cdot\frac{R}{r}+\frac{p^2}{(4R+r)^2}\geq 2\lambda+\frac{1}{3},\lambda\geq\frac{2}{15}}.$$

Proof.

*In **2.** we divide with k and we denote $\frac{n}{k}=\lambda$.*

The equality holds if and only if the triangle is equilateral.

*For $\lambda=\frac{1}{6}$ we obtain inequality **1.***

$\square$

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INEQUALITY IN TRIANGLE 295 ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. In ΔABC

$$\boldsymbol{\sum (b+c-a)m_a^2 \geq 18pr(R-r).}$$

Proposed by Abdikadir Altintas – Afyon – Turkey

Proof.

Using the known identity in triangle $\sum (p-a)m_a^2 = p(p^2-4r^2-7Rr)$

the inequality that we have to prove can be written:

$2p(p^2-4r^2-7Rr) \geq 18pr(R-r) \Leftrightarrow p^2 \geq 16Rr-5r^2$ *(Gerretsen's inequality)*

The equality holds if and only if the triangle is equilateral.

□

Remark

The inequality can be developed:

2. In ΔABC

$$\boldsymbol{\sum (b+c-na)m_a^2 \geq 9pr\Big[(3-n)R-2r\Big], \text{ where } n \leq 5.}$$

Proof.

Using the known identity in triangle $\sum a m_a^2 = \frac{1}{2}\cdot p(p^2+5r^2+2Rr)$*, we obtain:*

$$\sum (b+c-na)m_a^2 = \sum \Big[2p-(n+1)a\Big]m_a^2 = 2p\sum m_a^2 - (n+1)\sum a m_a^2 =$$

$$= 2p\sum m_a^2 - (n+1)\cdot\frac{1}{2}\cdot p(p^2+5r^2+2Rr) =$$

$$= 2p\cdot\frac{3}{4}\cdot 2(p^2-r^2-4Rr) - (n+1)\cdot\frac{1}{2}\cdot p(p^2+5r^2+2Rr) =$$

$$= \frac{p}{2}\Big[(5-n)p^2-(5n+11)r^2-(2n+26)Rr\Big].$$

the inequality that we have to prove can be written:

$$\frac{p}{2}\Big[(5-n)p^2-(5n+11)r^2-(2n+26)Rr\Big] \geq 9pr\Big[(3-n)R-2r\Big] \Leftrightarrow$$

$$(5-n)p^2-(5n+11)r^2-(2n+26)Rr \geq 18r\Big[(3-n)R-2r\Big],$$

which follows from Gerretsen's inequality $p^2 \geq 16Rr-5r^2$ *and the condition* $5-n \geq 0$.

The equality holds if and only if the triangle is equilateral.

□

Let's obtain an inequality that have an opposite sense.

3. In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{\sum (b+c-na)m_a^2 \le p\Big[(10-2n)R^2-(3n+3)Rr+(2-4n)r^2\Big],\ \textit{where}\ n \le 5.}$$

Proof.

Using the above proved inequality:

$$\sum (b+c-na)m_a^2 = \frac{p}{2}\Big[(5-n)p^2-(5n+11)r^2-(2n+26)Rr\Big]$$

Gerretsen's inequality $p^2 \le 4R^2+4Rr+3r^2$ *and the condition* $5-n \ge 0$*, we obtain:*

$$\sum (b+c-na)m_a^2 \le \frac{p}{2}\Big[(5-n)(4R^2+4Rr+3r^2)-(5n+11)r^2-(2n+26)Rr\Big] =$$

$$= p\Big[(10-2n)R^2-(3n+3)Rr+(2-4n)r^2\Big].$$

The equality holds if and only if the triangle is equilateral.

□

We can write the following double inequality:

4. In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{9pr\Big[(3-n)R-2r\Big] \le \sum (b+c-na)m_a^2 \le p\Big[(10-2n)R^2-(3n+3)Rr+(2-4n)r^2\Big], n \le 5}$$

Proposed by Marin Chirciu - Romania

Proof.

*See inequalities **2.** and **3.***

The equality holds if and only if the triangle is equilateral.

□

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INEQUALITY IN TRIANGLE 301
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MARIN CHIRCIU

1. In ΔABC

$$\sum \frac{a}{(b+c)(b+c-a)} \geq \frac{18r^2}{abc}$$

Proposed by Panagiote Ligouras - Florence - Italy

Proof.

We prove the following lemma:

Lemma

2. In ΔABC

$$\sum \frac{a}{(b+c)(b+c-a)} = \frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)}$$

We have

$$\sum \frac{a}{(b+c)(b+c-a)} = \sum\left(\frac{1}{b+c-a} - \frac{1}{b+c}\right) = \sum \frac{1}{b+c-a} - \sum \frac{1}{b+c} =$$

$$= \frac{4R+r}{2pr} - \frac{5p^2+r^2+4Rr}{2p(p^2+r^2+2Rr)} =$$

$$= \frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)}$$

Let's pass to solving the problem from the enunciation.

*Using the **Lemma**, the inequality that we have to prove can be written:*

$$\frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \geq \frac{18r^2}{abc} \Leftrightarrow \frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \geq \frac{18r^2}{4pRr} \Leftrightarrow$$

$$p^2(4R^2-4Rr-9r^2)+r(8R^3+2R^2r-18Rr^2-9r^3) \geq 0$$

We distinguish the following cases:

1. If $4R^2-4Rr-9r^2 \geq 0$, the inequality is obvious.

2. If $4R^2-4Rr-9r^2 < 0$, the inequality can be rewritten:

$$p^2(9r^2+4Rr-4R^2) \leq r(8R^3+2R^2r-18Rr^2-9r^3).$$

Using Gerretsen's inequality $p^2 \leq 4R^2+4Rr+3r^2$ *it is enough to prove that:*

$$(4R^2+4Rr+3r^2)(9r^2+4Rr-4R^2) \leq r(8R^3+2R^2r-18Rr^2-9r^3) \Leftrightarrow$$

$$\Leftrightarrow 8R^4+4R^3r-19R^2r^2-33Rr^3-18r^4 \geq 0 \Leftrightarrow (R-2r)(8R^3+20R^2r+21Rr^2+9r^3) \geq 0$$

which is obviously from Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark

*Inequality **1.** can be strengthened:*

3. In ΔABC

$$\boldsymbol{\sum \frac{a}{(b+c)(b+c-a)} \geq \frac{9Rr}{abc}}$$

Proof. *Using the **Lemma** the inequality can be written:*

$$\frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \geq \frac{9Rr}{abc} \Leftrightarrow \frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \geq \frac{9Rr}{4pRr} \Leftrightarrow$$

$$p^2(8R-17r)+r(16R^2-14Rr-9r^2) \geq 0.$$

We distinguish the cases:

1. If $8R-17r \geq 0$, the inequality is obvious.
2. If $8R-17r < 0$, the inequality can be rewritten:

$$\boldsymbol{p^2(17r-8R) \leq r(16R^2-14Rr-9r^2).}$$

Using Gerretsen's inequality $p^2 \leq 4R^2+4Rr+3r^2$ it suffices to prove that:

$$(4R^2+4Rr+3r^2)(17r-8R) \leq r(16R^2-14Rr-9r^2) \Leftrightarrow$$

$$\Leftrightarrow 16R^3-10R^2r-29Rr^2-30r^3 \geq 0 \Leftrightarrow (R-2r)(16R^2+22Rr+15r^2) \geq 0,$$

which is obvious from Euler's inequality $R \geq 2r$.

The inequality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **3.** is stronger than inequality **1.***

4. In ΔABC

$$\boldsymbol{\sum \frac{a}{(b+c)(b+c-a)} \geq \frac{9Rr}{abc} \geq \frac{18r^2}{abc}.}$$

Proof.

*See inequality **3.** and Euler's inequality $R \geq 2r$.*

The inequality holds if and only if the triangle is equilateral.

□

Remark.

*Also inequality **3.** can be developed:*

5. In ΔABC

$$\boldsymbol{\sum \frac{a}{(b+c)(b+c-a)} \geq \frac{9R^2}{2abc}}$$

Proof.

*Using the **Lemma**, the inequality that we have to prove can be written:*

$$\frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \geq \frac{9R^2}{2abc} \Leftrightarrow \frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \geq \frac{9R^2}{9pRr} \Leftrightarrow$$

$$p^2(7R-16r)+Rr(14R-r) \geq 0.$$

We distinguish the cases:

1. If $7R-16r \geq 0$, the inequality is obvious.
2. If $7R-16r < 0$, the inequality can be rewritten:

$$\boldsymbol{p^2(16r-7R) \leq Rr(14R-r).}$$

Using Gerretsen's inequality $p^2 \leq 4R^2+4Rr+3r^2$ it suffices to prove that:

$$(4R^2+4Rr+3r^2)(16r-7R) \leq Rr(14R-r) \Leftrightarrow$$

$$\Leftrightarrow 14R^3-11R^2r-22Rr^2-24r^3 \geq 0 \Leftrightarrow (R-2r)(14R^2+17Rr+12r^2) \geq 0,$$

which is obvious from Euler's inequality $R \geq 2r$.

The inequality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **5.** is stronger than inequality **3.***

6. In ΔABC

$$\boldsymbol{\sum \frac{a}{(b+c)(b+c-a)} \geq \frac{9R^2}{2abc} \geq \frac{9Rr}{abc}}$$

Proof.

*See inequality **5.** and Euler's inequality $R \geq 2r$.*

The inequality holds if and only if the triangle is equilateral.

□

Remark.

The inequalities can be written:

7. In ΔABC

$$\boldsymbol{\sum \frac{a}{(b+c)(b+c-a)} \geq \frac{9R^2}{2abc} \geq \frac{9Rr}{abc} \geq \frac{18r^2}{abc}}$$

Remark.

Let's obtain an inequality having an opposite sense.

8. In ΔABC

$$\boldsymbol{\sum \frac{a}{(b+c)(b+c-a)} \leq \frac{9R^2}{16Sr}.}$$

Proof.

Using the ***Lemma****, the inequality that we have to prove can be written:*

$$\frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \le \frac{9R^2}{16Sr} \Leftrightarrow \frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \le \frac{9R^2}{16r^2p} \Leftrightarrow$$

$$p^2(9R^2-32Rr+32r^2)+Rr(18R^2-55Rr-16r^2) \ge 0.$$

Because $9R^2-32Rr+32r^2 > 0(\Delta < 0,$ *for the trinomial* $9x^2-32x+32),$ *using Gerretsen's inequality* $p^2 \ge 16Rr-5r^2$ *is suffices to prove that:*

$$(16Rr-5r^2)(9R^2-32Rr+32r^2)+Rr(18R^2-55Rr-16r^2) \ge 0 \Leftrightarrow$$

$$\Leftrightarrow 81R^3-306R^2r+328Rr^2-80r^3 \ge 0 \Leftrightarrow (R-2r)(80R^2-144Rr+40r^2) \ge 0,$$

which is obviously from Euler's $R \ge 2r$.

The inequality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

9. In ΔABC

$$\boldsymbol{\frac{9R}{8S} \le \sum \frac{a}{(b+c)(b+c-a)} \le \frac{9R^2}{16Sr}}.$$

Proposed by Marin Chirciu - Romania

Proof.

See inequalities ***2.*** *and 8.*

The inequality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

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PROBLEM 298
TRIANGLE MARATHON 201 - 300
ROMANIAN MATHEMATICAL MAGAZINE
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MARIN CHIRCIU

1. In ΔABC

$$m_a \cos\frac{A}{2} + m_b \cos\frac{B}{2} + m_c \cos\frac{C}{2} \geq \frac{9r\sqrt{3}}{2}$$

Proposed by Kevin Soto Palacios - Huarmey - Peru

Remark.

The inequality can be strengthened:

2. In ΔABC

$$m_a \cos\frac{A}{2} + m_b \cos\frac{B}{2} + m_c \cos\frac{C}{2} \geq \frac{3p}{2}$$

Proposed by Marin Chirciu - Romania

Proof.

We use the remarkable inequality $m_a \geq \frac{b+c}{2}\cos\frac{A}{2}$

We obtain:

$$\sum m_a \cos\frac{A}{2} \geq \sum m_a \cos^2\frac{A}{2} = \sum \frac{b+c}{2} \cdot \frac{p(p-a)}{bc} = \frac{p}{2} \cdot \frac{a(b+c)(p-a)}{abc} =$$
$$= \frac{p}{2} \cdot \frac{12pRr}{4pRr} = \frac{3p}{2}.$$

The equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **2.** is stronger then Inequality **1.**:*

3. In ΔABC

$$m_a \cos\frac{A}{2} + m_b \cos\frac{B}{2} + m_c \cos\frac{C}{2} \geq \frac{3p}{2} \geq \frac{9r\sqrt{3}}{2}.$$

Proof.

*See inequality **2.** and Mitrinović's inequality:* $p \geq 3r\sqrt{3}$.

The equality holds if and only if the triangle is equilateral.

□

In the same mode can be proposed:

4. In ΔABC

$$\boldsymbol{m_a \sin\frac{A}{2} + m_b \sin\frac{B}{2} + m_c \sin\frac{C}{2} \geq \frac{ab+bc+ca}{4R}}$$

Proof.

Using the remarkable inequality $m_a \geq \frac{b+c}{2}\cos\frac{A}{2}$, *we obtain:*

$$\sum m_a \sin\frac{A}{2} \geq \sum \frac{b+c}{2}\cos\frac{A}{2}\sin\frac{A}{2} = \frac{1}{4}\sum(b+c)\sin A = \frac{1}{4}\sum(b+c)\cdot\frac{a}{2R} = \frac{ab+bc+ca}{4R}.$$

The equality holds if and only if the triangle is equilateral.

□

5. In ΔABC

$$\boldsymbol{m_a \sin\frac{A}{2} + m_b \sin\frac{B}{2} + m_c \sin\frac{C}{2} \geq \frac{S\sqrt{3}}{R}}$$

Proof.

See 4. and the remarkable inequality $ab+bc+ca \geq 4S\sqrt{3}$.

The equality holds if and only if the triangle is equilateral.

□

6. In ΔABC

$$\boldsymbol{m_a \sin\frac{A}{2} + m_b \sin\frac{B}{2} + m_c \sin\frac{C}{2} \geq \frac{r(5R-r)}{R}}$$

Proof.

See 4., the identity $ab+bc+ca = p^2+r^2+4Rr$ *and Gerretsen's inequality*

$$p^2 \geq 16Rr - 5r.$$

The equality holds if and only if the triangle is equilateral.

□

The following inequalities can be written:

7. In ΔABC

$$\boldsymbol{m_a \sin\frac{A}{2} + m_b \sin\frac{B}{2} + m_c \sin\frac{C}{2} \geq \frac{p^2+r^2+4Rr}{4R} \geq \frac{r(5R-r)}{R} \geq \frac{S\sqrt{3}}{R} \geq \frac{9r^2}{R}}$$

Proposed by Marin Chirciu - Romania

Proof.
See ***4.*** *the identity* $ab + bc + ca = p^2 + r^2 + 4Rr$ *, Gerretsen's inequality* $p^2 \geq 16Rr - 5r$*, Doucet's inequality* $4R + r \geq p\sqrt{3}$ *and Mitrinović's inequality:* $p \geq 3r\sqrt{3}$.

The equality holds if and only if the triangle is equilateral.

□

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MARIN CHIRCIU

1. In ΔABC

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+\frac{2r}{R}\geq 4.$$

Proposed by Adil Abdullayev - Baku - Azerbaidian

Proof.

We have $\sum \frac{r_a}{r_b}=\sum \frac{r_a^2}{r_ar_b}\overbrace{\geq}^{Bergstrom}\frac{(r_a+r_b+r_c)^2}{r_ar_b+r_br_c+r_cr_a}=\frac{(4R+r)^2}{p^2}\overbrace{\geq}^{(1)}4-\frac{2r}{R}$, *where (1) follows from Blundon - Gerretsen's inequality* $p^2\leq\frac{R(4R+r)^2}{2(2R-r)}$ *(true from Gergonne's identity:* $H\Gamma^2=4R^2\left[1-\frac{2p^2(2R-r)}{r(4R+r)^2}\right],\Gamma$ *is Gergonne's point, namely the lines intersections* AA_1,BB_1,CC_1*, where* A_1,B_1,C_1 *are the tangent point of incircle in* ΔABC with the sides BC,CA,AB).

The equality holds if and only if the triangle is equilateral.

□

Remark.

The inequality can be developed:

2. In ΔABC

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+n\cdot\frac{r}{R}\geq 3+\frac{n}{2},\ \textit{where } n\leq 2.$$

Proposed by Marin Chirciu - Romania

Proof.

We have $\sum \frac{r_a}{r_b}=\sum \frac{r_a^2}{r_ar_b}\overbrace{\geq}^{Bergstrom}\frac{(r_a+r_b+r_c)^2}{r_ar_b+r_br_c+r_cr_a}=\frac{(4R+r)^2}{p^2}\overbrace{\geq}^{(1)}3+\frac{n}{2}-\frac{nr}{R}$, *where (1) follows from Blundon-Gerretsen's inequality* $p^2\leq\frac{R(4R+r)^2}{2(2R-r)}$ *(true from Gergonne's identity:* $H\Gamma^2=4R^2\left[1-\frac{2p^2(2R-r)}{r(4R+r)^2}\right],\Gamma$ *is Gergonne's point, namely the intersection lines* AA_1,BB_1,CC_1*, where* A_1,B_1,C_1 *are the tangent points of the incircle in* ΔABC *with the sides* BC,CA,AB*).*
It remains to prove that:

$$\frac{(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\overbrace{\geq}^{(1)}3+\frac{n}{2}-\frac{nr}{R}\Leftrightarrow\frac{4R-2r}{R}+\frac{nr}{R}\geq\frac{n+6}{2}\Leftrightarrow(2-n)(R-2r)\geq 0$$

obviously from Euler's inequality $R \geq 2r$ and the condition $2-n \geq 0$.
The equality holds if and only if the triangle is equilateral.

□

Remark.

*For $n=2$ we obtain inequality **1.***
For $n=0$ we obtain the well known inequality $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a} \geq 3$.
For $n=-2$ we obtain the known inequality $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a} \geq 2+\frac{2r}{R}$.
Let's notice that for $n \leq 0$ the obtained inequalities are very weak, the inequality is interesting for $n>0$, being the strongest for $n=2$.

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PROBLEM 288
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MARIN CHIRCIU

1. In ΔABC

$$\frac{r_a^2}{h_a}+\frac{r_b^2}{h_b}+\frac{r_c^2}{h_c}\geq 9r.$$

Proposed by Mehmet Şahin - Ankara - Turkey

Proof.

We prove the following lemma:

Lemma.
2. In ΔABC

$$\frac{r_a^2}{h_a}+\frac{h_b^2}{h_b}+\frac{r_c^2}{h_c}=\frac{2R(4R+r)-p^2}{r}.$$

Proof.

We have $\sum\frac{r_a^2}{h_a}=\sum\frac{(\frac{S}{p-a})^2}{\frac{2S}{a}}=\frac{S}{2}\sum\frac{a}{(p-a)^2}=\frac{rp}{2}\cdot\frac{4R(4R+r)-2p^2}{r^2p}=\frac{2R(4R+r)-p^2}{r}$

□

Let's prove inequality ***1.***

Using the ***Lemma****, inequality* ***1****, can be written:*

$\frac{2R(4R+r)-p^2}{r}\geq 9r\Leftrightarrow p^2\leq 2R(4R+r)-9r^2$, *which follows from Gerretsen's inequality:*

$$p^2\leq 4R^2+4Rr+3r^2$$

It remains to prove that:

$4R^2+4Rr+3r^2\leq 2R(4R+r)-9r^2\Leftrightarrow 2R^2-Rr-6r^2\geq 0\Leftrightarrow (R-2r)(2R+3r)\geq 0,$

obviously from Euler's inequality: $R\geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality ***1.*** *can be strengthened:*

3. In ΔABC

$$\frac{r_a^2}{h_a}+\frac{r_b^2}{h_b}+\frac{r_c^2}{h_c}\geq\frac{9R}{2}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma**, inequality **3** can be written:*

$\frac{2R(4R+r)-p^2}{r} \geq \frac{9R}{2} \Leftrightarrow 2p^2 \leq 2R(4R+r)-9Rr$, *which follows from Gerretsen's inequality:*

$$p^2 \leq 4R^2+4Rr+3r^2.$$

It remains to prove that:

$2(4R^2+4Rr+3r^2) \leq 2R(4R+r)-9Rr \Leftrightarrow 8R^2-13Rr-6r^2 \geq 0 \Leftrightarrow (R-2r)(8R+3r) \geq 0,$

obviously from Euler's inequality: $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality 3. is stronger than inequality **1.***

4. In ΔABC

$$\boldsymbol{\frac{r_a^2}{h_a}+\frac{r_b^2}{h_b}+\frac{r_c^2}{h_c} \geq \frac{9R}{2} \geq 9r.}$$

Proof.

*See inequality **3.** and Euler's inequality.*

The equality holds if and only if the triangle is equilateral.

□

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PROBLEM 354 INEQUALITY IN TRIANGLE ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. In ΔABC

$$\sum \frac{a^3}{b+c-a} \geq 4S\sqrt{3}$$

Proposed by D.M. Bătinețu - Giurgiu, Neculai Stanciu - Romania

Proof.

We prove that following lemma:

Lemma.
2. In ΔABC

$$\sum \frac{a^3}{b+c-a} = \frac{p^2(2R-3r)+r^2(4R+r)}{r}.$$

Proof.

$$\sum \frac{a^3}{b+c-a} = \frac{1}{2}\sum \frac{a^3}{p-a} = \frac{1}{2}\cdot\frac{2p^2(2R-3r)+2r^2(4R+r)}{r} = \frac{p^2(2R-3r)+r^2(4R+r)}{r}.$$

□

Let's pass to solving the problem from enunciation.
*Base on the **Lemma** we write the following inequality:*

$$\frac{p^2(2R-3r)+r^2(4R+r)}{r} \geq 4rp\sqrt{3} \Leftrightarrow p^2(2R-3r)+r^2(4R+r) \geq 4r\cdot p\sqrt{3}$$

which follows from Gerretsen's inequality $p^2 \geq 16Rr-5r^2$ and Doucet's inequality $4R+r \geq p\sqrt{3}$
It remains to prove that:

$$(16Rr-5r^2)(2R-3r)+r^2(4R+r) \geq 4r\cdot(4R+r) \Leftrightarrow 16R^2-35Rr+6r^2 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)(16R-3r) \geq 0, \text{ obviously from Euler's inequality } R \geq 2r.$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **1.** can be strengthened:*

3. In ΔABC

$$\sum \frac{a^3}{b+c-a} \geq \frac{4p^2}{3}.$$

Proof.

Base on the ***Lemma*** *the inequality can be written:*

$$\frac{p^2(2R-3r)+r^2(4R+r)}{r} \geq \frac{4p^2}{3} \Leftrightarrow p^2(6R-13r)+3r^2(4R+r) \geq 0$$

We distinguish the cases:

1. If $6R-13r \geq 0$, *the inequality is obvious.*
2. If $6R-13r < 0$, *inequality can be rewritten:*

$$\boldsymbol{p^2(13r-6R) \leq 3r^2(4R+r)}$$

which follows from Gerretsen's inequality $p^2 \leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$(4R^2+4Rr+3r^2)(13r-6R) \leq 3r^2(4R+r) \Leftrightarrow 12R^3-14R^2r-11Rr^2-18r^3 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)(12R^2+10Rr+9r^2) \geq 0,$$ *obviously from Euler's inequality* $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality ***3.*** *is stronger than inequality* ***1.***

4. In ΔABC

$$\boldsymbol{\sum \frac{a^3}{b+c-a} \geq \frac{4p^2}{3} \geq 4S\sqrt{3}}$$

Proof.

See inequality ***3.*** *and Mitrinović's inequality* $p \geq 3r\sqrt{3}$.

The equality holds if and only if the triangle is equilateral.

□

Inequality ***3.*** *can be also strengthened:*

5. In ΔABC

$$\boldsymbol{\sum \frac{a^3}{b+c-a} \geq 2Rp\sqrt{3}}$$

Proposed by Marin Chirciu - Romania

Proof.

Base on the ***Lemma*** *we write the inequality:*

$$\frac{p^2(2R-3r)+r^2(4R+r)}{r} \geq 2Rp\sqrt{3} \Leftrightarrow p^2(2R-3r)+r^2(4R+r) \geq 2R \cdot p\sqrt{3}$$

which follows from Gerretsen's inequality $p^2 \geq 16Rr-5r^2$ *and Doucet's inequality* $4R+r \geq p\sqrt{3}$.

It remains to prove that:

$$(16Rr-5r^2)(2R-3r)+r^2(4R+r) \geq 2R \cdot (4R+r) \Leftrightarrow 3R^2-7Rr+2r^2 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)(3R-r) \geq 0,$$ *obviously from Euler's inequality* $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality ***5.*** *is stronger than inequality* ***3.****:*

6. In ΔABC

$$\boldsymbol{\sum \frac{a^3}{b+c-a} \geq 2Rp\sqrt{3} \geq \frac{4p^2}{3}}$$

Proof.

See inequality ***5.*** *and Mitrinović's inequality* $p \leq \dfrac{3R\sqrt{3}}{2}$

The equality holds if and only if the triangle is equilateral.

□

Remark.

We write the following inequalities:

7. In ΔABC

$$\boldsymbol{\sum \frac{a^3}{b+c-a} \geq 2Rp\sqrt{3} \geq \frac{4p^2}{3} \geq 4S\sqrt{3}}$$

Proof.

See inequality ***5.*** *and Mitrinović's inequality* $3r\sqrt{3} \leq p \leq \dfrac{3R\sqrt{3}}{2}$

The equality holds if and only if the triangle is equilateral.

□

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COMMENTED PROBLEM 30

MARIN CHIRCIU

1. Let be $a, b, c \in (0, \infty)$. Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + \frac{30abc}{(a+b)(b+c)(c+a)} \geq \frac{27}{4}.$$

Proposed by Costel Anghel – Romania

Proof.

Denoting $\frac{a}{b} = x, \frac{b}{c} = y, \frac{c}{a} = z$, *we have* $x, y, z > 0$ *and* $xyz = 1$.

We write $\frac{abc}{(a+b)(b+c)(c+a)} = \frac{1}{(\frac{a+b}{b})(\frac{b+c}{c})(\frac{c+a}{a})} = \frac{1}{(x+1)(y+1)(z+1)}$

We reformulate the problem:

If $x, y, z > 0$ *and* $xyz = 1$, *prove that* $x^2 + y^2 + z^2 + \frac{30}{(x+1)(y+1)(z+1)} \geq \frac{27}{4}$.

Denoting $x + y + z = t$, *we have* $t \geq 3\sqrt[3]{xyz} = 3; x^2 + y^2 + z^2 \geq \frac{(x+y+z)^2}{3} = \frac{t^2}{3}$

$(x+y+z)^2 \geq 3(xy+yz+zx) \Rightarrow xy+yz+zx \leq \frac{t^2}{3}; (x+y+z)^3 \geq 27xyz \Rightarrow xyz \leq \frac{t^3}{27};$

We have $(x+1)(y+1)(z+1) = xyz+xy+yz+zx+x+y+z+1 \leq \frac{t^3}{27} + \frac{t^2}{3} + t + 1.$

It suffices to prove that:

$$\frac{t^2}{3} + \frac{30}{\frac{t^3}{27} + \frac{t^2}{3} + t + 1} \geq \frac{27}{4} \Leftrightarrow \frac{t^2}{3} + \frac{810}{t^3 + 9t^2 + 27t + 27} \geq \frac{27}{4} \Leftrightarrow$$

$$4t^2(t^3 + 9t^2 + 27t + 27) + 9720 \geq 81(t^3 + 9t^2 + 27t + 27) \Leftrightarrow$$

$$4t^5+36t^4+27t^3-621t^2-2187t+7533 \geq 0 \Leftrightarrow (t-3)(4t^2+48t^3+171t^2-108t-2511) \geq 0,$$

Which follows from $t - 3 \geq 0$ *and* $4t^2 + 48t^3 + 171t^2 - 108t - 2511 \geq 324 > 0$.

Equality holds for $x = y = z = 1$, *namely for* $a = b = c$.

□

Remark.

The inequality can be devoloped:

2. Let be $a, b, c \in (0, \infty)$ and $n \leq 32$. Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + n \cdot \frac{abc}{(a+b)(b+c)(c+a)} \geq 3 + \frac{n}{8}$$

Proposed by Marin Chirciu - Romania

Proof.

$$\text{Denoting } \frac{a}{b}=x, \frac{b}{c}=y, \frac{c}{a}=z, \text{ we have } x,y,z>0 \text{ and } xyz=1.$$

$$\text{We write } \frac{abc}{(a+b)(b+c)(c+a)}=\frac{1}{(\frac{a+b}{b})(\frac{b+c}{c})(\frac{c+a}{a})}=\frac{1}{(x+1)(y+1)(z+1)}.$$

The problem can be reformulated:

$$\text{If } x,y,z>0 \text{ and } xyz=1, \text{ prove that } x^2+y^2+z^2+\frac{n}{(x+1)(y+1)(z+1)}\geq 3+\frac{n}{8}.$$

Denoting $x+y+z=t$, *we have* $t\geq 3\sqrt[3]{xyz}=3; x^2+y^2+z^2\geq \frac{(x+y+z)^2}{3}=\frac{t^2}{3}$

$(x+y+z)^2\geq 3(xy+yz+zx)\Rightarrow xy+yz+zx\leq \frac{t^2}{3}; (x+y+z)^3\geq 27xyz\Rightarrow xyz\leq \frac{t^3}{27};$

We have $(x+1)(y+1)(z+1)=xyz+xy+yz+zx+x+y+z+1\leq \frac{t^3}{27}+\frac{t^2}{3}+t+1.$

It suffices to prove that:

$$\frac{t^2}{3}+\frac{n}{\frac{t^3}{27}+\frac{t^2}{3}+t+1}\geq 3+\frac{n}{8}\Leftrightarrow \frac{t^2}{3}+\frac{27n}{t^3+9t^2+27t+27}\geq \frac{n+24}{8}\Leftrightarrow$$

$$8t^2(t^3+9t^2+27t+27)+648n\geq (3n+72)(t^3+9t^2+27t+27)\Leftrightarrow$$

$$8t^5+72t^4+(144-3n)t^3-(27n+432)t^2-(81n+1944)t+567n-1944\geq 0$$

$$\Leftrightarrow (t-3)(8t^4+96t^3+(432-3n)t^2+(864-36n)t+648-189)\geq 0,$$

Which follows from $t-3\geq 0$ *and* $8t^4+96t^3+(432-3n)t^2+(864-36n)t+648-189n\geq 0$, *for* $n\leq 32$.

The equality holds for $x=y=z=1$, *namely for* $a=b=c$.

□

We can formulate the following problem:

3. Let be $a,b,c\in(0,\infty)$. Prove that

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{32abc}{(a+b)(b+c)(c+a)}\geq 7.$$

Proof 1.

$$\text{Denoting } \frac{a}{b}=x, \frac{b}{c}=y, \frac{c}{a}=z, \text{ we have } x,y,z>0 \text{ and } xyz=1.$$

$$\text{We write } \frac{abc}{(a+b)(b+c)(c+a)}=\frac{1}{(\frac{a+b}{b})(\frac{b+c}{c})(\frac{c+a}{a})}=\frac{1}{(x+1)(y+1)(z+1)}.$$

The problem can be reformulated:

$$\text{If } x,y,z>0 \text{ and } xyz=1, \text{ prove that } x^2+y^2+z^2+\frac{30}{(x+1)(y+1)(z+1)}\geq \frac{27}{4}.$$

Denoting $x+y+z=t$, *we have* $t\geq 3\sqrt[3]{xyz}=3; x^2+y^2+z^2\geq \frac{(x+y+z)^2}{3}=\frac{t^2}{3}$

$(x+y+z)^2\geq 3(xy+yz+zx)\Rightarrow xy+yz+zx\leq \frac{t^2}{3}; (x+y+z)^3\geq 27xyz\Rightarrow xyz\leq \frac{t^3}{27};$

We have $(x+1)(y+1)(z+1)=xyz+xy+yz+zx+x+y+z+1\leq \frac{t^3}{27}+\frac{t^2}{3}+t+1$

It suffices to prove that:

$$\frac{t^2}{3}+\frac{32}{\frac{t^3}{27}+\frac{t^2}{3}+t+1}\geq 7 \Leftrightarrow \frac{t^2}{3}+\frac{864}{t^3+9t^2+27t+27}\geq 7 \Leftrightarrow$$

$$t^2(t^3+9t^2+27t+27)+2592\geq 21(t^3+9t^2+27t+27) \Leftrightarrow$$

$$t^5+9t^4+6t^3-162t^2-567t+2025\geq 0 \Leftrightarrow (t-3)(t^4+12t^3+42t^2-36t-675)\geq 0,$$

Which follows from $t-3\geq 0$ *and* $t^4+12t^3+42t^2-36t-675>0$.

The equality holds for $x=y=z=1$, *namely for* $a=b=c$.

□

Proof 2.

We put $n=32$ *in* ***2.***

□

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PROBLEM 356 INEQUALITY IN TRIANGLE ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. In ΔABC

$$\frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}+\frac{p^2}{p^2+r(R-2r)}\geq 2.$$

Proposed by Adil Abdullayev - Baku - Azerbaidian

Proof.

We prove the following lemma:

Lemma.
2. In ΔABC

$$\frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}\geq\frac{6(p^2-r^2-4Rr)}{5p^2-3r^2-12Rr}$$

Proof.

$$\sum m_a^2=\frac{3}{4}\sum a^2=\frac{3}{4}\cdot 2(p^2-r^2-4Rr)=\frac{3}{2}(p^2-r^2-4Rr)$$

Using the known inequality in triangle $4m_bm_c\leq 2a^2+bc$, wherefrom:

$$\sum m_bm_c\leq\frac{1}{4}\sum(2a^2+bc)=\frac{1}{4}\Big(2\sum a^2+\sum bc\Big)=\frac{1}{2}\cdot(p^2-r^2-4Rr)+\frac{1}{4}\cdot(p^2+r^2+4Rr)=$$
$$=\frac{1}{4}(5p^2-3r^2-12Rr).$$

We obtain $\dfrac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}\geq\dfrac{\frac{3}{2}(p^2-r^2-4Rr)}{\frac{1}{4}(5p^2-3r^2-12Rr)}=\dfrac{6(p^2-r^2-4Rr)}{5p^2-3r^2-12Rr}$

□

Let's pass to solving the problem from enuntiation.

Based on ***Lemma*** *it is enough to prove that:*

$$\frac{6(p^2-r^2-4Rr)}{5p^2-3r^2-12Rr}+\frac{p^2}{p^2+r(R-2r)}\geq 2\Leftrightarrow p^4+5p^2r^2-16p^2Rr\geq 0\Leftrightarrow p^2\geq 16Rr-5r^2$$

which is Gerretsen's inequality.

The equality holds if and only if the triangle is equilateral.

Remark.

Inequality ***1.*** *can be devoloped:*

3. In ΔABC

$$\frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}+n\cdot\frac{p^2}{p^2+r(R-2r)}\geq n+1,\ \textit{where}\ n\leq 1$$

Proposed by Marin Chirciu - Romania

Base on ***Lemma*** *it is enough to prove that:*

$$\frac{6(p^2-r^2-4Rr)}{5p^2-3r^2-12Rr}+\frac{np^2}{p^2+r(R-2r)}\geq n+1\Leftrightarrow$$

$$p^4+p^2\Big[(10n-5)r^2-(5n+11)Rr\Big]+(12n-12)R^2r^2+(21-21n)Rr^3+(6-6n)r^4\geq 0\Leftrightarrow$$

$$p^2\Big[p^2+(10n-5)r^2-(5n+11)Rr\Big]+(12n-12)R^2r^2+(21-21n)Rr^3+(6-6n)r^4\geq 0.$$

We have $p^2+(10n-5)r^2-(5n+11)Rr\geq 0$, *for* $n\leq 1$, *because using Gerretsen's inequality* $p^2\geq 16Rr-5r^2$, *we obtain:*

$$p^2+(10n-5)r^2-(5n+11)Rr\geq 16Rr-5r^2+(10n-5)r^2-(5n+11)Rr=$$

$=(5-5n)Rr+(10n-10)r^2=5(1-n)r(R-2r)\geq 0$, *obviously from* $R-2r\geq 0$ *and* $1-n\geq 0$.

It suffices to prove that:

$$(16Rr-5r^2)5(1-n)r(R-2r)+(12n-12)R^2r^2+(21-21n)Rr^3+(6-6n)r^4\geq 0\Leftrightarrow$$

$$(1-n)\Big[(16R-5r)5(R-2r)+12R^2-21Rr-6r^2\Big]\geq\Leftrightarrow(1-n)(R-2r)(17R-7r)\geq 0,$$

obviously from Euler's inequality and condition $n\leq 1$.

Equality holds if and only if the triangle is equilateral.

□

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SOLUTION
INEQUALITY IN TRIANGLE - 413
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. In ΔABC

$$\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3 \geq \frac{27(a+b)(b+c)(c+a)}{8abc}$$

Proposed by Abdullayev - Baku - Azerbaidian

Remark.

The inequality can be strengthened:

2. In ΔABC

$$\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3 \geq \frac{2p^2}{Rr}$$

Marin Chirciu - Romania

Proof.

We prove that following Lemma.

Lemma 1.
3. In ΔABC

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c} \geq \frac{p^2+r^2-2Rr}{4Rr}.$$

Proof.

Using Tereşin's inequality $m_a \geq \frac{b^2+c^2}{4R}$, *formula* $h_a = \frac{bc}{2R}$ *and the known inequality in triangle* $\sum \frac{b^2+c^2}{bc} = \frac{p^2+r^2-2Rr}{2Rr}$, *we obtain:*

$$\sum \frac{m_a}{h_a} \geq \sum \frac{\frac{b^2+c^2}{4R}}{\frac{bc}{2R}} = \frac{1}{2}\sum \frac{b^2+c^2}{bc} = \frac{p^2+r^2-2Rr}{4Rr}$$

Equality holds if and only if the triangle is equilateral.

□

□

Remark.

We can write the inequalities:

4. In ΔABC

$$\boldsymbol{\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\geq\frac{p^2+r^2-2Rr}{4Rr}\geq\frac{7R-2r}{2R}\geq 3.}$$

Proof.
*The first inequality is **Lemma 1**, the second inequality follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$, and the third inequality follows from Euler's inequality $R\geq 2r$.*

Equality holds if and only if the triangle is equilateral.

□

Lemma 2.
5. In ΔABC

$$\boldsymbol{\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^2\geq\frac{2p^2}{3Rr}}$$

Proof.

*Using **Lemma 1**, is enough to prove that:* $\left(\frac{p^2+r^2-2Rr}{4Rr}\right)^2\geq\frac{2p^2}{3Rr}\Leftrightarrow$

$3p^4+p^2(6r^2-44Rr)+12R^2r^2-12Rr^3+3r^4\geq 0\Leftrightarrow p^2(3p^2+6r^2-44Rr)+3r^2(2R-r)^2\geq 0$

□

We distinguish the cases:
1) If $\boldsymbol{3p^2+6r^2-44Rr\geq 0}$*, the inequality is obvious.*
2) If $\boldsymbol{3p^2+6r^2-44Rr<0}$*, inequality we can rewrite:*

$p^2(44Rr-6r^2-3p^2)\leq 3r^2(2R-r)^2$*, true from Gerretsen's inequality:*

$$(4R^2+4Rr+3r^2)\Big[44Rr-6r^2-3(16Rr-5r^2)\Big]\leq 3r^2(2R-r)^2\Leftrightarrow$$

$$\Leftrightarrow 4R^3-2R^2r-9Rr^2-6r^3\geq 0\Leftrightarrow(R-2r)(4R^2+6Rr+3r^2)\geq 0$$

obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

Remark 2.

We can rewrite the inequalities:

6. In ΔABC

$$\boldsymbol{\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^2\geq\frac{2p^2}{3Rr}\geq\frac{9(a+b)(b+c)(c+a)}{8abc}\geq 9.}$$

Proof.

*First inequality is **Lemma 2.***

Let's prove the second inequality.

Using the known identities in triangle: $(a+b)(b+c)(c+a) = 2p(p^2+r^2+2Rr)$

and $abc = 4Rrp$*, the second inequality:*

$$\frac{2p^2}{3Rr} \geq \frac{9 \cdot 2p(p^2+r^2+2Rr)}{8 \cdot 4Rrp} \Leftrightarrow 32p^2 \geq 27(p^2+r^2+2Rr) \Leftrightarrow 5p^2 \geq 27(r^2+2Rr)$$

which follows from Gerretsen's inequality: $p^2 \geq 16Rr - 5r^2$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

The third inequality is the well known inequality $(a+b)(b+c)(c+a) \geq 8abc$ *(Cesaro)*

We've obtained a strengthened inequality in triangle $\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \geq 3.$

*Let's pass to solving inquality **2**:* $\left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^3 \geq \frac{2p^2}{Rr}$

*Base on **Lemma 2** and the the last inequality from **Remark 1** we obtain:*

$$\left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^3 = \left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^2 \cdot \left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right) \geq \frac{2p^2}{3Rr} \cdot 3 = \frac{2p^2}{Rr}$$

Equality holds if and only if the triangle is equilateral.

Remark 3.

*Inequality **2** is stronger then inequality **1**:*

7. In ΔABC

$$\boldsymbol{\left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^3 \geq \frac{2p^2}{Rr} \geq \frac{27(a+b)(b+c)(c+a)}{8abc}}$$

Proof.

*The first inequality is **6**.*

Let's prove the second inequality.

Using the known identities in triangle: $(a+b)(b+c)(c+a) = 2p(p^2+r^2+2Rr)$

and $abc = 4Rrp$*, the second inequality:*

$$\frac{2p^2}{Rr} \geq \frac{27 \cdot 2p(p^2+r^2+2Rr)}{8 \cdot 4Rrp} \Leftrightarrow 32p^2 \geq 27(p^2+r^2+2Rr) \Leftrightarrow 5p^2 \geq 27(r^2+2Rr)$$

which follows from Gerretsen's inequality: $p^2 \geq 16Rr - 5r^2$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark 4.

We can write the inequalities:

8. In ΔABC

$$\boldsymbol{\left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^3 \geq \frac{2p^2}{Rr} \geq \frac{27(a+b)(b+c)(c+a)}{8abc} \geq 27}$$

Proof.

*See **7** and Cesaro's inequality* $(a+b)(b+c)(c+a) \geq 8abc$

□

Remark 5.

*Inequality **2** can also be strengthened:*

9. In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3 \geq \frac{p^2}{3Rr}\left(7-\frac{2r}{R}\right)}$$

Proof.

*Base on **Lemma 2** and on the second inequality from **Remark 1** we obtain:*

$$\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3 = \left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^2 \cdot \left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right) \geq \frac{2p^2}{3Rr}\cdot\frac{7R-2r}{2R} = \frac{p^2}{3Rr}\left(7-\frac{2r}{R}\right)$$

Equality holds if and only if the triangle is equilateral.

□

Remark 6.

*Inequality **9.** is stronger then inequality **2.**:*

10. In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3 \geq \frac{p^2}{3Rr}\left(7-\frac{2r}{R}\right) \geq \frac{2p^2}{Rr}}$$

Proof.

*See inequality **9.** and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark 7.

We can write the inequalities:

11. In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3 \geq \frac{p^2}{3Rr}\left(7-\frac{2r}{R}\right) \geq \frac{2p^2}{Rr} \geq \frac{27(a+b)(b+c)(c+a)}{8abc} \geq 27.}$$

Proof.

*See **10.** and **8.***

Equality holds if and only if the triangle is equilateral.

□

We've obtained again a strengthening of the well known inequality in triangle

$$\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \geq 3$$

*Finally we can propose a development of inequality **2.**:*

12. In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{\left(\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}\right)^n \geq 3^{n-3} \cdot \frac{2p^2}{Rr},\ \textit{where}\ n \geq 2.}$$

Proof.

*Base on **Lemma 2** and the last inequality from **Remark 1** we obtain:*

$$\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^n = \left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^2 \cdot \left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^{n-2} \geq \frac{2p^2}{3Rr} \cdot 3^{n-2} = 3^{n-3} \cdot \frac{2p^2}{Rr}.$$

Equality holds if and only if the triangle is equilateral.

□

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INEQUALITY IN TRIANGLE - 449
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. In ΔABC

$$\boldsymbol{\frac{1}{a(p-a)}+\frac{1}{b(p-b)}+\frac{1}{c(p-c)}\geq\frac{1}{2Rr}}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

We prove the following lemma:

Lemma
2. In ΔABC

$$\boldsymbol{\frac{1}{a(p-a)}+\frac{1}{b(p-b)}+\frac{1}{c(p-c)}=\frac{p^2+(4R+r)^2}{4Rrp^2}}$$

Proof.

We have $\sum\frac{1}{a(p-a)}=\frac{\sum bc(p-b)(p-c)}{abc(p-a)(p-b)(p-c)}=\frac{r^2[p^2+(4R+r)^2]}{4Rrp\cdot r^2p}=\frac{p^2+(4R+r)^2}{4Rrp^2}$

□

Let's pass to solving the inequality from enuntiation.

*Using the **Lemma** we write the inequality:*

$$\frac{p^2+(4R+r)^2}{4Rrp^2}\geq\frac{1}{2Rr}\Leftrightarrow 3p^2\geq(4R+r)^2$$

(Doucet's inequality, which follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$

and Euler's inequality $R\geq 2r$*).*

The equality holds if and only if the triangle is equilateral.

Remark 1.

The inequality can be strengthened

3. In ΔABC

$$\boldsymbol{\frac{1}{a(p-a)}+\frac{1}{b(p-b)}+\frac{1}{c(p-c)}\geq\frac{5R-2r}{4R^2r}}$$

Proof.

*Using the **Lemma** we write the inequality:*

$$\frac{p^2+(4R+r)^2}{4Rrp^2} \geq \frac{5R-2r}{4R^2r} \Leftrightarrow p^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$$

(Blundon-Gerretsen's inequality, which follows from Gergonne's identity

$$H\Gamma^2 = 4R^2\left[1-\frac{2p^2(2R-r)}{R(4R+r)^2}\right],$$ *where* Γ *is Gergonne's point).*

The equality holds if and only if the triangle is equilateral.

□

Remark 2.

*Inequality **3.** is stronger then inequality **1.**:*

4. In ΔABC

$$\boldsymbol{\frac{1}{a(p-a)}+\frac{1}{b(p-b)}+\frac{1}{c(p-c)} \geq \frac{5R-2r}{4R^2r} \geq \frac{1}{2Rr}}$$

Proof.

*The first inequality is inequality **3.***

The second inequality is equivalent with $R \geq 2r$ (Euler's inequality).

The equality holds if and only if the triangle is equilateral.

□

Remark 3.

Let's find an inequality having an opposite sense.

5. In ΔABC

$$\boldsymbol{\frac{1}{a(p-a)}+\frac{1}{b(p-b)}+\frac{1}{c(p-c)} \leq \frac{1}{2r^2}.}$$

Proof.

*Using the **Lemma** we write the inequality:*

$$\frac{p^2+(4R+r)^2}{4Rrp^2} \leq \frac{1}{2r^2} \Leftrightarrow p^2(2R-r) \geq r(4R+r)^2$$

which follows from Gerretsen's inequality $p^2 \geq 16Rr-5r^2$.

It remains to prove that:

$$(16Rr-5r^2)(2R-r) \geq r(4R+r)^2 \Leftrightarrow 16R^2-17Rr+2r^2 \geq 0 \Leftrightarrow (R-2r)(8R-r) \geq 0,$$

obviously form Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark 6.

The double inequality take place:

6. In ΔABC

$$\frac{5R-2r}{4R^2r} \leq \frac{1}{a(p-a)} + \frac{1}{b(p-b)} + \frac{1}{c(p-c)} \leq \frac{1}{2r^2}$$

Marin Chirciu - Romania

Proof.

*See inequalities **3.** and **5.***

The equality holds if and only if the triangle is equilateral.

□

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INEQUALITY IN TRIANGLE - 447
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. In ΔABC

$$\frac{2m_am_bm_c}{h_ah_bh_c}+1\geq\frac{(r_a+r_b+r_c)^2}{r_ar_b+r_br_c+r_cr_a}$$

Proposed by Adil Abdullayev - Baku - Azerbaidian

We prove the following lemmas:

Lemma 1.
2. In ΔABC

$$\frac{m_am_bm_c}{h_ah_bh_c}\geq\frac{R}{2r}.$$

Proof.

(1) $$\text{From } m_a\geq\sqrt{p(p-a)}\Rightarrow m_am_bm_c\geq pS=rp^2$$

(2) $$\text{and from } h_a=\frac{2S}{a}\Rightarrow h_ah_bh_c=\frac{2r^2p^2}{R}$$

$$\text{From (1) and (2) it follows } \frac{m_am_bm_c}{h_ah_bh_c}\geq\frac{rp^2}{\frac{2r^2p^2}{R}}=\frac{R}{2r}.$$

□

Lemma 2.
3. In ΔABC

$$p^2\geq\frac{r(4R+r)^2}{R+r}$$

Proof.

$$\text{Using Yang Xue Zhi's inequality } p^2\geq 16Rr-5r^2+\frac{r^2(R-2r)}{R-r}$$

(stronger inequality then Gerretsen's inequality $p^2\geq 16Rr-5r^2$), it remains to prove that:

$$16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\geq\frac{r(4R+r)^2}{R+r}\Leftrightarrow\frac{r(16R^2-20Rr+3r^2)}{R-r}\geq\frac{r(4R+r)^2}{R+r}\Leftrightarrow$$

$2R^2-5Rr+2r^2\geq 0\Leftrightarrow(R-2r)(2R-r)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$.

□

Let's pass to solving the inequality from enunciation.

*Using **Lemma 1** and the known identities in triangle*

$r_a + r_b + r_c = 4R + r$ *and* $r_a r_b + r_b r_c + r_c r_a = p^2$ *it suffices to prove that*

$$\frac{R}{r} \geq \frac{(4R+r)^2}{p^2} \Leftrightarrow p^2 \geq p^2 \geq \frac{r(4R+r)^2}{R+r} \text{ (Lemma 2).}$$

Equality holds if and only if the triangle is equilateral.

Remark.

Inequality can be devoloped:

4. In ΔABC

$$\boldsymbol{\lambda \cdot \frac{m_a m_b m_c}{h_a h_b h_c} + 3 - \lambda \geq \frac{(r_a + r_b + r_c)^2}{r_a r_b + r_b r_c + r_c r_a}, \textit{ where } \lambda \geq 2.}$$

Proof.

$$\text{We have } \lambda \cdot \frac{m_a m_b m_c}{h_a h_b h_c} + 3 - \lambda \cdot \frac{m_a m_b m_c}{h_a h_b h_c} + 1 \geq \frac{(r_a + r_b + r_c)^2}{r_a r_b + r_b r_c + r_c r_a}$$

where the first inequality is equivalent with $(\lambda - 2)\left(\frac{m_a m_b m_c}{h_a h_b h_c} - 1\right) \geq 0$

obviously from $\frac{m_a m_b m_c}{h_a h_b h_c} \geq 1$ *and the condition* $\lambda \geq 2$*, and the second inequality is inequality* ***1.***

Equality holds if and only if the triangle is equilateral.

□

Remark.

For $\lambda = 2$ *in inequality* ***4.*** *we obtain inequality* ***1.***

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INEQUALITY IN TRIANGLE - 435
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. $\boldsymbol{\Delta ABC}$

$$\sum \frac{a^2 \sin^2 A}{\sin B \sin C} \geq 36r^2.$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

We prove the following lemma:

Lemma.
2. In $\boldsymbol{\Delta ABC}$

$$\sum \frac{a^4}{bc} = \frac{p^4 - 10p^2(r^2 + Rr) + 5r^2(r^2 + 6Rr + 8R^2)}{2Rr}$$

Proof.

$$\sum \frac{a^4}{bc} = \frac{\sum a^5}{abc} = \frac{2p[p^4 - 10p^2(r^2 + Rr) + 5r^2(r^2 + 6Rr + 8R^2)]}{4Rrp} =$$

$$= \frac{p^4 - 10p^2(r^2 + Rr) + 5r^2(r^2 + 6Rr + 8R^2)}{2Rr}$$

□

Let's pass to solving the inequality from enuntiation.

*Using the sines theorem and the **Lemma** above, the inequality from enunciation can be written:*

$$\frac{p^4 - 10p^2(r^2 + Rr) + 5r^2(r^2 + 6Rr + 8R^2)}{2Rr} \geq 36r^2 \Leftrightarrow$$

$$\Leftrightarrow p^2(p^2 - 10r^2 - 10Rr) + r^2(5r^2 - 42Rr + 40R^2) \geq 0$$

We distinguish the following cases:

1. If $p^2 - 10r^2 - 10Rr \geq 0$ *, the inequality is obviously.*
2. If $p^2 - 10r^2 - 10Rr < 0$ *, the inequality can be rewritten:*
$p^2(10Rr + 10r^2 - p^2) \leq r^2(5r^2 - 42Rr + 40R^2)$ *, which follows from Gerretsen's inequality:* $16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$ *. It remains to prove that:*

$$(4R^2 + 4Rr + 3r^2)(10Rr + 10r^2 - 16Rr + 5r^2) \leq r^2(5r^2 - 42Rr + 40R^2) \Leftrightarrow$$

$$6R^3 + R^2r - 21Rr^2 - 10r^3 \geq 0 \Leftrightarrow (R - 2r)(6R^2 + 13Rr + 5r^2) \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

The inequality holds if and only if the triangle is equilateral.

Remark.

Inequality can be strengthened:

3. In ΔABC

$$\sum \frac{a^2 \sin^2 A}{\sin B \sin C} \geq 18Rr.$$

Marin Chirciu - Romania

Proof. *Using the sines theorem and the* ***Lemma*** *above, the inequality from enunciation can be written:*

$$\frac{p^2 - 10p^2(r^2 + Rr) + 5r^2(r^2 + 6Rr + 8R^2)}{2Rr} \geq 18Rr \Leftrightarrow$$

$$\Leftrightarrow p^2(p^2 - 10r^2 - 10Rr) + r^2(5r^2 + 30Rr + 4R^2) \geq 0.$$

We distinguish the following cases:

1. If $p^2 - 10r^2 - 10Rr \geq 0$, *the inequality is obviously.*
2. If $p^2 - 10r^2 - 10Rr < 0$, *inequality can be rewritten:*
$p^2(10Rr + 10r^2 - p^2) \leq r^2(5r^2 + 30Rr + 4R^2)$, *which follows from Gerretsen's inequality:* $16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$$(4R^2 + 4Rr + 3r^2)(10Rr + 10r^2 - 16Rr + 5r^2) \leq r^2(5r^2 + 30Rr + 4R^2) \Leftrightarrow$$

$6R^3 - 8R^2r - 3Rr^2 - 10r^3 \geq 0 \Leftrightarrow (R - 2r)(6R^2 - 4Rr + 5r^2) \geq 0$, *obviously form Euler's inequality* $R \geq 2r$.

The inequality holds if and only if the triangle is equilateral.

$\square$

Remark.

Inequality ***3.*** *is stronger then inequality* ***1.***:

4. In ΔABC

$$\sum \frac{a^2 \sin^2 A}{\sin B \sin C} \geq 18Rr \geq 36r^2.$$

Proof. *The first inequality is inequality* ***3.***, *and the second inequality is equivalent with* $R \geq 2r$ (Euler's inequality).
Equality holds if and only if the triangle is equilateral. $\square$

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PROBLEM TRIANGLE MARATHON - 377
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. In ΔABC

$$\boldsymbol{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq 4r(4R+r)}$$

Proposed by Kevin Soto Palacios - Huarmey - Peru

Proof.

Using the means inequality we obtain:

$$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq 3\sqrt[3]{(abc)^2}=3\sqrt[3]{(4Rrp)^2}\geq 4r(4R+r),$$

where the last inequality is equivalent with:

$$27\cdot(4Rrp)^2\geq 64r^3(4Rr+r)^3\Leftrightarrow 27R^2p^2\geq 4r(4R+r)^3$$

which follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$. It remains to prove that:

$$27R^2(16Rr-5r^2)\geq 4r(4R+r)^3\Leftrightarrow 176R^3-327R^2r-48Rr^2-4r^3\geq 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)(176R^2+25Rr+2r^2)\geq 0,$$ *obviously from Euler's inequality $R\geq 2r$.*

The equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **1.** can be strengthened:*

2. In ABC

$$\boldsymbol{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq 18Rr.}$$

Proof.

Using means inequality we obtain:

$$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq 3\sqrt[3]{(abc)^2}=3\sqrt[3]{(4Rrp)^2}\geq 18Rr,$$

where the last inequality is equivalent with:

$$(4Rrp)^2\geq(6Rr)^3\Leftrightarrow 16R^2r^2p^2\geq 216R^3r^3\Leftrightarrow 2p^2\geq 27Rr$$

which follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$. It remains to prove that:

$$2(16Rr-5r^2)\geq 27Rr\Leftrightarrow R\geq 2r \text{ (Euler's inequality).}$$

The equality holds if and only if the triangle is equilateral.

□

Remark 2.

*Inequality **1.** is stronger then inequality **2.**:*

3. In ΔABC

$$\boldsymbol{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq 18Rr\geq 4r(4R+r)}$$

Proof.

*See inequality **2.** and Euler's inequality $R\geq 2r$.*

The equality holds if and only if the triangle is equilateral.

□

Remark 3.

*Inequality **1.** can be developed:*

4. In ΔABC

$$\boldsymbol{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq r\Big[nR+(36-2n)r\Big]}, \textbf{ where } \boldsymbol{n\leq 19}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the means inequality we obtain:

$$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq 3\sqrt[3]{(abc)^2}=3\sqrt[3]{(4Rrp)^2}\geq r\Big[nR+(36-2n)r\Big],$$

where the last inequality is equivalent with:

$$27\cdot(4Rrp)^2\geq r^3\Big[nR+(36-2n)r\Big]^3\Leftrightarrow 27\cdot 16R^2r^2p^2\geq 64r^3\Big[nR+(36-2n)r\Big]^3\Leftrightarrow$$

$$432R^2p^2\geq r\Big[nR+(36-2n)r\Big]^3$$

which follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$. It remains to prove that:

$$432R^2(16Rr-5r^2)\geq 4\Big[nR+(36-2n)r\Big]^3\Leftrightarrow 432R^2(16R-5r)\geq\Big[nR+(36-2n)r\Big]^3\Leftrightarrow$$

$$6912R^3-2160R^2r\geq(nR)^3+3\cdot(nR)^2\cdot(36-2n)r+3(nR)\Big[(36-2n)r\Big]^2+\Big[(36-2n)r\Big]^3\Leftrightarrow$$

$$(6912-n^3)R^3+(6n^3-108n^2-2160)R^2r+(-12n^3+432n^2-3888n)Rr^2+$$
$$+(8n^3-432n^2+7776n-46656)r^3\geq 0\Leftrightarrow$$

$$\Leftrightarrow(R-2r)\Big[(6912-n^3)R^2+(4n^3-108n^2+11664)Rr+(-4n^3+216n^2+23328)r^2\Big]\geq 0$$

obviously from Euler's inequality $R\geq 2r$ and the condition $n\geq 19$

which assures the positivity of the right parenthesis.

The equality holds if and only if the triangle is equilateral.

□

Note.

*For $n=16$ we obtain inequality **1.**, and for $n=18$ we obtain inequality **2.***

Let's find an inequality having an opposite sense.

5. In ΔABC

$$\boldsymbol{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\leq 4(R+r)^2}.$$

Proof.

Using inequality $xy+yz+zx \leq x^2+y^2+z^2$ *for* $x=\sqrt{bc}, y=\sqrt{ca}, z=\sqrt{ab}$ *we obtain:*
$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab} \leq ab+bc+ca = p^2+r^2+4Rr \leq 4R^2+4Rr+3r^2+r^2+4Rr = 4(R+r)^2.$
where the last inequality follows from Gerretsen's inequality $p^2 \leq 4R^2+4Rr+3r^2$.

The equality holds if and only if the triangle is equilateral.

□

We can write the double inequality:

6. In ΔABC

$$\mathbf{18}\boldsymbol{Rr} \leq \boldsymbol{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}} \leq \mathbf{4}\boldsymbol{(R+r)}^{\mathbf{2}}$$

Proof.

See inequalities ***2.*** *and* ***5.***

The equality holds if and only if the triangle is equilateral.

□

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Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
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PROBLEM JP.080 RMM
NUMBER 6 AUTUMN 2017
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. Prove that in any triangle ABC,

$$\frac{a^2+b^2+c^2}{a+b+c}\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\geq 2\sqrt{3}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

Proof.

We use the following lemma:

Lemma 1.
In ΔABC

$$\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)^2\geq\frac{108}{5p^2-3r^2-12Rr}$$

Proof.

Using the inequality $(x+y+z)^2\geq 3(xy+yz+zx)$, *with* $x=\frac{1}{m_a}, y=\frac{1}{m_b}, z=\frac{1}{m_c}$ *we obtain*

$$\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)^2\geq 3\left(\frac{1}{m_am_b}+\frac{1}{m_bm_c}+\frac{1}{m_cm_a}\right)\overbrace{\geq}^{(1)}3\left(\frac{4}{2a^2+bc}+\frac{4}{2b^2+ca}+\frac{4}{2c^2+ab}\right)\geq$$

$$\overbrace{\geq}^{Bergstrom} 12\cdot\frac{9}{2a^2+bc+2b^2+ca+2c^2+ab}=\frac{108}{2(a^2+b^2+c^2)+ab+bc+ca}=$$

$$=\frac{108}{2\cdot 2(p^2-r^2-4Rr)+p^2+r^2+4Rr}=\frac{108}{5p^2-3r^2-12Rr}$$

where inequality (1) follows from $4m_bm_c\leq 2a^2+bc$ *and analogs.*

Equality holds if and only if the triangle is equilateral.

□

Let's pass to solving the inequality from enuntiation.

*Using **Lemma 1** and* $\sum a=2p, \sum a^2=2(p^2-r^2-4Rr)$ *it is enough to prove that*

$$\left(\frac{2(p^2-r^2-4Rr)}{2p}\right)^2\cdot\frac{108}{5p^2-3r^2-12Rr}\geq 12\Leftrightarrow 9(p^2-r^2-4Rr)^2\geq p^2(5p^2-3r^2-12Rr)\Leftrightarrow$$

$$\Leftrightarrow p^2(4p^2-15r(4R+r))+9r^2(4R+2)^2\geq 0.$$ *We distinguish the cases:*

Case 1. If $4p^2-15r(4R+r)\geq 0$ *the inequality is obvious.*
Case 2. If $4p^2-15r(4R+r)<0$ *we write the inequality*

$p^2(15r(4R+r)-4p^2) \le 9r^2(4R+r)^2$, *which follows from Gerretsen's inequality* $16Rr-5r^2 \le p^2 \le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$(4R^2+4Rr+3r^2)(15r(4R+r)-4(16Rr-5r^2)) \le 9r^2(4R+r)^2 \Leftrightarrow$$

$$\Leftrightarrow (4R^2+4Rr+3r^2)(-4Rr+35r^2) \le 9r^2(4R+r)^2 \Leftrightarrow 4R^3+5R^2r-14Rr^2-24r^3 \ge 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)(4R^2+13Rr+12r^2) \ge 0, \textit{ obviously from Euler's inequality } R \ge 2r$$

Equality holds if and only if the triangle is equilateral.

□

Remark 1.

*Inequality **1** can be developed:*

3. In ΔABC

$$\frac{a^2+b^2+c^2}{a+b+c}\left(\frac{1}{m_a+m_b}+\frac{1}{m_b+m_c}+\frac{1}{m_c+m_a}\right) \ge \sqrt{3}$$

Proof.

Using the following lemma:

Lemma 2.

4. In ΔABC

$$\left(\frac{1}{m_a+m_b}+\frac{1}{m_b+m_c}+\frac{1}{m_c+m_a}\right)^2 \ge \frac{36}{7p^2-5r^2-20Rr}.$$

Proof.

Using the inequality $(x+y+z)^2 \ge 3(xy+yz+zx)$,

with $x=\dfrac{1}{m_a+m_b}, y=\dfrac{1}{m_b+m_c}, z=\dfrac{1}{m_c+m_a}$ *we obtain*

$$\left(\sum\frac{1}{m_b+m_c}\right)^2 \ge 3\left(\sum\frac{1}{(m_a+m_b)(m_a+m_c)}\right) \overbrace{\ge}^{Bergstrom} 3\cdot\frac{9}{\sum(m_a+m_b)(m_a+m_c)} =$$

$$= \frac{27}{\sum(m_a^2+m_am_b+m_bm_c+m_cm_a)} = \frac{27}{\sum m_a^2+3\sum m_bm_c} \overbrace{\ge}^{(1)} \frac{27}{\frac{3}{4}\sum a^2+\frac{3}{4}\sum(2a^2+bc)} =$$

$$= \frac{36}{3\sum a^2+\sum bc} = \frac{36}{3\cdot 2(p^2-r^2-4Rr)+p^2+r^2+4Rr} = \frac{36}{7p^2-5r^2-20Rr}$$

where inequality (1) follows from $4m_bm_c \le 2a^2+bc$ *and analogs.*

Equality holds if and only if the triangle is equilateral.

□

*Let's pass to solving inequality **3.***

*Using **Lemma 2** and* $\sum a = 2p, \sum a^2 = 2(p^2-r^2-4Rr)$ *it is enough to prove that*

$$\left(\frac{2(p^2-r^2-4Rr)^2}{2p}\right)^2 \cdot \frac{36}{7p^2-5r^2-20Rr} \ge 3 \Leftrightarrow 12(p^2-r^2-4Rr)^2 \ge p^2(7p^2-5r^2-20Rr) \Leftrightarrow$$

$\Leftrightarrow p^2(5p^2-19r(4R+r))+12r^2(4R+r)^2 \ge 0$. *We distinguish the following cases:*

Case 1. If $5p^2 - 19r(4R+r) \geq 0$ inequality is obvious.
Case 2. If $5p^2 - 19r(4R+r) < 0$ inequality can be rewritten
$p^2(19r(4R+r) - 4p^2) \leq 12r^2(4R+r)^2$, which follows from Gerretsen's inequality

$16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$. It remains to prove that:

$$(4R^2 + 4Rr + 3r^2)(19r(4R+r) - 5(16Rr - 5r^2)) \leq 9r^2(4R+r)^2 \Leftrightarrow$$
$$\Leftrightarrow (4R^2+4Rr+3r^2)(-4Rr+44r^2) \leq 12r^2(4R+r)^2 \Leftrightarrow 4R^3+8R^2r-17Rr^2-30r^3 \geq 0 \Leftrightarrow$$
$$\Leftrightarrow (R-2R)(4R^2 + 16Rr + 15r^2) \geq 0,$$ obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark 2.

*Inequality **3.** can be developed:*

5. In ΔABC

$$\frac{a^2+b^2+c^2}{a+b+c}\left(\frac{1}{m_a+\lambda m_b}+\frac{1}{m_b+\lambda m_c}+\frac{1}{m_c+\lambda m_a}\right) \geq \frac{2\sqrt{3}}{1+\lambda}, \text{ where } \lambda \geq 0$$

Proposed by Marin Chirciu - Romania

Proof.

We use the following lemma:

Lemma 3.

6. In ΔABC, for $\lambda \geq 0$,

$$\left(\frac{1}{m_a+\lambda m_b}+\frac{1}{m_b+\lambda m_c}+\frac{1}{m_c+\lambda m_a}\right)^2 \geq$$
$$\geq \frac{108}{(5\lambda^2+11\lambda+5)p^2-(3\lambda^2+9\lambda+3)r^2-(12\lambda^2+36\lambda+12)Rr}$$

Proof.

Using the inequality $(x+y+z)^2 \geq 3(xy+yz+zx)$,

with $x = \frac{1}{m_a+\lambda m_b}, y = \frac{1}{m_b+\lambda_c}, z = \frac{1}{m_c+\lambda m_a}$ *we obtain*

$$\left(\sum \frac{1}{m_b+\lambda m_c}\right)^2 \geq 3\left(\sum \frac{1}{(m_a+\lambda m_b)(\lambda m_a+m_c)}\right) \overbrace{\geq}^{Bergstrom} 3 \cdot \frac{9}{\sum(m_a+\lambda m_b)(\lambda m_a+m_c)} =$$
$$= \frac{27}{\sum(\lambda m_a^2+\lambda^2 m_a m_b+\lambda m_b m_c+m_c m_a)} = \frac{27}{\lambda\sum m_a^2+(\lambda^2+\lambda+1)\sum m_b m_c} \overbrace{\geq}^{(1)}$$
$$\overbrace{\geq}^{(1)} \frac{27}{\frac{3\lambda}{4}\sum a^2+\frac{\lambda^2+\lambda+1}{4}\sum(2a^2+bc)} = \frac{108}{(2\lambda^2+5\lambda+2)\sum a^2+(\lambda^2+\lambda+1)\sum bc} =$$
$$= \frac{108}{(2\lambda^2+5\lambda+2)\cdot 2(p^2-r^2-4Rr)+(\lambda^2+\lambda+1)(p^2+r^2+4Rr)} =$$
$$= \frac{108}{(5\lambda^2+11\lambda+5)p^2-(3\lambda^2+9\lambda+3)r^2-(12\lambda^2+36\lambda+12)Rr}$$

where inequality (1) follows from $4m_b m_c \leq 2a^2+bc$ *and analogs.*

Equality holds if and only if the triangle is equilateral.

□

*Let's pass to solve inequality **5.***

*Using **Lemma 3** and* $\sum a = 2p, \sum a^2 = 2(p^2 - r^2 - 4Rr)$ *it is enough to prove that*

$$\left(\frac{2(p^2-r^2-4Rr)}{2p}\right)^2 \cdot \frac{108}{(5\lambda^2+11\lambda+5)p^2-(3\lambda^2+9\lambda+3)r^2-(12\lambda^2+36\lambda+12)Rr} \geq \frac{12}{1+\lambda}$$

$$\Leftrightarrow 9(\lambda+1)^2(p^2-r^2-4Rr)^2 \geq p^2((5\lambda^2+11\lambda+5)p^2-(36\lambda^2+9\lambda+3)r^2-(12\lambda^2+36\lambda+12))$$

$$\Leftrightarrow p^2((4\lambda^2+7\lambda+4)p^2-3(5\lambda^2+9\lambda+5)r(4Rr+r))+9(\lambda+1)^2r^2(4R+r)^2 \geq 0$$

We distinguish the following cases:

Case 1. If $(4\lambda^2+7\lambda+4)p^2-3(5\lambda^2+9\lambda+5)r(4R+r) \geq 0$ *inequality is obvious.*
Case 2. If $(4\lambda^2+7\lambda+4)p^2-3(5\lambda^2+9\lambda+5)r(4R+r) < 0$ *we write the following inequality:*

$$p^2(3(5\lambda^2+9\lambda+5)r(4R+r)-(4\lambda^2+7\lambda+4)p^2) \leq 9(\lambda+1)^2r^2(4R+r)^2$$

which follows from Blundon-Gerretsen's inequality $16Rr-5r^2 \leq p^2 \leq \dfrac{R(4R+r)^2}{2(2R-r)}$

It remains to prove that:

$$\frac{R(4R+r)^2}{2(2R-r)} \cdot (3(5\lambda^2+9\lambda+5)r(4R+r)-(4\lambda^2+7\lambda+4)p^2) \leq 9(\lambda+1)^2r^2(4R+r)^2 \Leftrightarrow$$

$$\Leftrightarrow (4\lambda^2+4\lambda+4)R^2+(\lambda^2+10\lambda+1)Rr-(18\lambda^2+36\lambda+18)r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)((4\lambda^2+4\lambda+4)R+(9\lambda^2+18\lambda+9)r) \geq 0$, *obviously from* $R \geq 2r$ *(Euler).*

Equality holds if and only if the triangle is equialteral.

□

Note.

For $\lambda = 0$ *in inequality **5.** we obtain inequality **1.**, and for the case* $\lambda = 1$ *we obtain **3.***

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Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
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PROBLEM JP.088 RMM
NUMBER 6 AUTUMN 2017
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. Let ABC be an acute triangle. Prove that

$$\sum \sqrt{\cos A \sin B \sin C} \le \frac{3}{2}\sqrt{\frac{3}{2}}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Proof.

Using CBS inequality we obtain

$$\left(\sum \sqrt{\cos A \cdot \sin B \cdot \sin C}\right)^2 \le \sum \cos A \cdot \sum \sin B \sin C = \left(1+\frac{r}{R}\right)\cdot \frac{p^2+r^2+4Rr}{4R^2} < \frac{3}{2}\cdot\frac{9}{4} = \left(\frac{3}{2}\sqrt{\frac{3}{2}}\right)^2,$$

where the last inequality follows from:

1) $1+\frac{r}{R} \le \frac{3}{2} \Leftrightarrow R \ge 2r$ *(Euler's inequality).*
2) $\frac{p^2+r^2+4Rr}{4R^2} \le \frac{9}{4} \Leftrightarrow p^2 \le 9R^2-4Rr-r^2$ *, true from Gerretsen's inequality* $p^2 \le 4R^2+4Rr+3r^2$ *. It remains to prove that:*

$$4R^2+4Rr+3r^2 \le 9R^2-4Rr-r^2 \Leftrightarrow 5R^2-8Rr-4r^2 \ge 0 \Leftrightarrow (R-2r)(5R+2r) \ge 0,$$

obviously from Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

In the same way it can be proposed:

2. In ΔABC

$$\sum \sqrt{\sin A \cos B \cos C} \le \frac{3}{2}\sqrt{\frac{\sqrt{3}}{2}}$$

Proposed by Marin Chirciu

Proof.

Using CBS inequality we obtain

$$\left(\sum \sqrt{\sin A \cdot \cos B \cos C}\right)^2 \le \sum \sin A \cdot \sum \cos B \cos C = \frac{rp}{2R^2}\cdot\frac{p^2+r^2-4R^2}{4R^2} \le \frac{3\sqrt{3}}{8}\cdot\frac{3}{4} = \left(\frac{3}{2}\sqrt{\frac{\sqrt{3}}{2}}\right)^2$$

where the last inequality follows from:

1) $\frac{rp}{2R^2} \le \frac{3\sqrt{3}}{8} \Leftrightarrow p \le \frac{3R^2\sqrt{3}}{4r}$ *, true from Mitrinović's inequality* $p \le \frac{3R\sqrt{3}}{2}$ *and Euler's inequality* $R \ge 2r$

2) $\frac{p^2+r^2-4R^2}{4R^2} \leq \frac{3}{4} \Leftrightarrow p^2 \leq 7R^2 - r^2$ *, true from Gerretsen's inequality* $p^2 \leq 4R^2 + 4Rr + 3r^2$. It remains to prove that:

$$4R^2 + 4Rr + 3r^2 \leq 7R^2 - r^2 \Leftrightarrow 3R^2 - 4Rr - 4r^2 \geq 0 \Leftrightarrow (R-2r)(3R+2r) \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

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Romanian Mathematical Magazine
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PROBLEM SP.083 RMM
NUMBER 6 AUTUMN 2017
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. In ΔABC

$$\boldsymbol{\left(1+\frac{1}{m_a}\right)\left(1+\frac{1}{m_b}\right)\left(1+\frac{1}{m_c}\right)\geq\left(1+\frac{2}{3R}\right)^3.}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Proof.

Using Huygens' inequality we obtain

$$\left(1+\frac{1}{m_a}\right)\left(1+\frac{1}{m_b}\right)\left(1+\frac{1}{m_c}\right)\geq\left(1+\sqrt[3]{\frac{1}{m_am_bm_c}}\right)^3\geq\left(1+\frac{2}{3R}\right)^3,$$

where the last inequality is equivalent with:

$$\sqrt[3]{\frac{1}{m_am_bm_c}}\geq\frac{2}{3R}\Leftrightarrow\frac{1}{m_am_bm_c}\geq\left(\frac{2}{3r}\right)^3\Leftrightarrow m_am_bm_c\leq\left(\frac{3R}{2}\right)^3$$

which follows from means inequality and the known inequality in triangle $\sum m_a\leq 4R+r\leq\frac{9R}{2}$;

indeed: $m_am_bm_c\leq\left(\frac{m_a+m_b+m_c}{3}\right)^3\leq\left(\frac{4R+r}{3}\right)^3\leq\left(\frac{3R}{2}\right)^3$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

In the same way it can be proposed:

2. In ΔABC

$$\boldsymbol{\left(1+\frac{1}{m_a+m_b}\right)\left(1+\frac{1}{m_b+m_c}\right)\left(1+\frac{1}{m_c+m_a}\right)\geq\left(1+\frac{1}{3R}\right)^3}$$

Proof.

Using Huygens' inequality we obtain

$$\left(1+\frac{1}{m_a+m_b}\right)\left(1+\frac{1}{m_b+m_c}\right)\left(1+\frac{1}{m_c+m_a}\right)\geq\left(1+\sqrt[3]{\frac{1}{(m_a+m_b)(m_b+m_c)(m_c+m_a)}}\right)^3\geq\left(1+\frac{1}{3R}\right)^3$$

where the last inequality is equivalent with:

$$\sqrt[3]{\frac{1}{(m_a+m_b)(m_b+m_c)(m_c+m_a)}}\geq\frac{1}{3R}\Leftrightarrow\frac{1}{(m_a+m_b)(m_b+m_c)(m_c+m_a)}\geq\left(\frac{1}{3R}\right)^3\Leftrightarrow$$

$$(m_a+m_b)(m_b+m_c)(m_c+m_a)\leq(3R)^3$$

which follows from means inequality and the known inequality in triangle

$$\sum m_a \leq 4R + r \leq \frac{9R}{2}; \textit{ indeed:}$$

$$(m_a+m_b)(m_b+m_c)(m_c+m_a) \leq \Big(\frac{2(m_a+m_b+m_c)}{3}\Big)^3 \leq \Big(\frac{2(4R+r)}{3}\Big)^3 \leq (3R)^3$$

Equality holds if and only if the triangle is equilateral.

□

3. In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{\Big(1+\frac{1}{m_a+\lambda m_b}\Big)\Big(1+\frac{1}{m_b+\lambda m_c}\Big)\Big(1+\frac{1}{m_c+\lambda m_a}\Big) \geq \Big(1+\frac{2}{3(\lambda+1)R}\Big)^3}, \textit{\textbf{ where }} \boldsymbol{\lambda \geq 0}$$

Proposed by Marin Chirciu - Romania

Proof.

Using Huygens' inequality we obtain

$$\Big(1+\frac{1}{m_a+\lambda m_b}\Big)\Big(1+\frac{1}{m_b+\lambda m_c}\Big)\Big(1+\frac{1}{m_c+\lambda m_a}\Big) \geq \Big(1+\sqrt[3]{\frac{1}{(m_a+\lambda m_b)(m_b+\lambda m_c)(m_c+\lambda m_a)}}\Big)^3 \geq$$

$$\geq \Big(1+\frac{2}{3(\lambda+1)R}\Big)^3$$

where the last inequality is equivalent with:

$$\sqrt[3]{\frac{1}{(m_a+\lambda m_b)(m_b+\lambda m_c)(m_c+\lambda m_a)}} \geq \frac{1}{3R} \Leftrightarrow \frac{1}{(m_a+\lambda m_b)(m_b+\lambda m_c)(m_c+\lambda m_a)} \geq \Big(\frac{2}{3(\lambda+1)R}\Big)^3 \Leftrightarrow$$

$$(m_a+\lambda m_b)(m_b+\lambda m_c)(m_c+\lambda m_a) \leq \Big(\frac{3(\lambda+1)R}{2}\Big)^3$$

which follows from means inequality and the known inequality in triangle

$$\sum m_a \leq 4R + r \leq \frac{9R}{2}; \textit{ indeed:}$$

$$(m_a+\lambda m_b)(m_b+\lambda m_c)(m_c+\lambda m_a) \leq \Big(\frac{(1+\lambda)(m_a+m_b+m_c)}{3}\Big)^3 \leq \Big(\frac{(1+\lambda)(4R+r)}{3}\Big)^3 \leq \Big(\frac{3(\lambda+1)R}{2}\Big)^3$$

Equality holds if and only if the triangle is equilateral.

□

Note

*For $\lambda = 0$ we obtain inequality **1.**, and for $\lambda = 1$ we obtain inequality **2.***

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Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

PROBLEM SP.077 RMM
NUMBER 6 AUTUMN 2017
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. Let ABC be an acute triangle. Prove that

$$\frac{m_a}{h_a}\cdot\cos A+\frac{m_a}{h_c}\cdot\cos B+\frac{m_c}{h_c}\cdot\cos C\geq\frac{3}{2}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

Proof.

Using $m_a\geq\frac{b^2+c^2}{4R}, h_a=\frac{bc}{2R}$ *and* $\cos A=\frac{b^2+c^2-a^2}{2bc}$
we obtain $\frac{m_a}{h_a}\cdot\cos A\geq\frac{(b^2+c^2)(b^2+c^2-a^2)}{4b^2c^2}$

It follows $\sum\frac{m_a}{h_a}\cdot\cos A\geq\sum\frac{(b^2+c^2)(b^2+c^2-a^2)}{4b^2c^2}=\frac{\sum a^2(b^2+c^2)(b^2+c^2-a^2)}{4a^2b^2c^2}=\frac{6a^2b^2c^2}{4a^2b^2c^2}=\frac{3}{2}$

Equality holds if and only if the triangle is equilateral.

□

Note.

From the above proof the condition of acute-angled triangle is not necessary.

Remark.

In the same way it can be proposed:

2. In ΔABC

$$\frac{m_a}{h_a}\cdot(\cos B+\cos C)+\frac{m_a}{h_b}\cdot(\cos C+\cos A)+\frac{m_c}{h_c}\cdot(\cos A+\cos B)\leq\frac{2R}{r}-1$$

Proposed by Marin Chirciu - Romania

Proof.

We have

$$\sum\frac{m_a}{h_a}\cdot(\cos B+\cos C)=\sum\frac{m_a}{h_a}\cdot(\cos A+\cos B+\cos C-\cos A)=\sum\frac{m_a}{h_a}\sum\cos A-\sum\frac{m_a}{h_a}\cos A\leq$$

$$\leq\left(1+\frac{r}{R}\right)\sum\frac{m_a}{h_a}-\frac{3}{2}\leq\frac{3}{2}\sum\frac{m_a}{h_a}-\frac{3}{2}=\frac{3}{2}\left(\sum\frac{m_a}{h_a}-1\right)\leq\frac{3}{2}\left(\frac{4R+r}{3r}-1\right)=\frac{2R}{r}-1$$

where we've used:

$\frac{m_a}{h_a}\cdot\cos A\geq\frac{3}{2}$ *(inequality **1.**),* $\sum\cos A=1+\frac{r}{R}\leq\frac{3}{2}$ *(Euler's inequality) and*

$\sum\frac{m_a}{h_a}\leq\frac{4R+r}{3r}$*, which follows from Cebyshev's inequality:*

The triplets (m_a,m_b,m_c) *and* $\left(\frac{1}{h_a},\frac{1}{h_b},\frac{1}{h_c}\right)$ *are reversed ordered, and* $\sum m_a\leq 4R+r$ *and* $\sum\frac{1}{h_a}=\frac{1}{r}$

$$\textit{wherefrom } \sum \frac{m_a}{h_a} \le \frac{1}{3} \cdot \sum m_a \sum \frac{1}{h_a} \le \frac{1}{3} \cdot (4R + r) \cdot \frac{1}{r} = \frac{4R + r}{3r}$$

The equality holds if and only if the triangle is equilateral.

□

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Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
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PROBLEM JP.090 RMM
NUMBER 6 AUTUMN 2017
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. Prove that in any triangle ABC

$$\frac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C}\geq\frac{p}{3r}$$

Proposed by Martin Lukarevski - Skopje - Macedonia

Proof.

Using the known identities known in triangle: $\prod\cos A=\frac{p^2-(2R+r)^2}{4R^2}$ *and*

$$\prod\sin A=\frac{rp}{2R^2}$$

We write the inequality $\frac{1+\frac{p^2-(2R+r)^2}{4R^2}}{\frac{rp}{2R^2}}\geq\frac{p}{3r}\Leftrightarrow\frac{p^2-r^2-4Rr}{2rp}\geq\frac{p}{3r}\Leftrightarrow p^2\geq 12Rr+3r^2$, *which follows from Gerretsen's inequality* $p^2\geq 16Rr-5r^2$ *. It remains to prove that:* $16Rr-5r^2\geq 12Rr+3r^2\Leftrightarrow R\geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

□

Remark 1.

*Inequality **1.** can be strengthened:*

2. In ΔABC

$$\frac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C}\geq\frac{p}{3r}+\frac{p}{24}\left(\frac{1}{r}-\frac{2r}{R}\right)$$

Proof.

Using the known identities in triangle: $\prod\cos A=\frac{p^2-(2R+r)^2}{4R^2}$ *and*

$$\prod\sin A=\frac{rp}{2R^2}$$

We write the inequality:

$$\frac{1+\frac{p^2-(2R+r)^2}{4R^2}}{\frac{rp}{2R^2}}\geq\frac{p}{3r}+\frac{p}{24}\left(\frac{1}{r}-\frac{2r}{R}\right)\Leftrightarrow\frac{p^2-r^2-4Rr}{2rp}\geq\frac{p}{3r}+\frac{p}{24}\left(\frac{1}{r}-\frac{2r}{R}\right)\Leftrightarrow$$

$\Leftrightarrow p^2(3R+2r)\geq 12Rr(4R+r)$, *which follows from Gerretsen's inequality* $p^2\geq 16Rr-5r^2$.

It remains to prove that:

$$(16Rr-5r^2)(3R+2)\geq 12Rr(4R+r)\Leftrightarrow R\geq 2r \text{ (Euler's inequality).}$$

The equality holds if and only if the triangle is equilateral.

□

Remark 2.

*Inequality **2.** is stronger then inequality **1.**:*

3. In ΔABC

$$\frac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C}\geq\frac{p}{3r}+\frac{p}{24}\left(\frac{1}{r}-\frac{2r}{R}\right)\geq\frac{p}{3r}.$$

Proof.

*See inequality **2.** and Euler's inequality $R\geq 2r$.*

The equality holds if and only if the triangle is equilateral.

□

Remark 3.

*Inequality **2.** can be developed*

4. In ΔABC

$$\frac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C}\geq\frac{p}{3r}+\lambda p\left(\frac{1}{r}-\frac{2r}{R}\right),\ \text{where } \lambda\leq\frac{1}{24}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the known identities in triangle: $\prod\cos A=\dfrac{p^2-(2R+r)^2}{4R^2}$ *and*

$$\prod\sin A=\frac{rp}{2R^2}$$

We write the inequality:

$$\frac{1+\frac{p^2-(2R+r)^2}{4R^2}}{\frac{rp}{2R^2}}\geq\frac{p}{3r}+\lambda p\left(\frac{1}{r}-\frac{2r}{R}\right)\Leftrightarrow\frac{p^2-r^2-4Rr}{2rp}\geq\frac{p}{3r}+\lambda p\left(\frac{1}{r}-\frac{2r}{R}\right)\Leftrightarrow$$

$p^2\Big[(1-6\lambda)R+12\lambda r\Big]\geq 3Rr(4R+r)$, *which follows from Gerretsen's inequality* $p^2\geq 16Rr-5r^2$ *and the condition* $1-6\lambda\geq 0$. *It remains to prove that:*

$$(16Rr-5r^2)\Big[(1-6\lambda)R+12\lambda r\Big]\geq 3Rr(4R+r)\Leftrightarrow(2-48\lambda)R^2+(111\lambda-4)Rr-30\lambda r^2\geq 0$$

$\Leftrightarrow(R-2r)\Big[(2-48\lambda)R+15\lambda r\Big]\geq 0$, *obviously from Euler's inequality* $R\geq 2r$ *and the condition* $2-48\lambda\geq 0$ □

The equality holds if and only if the triangle is equialateral.

Remark 4.

*For $\lambda\geq 0$ inequality **4.** is stronger then inequality **1.**:*

5. In ΔABC

$$\frac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C}\geq\frac{p}{3r}+\lambda p\left(\frac{1}{r}-\frac{2r}{R}\right)\geq\frac{p}{3r},\ \text{where } 0\leq\lambda\leq\frac{1}{24}.$$

Proof.

*See inequality **4.**, Euler's inequality $R \geq 2r$ and the condition $\lambda \geq 0$.*

$\square$

Note.

*In inequality **4.** for $\lambda = 0$ we obtain inequality **1**, and for $\lambda = \frac{1}{24}$ we obtain **2**.*

Remark.

Taking into account that $\frac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C} = \frac{a^2+b^2+c^2}{4S}$

*inequalities **1.**, **2.**, **3.**, **4.**, **5.** can be reformulated*

1.a. In ΔABC

$$\frac{a^2+b^2+c^2}{4S} \geq \frac{p}{3r}$$

2.a. In ΔABC

$$\frac{a^2+b^2+c^2}{4S} \geq \frac{p}{3r} + \frac{p}{24}\left(\frac{1}{r} - \frac{2r}{R}\right)$$

3.a. In ΔABC

$$\frac{a^2+b^2+c^2}{4S} \geq \frac{p}{3r} + \frac{p}{24}\left(\frac{1}{r} - \frac{2r}{R}\right) \geq \frac{p}{3r}.$$

4.a. In ΔABC

$$\frac{a^2+b^2+c^2}{4S} \geq \frac{p}{3r} + \lambda p\left(\frac{1}{r} - \frac{2r}{R}\right), \text{ where } \lambda \leq \frac{1}{24}.$$

5.a. In ΔABC

$$\frac{a^2+b^2+c^2}{4S} \geq \frac{p}{3r} + \lambda p\left(\frac{1}{r} - \frac{2r}{R}\right), \text{ where } \lambda \leq \frac{1}{24}.$$

Remark 5.

*Inequality **1.** can be developed also in the following way:*

6. In ΔABC,

$$\frac{n+\cos A\cos B\cos C}{\sin A\sin B\sin C} \geq \frac{(8n+1)p}{27r}, \text{ where } 1 \leq n \leq \frac{25}{16}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the known identities in triangle: $\prod \cos A = \frac{p^2-(2R+r)^2}{4R^2}$ *and*

$$\prod \sin A = \frac{rp}{2R^2}$$

We write the inequality $\frac{n+\frac{p^2-(2R+r)^2}{4R^2}}{\frac{rp}{2R^2}} \geq \frac{(8n+1)p}{27r} \Leftrightarrow \frac{4nR^2+p^2-(2R+r)^2}{2rp} \geq \frac{(8n+1)p}{27r} \Leftrightarrow$

$108nR^2+27p^2-27(2R+r)^2 \geq (16n+2)p^2 \Leftrightarrow p^2(25-16n) \geq 27(2R+r)^2-108nR^2$,
which follows from Gerretsen's inequality $p^2 \geq 16Rr-5r^2$ and the condition
$25-16n \geq 0$

It remains to prove that:

$(16Rr-5r^2)(25-16n) \geq 27(2Rr+r)^2-108nR^2 \Leftrightarrow (27n-27)R^2+(73-64n)Rr+(38-20n) \geq 0$

$\Leftrightarrow (R-2r)\Big[(27n-27)R+(10n-19)r\Big] \geq 0$, *obviously from Euler's inequality* $R \geq 2r$ *and the condition* $27n-27 \geq 0$.

The equality holds if and only if the triangle is equilateral.

□

Note.

For $n = 1$ *we obtain inequality* ***1.*** *from enunciation.*

Remark 6.

Inequality ***5.*** *can be reformulated:*

7. In ΔABC,

$$\boldsymbol{\frac{1+k\cos A\cos B\cos C}{\sin A\sin B\sin C} \geq \frac{(k+8)p}{27r}}, \text{ ***where*** } \boldsymbol{\frac{16}{25} \leq k \leq 1}.$$

Proof.

In inequality ***6.*** *we put* $n = \frac{1}{k}$

The equality holds if and only if the triangle is equilateral.

□

Note.

For $k = 1$ *we obtain inequality* ***1.*** *from enunciation.*

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Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro

PROBLEM UP.083 RMM
NUMBER 6 AUTUMN 2017
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. In ΔABC

$$R\sum(b+c-2a)^2 \le 4(R-2r)\sum a^2$$

Proposed by Daniel Sitaru - Romania

Proof.

Avem $\sum(b+c-2a)^2 = \sum(2p-3a)^2 = \sum(4p^2-12pa+9a^2) = 12p^2-12p\sum a+9\sum a^2 =$

$$= 12p^2 - 12p\cdot 2p + 9\cdot 2(p^2-r^2-4Rr) = 6p^2-18r^2-72Rr$$

Using the identities $\sum a(b+c-2a)^2 = 6p^2-18r^2-72Rr$ *and* $\sum a^2 = 2(p^2-r^2-4Rr)$

inequality that we have to prove:

$$R\cdot(6p^2-18r^2-72Rr) \le 4(R-2r)\cdot 2(p^2-r^2-4Rr)$$

$$\Leftrightarrow (R-8r)p^2 + r(20R^2+37Rr+8r^2) \ge 0$$

Distinguish the cases:

Case 1. If $R-8r\ge 0$ *inequality is obviously.*
Case 2. If $R-8r<0$ *inequality can be rewritten*
$(8r-R)p^2 \le r(20R^2+37Rr+8r^2)$ which follows from Gerretsen's inequality: $p^2\le 4R^2+4Rr+3r^2$. It remains to prove that:

$(8r-R)(4R^2+4Rr+3r^2) \le r(20R^2+37Rr+8r^2) \Leftrightarrow R^3-2R^2r+2Rr^2-4r^3\ge 0 \Leftrightarrow$

$\Leftrightarrow (R-2r)(R^2+r^2)\ge 0$, *obviously from Euler's inequality* $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark. 1 *Inequality can be developed:*
2. In ΔABC

$$R\sum(b+c-2a)^2 \le n(R-2r)\sum a^2, \text{ where } n\ge 3.$$

Proposed by Marin Chirciu - Romania

Proof.

Using the identities $\sum(b+c-2a)^2 = 6p^2-18r^2-72Rr$ *and* $\sum a^2 = 2(p^2-r^2-4Rr)$

inequality that we have to prove can be written:

$$R\cdot(6p^2-18r^2-72Rr) \le n(R-2r)\cdot 2(p^2-r^2-4Rr)$$

$$\Leftrightarrow \Big[(n-3)R-2nr\Big]p^2 + r\Big[(36-4n)R^2+(7n+9)Rr+2nr^2\Big] \ge 0$$

We distinguish the cases:

Case 1. If $(n-3)R-2nr \geq 0$ *we use Gerretsen's inequality* $p^2 \geq 16Rr-5r^2$. *It remains to prove that:*

$$\Leftrightarrow \Big[(n-3)R-2nr\Big](16Rr-5r^2)+r\Big[(36-4n)R^2+(7n+9)Rr+2nr^2\Big] \geq 0 \Leftrightarrow$$

$\Leftrightarrow (2n-2)R^2+(4-5n)Rr+2nr^2 \geq 0$, *obviously from Euler's inequality* $R \geq 2r$ *and* $n \geq 3$.

Case 2. If $(n-3)R-2nr < 0$ *inequality can be rewritten*

$\Big[2nr+(3-n)R\Big]p^2 \leq r\Big[(36-4n)R^2+(7n+9)Rr+2nr^2\Big]$, *which follows from Gerretsen's inequality:*

$p^2 \leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$[2nr+(3-n)R](4R^2+4Rr+3r^2) \leq r\Big[(36-4n)R^2+(7n+9)Rr+2nr^2\Big] \Leftrightarrow$$

$$\Leftrightarrow (2n-6)R^3+(12-4n)R^2r+nRr^2-4nr^3 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)\Big[(2n-6)R^2+nr^2\Big] \geq 0$$

obviously from Euler's inequality $R \geq 2r$ *and the condition* $n \geq 3$.

Equality holds if and only if the triangle is equialteral.

□

Note.

For $n=4$ *we obtain inequality* ***1.***

Remark 2.

In the same way we can propose:

3. In ΔABC

$$\sum a(b+c-2a)^2 \leq n\sqrt{3}(R-2r)R^2, \text{ where } n \geq 21.$$

Proposed by Marin Chirciu - Romania

Proof.

We have

$$\sum a(b+c-2a)^2 = \sum a(2p-3a)^2 = \sum a(4p^2-12pa+9a^2) = 4p^2\sum a - 12p\sum a^2 + 9\sum a^3 =$$

$$= 4p^2 \cdot 2p - 12p \cdot 2(p^2-r^2-4Rr) + 9 \cdot 2p(p^2-r^2-6Rr) = 2p(p^2-15r^2-6Rr).$$

Inequality that we have to prove can be written:

$$2p(p^2-15r^2-6Rr) \leq n\sqrt{3}(R-2r)R^2 \Leftrightarrow 2p\sqrt{3}(p^2-15r^2-6Rr) \leq 3n(R-2r)R^2,$$

which follows from Doucet's inequality $p\sqrt{3} \leq 4R+r$

and Gerretsen's inequality $p^2 \leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$2(4R+r)(4R^2+4Rr+3r^2-15r^2-6Rr) \leq 3n(R-2r)R^2 \Leftrightarrow$$

$$\Leftrightarrow (3n-32)R^3+(8-6n)R^2r+100Rr^2+24r^3 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)[(3n-32)R^2-56Rr-12r^2] \geq 0$, *obviously from Euler's inequality* $R \geq 2r$ *and the condition* $n \geq 21$, *which assures the positivity of the right parentheses.*

The equality holds if and only if the triangle is equilateral.

□

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