RMM Commented Problems Marathon 1 - 20

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COMMENTED SOLUTION

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ABSTRACT. In this article we comment problem nr.22972 from GM3/1994, which represents a good opportunity for a lesson synthesis for the class or for the circle of students.

In GM3/1994 Răzvan Satnoianu proposes the following problem:
22972: Let $A, B \in M_n(\mathbb{R})$ be invertible, having the property $A^{-1} + B^{-1} = I_n$. Prove that:

$$\det\Big[I_n - A^{2m+1} - B^{2m+1} + (AB)^{2m+1}\Big] \geq 0, (\forall)m \in \mathbb{N}$$

We will comment the author's solution (published in GM1/1995):
Using the simplification rules on the left and on the right the author processes the relationship from the hypothesis like this

$$A^{-1} + B^{-1} = I_n | \cdot A$$
$$A(A^{-1} + B^{-1}) = AI_n \Rightarrow I_n + AB^{-1} = A$$
$$A^{-1} + B^{-1} = I_n | \cdot A$$
$$(A^{-1} + B^{-1})A = I_n \Rightarrow I_n + B^{-1}A = A$$

From $I_n + AB^{-1} = A$ și $I_n + B^{-1}A = A$ we obtain

$$I_n + AB^{-1} = I_n + B^{-1}A \Rightarrow AB^{-1} = B^{-1}A$$

We repeat the procedure processing the relationship from the hypothesis by multiplying on the left and on the right with AB.

$$A^{-1} + B^{-1} = I_n | \cdot AB$$
$$AB(A^{-1} + B^{-1}) = (AB)I_n \Rightarrow ABA^{-1} + A = AB$$
$$(A^{-1} + B^{-1})AB = I_n(AB) \Rightarrow B + B^{-1}AB = AB$$

We deduce $ABA^{-1} + A = B^{-1}AB + B$. We use the relationship $AB^{-1} = B^{-1}A$ that we've obtained previously and it follows:

$$ABA^{-1} + A = AB^{-1}B + B$$

$$ABA^{-1} + A = A + B \quad (0.1)$$

$$ABA^{-1} = B$$
$$ABA^{-1} + B^{-1}AB = AB$$
$$ABA^{-1} + AB^{-1}B = AB$$

$$ABA^{-1} + A = AB \quad (0.2)$$

From 0.1 și 0.2 we deduce:

$$A + B = AB \Rightarrow I_n = (I_n - A)(I_n - B)$$

Remark:
In the following reasoning the author uses the relationship:

$$A^{2n+1} - B^{2n+1} = (A - B)(A^{2n} + A^{2n-1}B + \ldots + B^{2n})$$
$$A, B \in M_n(\mathbb{R}); AB = BA$$

Using this relationship:

$$I_n - A^{2m+1} = (I_n - A)(I_n + A + \ldots + A^{2m})$$
$$I_n - B^{2m+1} = (I_n - B)(I_n + B + \ldots + B^{2m})$$

By multiplying:

$$(I_n - A^{2m+1})(I_n - B^{2m+1}) =$$
$$= (I_n - A)(I_n + A + \ldots + A^{2m})(I_n - B)(I_n + B + \ldots + B^{2m}) =$$
$$= (I_n + A + \ldots + A^{2m})(I_n + B + \ldots + B^{2m})$$

On the other hand:

$$(I_n - A^{2m+1})(I_n - B^{2m+1}) = I_n - A^{2m+1} - B^{2m+1} + A^{2m+1}B^{2m+1} =$$
$$= I_n - A^{2m+1} - B^{2m+1} + (AB)^{2m+1} \tag{0.3}$$

Remark:
The following results appeals to:
- the some of some terms in geometric progression:

$$1 + x + \ldots + x^{2m} = \frac{x^{2m+1} - 1}{x - 1}$$

- the factors decomposition formula of the polynoms:

$$1 + x + \ldots + x^{2m} = (x - \varepsilon_1)(x - \varepsilon_2) \cdot \ldots \cdot (x - \varepsilon_{2m})$$

where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2m}$ are the solution of the equation: $x^{2m+1} - 1 = 0$ except 1.
- the determinant of two matrixes product:

$$\det(AB) = \det A \cdot \det B; A, B \in M_n(\mathbb{R})$$

- complex numbers property:

$$z \cdot \overline{z} = |z|^2; (\forall) z \in \mathbb{C}$$

The "$2m + 1$" squares having the order "$2m + 1$" of the unit are:

$$\varepsilon_k = \cos \frac{2k\pi}{2m + 1} + i \sin \frac{2k\pi}{2m + 1}; k \in \overline{0, 2m}$$
$$1 + x + \ldots + x^{2m} = (x - \varepsilon_1)(x - \varepsilon_2) \cdot \ldots \cdot (x - \varepsilon_{2m})$$

On the other hand $\overline{\varepsilon_i} = \varepsilon_{2m-i}; i \in \{1, 2, \ldots, m\}$
In matrix writing:

$$I_n + X + \ldots + X^{2m} = (X - \varepsilon_1 I_n)(X - \varepsilon_2 I_n) \cdot \ldots \cdot (X - \varepsilon_{2m} I_n)$$

and passing to determinants:

$$\det(I_n + X + \ldots + X^{2m}) = \prod_{k=1}^{2m} \det(X - \varepsilon_k I_n) =$$
$$= \prod_{k=1}^{m} \det(X - \varepsilon_k I_n) \det(X - \overline{\varepsilon_k} I_n) =$$

$$= \prod_{k=1}^{n} \det(X - \varepsilon_k I_n) \cdot \overline{\det(X - \varepsilon_k I_n)} =$$
$$= \prod_{k=1}^{n} |\det(X - \varepsilon_k I_n)|^2 \geq 0; (\forall) m \in \mathbb{R}.$$

We replace X with A și B we consider 0.3. It follows:

$$\det(I_n - A^{2m+1} - B^{2m+1} + (AB)^{2m+1}) \geq 0$$

Remark:
The problem is particularly complex because it requires various knowledge about complex numbers, polynomials, matrix calculus and determinants. The links that can be made with previous chapters, are making from this problem a very good end point to a recap and systematisation lesson or to circles of students.

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COMMENTED PROBLEM - 3

MARIN CHIRCIU - ROMANIA

In Mathematical Gazette nr. 11/2016, problem 27298 has the following content: Prove that in any triangle ΔABC we have

$$\sum \frac{a}{b+c} + \frac{r}{R} \leq 2$$

Florin Stănescu, Găești, Dâmbovița

a)$\sum \frac{a}{b+c} + \frac{r}{R} \leq 2$.

Mathematical Reflections 4/2016, Florin Stănescu, Găești, Romania

Solution:

Using the known identity in triangle $\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}$, we write the inequality

$$\frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} + \frac{r}{R} \leq 2 \Leftrightarrow 2R(p^2-r^2-Rr) \leq (2R-r)(p^2+r^2+2Rr)$$

$\Leftrightarrow p^2 \leq 6R^2+2Rr-r^2$, which follows from Gerresten's inequality $\Leftrightarrow p^2 \leq 4R^2+4Rr+3r^2$. It remains to prove that $\Leftrightarrow 4R^2+4Rr+3r^2 \leq 6R^2+2Rr-r^2$ $\Leftrightarrow R^2-Rr-2r^2 \geq 0 \Leftrightarrow (R-2r)(R+r) \geq 0$, obviously from Euler's inequality $R \geq 2r$.
The equality holds if and only if the triangle is equilateral.
The article proposes to strengthen this inequality, and developments of some inequalities with sums having the form $\sum \frac{a}{b+c}$. □

b) $\sum \frac{a}{b+c} + \frac{3r}{2R+2r} \leq 2$.

Solution:

Using the known identity in triangle $\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2r}$, we write the inequality

$$\frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} + \frac{3r}{2R+2r} \leq 2 \Leftrightarrow 3p^2 \leq 12R^2+14Rr+5r^2,$$

which follows from Gerretsen's inequality $\Leftrightarrow p^2 \leq 4R^2+4Rr+3r^2$.
It remains to prove that $\Leftrightarrow 3(4R^2+4Rr+3r^2) \leq 12R^2+14Rr+5r^2 \Leftrightarrow R \geq 2r$ obviously from Euler's inequality.
The equality holds if and only if the triangle is equilateral. □

c) $\sum \frac{a}{b+c} + \frac{r}{R} \leq \sum \frac{a}{b+c} + \frac{3r}{2R+2r} \leq 2$.

Solution:

The first inequality is equivalent with Euler's inequality $R \geq 2r$, the second is b). Obviously b) is stronger than a).
The equality holds if and only if the triangle is equilateral. □

d) $\sum \frac{a}{b+c} + n \cdot \frac{r}{R} \leq \frac{n+3}{2}$, where $n \geq 1$.

Solution:

We use $\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}$, Gerretsen's inequality.

$$16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2 \text{and Euler's inequality} R \geq 2r.$$

The equality holds if and only if the triangle is equilateral. □

e) $\sum \frac{a}{b+c} + \frac{3n}{2} \cdot \frac{r}{R+r} \leq \frac{n+3}{2}$, unde $n \geq 1$.

Solution:

Analogous d).
The equality holds if and only if the triangle is equilateral. □

f) $\sum \frac{a}{b+c} + n \cdot \frac{r}{R} \leq \sum \frac{a}{b+c} + \frac{3n}{2} \cdot \frac{r}{R+r} \leq \frac{n+3}{2}$, where $n \geq 1$.

Developments, M. Chirciu

Solution:

Analogous c).
The equality holds if and only if the triangle is equilateral. □

g) $\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} \geq \frac{3}{2}$.

Solution:

$$\sum \frac{a}{b+c} = \frac{\sum a(a+b)(a+c)}{\prod(b+c)} = \frac{\sum a^3 + \sum a \sum bc}{\prod(b+c)} = \frac{2p(p^2-3r^2-6Rr)+2p(p^2+r^2+4Rr)}{2p(p^2+r^2+2Rr)} =$$

$$= \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}.$$

The inequality $\frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} \geq \frac{3}{2}$ is equivalent with $p^2 \geq 10Rr + 7r^2$, which follows

from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$ and Euler's inequality$R \geq 2r$.

The equality holds if and only if the triangle is equilateral.
Its Nesbitt's inequality in triangle. □

h) $\sum \frac{a}{b+c} = \frac{11p^2-15r^2-60Rr}{6p^2-6r^2-24Rr} \geq \frac{3}{2}$.

Mathematical Recreations 2/2009, Marius Olteanu, Rm. Vâlcea

Solutions:

See g). □

i) $\sum \frac{a}{b+c} \geq \frac{4p^2-6Rr}{2p^2+5Rr} \geq \frac{3}{2}$.

Solution:

We use $\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}$, Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$ and Euler's inequality $R \geq 2r$.
It is a strengthening of Nesbitt's inequality in triangle.
The equality holds if and only if the triangle is equilateral. □

j)$\sum \frac{a^2}{b^2+c^2} \geq 2 - \frac{r}{R} \geq \sum \frac{a}{b+c} \geq \frac{3}{2}$.

Solution:

For the first inequality we use Bergstrom, Gerretsen and Euler.
We obtain

$$\sum \frac{a^2}{b^2+c^2} = \sum \frac{a^4}{a^2b^2+a^2c^2} \geq \frac{(\sum a^2)^2}{2\sum b^2c^2} = \frac{[2(p^2-r^2-4Rr)]^2}{2[p^4-2p^2(4Rr-r^2)+r^2(4R+r)^2]} \geq$$

$\geq 2 - \dfrac{r}{R}$, the last inequality is equivalent to $p^4 + p^2(2r^2 - 16Rr) + r^2(4R+r)^2 \geq 0$, which follows from Gerretsen's inequality.

For the second inequality we use $\displaystyle\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}$ and Gerretsen.

The equality holds if and only if the triangle is equilateral. □

k) $\sum \frac{a}{b+c} + \frac{9r}{4R+r} \leq \frac{5}{2}$.

Solution:

$$\text{We use } \sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} \text{ and Gerretsen.}$$

The equality holds if and only if the triangle is equilateral. □

l) $\sum \frac{a}{b+c} + \frac{nr}{4R+r} \leq \frac{3}{2} + \frac{n}{9}$, unde $n \geq \frac{9}{2}$.

Solution:

$$\text{We use } \sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} \text{ and Gerretsen.}$$

The equality holds if and only if the triangle is equilateral. □

m) $\sum \frac{a}{b+c} + \frac{3abc}{\sum bc(b+c)} \geq 2$.

Solution:

We use $\displaystyle\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}, \sum bc(b+c) = 2p(p^2+r^2-2Rr)$ and Gerretsen.

The equality holds if and only if the triangle is equilateral. □

n) $\sum \frac{a}{b+c} + n \cdot \frac{abc}{\sum bc(b+c)} \geq \frac{n+9}{6}$, where $n \leq 3$.

Solution:

Analogous m). □

o) $\sum \frac{a}{b+c} + 4\prod \frac{a}{b+c} \geq 2$.

Solution:

We use $\sum \dfrac{a}{b+c} = \dfrac{2(p^2 - r^2 - Rr)}{p^2 + r^2 + 2Rr}, \prod \dfrac{a}{b+c} = \dfrac{2Rr}{p^2 + r^2 + 2Rr}$ and Gerretsen. The equality holds if and only if the triangle is equilateral. □

p) $\sum \frac{a}{b+c} + n \cdot \prod \frac{a}{b+c} \geq \frac{n+12}{8}$, where $n \leq 4$.

Solution:

Analogous o).

InеMath 10/2016, M. Chirciu

□

Other inequalities with sums having the form $\sum \frac{a}{b+c}$.

1) $\sum \frac{b+c}{a} \geq 4 \sum \frac{a}{b+c}$.
2) $\sum \frac{b+c}{a} \geq 6 - \frac{3n}{2} + n \sum \frac{a}{b+c}$, where $n \leq 4$.
3) $3 \sum \frac{a}{b+c} \geq \sum a \cdot \sum \frac{1}{b+c} \geq 9$.
4) $\sum a^2 \cdot \sum \frac{1}{a^2} \geq 6 \sum \frac{a}{b+c}$.
5) $\frac{a^2+b^2+c^2}{ab+bc+ca} \geq \frac{2}{3} \sum \frac{a}{b+c} \geq 1$.
6) $\frac{a^3+b^3+c^3}{abc} \geq 2 \sum \frac{a}{b+c} \geq 3$.
7) $\frac{a^3+b^3+c^3}{abc} + n \geq \frac{2}{3}(n+2) \sum \frac{a}{b+c}$, where $n \leq \frac{3}{4}$.
8) $\sum a \cdot \sum \frac{a}{bc} \geq 6 \sum \frac{a}{b+c} \geq 9$.
9) $2 \sum \frac{a}{b+c} \geq \frac{(a+b+c)^2}{ab+bc+ca} \geq 3$.
10) $\sum a \cdot \sum \frac{1}{a} \geq \frac{12r}{R} \sum \frac{a}{b+c}$.
11) $\sum \frac{b+c}{a} - 2 \sum \frac{a}{b+c} \geq 3$; b) $\sum \frac{b+c}{a} - \sum \frac{a}{b+c} \geq 6n - \frac{3}{2}$, where $n \geq \frac{1}{4}$.
12) $\sum \frac{a}{b+c} + \frac{4abc}{(a+b)(b+c)(c+a)} \geq 2$;
13) $\sum \frac{a}{b+c} + n \cdot \frac{abc}{(a+b)(b+c)(c+a)} \geq \frac{n+12}{8}$, where $n \leq 4$.
14) $\sum \frac{a}{b+c} + 3 \cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \leq \frac{9}{2}$.
15) $\sum \frac{a}{b+c} + n \cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \leq n + \frac{3}{2}$, where $n \geq 1$.
16) $\sum \frac{a}{b+c} + \frac{1}{3} \cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \geq n + \frac{3}{2}$.
17) $\sum \frac{a}{b+c} + n \cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \geq n + \frac{3}{2}$, where $n \leq \frac{1}{3}$.
18) $\sum \frac{a}{b+c} + 3 \cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \geq \frac{11}{2}$.
19) $\sum \frac{a}{b+c} + n \cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \geq n + \frac{3}{2}$, where $n \geq 3$.

IneMath 11/2016, M. Chirciu

Solution:

We use the known identities in triangle:

$$\sum \frac{b+c}{a} = \frac{p^2 + r^2 - 2Rr}{2Rr}; \sum \frac{1}{b+c} = \frac{5p^2 + r^2 + 4Rr}{2p(p^2 + r^2 + 2Rr)}; \sum a^2 = 2(p^2 - r^2 - 4Rr)$$

$$\sum \frac{1}{a^2} = \frac{p^2 - 2p^2(4Rr - r^2) + r^2(4R + r)^2}{16R^2r^2p^2}; \sum bc = p^2 + r^2 + 4Rr; abc = 4Rrp;$$

$$\sum a^3 = 2p(p^2 - 3r^2 - 6Rr); \sum \frac{a}{bc} = \frac{p^2 - r^2 - 4Rr}{2Rrp}; \sum \frac{1}{a} = \frac{p^2 + r^2 + 4Rr}{4Rrp}$$

$$\prod (b + c) = 2p(p^2 + r^2 + 2R).$$

Then we use like the proves before:
Gerresten's inequality: $16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$.
Euler's inequality: $R \geq 2r$.
To each proposed inequalities, the equality is realised if and only if the triangle is equilateral. □

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REGARDING PROBLEM S:L16.284 FROM SGM 11/2016 METHODS OF SOLVING AN INEQUALITY

MARIN CHIRCIU

In Mathematical Gazette Supplement nr. 11/2016 the following problem is proposed:
Let be $n \in N, n \geq 2$. Prove that for $a_1, a_2, \ldots, a_n \in \left(0, \sqrt{n}\right]$, with
$a_1 + a_2 + \ldots a_n = n$, the following inequality holds:

$$\frac{1}{a_1^2 + (a_2 + a_3 + \ldots + a_n)} + \frac{1}{a_2^2 + (a_1 + a_3 + \ldots + a_n)} + \ldots + \frac{1}{a_n^2 + (a_1 + a_3 + \ldots + a_{n-1})} \leq 1$$

Andra - Mălina Cardaş, student, Botoşani

The article presents a methodical treatment of this problem, descending it first to three variable, developing then this result and finishing with the developing of the general case.

Proof.

Case $n = 3$.
If $a, b, c > 0$ with $a + b + c = 3$, prove that $\frac{1}{a^2+b+c} + \frac{1}{b^2+c+a} + \frac{1}{c^2+a+b} \leq 1$. □

Proof. Because $a + b + c = 3$, we have $b + c = 3 - a$ and we write the inequality $\sum \frac{1}{a^2-a+3} \leq 1$.
In order to obtain this result we look for an inequality having the following form: $\frac{1}{a^2-a+3} \leq x \cdot a + y$ (Tangent Line Method) and we determine x and y such that the attached equation in the variable "a" to have double root on 1. We obtain $x = \frac{-1}{9}$ and $y = \frac{4}{9}$.
We have $\frac{1}{a^2-a+3} \leq \frac{4-a}{9} \Leftrightarrow (a-1)^2(3-a) \geq 0$, obviously from $a, b, c > 0$ and $a + b + c = 3$, with equality if and only if $a = 1$.
We obtain $\sum \frac{1}{a^2-a+3} \leq \sum \frac{4-a}{9} = \frac{12-\sum a}{9} = \frac{12-3}{9} = 1$.
The equality holds if and only if $a = b = c = 1$.

□

Development.
If $a, b, c > 0$ with $a + b + c = 3$, prove that

$$\frac{1}{a^2 + k(b+c)} + \frac{1}{b^2 + k(c+a)} + \frac{1}{c^2 + k(a+b)} \leq \frac{3}{2k+1}, \text{ where } 1 \leq k \leq 2.$$

Proof. Because $a + b + c = 3$, we have $b + c = 3 - a$ and we write the inequality

$$\sum \frac{1}{a^2 + k(3-a)} \leq \frac{3}{2k+1}$$

We look for an inequality having the form $\frac{1}{a^2+k(3-a)} \leq x \cdot a + y$ and we determine x and y such that the attached equation the variable "a" to have double root on 1.

We obtain $x = \frac{k-2}{(2k+1)^2}$ and $y = \frac{k+3}{(2k+1)^2}$.

We have $\frac{1}{a^2+k(3-a)} \leq \frac{k+3+(k-2)a}{(2k+1)^2} \Leftrightarrow (a-1)^2\Big[(k-2)a-k^2+5k-1\Big] \geq 0$, obviously from $a,b,c>0$ with $a+b+c=3$ and $1 \leq k \leq 2$, with equality if and only if $a=1$.

We obtain

$$\sum \frac{1}{a^2+k(3-a)} \leq \sum \frac{k+3+(k-2)a}{(2k+1)^2} = \frac{3(k+3)+(k-2)\sum a}{(2k+1)^2} =$$

$$= \frac{3(k+3)+(k-2)3}{(2k+1)^2} = \frac{3}{2k+1}$$

The equality holds if and only if $a=b=c=1$. □

Solving the general case.

Let be $n \in N, n \geq 2$. Prove that for $a_1, a_2, \ldots, a_n > 0$, with $a_1+a_2+\ldots+a_n = n$, then

$$\frac{1}{a_1^2+(a_2+a_3+\ldots+a_n)} + \frac{1}{a_2^2+(a_1+a_3+\ldots+a_n)} + \ldots + \frac{1}{a_n^2+(a_1+a_3+\ldots+a_{n-1})} \leq 1$$

Proof. Because $a_1+a_2+\ldots+a_n = n$, we have $a_2+a_3+\ldots+a_n = n-a_1$ and we write the inequality

$$\sum_{i=1}^{n} \frac{1}{a_i^2-a_i+n} \leq 1.$$

We look an inequality having the form $\frac{1}{a^2-a+n} \leq x \cdot a + y$, and we determine x and y such that the attached equation in variable "a" to have double root on 1.

We obtain $x = \frac{-1}{n^2}$ and $y = \frac{n+1}{n^2}$.

We have $\frac{1}{a_i^2-a+n} \leq \frac{n+1-a_i}{n^2} \Leftrightarrow (a_i-1)^2(n-a_i) \geq 0$, obviously from $\sum_{i=1}^n a_i = 1$ and $a_i > 0, i = \overline{1,n}$, with equality if and only if $a_i = 1, i = \overline{1,n}$.

We obtain $\sum_{i=1}^n \frac{1}{a_i^2-a_i+n} \leq \sum_{i=1}^n \frac{n+1-a_i}{n^2} = \frac{n(n+1)-n}{n^2} = 1$.

The equality holds if and only if $a_1 = a_2 = \ldots = a_n = 1$.

□

Observation.

The condition $a_1, a_2, \ldots, a_n \in \Big(0, \sqrt{n}\Big]$ is not necessary. It is sufficient to have $a_1, a_2, \ldots, a_n > 0$.

Development.

Let be $n \in N, n \geq 2$. Prove that for $a_1, a_2, \ldots, a_n > 0$, with $a_1+a_2+\ldots+a_n = n$ holds the following inequality:

$$\frac{1}{a_1^2+k(a_2+a_3+\ldots+a_n)} + \frac{1}{a_2^2+k(a_1+a_3+\ldots+a_n)} + \ldots +$$

$$+ \frac{1}{a_n^2+k(a_1+a_3+\ldots+a_{n-1})} \leq \frac{n}{1+k(n-1)}$$

where $1 \leq k \leq 2$.

Proof. Because $a_1+a_2+\ldots+a_n = n$, we have $a_2+a_3+\ldots+a_n = n-a_1$ and we write the inequality

$$\sum_{i=1}^{n} \frac{1}{a_i^2+k(n-a_i)} \leq \frac{3}{1+k(n-1)}.$$

We look for an inequality having the form $\frac{1}{a^2+k(n-a)} \leq x \cdot a + y$ and we determine x and y such that the attached equation in variable "a" to have double root on 1. We obtain

$$x = \frac{k-2}{[1+k(n-1)]^2} \text{ and } y = \frac{3+k(n-2)}{[1+k(n-1)]^2}.$$

We have $\frac{1}{a_i^2+k(n-a_i)} \leq \frac{kn+3-2k+(k-2)a_i}{[1+k(n-1)]^2} \Leftrightarrow (a_i-1)^2 \left[(k-2)a_i + kn - (k-1)^2\right] \geq 0$, obviously from $\sum_{i=1}^n a_i = 1$ and $a_i > 0, i = \overline{1,n}$, with inequality if and only if $a_i = 1, i = \overline{1,n}$.

We obtain $\sum_{i=1}^n \frac{1}{a_i^2+k(n-a_i)} \leq \sum_{i=1}^n \frac{kn+3-2k+(2-k)a_i}{[1+k(n-1)]^2} \leq \frac{n(kn+3-2k)+(k-2)n}{[1+k(n-1)]^2} =$
$= \frac{n}{1+k(n-1)}$.

The equality holds if and only if $a_1 = a_2 = \ldots = a_n = 1$. □

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SOLUTION TO THE PROBLEM UP.052 FROM ROMANIAN MATHEMATICAL MAGAZINE NUMBER 4, SPRING 2017

MARIN CHIRCIU

UP.052. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a}\geq\frac{3}{2}.$$

Proposed by George Apostolopoulos, Messolonghi, Greece

Proof.

With Hölder's inequality we have $\frac{A^3}{X}+\frac{B^3}{Y}+\frac{C^3}{Z}\geq\frac{(A+B+C)^3}{3(X+Y+Z)}, \forall A,B,C,X,Y,Z>0.$

We obtain $\sum\frac{a^6}{a^2+b}=\sum\frac{(a^2)^3}{a^2+b}\geq\frac{(\sum a^2)^3}{3\sum(a^2+b)}=\frac{t^3}{3(t+3)}\geq\frac{3}{2}$, *where* $t=\sum a^2$,

and the last inequality is equivalent with

$$2t^3\geq 9(t+3)\Leftrightarrow 2t^3-9t-27\geq 0\Leftrightarrow (t-3)(2t^2+6t+9)\geq 0\Leftrightarrow t\geq 3,$$

obviously from $a^2+b^2+c^2\geq\frac{(a+b+c)^2}{3}$.

The equality holds if and only if $a=b=c=1$.

□

The problem can be developed:

Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{a^6}{a^2+nb}+\frac{b^2}{b^2+nc}+\frac{c^6}{c^2+na}\geq\frac{3}{n+1}, \text{ where } n\geq 0.$$

Marin Chirciu - Romania

Proof.

With Hölder's inequality we have $\frac{A^3}{X}+\frac{B^3}{Y}+\frac{C^3}{Z}\geq\frac{(A+B+C)^3}{3(X+Y+Z)}, \forall A,B,C,X,Y,Z>0.$

We obtain $\frac{a^6}{a^2+nb}=\sum\frac{(a^2)^3}{a^2+nb}\geq\frac{(\sum a^2)^3}{3\sum(a^2+nb)}=\frac{t^3}{3(t+3n)}\geq\frac{3}{n+1}$, *where* $t=\sum a^2$,

and the last inequality is equivalent with

$$(n+t)t^3\geq 9(t+3n)\Leftrightarrow (n+1)t^3-9t-27n\geq 0\Leftrightarrow (t-3)\Big[(n+1)t^2+3(n+1)t+9n\Big]\geq 0\Leftrightarrow$$

$t\geq 3$, *obviously from* $a^2+b^2+c^2\geq\frac{(a+b+c)^2}{3}$.

The equality holds if and only if $a=b=c=1$.

□

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SOLUTION TO PROBLEM UP.048 FROM ROMANIAN MATHEMATICAL MAGAZINE NUMBER 4, SPRING 2017

MARIN CHIRCIU

UP.048. Let a, b, c be non-negative real numbers such that $a+b+c=1$. Prove that:

$$a^4+b^4+c^4+26abc\leq 1$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

Proof.

Homogenising the inequality we obtain

$$a^4+b^4+c^4+26abc(a+b+c)\leq (a+b+c)^4$$

As $(a+b+c)^4=\sum a^4+4\sum bc(b^2+c^2)+6\sum b^2c^2+12abc(a+b+c)$*, the above inequality can be written:*

$$\sum a^4+4\sum bc(b^2+c^2)+6\sum b^2c^2+12abc(a+b+c)\geq \sum a^4+26abc(a+b+c)\Leftrightarrow$$

$2\sum bc(b^2+c^2)+3\sum b^2c^2\geq 7abc(a+b+c)$*, which follows from means inequality and the inequality* $x^2+y^2+z^2\geq xy+yz+zx$*, with* $x=bc, y=ca, z=ab$.

Indeed:

$$2\sum bc(b^2+c^2)+3\sum b^2c^2\geq 4\sum b^2c^2+3\sum b^2c^2=7\sum b^2c^2\geq 7\sum bc\cdot ca=7abc(a+b+c).$$

The equality holds if and only if $a=b=c=\frac{1}{3}$.

□

The problem can be developed:

If $a, b, c>0, a+b+c+1$ then $a^4+b^4+c^4+\lambda abc\leq \frac{\lambda+1}{27}$, where $\lambda\geq 26$.

Proposed by Marin Chirciu - Romania

Proof.

Homogenising the inequality we obtain:

$$a^b+b^4+c^4+\lambda abc(a+b+c)\leq \frac{\lambda+1}{27}(a+b+c)^4.$$

As $(a+b+c)^4=\sum a^4+4\sum bc(b^2+c^2)+6\sum b^2c^2+12abc(a+b+c)$*, the above inequality can be written:*

$$\frac{\lambda+1}{27}\cdot\Big[\sum a^4+4\sum bc(b^2+c^2)+6\sum b^2c^2+12abc(a+b+c)\Big]\geq \sum a^4+\lambda abc(a+b+c)\Leftrightarrow$$

$$(\lambda-26)\sum a^4+(8\lambda+8)\sum bc(b^2+c^2)+(6\lambda+6)\sum b^2c^2\geq (15\sum \lambda-12)abc(a+b+c)$$

which follows from the condition $\lambda\geq 26$*, means inequality and the inequality*

$x^2+y^2+z^2 \geq xy+yz+zx$, *with* $x=a^2, y=b^2, z=c^2$, *then* $x=bc, y=ca, z=ab$.

Indeed:

$$(\alpha-26)\sum a^4+(4\lambda+4)\sum bc(b^2+c^2)+(6\lambda+6)\sum b^2c^2 \geq$$

$$\geq (\lambda-26)\sum b^2c^2+(8\lambda+8)\sum b^2c^2+(6\lambda+6)\sum b^2c^2=$$

$$=(15\lambda-12)\sum b^2c^2 \geq (15\lambda-12)\sum bc\cdot ca$$

$$=(15\lambda-12)abc(a+b+c).$$

The equality holds if and only if $a=b=c=\dfrac{1}{3}$.

□

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SOLUTION TO PROBLEM JP.060. FROM ROMANIAN MATHEMATICAL MAGAZINE NUMBER 4, SPRING 2017

MARIN CHIRCIU

JP.060. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b}+\frac{ab}{a+b}+\frac{ab}{a+b}\leq\frac{3\sqrt{3}}{2}R.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Proof.

We have $\sum\frac{bc}{b+c}\leq\sum\frac{b+c}{4}=p\leq\frac{3\sqrt{3}}{2}R$*, where the last inequality is Mitrinović's inequality.*

The equality holds if and only if the triangle is equilateral.

□

The inequality can be strengthened:

1. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b}+\frac{ab}{a+b}+\frac{ab}{a+b}\leq p.$$

Proof.

$$\sum\frac{bc}{b+c}\leq\sum\frac{b+c}{4}=p.$$

The equality holds if and only if the triangle is equilateral.

*Inequality **1.** is stronger then **JP.060.***

□

2. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b}+\frac{ab}{a+b}+\frac{ab}{a+b}\leq p\leq\frac{3\sqrt{3}}{2}R.$$

Proof.

We have $\sum \frac{bc}{b+c} \leq \sum \frac{b+c}{4} =\leq \frac{3\sqrt{3}}{2}R$, *where the last inequality is Mitrinonvić's inequality.*

The equality holds if and only if the triangle is equilateral.

□

*Inequality **1.** can also be strengthened:*

3. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b} + \frac{ab}{a+b} + \frac{ab}{a+b} \leq \frac{3(ab+bc+ca)}{2(a+b+c)}.$$

Proof 1.

We use the known identities in triangle

$$\sum \frac{bc}{b+c} = \frac{p^4 + 2p^2(8R+r^2) + (4R+r)^3}{2p(p^2+r^2+2Rr)} \text{ and } \sum bc = p^2 + r^2 + 4Rr.$$

We write the inequality:

$$\frac{p^4 + 2p^2(8R+r^2) + (4R+r)^3}{2p(p^2+r^2+2Rr)} \leq \frac{3(p^2+r^2+4Rr)}{2 \cdot 2p} \Leftrightarrow$$

$$p^2(p^2 - 14Rr + 2r^2) \geq r^2(8R^2 - 2Rr - r^2).$$

As $(p^2 - 14Rr + 2r^2) > 0$, *see Gerretsen's inquality* $p^2 \geq 16Rr - 5r^2$, *using again Gerretsen's inequality it suffices to prove that*

$$(16Rr - 5r^2)(16Rr - 5r^2 - 14Rr + 2r^2) \geq r^2(8R^2 - 2r - r^2) \Leftrightarrow$$

$$(16R-5r)(2R-3r) \geq r^2(8R^2-2Rr-r^2) \Leftrightarrow 3R^2-7Rr+2r^2 \geq 0 \Leftrightarrow (R-2r)(3R-r) \geq 0.$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Proof 2.

The triplets $(a+b, b+c, c+a)$ *and* $\left(\frac{ab}{a+b}, \frac{bc}{b+c}, \frac{ca}{c+a}\right)$ *are ordered the same.*

With Chebyshev's inequality we obtain:

$$(a+b)\cdot\frac{ab}{a+b}+(b+c)\cdot\frac{bc}{b+c}+(c+a)\cdot\frac{ca}{c+a} \geq \frac{1}{3}\Big[(a+b)+(b+c)+(c+a)\Big]\left[\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\right]$$

$$\Leftrightarrow (ab+bc+ca) \geq \frac{1}{3} \cdot 2(a+b+c)\cdot\left(\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{ab}{a+b} + \frac{ab}{a+b} + \frac{ab}{a+b} \leq \frac{3(ab+bc+ca)}{2(a+b+c)}.$$

The equality holds if and only if the triangle is equilateral.

*Inequality **3.** is stronger then Inequality **1.**:*

□

4. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b}+\frac{ab}{a+b}+\frac{ab}{a+b}\leq\frac{3(ab+bc+ca)}{2(a+b+c)}\leq p.$$

Proof.

We use inequality 3. and

$$\frac{3(ab+bc+ca)}{2(a+b+c)}\leq p\Leftrightarrow\frac{3(ab+bc+ca)}{2(a+b+c)}\leq\frac{a+b+c}{2}\Leftrightarrow(a+b+c)^2\geq 3(ab+bc+ca).$$

The equality holds if and only if the triangle is equilateral.

□

We can write the series of inequalities:

5. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b}+\frac{ab}{a+b}+\frac{ab}{a+b}\leq\frac{3(ab+bc+ca)}{2(a+b+c)}\leq\frac{a+b+c}{2}\leq\frac{3(a^2+b^2+c^2)}{2(a+b+c)}.$$

Proof.

We use inequality 4. and $\frac{a+b+c}{2}\leq\frac{3(a^2+b^2+c^2)}{2(a+b+c)}\Leftrightarrow a^2+b^2+c^2\geq ab+bc+ca.$

□

6. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b}+\frac{ab}{a+b}+\frac{ab}{a+b}\leq\frac{3(ab+bc+ca)}{2(a+b+c)}\leq p\leq\frac{3\sqrt{3}}{2}R.$$

Proof.

We use inequality 4. and Mitrinović's inequality.

The equality holds if and only if the triangle is equilateral.

□

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SOLUTION TO PROBLEM SP.055. FROM ROMANIAN MATHEMATICAL MAGAZINE NUMBER 4, SPRING 2017

MARIN CHIRCIU

SP.055. Let m_a, m_b, m_c be the lengths of medians of a triangle ABC with inradius r. Prove that

$$\frac{m_a + m_b + m_c}{\sin^2 A + \sin^2 B + \sin^2 C} \geq 4r.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Proof.

With sine theorem we write the inequality:

$$\frac{m_a + m_b + m_c}{a^2 + b^2 + c^2} \geq \frac{r}{R^2} \Leftrightarrow \sum m_a \geq \frac{r}{R^2} \cdot \sum a^2.$$

We use the known inequality $m_a \geq \frac{b^2 + c^2}{4R}$ *it follows:*

$\sum m_a \geq \sum \frac{b^2 + c^2}{4R} = \frac{2\sum a^2}{4R} = \sum \frac{\sum a^2}{2R} \geq \frac{r}{R^2} \cdot \sum a^2$, *where the last inequality is equivalent with* $R \geq 2r$, *namely Euler's inequality.*

The equality holds if and only if the triangle is equilateral

□

The inequality can be strengthened:

1. Let m_a, m_b, m_c be the lengths of medians of a triangle ABC with inradius r. Prove that

$$\frac{m_a + m_b + m_c}{\sin^2 A + \sin^2 B + \sin^2 C} \geq 2R.$$

Proof.

With sine theorem we write the inequality:

$$\frac{m_a + m_b + m_c}{a^2 + b^2 + c^2} \geq \frac{1}{2R} \Leftrightarrow \sum m_a \geq \frac{1}{2R} \cdot \sum a^2.$$

Using the known inequality $m_a \geq \frac{b^2 + c^2}{4R}$ *it follows:*

$$\sum m_a \geq \sum \frac{b^2 + c^2}{4R} = \frac{2\sum a^2}{4R} = \frac{1}{2R} \cdot \sum a^2.$$

Equality holds if and only if the triangle is equilateral.

□

*Inequality **1.** is stronger then **SP.055.***

2. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{m_a + m_b + m_c}{\sin^2 A + \sin^2 B + \sin^2 C} \geq 2R \geq 4r.$$

Proof.

*We use Inequality **1.** and Euler's inequality $R \geq 2r$.*

The equality holds if and only if the triangle is equilateral.

□

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SOLUTION TO THE PROBLEM 82
MATH ADEVENTURES
ON
CUTTHEKNOTMATH
51-100
BY ALEXANDER BOGOMOLNY AND DANIEL SITARU
ROMANIAN MATHEMATICAL MAGAZINE

MARIN CHIRCIU

1. Prove that in triangle ABC, with angles $A; B; C$ side lengths $a; b; c$ the following inequality holds:

$$\frac{a(b+c)}{bc\cdot\cos^2\frac{A}{2}}+\frac{b(c+a)}{ca\cdot\cos^2\frac{B}{2}}+\frac{c(a+b)}{ab\cdot\cos^2\frac{C}{2}}\geq 8$$

Proposed by Daniel Sitaru - Romania

Proof.

We have

$$\sum\frac{a(b+c)}{bc\cdot\cos^2\frac{A}{2}}=\sum\frac{a(b+c)}{bc\cdot\frac{p(p-a)}{bc}}=\sum\frac{a(b+c)}{p(p-a)}=\sum\frac{a(2p-a)(p-b)(p-c)}{p(p-a)(p-b)(p-c)}=\frac{4R}{r}\geq 8,$$

where the last inequality follows from Euler's inequality $R\geq 2r$.

The equality holds if and only if $a=b=c$.

□

Next, are proposed inequalities for sums having the form $\sum\frac{a(b+c)}{bc\cdot f(A)}$, *where f is one of the trigonometric functions.*

2. Prove that in any triangle ABC, with angles $A; B; C$ side lengths $a; b; c$ the following inequality holds:

$$\frac{a(b+c)}{bc\cdot\sin^2\frac{A}{2}}+\frac{b(c+a)}{ca\cdot\sin^2\frac{B}{2}}+\frac{c(a+b)}{ab\cdot\sin^2\frac{C}{2}}\geq 12.$$

Proposed by Marin Chirciu - Romania

Proof.

We have

$$\sum\frac{a(b+c)}{bc\cdot\sin^2\frac{A}{2}}=\sum\frac{a(b+c)}{bc\cdot\frac{(p-b)(p-c)}{bc}}=\sum\frac{a(b+c)}{(p-b)(p-c)}=\sum\frac{a(2p-a)(p-a)}{(p-a)(p-b)(p-c)}=\frac{12R}{r}\geq 24$$

where the last inequality follows from Euler's inequality $R\geq 2r$.

The equality holds if and only if $a=b=c$.

□

3. Prove that in triangle ABC, with angles $A; B; C$ side lengths $a; b; c$ the following inequality holds:

$$\frac{a(b+c)}{bc\cdot\sin^2 A}+\frac{b(c+a)}{ca\cdot\sin^2 B}+\frac{c(a+b)}{ab\cdot\sin^2 C}\geq 8.$$

Proof 1.

We have

$$\sum\frac{a(b+c)}{bc\cdot\sin^2 A}=\sum\frac{a(b+c)}{bc\cdot\frac{a^2}{4R^2}}=\frac{4R^2}{abc}\sum(b+c)=\frac{4R^2}{4pRr}\cdot=\frac{4R}{r}\geq 8,$$

where the last inequality follows from Euler's inequality $R\geq 2r$.

The equality holds if and only if $a=b=c$.

□

Proof 2.

*We add the inequalities **1.** and **2.***

□

4. Prove that in triangle ABC, with angles $A; B; C$ side lengths $a; b; c$ the following inequality holds:

$$\frac{a(b+c)}{bc\cdot\sin A}+\frac{b(c+a)}{cd\cdot\sin B}+\frac{c(a+b)}{ab\cdot\sin C}\geq 4\sqrt{3}.$$

Proof.

We have

$$\sum\frac{a(b+c)}{bc\cdot\sin A}=\sum\frac{a(b+c)}{bc\cdot\frac{a}{2R}}=\frac{2R}{abc}\sum a(b+c)=\frac{2R}{4pRr}\cdot 2\sum bc=\frac{p^2+r^2+4Rr}{pr}\geq 4\sqrt{3},$$

where the last inequality follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$,

Doucet's inequality $p\sqrt{3}\leq 4R+r$ and Euler's inequality $R\geq 2r$.

The equality holds if and only if $a=b=c$.

□

5. Prove that in triangle ABC, with angles $A; B; C$ side lengths $a; b; c$ the following inequality holds:

$$\frac{a(b+c)}{bc\cdot\sin^3 A}+\frac{b(c+a)}{ca\cdot\sin^3 B}+\frac{c(a+b)}{ab\cdot\sin^3 C}\geq\frac{16}{\sqrt{3}}$$

Proof.

We have

$$\sum\frac{a(b+c)}{bc\cdot\sin^3 A}=\sum\frac{a(b+c)}{bc\cdot\frac{a^3}{8R^3}}=\frac{8R^3}{abc}\sum\frac{b+c}{a}=\frac{8R^3}{4pRr}\cdot\frac{p^2+r^2-2Rr}{2Rr}=\frac{R(p^2+r^2-2Rr)}{pr^2}\geq\frac{16}{\sqrt{3}},$$

where the last inequality follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$,

Doucet's inequality $p\sqrt{3}\leq 4R+r$ and Euler's inequality $R\geq 2r$.

The equality holds if and only if $a=b=c$.

□

6. Prove that in triangle ABC, with angles $A;B;C$ side lengths $a;b;c$ the following inequality holds:

$$\frac{a(b+c)}{bc\cdot\sin^4 A}+\frac{b(c+a)}{ca\cdot\sin^4 B}+\frac{c(a+b)}{ab\cdot\sin^4 C}\geq\frac{32}{3}.$$

Proposed by Marin Chirciu - Romania

Proof.

We have

$$\sum\frac{a(b+c)}{bc\cdot\sin^4 A}=\sum\frac{a(b+c)}{bc\cdot\frac{a^4}{16R^4}}=\frac{16R^4}{abc}\sum\frac{b+c}{a^2}=$$

$$=\frac{16R^4}{4pRr}\cdot\frac{p^4+p^2(2r^2-10Rr)+r^2(4R+r)(2R+r)}{8pR^2r^2}=$$

$$=\frac{R}{2p^2r^3}\Big[p^4+p^2(2r^2-10Rr)+r^2(4R+r)(2R+r)\Big]\geq\frac{32}{3},$$

where the last inequality holds if

$$3R[p^4+p^2(2r^2-10Rr)+r^2(4R+r)(2R+r)]\geq 64p^2r^3\Leftrightarrow$$

$$p^2(3Rp^2-30R^2r+6Rr^2-64r^3)+3Rr^2(8R^2+6Rr+r^2)\geq 0.$$

We distinguish the cases:

1. If $3Rp^2-30R^2r+6Rr^2-6r^3\geq 0$*, the inequality is equivalent.*

2. If $3Rp^2-30R^2r+6Rr^2-64r^3<0$*, we rewrite the inequality:*

$p^2(30R^2r-6Rr^2+64r^3-3Rp^2)\leq 3Rr^2(8R^2+6Rr+r^2)$*, which follows from Gerretsen's inequality* $16Rr-5r^2\leq p^2\leq 4R^2+4Rr+3r^2$*. It remains to prove that:*

$$(4R^2+4Rr+3r^2)[30R^2r-6Rr^2+64r^3-3R(16Rr-5r^2)]\leq 3Rr^2(8R^2+6Rr+r^2)\Leftrightarrow$$

$$\Leftrightarrow 18R^4+15R^3r-55R^2r^2-70Rr^3-48r^4\geq 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(18R^3+51R^2r+47Rr^2+24r^3)\geq 0$*, obviously from Euler's inequality* $R\geq 2r$*.*

The equality holds if and only if $a=b=c$*.*

□

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ROMANIAN MATHEMATICAL MAGAZINE
TRIANGLE MARATHON 101 - 200
PROBLEM 177

MARIN CHIRCIU

1. In ΔABC

$$\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-c}}\le\frac{3R}{2r}.$$

Proposed by George Apostolopoulos - Messolonghi - Grece

Proof.

Using Hölder's inequality, we obtain

$$\left(\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-c}}\right)^3\le$$

$$\le(a+b+c)\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)(1+1+1)=$$

$=2p\cdot\frac{4R+r}{2pr}\cdot3=3\left(1+\frac{4R}{r}\right)\le\left(\frac{3R}{2r}\right)^3$, *where the last inequality is equivalent with*

$$9R^3\ge8r^2(4R+r)\Leftrightarrow9R^3-32Rr^2-8r^3\ge0\Leftrightarrow(R-2r)(9R^2+18Rr+4r^2)\ge0$$

true from Euler's inequality: $R\ge2r$.

The equality holds for an equilateral triangle.

□

Remark

*Inequality **1.** can be strengthened:*

2. In ΔABC

$$\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-c}}\le1+\frac{R}{r}$$

Proposed by Marin Chirciu - Romania

Proof.

Using Hölder's inequality we obtain

$$\left(\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-c}}\right)^3\le$$

$$\le(a+b+c)\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)(1+1+1)=$$

$=2p\cdot\frac{4R+r}{2pr}\cdot3=3\left(1+\frac{4R}{r}\right)\le\left(1+\frac{R}{r}\right)^3$, *where the last inequality is equivalent with*

$$(R+r)^3 \geq 3r^2(4R+r) \Leftrightarrow R^3+3R^2r-9Rr^2-2r^3 \geq 0 \Leftrightarrow (R-2r)(R^2+5r+r^2) \geq 0$$

true from Euler's inequality: $R \geq 2r$.

The equality holds for an equilateral triangle.

□

Remark

*Inequality **2.** is stronger the inequality **1.***

3. In ΔABC

$$\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-a}} \leq 1+\frac{R}{r} \leq \frac{3R}{2r}.$$

Proof.

*See inequality **2.** and* $1+\frac{R}{r} \leq \frac{3R}{2r} \Leftrightarrow R \geq 2r$ *(Euler's inequality)*

Equality holds for an equilateral triangle.

□

*Inequality **2** can be developed*

4. In ΔABC

$$\sqrt[4]{\frac{a}{b+c-a}}+\sqrt[4]{\frac{b}{c+a-b}}+\sqrt[4]{\frac{c}{a+b-c}} \leq 1+\frac{R}{r}.$$

Proof.

Using Hölder's inequality we obtain

$$\left(\sqrt[4]{\frac{a}{b+c-a}}+\sqrt[4]{\frac{b}{c+a-b}}+\sqrt[4]{\frac{c}{a+b-c}}\right)^4 \leq$$

$$\leq (a+b+c)\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)(1+1+1)(1+1+1)=$$

$= 2p\cdot\frac{4R+r}{2pr}\cdot 3\cdot 3 = 9\left(1+\frac{4R}{r}\right) \leq \left(1+\frac{R}{r}\right)^4$*, where the last inequality is equivalent with*

$$(R+r)^4 \geq 9r^3(4R+r) \Leftrightarrow R^4+4R^3r+6R^2r^2-32Rr^3-8r^4 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)(R^3+6R^3r+18Rr^2+4r^3) \geq 0$$

which is true form Euler's inequality: $R \geq 2r$

The equality holds for an equilateral triangle.

□

5. In ΔABC

$$\sqrt[4]{\frac{a}{b+c-a}}+\sqrt[4]{\frac{b}{c+a-b}}+\sqrt[4]{\frac{c}{a+b-c}} \leq 1+\frac{R}{r} \leq \frac{3R}{2r}.$$

Proof.

*See **4.** and Euler's inequality $R \geq 2r$.*

□

Let's generalise inequality ***1.***

6. In ΔABC

$$\sqrt[n]{\frac{a}{b+c-a}}+\sqrt[n]{\frac{b}{c+a-b}}+\sqrt[n]{\frac{c}{a+b-c}}\leq\frac{3R}{2r},\ \text{where } n\in\mathbb{N}, n\geq 2$$

Proposed by Marin Chirciu - Romania

Proof.

Using Hölder's inequality we obtain

$$\left(\sqrt[n]{\frac{a}{b+c-a}}+\sqrt[n]{\frac{b}{c+a-b}}+\sqrt[n]{\frac{c}{a+b-c}}\right)^n\leq$$

$$\leq(a+b+c)\left(\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}\right)(1+1+1)\ldots(1+1+1)$$

$$=2p\cdot\frac{4R+r}{2pr}\cdot 3^{n-2}=3^{n-2}\cdot\left(1+\frac{4R}{r}\right)\leq\left(\frac{3R}{2r}\right)^n,$$ *where the last inequality is equivalent with*

$$9R^n\geq 2^n r^{n-1}\left(4R+r\right)\Leftrightarrow 9R^n-2^{n+2}Rr^{n-1}-2^n r^n\geq 0$$

Denoting $\frac{R}{r}=t\geq 2$ *it remains to prove that*

$$9t^n-2^{n+2}t-2^n\geq 0\Leftrightarrow(t-2)(9t^{n-1}+9\cdot 2t^{n-2}+9\cdot 2^2\cdot t^{n-3}+\ldots+9\cdot 2^{n-3}t^2+9\cdot 2^{n-2}t+2^{n-1})\geq 0,$$

Obviously because $t\geq 2$.

The equality holds for an equilateral triangle.

□

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INEQUALITY IN TRIANGLE - 242

MARIN CHIRCIU

Prove that in any triangle:

$$\frac{R}{r} \geq \frac{r_a}{r_b+r_c}+\frac{r_b}{r_c+r_a}+\frac{r_c}{r_a+r_b}+\frac{1}{2}$$

Proposed by Adil Abdulallayev - Baku - Azerbaidian, Marian Ursarescu - Romania

Proof.

Using $r_a=\frac{S}{p-a}$ *we obtain* $\sum\frac{r_a}{r_b+r_c}=\sum\frac{(p-b)(p-c)}{a(p-a)}=\frac{(4R+r)^3-p^2(8R-r)}{4p^2R}$

We write the inequality $\frac{R}{r}\geq\frac{(4R+r)^3-p^2(8R-r)}{4p^2R}+\frac{1}{2}\Leftrightarrow p^2(4R^2+6Rr-r^2)\geq r(4R+r)^3$,

which follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$. *It remains to prove that*

$$\Leftrightarrow (16Rr-5r^2)(4R^2+6Rr-r^2)\geq r(4R+r)^3 \Leftrightarrow 14R^2-29Rr+2r^2\geq 0\Leftrightarrow$$

$(R-2r)(14R-r)\geq 0$, *obviously from Euler's inequality:* $R\geq 2r$.

The equality holds for an equilateral triangle

□

Remark

The inequality can be devoloped

Prove that in any triangle:

$$\frac{R}{r}\geq n\Big(\frac{r_a}{r_b+r_c}+\frac{r_b}{r_c+r_a}+\frac{r_c}{r_a+r_b}\Big)+\frac{4-3n}{2}, \text{ where } 0\leq n\leq 1.$$

Proposed by Marin Chirciu - Romania

Proof.

Using $r_a=\frac{S}{p-a}$ *we obtain* $\sum\frac{r_a}{r_b+r_c}=\sum\frac{(p-b)(p-c)}{a(p-a)}=\frac{(4R+r)^3-p^2(8R-r)}{4p^2R}$

We write the inequality:

$$\frac{R}{r}\geq n\cdot\frac{(4R+r)^3-p^2(8R-r)}{4p^2R}+\frac{4-3n}{2}\Leftrightarrow p^2(4R^2+14nRr-8Rr-nr^2)\geq nr(4R+r)^3$$

which follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$.

It remains to prove that:

$$\Leftrightarrow (16Rr-5r^2)(4R^2+14nRr-8Rr-nr^2)\geq nr(4R+r)^3$$

$$\Leftrightarrow (32-32n)R^3+(88n-74)R^2r+(20-49n)Rr^2+2nr^2\geq 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)[(32-32n)R^2+(24n-10)Rr-nr^2]\geq 0,$$

obviously from Euler's inequality: $R\geq 2r$ *and the condition from the hypothesis* $0\leq n\leq 1$

The equality holds for an equilateral triangle.

□

Remark

For $n = 1$ we obtain ***INEQUALITY IN TRIANGLE - 242.***

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PROBLEM 175 - TRIANGLE MARATHON 101 - 200

MARIN CHIRCIU

1. In ΔABC

$$\sum \frac{1}{\sin^4 \frac{A}{2}} \geq \frac{(12r)^4}{\sum a^4}.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Remark

*Inequality **1** can be strengthened:*

2. In ΔABC

$$\sum \frac{1}{\sin^4 \frac{A}{2}} \geq \frac{(72Rr)^2}{\sum a^4}.$$

Proposed by Marin Chirciu - Romania

Proof.

In order to prove this inequality we will first present two additional results.

Lemma 1

3. In ABC

$$\sum \frac{1}{\sin^4 \frac{A}{2}} = \frac{p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4}{r^4}$$

Proof.

$$\sum \frac{1}{\sin^4 \frac{A}{2}} = \sum \frac{b^2c^2}{(p-b)^2(p-c)^2} = \frac{\sum b^2c^2(p-a)^2}{\prod(p-a)^2} = \frac{p^6 + p^4(2r^2 - 16Rr) + p^2(32R^2r^2 + r^4)}{p^2r^4}$$
$$= \frac{p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4}{r^4}.$$

□

Lamma 2

4. In ΔABC

$$\sum \frac{1}{\sin^4 \frac{A}{2}} \geq \frac{12R^2}{r^2}.$$

Proof.

*Using **Lemma 1** the inequality to prove can be written:*

$$\frac{p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4}{r^4} \geq \frac{12R^2}{r^2} \Leftrightarrow p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4 \geq 12R^2r^2$$
$$\Leftrightarrow p^2(p^2 + 2r^2 - 16Rr) + 20R^2r^2 + r^4 \geq 0.$$

We distinguish the following cases:

Case 1. *If $p^2 + 2r^2 - 16Rr \geq 0$, the inequality is obvious.*

Case 2. *If* $p^2+2r^2-16Rr<0$*, the inequality can be rewritten:*
$p^2(16Rr-2r^2-p^2)\le 20R^2r^2+r^4$*, which follows form Gerretsen's inequality:*
$16Rr-5r^2\le p^2\le 4R^2+4Rr+3r^2$. *It remains to prove that:*
$(4R^2+4Rr+3r^2)(16Rr-2r^2-16Rr+5r^2)\le 20R^2r^2+r^4\Leftrightarrow 3(4R^2+4Rr+3r^2)\le 20R^2+r^2$
$\Leftrightarrow 2R^2-3Rr-2r^2\ge 0\Leftrightarrow (R-2r)(2R+r)\ge 0$*, which is obvious from Euler's inequality:* $R\ge 2r$.
The equality holds for an equilateral triangle.

□

*Let's pass to solving inequality **2.***

In ΔABC

$$\sum \frac{1}{\sin^4\frac{A}{2}}\ge \frac{(72Rr)^2}{\sum a^4}.$$

*Inequality **2.** is equivalent with:*
$\sum a^4\cdot\sum \frac{1}{\sin^4\frac{A}{2}\ge (72Rr)^2}$*, which follows from using the known identity in triangle*
$\sum a^4=2\Big[p^4-2p^2(4Rr+3r^2)+r^2(4R+r)^2\Big]$ *and the inequality* $\sum \frac{1}{\sin^4\frac{A}{2}\ge\frac{12R^2}{r^2}}$
*which we've proved in **Lemma 2.***
It is enough to prove that:

$$2\Big[p^4-2p^2(4Rr+3r^2)+r^2(4R+r)^2\Big]\cdot\frac{12R^2}{r^2}\ge (72Rr)^2\Leftrightarrow$$

$p^4-2p^2(4Rr+3r^2)+r^2(4R+r)^2\ge 216r^4\Leftrightarrow p^2(p^2-8Rr-6r^2)+r^2(4R+r)^2\ge 216r^4$
which follows from Gerretsen's inequality $p^2\ge 16Rr-5r^2$ *and the remark that* $p^2-8Rr-6r^2>0$.
It remains to prove that:

$$(16Rr-5r^2)(16Rr-5r^2-8Rr-6r^2)+r^2(4R+r)^2\ge 216r^4\Leftrightarrow$$
$$(16Rr-5r^2)(8Rr-11r^2)+r^2(4R+r)^2\ge 216r^4\Leftrightarrow$$
$$(16R-5r)(8R-11r)+(4R+r)^2\ge 216r^2\Leftrightarrow 9R^2-13Rr-10r^2\ge 0\Leftrightarrow$$
$$\Leftrightarrow (R-2r)(9R+5r)\ge 0,$$ *obvious from Euler's inequality:* $R\ge 2r$.

The inequality holds for an equilateral triangle.

Remark

*Inequality **2.** is stronger than inequality **1.**:*

□

5. In ΔABC

$$\sum \frac{1}{\sin^4\frac{A}{2}}\ge \frac{(72Rr)^2}{\sum a^4}\ge\frac{(12r)^4}{\sum a^4}$$

Proof.

*See inequality **2.** and Euler's inequality* $R\ge 2r$.
The inequality holds for an equilateral triangle.

□

6. If $a, b, c > 0$ and $ab + bc + ca = 3$ prove that

$$\sum \frac{a^3 + b^3}{a^2 + ab + b^2} \geq 2$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

Remark

The inequality can be developed:

If $a, b, c > 0$ and $ab + bc + ca = 3$ prove that

$$\sum \frac{a^3 + b^3}{a^2 + nab + b^2} \geq \frac{6}{n+2}, \text{ where } n \geq 0.$$

Proposed by Marin Chirciu - Romania

Proof.

We have $\frac{a^2 - ab + b^2}{a^2 + nab + b^2} \geq \frac{1}{n+2} \Leftrightarrow (n+1)(a-b)^2 \geq 0$, *obvious, with equality for* $a = b$.

We obtain $\sum \frac{(a+b)(a^2 - ab + b^2)}{a^2 + nab + b^2} \geq \sum (a+b) \cdot \frac{1}{n+2} = \frac{2\sum a}{n+2} \geq \frac{6}{n+2}$, *wherefrom the last inequality is equivalent with* $\sum a \geq 3 \Leftrightarrow (\sum a)^2 \geq 9$, *which is true from*

$$(a + b + c)^2 \geq 3(ab + bc + ca) = 9.$$

The equality holds if and only if $a = b = c = 1$.

□

Remark

For $n = 1$ *we obtain* ***Problem 171*** *from TRIANGLE MARATHON 101 -200, proposed by Nguyen Viet Hung - Hanoi - Vietnam*

7. In ΔABC

$$\sum \frac{b^4 + c^4}{\tan^2 \frac{B}{2} + \tan^2 \frac{C}{2}} \geq 48S^2.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Remark

The inequality can be developed:

In ΔABC

$$\sum \frac{b^4 + nc^4}{\tan^2 \frac{B}{2} + n\tan^2 \frac{C}{2}} \geq 48S^2, \text{ where } n \geq 0.$$

Proposed by Marin Chirciu - Romania

Proof.

We have $a^2 \geq 4(p-b)(p-c) \Leftrightarrow a^2 \geq (a+b-c)(a+c-b) \Leftrightarrow a^2 \geq a^2-(b-c)^2 \Leftrightarrow (b-c)^2 \geq 0$

We obtain:

$$b^4 + nc^4 \geq 16(p-a)^2(p-c)^2 + n \cdot 16(p-a)^2(p-b)^2 = 16(p-a)^2\left[(p-c)^2 + n(p-b)^2\right];$$

$$\tan^2 \frac{B}{2} + n\tan^2 \frac{C}{2} = \frac{(p-a)(p-c)}{p(p-b)} + n \cdot \frac{(p-a)(p-b)}{p(p-c)} = \frac{p-a}{p(p-b)(p-c)}\left[(p-c)^2 + n(p-b)^2\right]$$

It follows

$$\frac{b^4+nc^4}{\tan^2\frac{B}{2}+n\tan^2\frac{C}{2}} \geq \frac{16(p-a)^2[(p-c)^2+n(p-b)^2]}{\frac{p-a}{p(p-b)(p-c)}[(p-c)^2+n(p-b)^2]} = 16p(p-a)(p-b)(p-c) = 16S^2$$

We deduce that $\sum \frac{b^4+nc^4}{\tan^2\frac{B}{2}+n\tan^2\frac{C}{2}} \geq \sum 16S^2 = 48S^2.$

The equality holds for an equilateral triangle.

□

Remark

For $n = 1$ *we obtain* ***Problem 137*** *from TRIANGLE MARATHON 101 - 200, proposed by George Apostolopoulos - Messolonghi - Greece.*

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PROBLEM 135 - TRIANGLE MARATHON 101 - 200

MARIN CHIRCIU

1. In ΔABC

$$a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\leq 3R\sqrt{2p}$$

Proposed by Daniel Sitaru - Romania

Remark

2. In ΔABC

$$a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\leq 2(R+r)\sqrt{2p}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the CBS inequality, we have:

$$\left(\sum a\sqrt{b}\right)^2=\left(\sum\sqrt{a}\sqrt{ab}\right)^2\leq\sum a\cdot\sum ab=2p\cdot(p^2+r^2+4Rr)\underbrace{\leq}^{Gerretsen}$$

$$\leq 2p\cdot(4R^2+4Rr+3r^2+r^2+4Rr)=$$

$$=2p\cdot 4(R+r)^2,\ \text{wherefrom}\ \sum a\sqrt{b}\leq 2(R+r)\sqrt{2p}$$

The equality holds if and only if the triangle is equilateral.

□

Remark

We can write the double inequality:

3. In ΔABC

$$a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\leq 2(R+r)\sqrt{2p}\leq 3R\sqrt{2p}$$

Proof.

Taking into account Euler's inequality we obtain $2(R+r)\sqrt{2p}\leq 3R\sqrt{2p}$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

In the same note we can propose:

4. In ΔABC

$$a\sqrt{b+c}+b\sqrt{c+a}+c\sqrt{a+b}\leq 4(R+r)\sqrt{p}\leq 6R\sqrt{p}$$

Proof.

Using CBS inequality we have:

$$\left(\sum a\sqrt{b+c}\right)^2 = \left(\sum \sqrt{a}\sqrt{a(b+c)}\right)^2 \sum a\cdot\sum 2bc = 4p\cdot(p^2+r^2+4Rr) \overbrace{\leq}^{Gerretsen}$$

$$4p\cdot(4R^2+4Rr+3r^2+r^2+4Rr) = 4p\cdot 4(R+r)^2, \text{ wherefrom } \sum a\sqrt{b+c} \leq 4(R+r)\sqrt{p}.$$

Taking into account Euler's inequality we obtain $4(R+r)\sqrt{p} \leq 6R\sqrt{p}$.

The equality holds if and only if the triangle is equilateral.

□

Remark

The inequality can be strengthened

5. In ΔABC

$$\boldsymbol{a\sqrt{b+nc}+b\sqrt{c+na}+c\sqrt{a+nb} \leq 2(R+r)\sqrt{2(n+1)p} \leq 3R\sqrt{2(n+1)p}, \text{ where } n \geq 0.}$$

Proposed by Marin Chirciu - Romania

Proof.

Using CBS inequality we have:

$$\left(\sum a\sqrt{b+nc}\right)^2 = \left(\sum \sqrt{a}\sqrt{a(b+nc)}\right)^2 \leq \sum a\cdot\sum(n+1)bc = 2(n+1)p\cdot(p^2+r^2+4Rr) \overbrace{\leq}^{Gerretsen}$$

$$= 2(n+1)p\cdot(4R^2+4Rr+3r^2+r^2+4Rr) = 2(n+1)p\cdot 4(R+r)^2, \text{ wherefrom}$$

$$\sum a\sqrt{b+nc} \leq 2(R+r)\sqrt{2(n+1)p}.$$

Taking into account Euler's inequality we obtain $2(R+r)\sqrt{2(n+1)p} \leq 3R\sqrt{2(n+1)p}$.

□

Remark

*For $n=0$ in **5.** we obtain **3.**, and for $n=1$ in **5.** we obtain **4.***

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PROBLEM 175 - TRIANGLE MARATHON 101 - 200

MARIN CHIRCIU

1. In ΔABC

$$\sum \frac{1}{\sin^4 \frac{A}{2}} \geq \frac{(12r)^4}{\sum a^4}.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Remark

*Inequality **1.** can be strengthened:*

2. In ΔABC

$$\sum \frac{1}{\sin^4 \frac{A}{2}} \geq \frac{(72Rr)^2}{\sum a^4}.$$

Proposed by Marin Chirciu - Romania

Proof.

In order to prove this inequality we will first present two additional results.

Lemma 1
3. In ΔABC

$$\sum \frac{1}{\sin^4 \frac{A}{2}} = \frac{p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4}{r^4}.$$

Proof.

$$\sum \frac{1}{\sin^4 \frac{A}{2}} = \sum \frac{b^2c^2}{(p-b)^2(p-c)^2} = \frac{\sum b^2c^2(p-a)^2}{\prod(p-a)^2} = \frac{p^6 + p^4(2r^2 - 16Rr) + p^2(32R^2r^2 + r^4)}{p^2r^4}$$
$$= \frac{p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4}{r^4}.$$

□

Lemma 2
4. In ΔABC

$$\sum \frac{1}{\sin^4 \frac{A}{2}} \geq \frac{12R^2}{r^2}.$$

Proof.

*Using **Lemma 1** the inequality we have to prove can be written:*

$$\frac{p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4}{r^4} \geq \frac{12R^2}{r^2} \Leftrightarrow p^4 + p^2(2r^2 - 16Rr) + 32R^2r^2 + r^4 \geq 12R^2r^2$$
$$\Leftrightarrow p^2(p^2 + 2r^2 - 16Rr) + 20R^2r^2 + r^4 \geq 0$$

We distinguish the cases:

Case 1. *If* $p^2+2r^2-16Rr\geq 0$*, the inequality is obvious.*

Case 2. *If* $p^2+2r^2-16Rr<0$*, the inequality can be rewritten:*

$p^2(16Rr-2r^2-p^2)\leq 20R^2r^2+r^4$*, which follows from Gerretsen's inequality:*

$16Rr-5r^2\leq p^2\leq 4R^2+4Rr+3r^2$*. It remains to prove that:*

$(4R^2+4Rr+3r^2)(16Rr-2r^2-16Rr+5r^2)\leq 20R^2r^2+r^4 \Leftrightarrow 3(4R^2+4Rr+3r^2)\leq 20R^2+r^2$

$\Leftrightarrow 2R^2-3Rr-2r^2\geq 0\Leftrightarrow (R-2r)(2R+r)\geq 0$*, obvious from Euler's inequality:* $R\geq 2r$*.*

The equality holds for an equilateral triangle.

□

Let's pass to solving inequality ***2.***

In ΔABC

$$\sum\frac{1}{\sin^4\frac{A}{2}}\geq\frac{(72Rr)^2}{\sum a^4}.$$

Inequality ***2.*** *is equivalent with:*

$\sum a^4\cdot\sum\frac{1}{\sin^4\frac{A}{2}}\geq(72Rr)^2$*, which follows from using the known identity in triangle.*

$\sum a^4=2[p^4-2p^2(4Rr+3r^2)+r^2(4R+r)^2]$ *and the inequality* $\sum\frac{1}{\sin^4\frac{A}{2}}\geq\frac{12R^2}{r^2}$

which we have proved it in ***Lemma 2****.*

It is enough to prove that:

$$2[p^4-2p^2(4Rr+3r^2)+r^2(4R+r)^2]\cdot\frac{12R^2}{r^2}\geq(72Rr)^2\Leftrightarrow$$

$p^4-2p^2(4Rr+3r^2)+r^2(4R+r)^2\geq 216r^4\Leftrightarrow p^2(p^2-8Rr-6r^2)+r^2(4R+r)^2\geq 216r^4$,

which follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$ *and the remark that* $p^2-8Rr-6r^2>0$.

It remains to prove that:

$$(16Rr-5r^2)(16Rr-5r^2-8Rr-6r^2)+r^2(4R+r)^2\geq 216r^4\Leftrightarrow$$

$$(16Rr-5r^2)(8Rr-11r^2)+r^2(4R+r)^2\geq 216r^4\Leftrightarrow$$

$$(16R-5r)(8R-11r)+(4R+r)^2\geq 216r^2\Leftrightarrow 9R^2-13Rr-10r^2\geq 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(9R+5r)\geq 0$*, obvious from Euler's inequality:* $R\geq 2r$*.*

The equality holds for an equilateral triangle.

□

Remark

Inequality ***2.*** *is stronger than inequality* ***1.****:*

5. In ΔABC

$$\sum\frac{1}{\sin^4\frac{A}{2}}\geq\frac{(72Rr)^2}{\sum a^4}\geq\frac{(12r)^4}{\sum a^4}.$$

Proof.

See inequality ***2.*** *and Euler's inequality* $R\geq 2r$*.*

The equality holds for an equilateral triangle.

□

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PROBLEM 127 - TRIANGLE MARATHON 101 - 200

MARIN CHIRCIU

1. In ΔABC

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\leq\frac{R}{2r}\sqrt{\frac{2R}{r}-1}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Proof.

Using the known identity in triangle $\sum\tan\frac{A}{2}=\frac{4R+r}{p}$ *we write the inequality:*

$$\frac{4R+r}{p}\leq\frac{R}{2r}\sqrt{\frac{2R}{r}-1}\Leftrightarrow\left(\frac{4R+r}{p}\right)^2\leq\left(\frac{R}{2r}\right)^2\left(\frac{2R}{r}-1\right)\Leftrightarrow p^2R^2(2R-r)\geq 4r^3(4R+r)^2$$

which follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$. *It remains to prove that:*

$\Leftrightarrow(16R-5r^2)\cdot R^2(2R-r)\geq 4r^3(4R+r)^2\Leftrightarrow 34R^4-26R^3r-59R^2r^2-32Rr^3-4r^4\geq 0\Leftrightarrow$

$\Leftrightarrow(R-2r)(32R^3+38Rr^2+17Rr^2+2r^3)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark

The inequality can be developed:

2. In ΔABC

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\leq\frac{R}{r}\sqrt{n\cdot\frac{R}{r}-2n+\frac{3}{4}},\ \textbf{where}\ n\geq 0.$$

Proposed by Marin Chirciu - Romania

Proof.

Using the known identity in triangle $\sum\tan\frac{A}{2}=\frac{4R+r}{p}$ *we write the inequality:*

$$\frac{4R+r}{p}\leq\frac{R}{r}\sqrt{n\cdot\frac{R}{r}-2n+\frac{3}{4}}\Leftrightarrow\left(\frac{4R+r}{p}\right)^2\leq\left(\frac{R}{r}\right)^2\left(n\cdot\frac{R}{r}-2n+\frac{3}{4}\right)\Leftrightarrow$$

$\Leftrightarrow p^2R^2\Big[4nR+(3-8n)r\Big]\geq 4r^3(4R+r)^2$, *which follows from Gerretsen's inequality:*

$p^2\geq 16Rr-5r^2$. *It remains to prove that:*

$$\Leftrightarrow(16Rr-5r^2)\cdot R^2\Big[4nR+(3-8n)r\Big]\geq 4r^3(4R+r)^2$$

$$\Leftrightarrow 64nR^4+(48-148n)R^3r+(40n-79)R^2r^2-32Rr^3-4R^4\geq 0\Leftrightarrow$$

$$\Leftrightarrow(R-2r)\Big[64nR^3+(48-20n)Rr^2+17Rr^2+2r^3\Big]\geq 0$$

obviously from Euler's inequality $R\geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark

For $n = \frac{1}{2}$ *in inequality* ***2.*** *we obtain inequality* ***1.****, meaning* ***Problem 127*** *from TRIANGLE MARATHON 101-200 proposed by George Apostolopoulos - Messolonghi - Greece.*

Remark

We can write the double inequality:

3. In ΔABC

$$\sqrt{3} \le \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \le \frac{R}{r}\sqrt{n \cdot \frac{R}{r} - 2n + \frac{3}{4}},\ \textit{where } n \ge 0.$$

Proof.

The first inequality follows from the identity $\sum \tan\frac{A}{2} = \frac{4R+r}{p}$ *and Doucet's inequality* $4R + r \ge p\sqrt{3}$*, the second inequality is* ***2.***

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

Remark

We can propose inequalities in the same format:

4. In ΔABC

$$\mathbf{1 \le \tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \le \left(\frac{R}{2r}\right)^2}$$

Proof.

The first inequality follows from the identity $\sum \tan^2\frac{A}{2} = \frac{(4R+r)^2 - 2p^2}{p^2}$ *and from Doucet's inequality:* $(4R+r)^2 \ge 3p^2$.

The second inequality, taking into account the above identity, can be written:

$$\frac{(4R+r)^2 - 2p^2}{p^2} \le \left(\frac{R}{2r}\right)^2 \Leftrightarrow 4r^2(4R+r)^2 - 8r^2p^2 \ge p^2R^2 \Leftrightarrow p^2(R^2+8r^2) \ge 4r^2(4R+r)^2,$$

Which follows from Gerretsen's inequality: $p^2 \ge 16Rr - 5r^2$*. It remains to prove that:*

$$(16Rr - 5r^2)(R^2 + 8r^2) \ge 4r^2(4R+r)^2 \Leftrightarrow 16R^3 - 69R^2r + 96Rr^2 - 44r^3 \ge 0 \Leftrightarrow$$

$(r - 2r)(16R^2 - 37Rr + 22r^2) \ge 0$*, obviously from Euler's inequality* $R \ge 2r$.

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

5. In ΔABC:

$$\mathbf{\frac{3r}{p} \le \tan^3\frac{A}{2} + \tan^3\frac{B}{2} + \tan^3\frac{C}{2} \le \frac{3R}{2p}\left[\left(\frac{3R}{2r}\right)^2 - 8\right].}$$

Proposed by Marin Chirciu - Romania

Proof.

First we prove the following identity:

Lemma

6. In ΔABC

$$\sum \tan^3 \frac{A}{2} = \frac{(4R+r)^3 - 12p^2R}{p^3}$$

Proof.

We use the identity $(x+y+z)^3 = x^3+y^3+z^3+3(x+y)(y+z)(z+x)$

we put $x = \tan\frac{A}{2}, y = \tan\frac{B}{2}, z = \tan\frac{C}{2}$ *and then we take into account that*

$$x+y+z = \sum \tan\frac{A}{2} = \frac{4R+r}{p},$$

$$(x+y)(y+z)(z+x) = \prod\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right) = \frac{4R}{p}.$$

Let's pass to solving the double inequality ***5.****:*

We write the first inequality:

$\frac{(4R+r)^3-12p^2R}{p^3} \geq \frac{3r}{p}$, *which follows from Doucet's inequality:* $(4R+r)^2 \geq 3p^2$.

We obtain $\frac{(4R+r)^3-12p^2R}{p^3} \geq \frac{(4R+r)\cdot 3p^2-12p^2R}{p^3} = \frac{3r}{p}$

We write the second inequality:

$\frac{(4R+r)^3-12p^2R}{p^3} \leq \frac{3R}{2p}\left[\left(\frac{3R}{2r}\right)^2-8\right] \Leftrightarrow 8r^2(4R+r)^3-96p^2Rr^2 \leq 3p^2R(9R^2-32r^2) \Leftrightarrow$

$27p^2R^2 \geq 8r^2(4R+r)^3$, *which follows from Gerretsen's inequality:* $p^2 \geq 16Rr-5r^2$.

It remains to prove that:

$27(16Rr-5r^2)R^2 \geq 8r^2(4R+r)^3 \Leftrightarrow 432R^4-647R^3r-384R^2r^2-96Rr^3-8r^4 \geq 0 \Leftrightarrow$

$(R-2r)(432R^3+217R^2r+50Rr^2+4r^3) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

□

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PROBLEM 125 - TRIANGLE MARATHON 101 - 200

MARIN CHIRCIU

1. In ΔABC

$$\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2} \geq \frac{8r}{3R}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Proof.

Using the known identity in triangle $\sum a^2 = 2(p^2-r^2-4Rr)$ *and the remarkable inequality* $\sum l_a^2 \leq p^2$, *which follows from* $l_a \leq \sqrt{p(p-a)}$, *we obtain*

$\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2} \geq \frac{2(p^2-r^2-4Rr)}{p^2} \geq \frac{8r}{3R}$, *where the last inequality is equivalent with:*

$3R(p^2-r^2-4Rr) \geq p^2 r \Leftrightarrow p^2(3R-4r) \geq 3R(r^2+4Rr)$, *true from Gerretsen's inequality* $p^2 \geq 16Rr-5r^2$. *It remains to prove that:*

$(16Rr-5r^2)(3R-4r) \geq 3R(r^2+4Rr) \Leftrightarrow 18R^2-41Rr+10r^2 \geq 0 \Leftrightarrow (R-2r)(18R-5r) \geq 0$,

obviously from Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark

The inequality can be strengthened:

2. In ΔABC

$$\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2} \geq \frac{18Rr}{p^2}.$$

Proof.

Using the known identity in triangle $\sum a^2 = 2(p^2-r^2-4Rr)$, *and the remarkable inequality* $\sum l_a^2 \leq p^2$, *which follows from* $l_a \leq \sqrt{p(p-a)}$, *we obtain*

$$\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2} \geq \frac{2(p^2-r^2-4Rr)}{p^2} \geq \frac{18Rr}{p^2},$$

where the last inequality is equivalent with: $p^2 \geq r^2+13Rr$, *true from Gerretsen's inequality* $p^2 \geq 16Rr-5r^2$.

It remains to prove that:

$16Rr-5r^2 \geq r^2+13Rr \Leftrightarrow 3Rr \geq 6r^2 \Leftrightarrow R \geq 2r$, *(Euler's inequality).*

The equality holds if and only if the triangle is equilateral.

□

Remark

Inequality 2. is stronger then inequality 1.:

3. In ΔABC

$$\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2} \geq \frac{18Rr}{p^2} \geq \frac{8r}{3R}$$

Proof.

See inequality 2. and Mitrinović's inequality: $p^2 \leq \dfrac{27R^2}{4}$.

The equality holds if and only if the triangle is equilateral.

□

Remark

Also, inequality 2. can be strengthened:

4. In ΔABC

$$\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2} \geq \frac{4}{3}.$$

Proof.

Using the known identity known in triangle $\sum a^2 = 2(p^2 - r^2 - 4Rr)$ *and the remarkable inequality* $\sum l_a^2 \leq p^2$, *which follows from* $l_a \leq \sqrt{p(p-a)}$, *we obtain* $\dfrac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2} \geq \dfrac{2(p^2-r^2-4Rr)}{p^2} \geq \dfrac{4}{3}$, *where the last inequality is equivalent with:*

$$3(p^2 - r^2 - 4Rr) \geq 2p^2 \Leftrightarrow p^2 \geq 3r^2 + 12Rr,$$

true from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$.

It remains to prove that:

$$16Rr - 5r^2 \geq 3r^2 + 12Rr \Leftrightarrow 4Rr \geq 8r^2 \Leftrightarrow R \geq 2r, \text{ (Euler's inequality).}$$

The equality holds if and only if the triangle is equilateral.

□

Remark

Inequality 4. is stronger than inequality 2.:

5. In ΔABC

$$\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2} \geq \frac{4}{3} \geq \frac{18Rr}{p^2}.$$

Proposed by Marin Chirciu - Romania

Proof.

See inequality 4. and inequality: $2p^2 \geq 27Rr$, *true from Gerretsen's inequality* $p^2 \geq 16Rr - 5r^2$. *It remains to prove that:* $2(16Rr - 5r^2) \geq 27Rr \Leftrightarrow 5Rr \geq 10r^2 \Leftrightarrow R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

We can write the triple inequality:

6. In ΔABC

$$\frac{a^2+b^2+c^2}{l_a^2+l_b^2+l_c^2} \geq \frac{4}{3} \geq \frac{18Rr}{p^2} \geq \frac{8r}{3R}$$

Proof.

*See inequality **5.** and inequality **3.***

Equality holds if and only if the triangle is equilateral.

□

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INEQUALITY IN TRIANGLE 295

MARIN CHIRCIU

1. Prove that in any triangle ABC

$$\sum \frac{1}{(II_a)^2} + \sum \frac{1}{(I_bI_c)^2} \leq \frac{1}{4r^2}$$

Proposed by Daniel Sitaru - Romania

Proof.

Using the formulas $II_a = 4R\sin\frac{A}{2}, I_bI_c = 4R\cos\frac{A}{2}$ *and the known identities in triangle:*

$$\sum \frac{1}{\sin^2\frac{A}{2}} = \frac{p^2+r^2-8Rr}{r^2}, \sum \frac{1}{\cos^2\frac{A}{2}} = \frac{p^2+(4R+r)^2}{p^2}, \text{ we obtain:}$$

$$\sum \frac{1}{(II_a)^2} = \frac{p^2+r^2-8Rr}{16R^2r^2} \text{ and } \sum \frac{1}{(I_bI_c)^2} = \frac{p^2+(4R+r)^2}{16R^2p^2}.$$

We write the inequality:

$$\frac{p^2+r^2-8Rr}{16R^2r^2} + \frac{p^2+(4R+r)^2}{16R^2p^2} \leq \frac{1}{4r^2} \Leftrightarrow p^2(4R^2+8Rr-2r^2-p^2) \geq r^2(4R+r)^2,$$

which follows from Gerretsen's inequality $16Rr-5r^2 \leq p^2 \leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$(16Rr-5r^2)(4R^2+8Rr-2r^2-4R^2-4Rr-3r^2) \geq r^2(4R+r)^2 \Leftrightarrow$$

$$\Leftrightarrow (16R-5r)(4R-5r) \geq (4R+r)^2 \Leftrightarrow 4R^2-9Rr+2r^2 \geq 0 \Leftrightarrow (R-2r)(4R-r) \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

It can also be shown an inequality having an opposite sense for the above sum:

2. Prove that in any triangle ABC

$$\sum \frac{1}{(II_a)^2} + \sum \frac{1}{(I_bI_c)^2} \geq \frac{1}{R^2}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the above identities $\sum \frac{1}{(II_a)^2} = \frac{p^2+r^2-8Rr}{16R^2r^2}$ *and* $\sum \frac{1}{(I_bI_c)^2} = \frac{p^2+(4R+r)^2}{16R^2p^2}$,

We obtain:

$$\sum \frac{1}{(II_a)^2} = \frac{p^2+r^2-8Rr}{16R^2r^2} \geq \frac{16Rr-5r^2+r^2-8Rr}{16R^2r^2} = \frac{8Rr-4r^2}{16R^2r^2} =$$
$$= \frac{2R-r}{4R^2r} \geq \frac{3r}{4R^2r} = \frac{3}{4R^2},$$

where the first inequality follows from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$,
and the second from Euler's inequality $R \geq 2r$.
The equality holds if and only if the triangle is equilateral.
We've obtained the helpful result:

Lemma 1.
Prove that in any triangle *ABC*

$$\sum \frac{1}{(II_a)^2} \geq \frac{3}{4R^2}.$$

Then $\sum \frac{1}{(I_bI_c)^2} = \frac{p^2+(4R+r)^2}{16R^2p^2} \geq \frac{p^2+3p^2}{16R^2p^2} = \frac{1}{4R^2}$,
which follows from Doucet's inequality: $(4R+r)^2 \geq 3p^2$.
The equality holds if and only if the triangle is equilateral.
We've obtained the following helpful result:

Lemma 2.
Prove that in any triangle *ABC*

$$\sum \frac{1}{(I_bI_c)^2} \geq \frac{1}{4R^2}.$$

Adding the inequality obtained from ***Lemma 1*** *and* ***Lemma 2*** *we obtain conclusion* ***2***.

□

Remark.

Finally it can be written the double inequality:

Prove that in any triangle *ABC*

$$\frac{1}{R^2} \leq \sum \frac{1}{(II_a)^2} + \sum \frac{1}{(I_bI_c)^2} \leq \frac{1}{4r^2}.$$

Proof.

See ***1*** *and* ***2***.
The equality holds if and only if the triangle is equilateral.
We've obtained a refinement of Euler's inequality.

□

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134 INEQUALITY IN TRIANGLE MATH ADVENTURES ON CUTTHEKNOT MATH 101-150

MARIN CHIRCIU

1. Prove that in any acute triangle the following relationship holds

$$\frac{a^2}{\tan B+\tan C}+\frac{b^2}{\tan C+\tan A}+\frac{c^2}{\tan A+\tan B}\le pR.$$

Proposed by Daniel Sitaru - Romania

Proof.

We have

$$\sum\frac{a^2}{\tan B+\tan C}=\sum\frac{a^2}{\frac{\sin(B+C)}{\cos B\cos C}}=\sum\frac{a^2\cos B\cos C}{\sin A}=\sum\frac{a^2\cos B\cos C}{\frac{a}{2R}}=$$

$$=2R\sum a\cos B\cos C=2R\cdot\frac{pr}{R}=2pr\overbrace{\le}^{Euler}pR.$$

From the above proof it follows that the relationship holds for any non-right angled triangle.

The equality holds if and only if the triangle is equilateral.

□

Remark.

In the same way we can propose the following:

2. Prove that in any triangle the following relationship holds:

$$\frac{a^2}{\cot B+\cot C}+\frac{b^2}{\cot C+\cot A}+\frac{c^2}{\cot A+\cot B}\le 3pR$$

Proposed by Marin Chirciu - Romania

Proof.

We have $\sum\frac{a^2}{\cot B+\cot C}=\sum\frac{a^2}{\frac{\sin(B+C)}{\sin B\sin C}}=\sum\frac{a^2\sin B\sin C}{\sin A}=\sum\frac{a^2\cdot\frac{a}{2R}\cdot\frac{b}{2R}}{\frac{a}{2R}}=$

$$=\frac{1}{2R}\sum abc=\frac{1}{2R}\cdot 3abc=\frac{1}{2R}\cdot 12pRr=6pr\overbrace{\le}^{Euler}3pR.$$

The equality holds if and only if the triangle is equilateral.

□

Remark.

Adding the two inequalities we obtain:

3. Prove that in any triangle the following relationship holds:

$$\sum \frac{a^2 \cos(B-C)}{\sin A} \leq 4pR.$$

Proposed by Daniel Sitaru - Romania and Marin Chirciu - Romania

Proof 1.

We have $\dfrac{a^2}{\tan B + \tan C} + \dfrac{a^2}{\cot B + \cot C} = \dfrac{a^2 \cos B \cos C}{\sin A} + \dfrac{a^2 \sin B \sin C}{\sin A} = \dfrac{a^2 \cos(B-C)}{\sin A}$.

Then $\sum \dfrac{a^2}{\tan B + \tan C} = 2S$ *and* $\sum \dfrac{a^2}{\cot B + \cot C} = 6S$.

It follows $\sum \dfrac{a^2 \cos(B-C)}{\sin A} = 8S$.

We write the inequality $8S \leq 4pR \Leftrightarrow 8pr \leq 4pR \Leftrightarrow 2r \leq R$ *(Euler's Inequality).*

The equality holds if and only if the triangle is equilateral.

□

Proof 2.

We have $\sum \dfrac{a^2 \cos(B-C)}{\sin A} = \sum \dfrac{a^2 \cos(B-C)}{\frac{a}{2R}} = 2R \sum \cos(B-C) = 2R \cdot \dfrac{4pr}{R} =$

$= 8pr \leq 4pR.$

*From the above proof it follows that the inequality from **3.** is true in any triangle.*

□

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PROBLEM 120
RMM TRIANGLE MARATHON
101-200

MARIN CHIRCIU

1. In ΔABC

$$\frac{1}{r_a - r} + \frac{1}{r_b - r} + \frac{1}{r_c - r} \geq \frac{7}{R} - \frac{2}{r}$$

Proposed by Mehmet Şahin - Ankara - Turkey

Remark.

*Inequality **1** can be developed:*

2. In ΔABC

$$\frac{1}{r_a - r} + \frac{1}{r_b - r} + \frac{1}{r_c - r} \geq \frac{\alpha}{R} - \frac{\beta}{r}, \text{ where } \alpha - 2\beta = 3 \text{ and } \beta \geq -2.$$

Proposed by Marin Chirciu - Romania

Proof.

Using the forumulas $r_a = \frac{S}{p-a}$ *and* $r = \frac{S}{p}$ *we obtain*

$$\sum \frac{1}{r_a - r} = \frac{1}{r} \sum \frac{p-a}{a} = \frac{1}{r} \cdot \frac{p^2 + r^2 - 8Rr}{4Rr} = \frac{p^2 + r^2 - 8Rr}{4Rr^2}.$$

The inequality can be written

$$\frac{p^2 + r^2 - 8Rr}{4Rr^2} \geq \frac{\alpha}{R} - \frac{\beta}{r} \Leftrightarrow p^2 + r^2 - 8Rr \geq 4r(\alpha r - \beta R), \text{ which follows from}$$

Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$. *It remains to prove that:*

$$16Rr - 5r^2 + r^2 - 8Rr \geq 4r(\alpha r - \beta R) \Leftrightarrow (\beta + 2)R \geq (\alpha + 1)r \Leftrightarrow R \geq 2r,$$

because $\alpha - 2\beta = 3$ *and* $\beta \geq -2$.

The equality holds if and only if the triangle is equilateral.

For $\alpha = 7$ *and* $\beta = 2$ *we obtain inequality **1**, namely **Problem 120** from **RMM Triangle Marathon 101-200**, proposed by Mehmet Şahin - Ankara - Turkey.*

□

Remark.

*Inequality **1** can be strengthened:*

3. In ΔABC

$$\frac{1}{r_a - r} + \frac{1}{r_b - r} + \frac{1}{r_c - r} \geq \frac{9}{4R - 2r}.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Proof.

*Using the proven inequality at **2**:* $\sum \frac{1}{r_a - r} = \frac{p^2 + r^2 - 8Rr}{4Rr^2}$, *inequality can be written:*

$\frac{p^2 + r^2 - 8Rr}{4Rr^2} \geq \frac{9}{4R - 2r}$, *which follows from Gerretsen's inequality* $p^2 \geq 16Rr - 5r^2$.

It remains to prove that: $\frac{16Rr - 5r^2 + r^2 - 8Rr}{4Rr^2} \geq \frac{9}{4R - 2r} \Leftrightarrow \frac{2R - r}{Rr} \geq \frac{9}{4R - 2r} \Leftrightarrow$

$\Leftrightarrow 8r^2 - 17Rr + 2r^2 \geq 0 \Leftrightarrow (R - 2r)(8R - r) \geq 0$, *obviously from Euler's inequality:* $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **3.** is stronger then inequality **1.**:*

4. In ΔABC

$$\frac{1}{r_a - r} + \frac{1}{r_b - r} + \frac{1}{r_c - r} \geq \frac{9}{4R - 2r} \geq \frac{7}{R} - \frac{2}{r}.$$

Proof.

*The first inequality is **3.**, and the second inequality is equivalent with:*

$$\frac{9}{4R - 2r} \geq \frac{7r - 2R}{Rr} \Leftrightarrow 8R^2 - 23Rr + 14r^2 \geq 0 \Leftrightarrow (R - 2r)(8R - 7r) \geq 0,$$

obviously from Euler's inequality: $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **3.** can be developed:*

5. In ΔABC

$$\frac{1}{r_a - r} + \frac{1}{r_b - r} + \frac{1}{r_c - r} \geq \frac{1}{xR - yr}, \text{ where } 2x - y = \frac{2}{3} \text{ and } x \geq 0.$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the proven identity at **2.**:* $\sum \frac{1}{r_a - r} = \frac{p^2 + r^2 - 8Rr}{4Rr^2}$, *the inequality can be written:*

$\frac{p^2 + r^2 - 8Rr}{4Rr^2} \geq \frac{1}{xR - yr}$, *which follows from Gerretsen's inequality* $p^2 \geq 16Rr - 5r^2$ *and*

the observation that $xR - yr > 0$, *for* $2x - y = \frac{2}{3}$ *and* $x \geq 0$.

It remains to prove that: $\frac{16Rr - 5r^2 + r^2 - 8Rr}{4Rr^2} \geq \frac{1}{xR - yr} \Leftrightarrow \frac{2R - r}{Rr} \geq \frac{1}{xR - yr} \Leftrightarrow$

$\Leftrightarrow (2R - r)(xR - yr) \geq Rr \Leftrightarrow 2xR^2 - (x + 2y + 1)Rr + yr^2 \geq 0 \Leftrightarrow (R - 2r)(4xR - yr) \geq 0$,

obviously from Euler's inequality: $R \geq 2r$ *and* $2x - y = \frac{2}{3}, x \geq 0$.

The equality holds if and only if the triangle is equilateral.

*For $x = \frac{4}{9}$ and $y = \frac{2}{3}$ we obtain inequality **3.** proposed by George Apostolopoulos - Messolonghi - Greece*

□

Remark.

*Inequality **5.** is stronger than inequality **2.**:*

6. In ΔABC

$$\frac{1}{r_a - r} + \frac{1}{r_b - r} + \frac{1}{r_c - r} \geq \frac{1}{xR - yr} \geq \frac{\alpha}{R} - \frac{\beta}{r},$$

$$\text{where } 2x - y = \frac{2}{3}, x \geq 0 \text{ and } \alpha - 2\beta = 3, \beta \geq 0.$$

Proof.

*First inequality is **5.**, and the second inequality is equivalent with:*

$\frac{1}{xR - yr} \geq \frac{\alpha}{R} - \frac{\beta}{r} \Leftrightarrow Rr \geq (xR - yr)(\alpha r - \beta R) \Leftrightarrow \beta x R^2 + (1 - \alpha x - \beta y)Rr + \alpha y r^2 \geq 0 \Leftrightarrow$

$\Leftrightarrow (R - 2r)(2\beta x R - \alpha y r) \geq 0$, *obviously from Euler's inequality:* $R \geq 2r$ *and* $2x - y = \frac{2}{3}$, $x \geq 0$, *and* $\alpha - 2\beta = 3, \beta \geq 0$, *which lead to* $(2x - y)(\alpha - 2\beta) = 2$, *wherefrom* $-\alpha y - 4\beta x = 2(1 - \alpha x - \beta y)$, *thus motivating the last inequality.*

The equality holds if and only if the triangle is equilateral.

For $x = \frac{4}{9}, y = \frac{2}{3}, \alpha = 7$ *and* $\beta = 2$ *its obtained the double inequality **4.***

□

Remark.

We can propose inequalities with sums having the form $\sum \frac{a^n}{r_a - r}$, *where* $n = 1, 2, 3, 4, 5$.

7. In ΔABC

$$3\sqrt{3} \leq \sum \frac{a}{r_a - r} \leq 3\sqrt{3} \cdot \frac{R}{2r}$$

Proof.

Using the formulas $r_a = \frac{S}{p - a}$ *and* $r = \frac{S}{p}$ *we obtain* $\sum \frac{a}{r_a - r} = \frac{1}{r} \sum (p - a) = \frac{p}{r}$.

The double inequality follows from Mitrinović's inequalities: $3\sqrt{3} \cdot r \leq p \leq \frac{3\sqrt{3}}{2} \cdot R$.

The equality holds if and only if the triangle is equilateral.

□

8. In ΔABC

$$18r \leq \sum \frac{a^2}{r_a - r} \leq 9R.$$

Proof.

Using the formulas $r_a = \dfrac{S}{p-a}$ *and* $r = \dfrac{S}{p}$ *we obtain*

$$\sum \frac{a^2}{r_a - r} = \frac{1}{r}\sum a(p-a) = \frac{1}{r}\cdot 2r(4R+r) = 2(4R+r).$$

The double inequality follows from Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

9. In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{12pr \leq \sum \frac{a^3}{r_a - r} \leq 6pR.}$$

Proof.

Using the formulas $r_a = \dfrac{S}{p-a}$ *and* $r = \dfrac{S}{p}$ *we obtain*

$$\sum \frac{a^3}{r_a - r} = \frac{1}{r}\sum a^2(p-a) = \frac{1}{r}\cdot 4pr(R+r) = 4p(R+r).$$

The double inequality follows from Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

10. In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{(6r)^3 \leq \sum \frac{a^4}{r_a - r} \leq (3R)^3.}$$

Proof.

Using the formulas $r_a = \dfrac{S}{p-a}$ *and* $r = \dfrac{S}{p}$ *we obtain*

$$\sum \frac{a^4}{r_a - r} = \frac{1}{r}\cdot\sum a^3(p-a) = \frac{1}{r}\cdot 2r\left[p^2(2R+3r) - r(4R+r)^2\right] = 2p^2(2R+3r) - 2r(4R+r)^2.$$

The first inequality follows from the above identity, Gerretsen's inequality:

$p^2 \geq 16Rr - 5r^2$ *and Euler's inequality:* $R \geq 2r$.

We obtain $2p^2(2R+3r) - 2r(4R+r)^2 \geq 2(16Rr - 5r^2)(2R+3r) - 2r(4R+r)^2 =$

$= 4r(8R^2 + 15Rr - 8r^2) \geq 4r \cdot 54r^2 = 216r^3 = (6r)^3.$

For the second inequality we use the above identity, Gerretsen's inequality:

$p^2 \leq 4R^2 + 4Rr + 3r^2$ *and Euler's inequality:* $R \geq 2r$.

We obtain $2p^2(2R+3r) - 2r(4R+r)^2 \leq 2(4R^2+4Rr+3r^2)(2R+3r) - 2r(4R+r)^2 =$

$= 16R^3 + 8R^2r + 20Rr^2 + 16r^3 \leq 16R^3 + 4R^3 + 5R^3 + 2R^3 = 27R^3 = (3R)^3.$

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

11. In ΔABC

$$\mathbf{18p\cdot(2r)^3\le\sum\frac{a^5}{r_a-r}\le 18p\cdot R^3.}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the formulas $r_a=\dfrac{S}{p-a}$ *and* $r=\dfrac{S}{r}$ *we obtain*

$$\sum\frac{a^5}{r_a-r}=\frac{1}{r}\cdot\sum a^4(p-a)=\frac{1}{r}\cdot 4pr\Big[p^2(R+2r)-r(12R^2+11Rr+2r^2)\Big]=$$

$$=4p\Big[p^2(R+2r)-r(12R^2+11Rr+2r^2)\Big].$$

The first inequality follows from the above identity, Gerretsen's inequality: $p^2\ge 16Rr-5r^2$ *and Euler's inequality:* $R\ge 2r$.

We obtain

$$4p\Big[p^2(R+2r)-r(12R^2+11Rr+2r^2)\Big]\ge 4pr\Big[(16Rr-5r^2)(R+2r)-r(12R^2+11Rr+2r^2)\Big]$$

$$=16pr(R^2+4Rr-3r^2)\ge 16pr\cdot(4r^2+8r^2-3r^2)=16pr\cdot 9r^2=144pr^3=18p\cdot(2r)^3.$$

$$=4r(8R^2+15Rr-8r^2)\ge 4r\cdot 54r^2=216r^3=(6r)^3.$$

For the second inequality we use the above identity, Gerretsen's inequality: $p^2\le 4R^2+4Rr+3r^2$ *and Euler's inequality:* $R\ge 2r$.

We obtain

$$4p\Big[p^2(R+2r)-r(12R^2+11Rr+2r^2)\Big]\le 4p\Big[(4R^2+4Rr+3r^2)(R+2r)-r(12R^2+11Rr+2r^2)\Big]=$$

$$=16p(R^3+r^3)\le 16p\cdot\left(R^3+\frac{R^3}{8}\right)=18p\cdot R^3$$

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

Remark.

We can propose inequalities with sums having the form $\sum\dfrac{a^n(b+c)}{r_a-r}$, *where* $n=1,2,3,4$.

12. In ΔABC

$$\mathbf{(6r)^2\le r\sum\frac{a(b+c)}{r_a-r}\le(3R)^2.}$$

Proof.

Using the formulas $r_a=\dfrac{S}{p-a}$ *and* $r=\dfrac{S}{p}$ *we obtain*

$$\sum\frac{a(b+c)}{r_a-r}=\frac{1}{r}\sum(b+c)(p-a)=\frac{1}{r}\cdot 2(p^2-r^2-4Rr)=\frac{2(p^2-r^2-4Rr)}{r}.$$

The first inequality follows from the above identity, Gerretsen's inequality: $p^2\ge 16Rr-5r^2$ *and Euler's inequality:* $R\ge 2r$.

We obtain

$$r\sum\frac{a(b+c)}{r_a-r}=r\cdot\frac{2(p^2-r^2-4Rr)}{r}=2(p^2-r^2-4Rr)\ge 2(16Rr-5r^2-r^2-4Rr)=$$

$$= 12r(2R-r) \geq 12R \cdot 3r = (6r)^2.$$

For the second inequality we use the above identity, Gerretsen's inequality: $p^2 \leq 4R^2 + 4Rr + 3r^2$ *and Euler's inequality:* $R \geq 2r$.

We obtain

$$r\sum \frac{a(b+c)}{r_a - r} = r \cdot \frac{2(p^2 - r^2 - 4Rr)}{r} = 2(p^2 - r^2 - 4Rr) \geq 2(4R^2 + 4Rr + 3r^2 - r^2 - 4Rr) =$$

$$= 8R^2 + 4r^2 \leq 9R^2 = (3R)^2.$$

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

13. In ΔABC

$$\mathbf{36\sqrt{3} \cdot Rr \leq \sum \frac{a^2(b+c)}{r_a - r} \leq 18\sqrt{3} \cdot R^2.}$$

Proof.

Using the formulas $r_a = \dfrac{S}{p-a}$ *and* $r = \dfrac{S}{p}$ *we obtain*

$$\sum \frac{a^2(b+c)}{r_a - r} = \frac{1}{r}\sum a(b+c)(p-a) = \frac{1}{r} \cdot 12pRr = 12pR.$$

The double inequality follows from Mitrinović's inequalities: $3\sqrt{3} \cdot r \leq p \leq \dfrac{3\sqrt{3}}{2} \cdot R$.

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

14. In ΔABC

$$\mathbf{(6r)^4 \leq 3\sum \frac{a^3(b+c)}{r_a - r} \leq (3R)^4.}$$

Proof.

Using the formulas $r_a = \dfrac{S}{p-a}$ *and* $r = \dfrac{S}{p}$ *we obtain*

$$\sum \frac{a^3(b+c)}{r_a - r} = \frac{1}{r}\sum a^2(b+c)(p-a) = \frac{1}{r} \cdot \left[2p^2(2Rr + r^2) + 2r^2(4R+r)^2\right] =$$

$$= \frac{2p^2(2Rr + r^2) + 2r^2(4R+r)^2}{r}.$$

The first inequality follows from the above identity, Gerretsen's inequality: $p^2 \geq 16Rr - 5r^2$ *and Euler's inequality:* $R \geq 2r$.

We obtain

$$3\sum \frac{a^3(b+c)}{r_a - r} = 3 \cdot \frac{2p^2(2Rr + r^2) + 2r^2(4R+r)^2}{r} \geq 3 \cdot \frac{2(16Rr - 5r^2)(2Rr + r^2) + 2r^2(4R+r)^2}{r}$$

$$= 3 \cdot 4r^2(24R^2 + 7Rr - 2r^2) \geq 12r^2(24 \cdot 4r^2 + 7r \cdot 2r - 2r^2) = 12r^2 \cdot 108r^2 = 1296r^4 = (6r)^4.$$

For the second inequality we use the above identity, Gerretsen's inequality: $p^2 \leq 4R^2 + 4Rr + 3r^2$ *and Euler's inequality:* $R \geq 2r$.

We obtain

$$3\sum\frac{a^3(b+c)}{r_a-r}=3\cdot\frac{2p^2(2Rr+r^2)+2r^2(4R+r)^2}{r}\le 3\cdot\frac{2(4R^2+4Rr+3r^2)(2Rr+r^2)+2r^2(4R+r)^2}{r}$$

$$=3\cdot r(16R^3+56R^2r+36Rr^2+8r^3)\le 3r(16R^3+28R^3+9R^3+R^3)\le\frac{3R}{2}\cdot 54R^3=81R^4=(3R)^4$$

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

15. In ΔABC

$$p(12r)^3\le 6\sum\frac{a^4(b+c)}{r_a-r}\le p(6R)^3.$$

Proposed by Marin Chirciu - Romania

Proof.

Using the formulas $r_a=\dfrac{S}{p-a}$ *and* $r=\dfrac{S}{p}$ *we obtain*

$$\sum\frac{a^4(b+c)}{r_a-r}=\frac{1}{r}\sum a^3(b+c)(p-a)=\frac{1}{r}\cdot\left[4p^3(Rr+r^2)+4p(-4R^2r^2+3Rr^3+r^4)\right]=$$

$$=4p^3(R+r)+4p(-4R^2r+3Rr^2+r^3).$$

The first inequality follows from the above identity, Gerretsen's inequality: $p^2\ge 16Rr-5r^2$ *and Euler's inequality:* $R\ge 2r$.

We obtain

$$6\sum\frac{a^4(b+c)}{r_a-r}=6\left[4p^3(R+r)+4p(-4R^2r+3Rr^2+r^3)\right]=24p\left[p^2(R+r)-4R^2r+3Rr^2+r^3\right]$$

$$\ge 24p\left[(16Rr-5r^2)(R+r)-4R^2r+3Rr^2+r^3\right]=48pr(12R^2+14Rr-4r^2)\ge$$

$$\ge 48pr(24r^2+14r^2-2r^2)=48pr\cdot 36r^2=1728pr^3=p(12r)^3.$$

For the second inequality we use the above identity, Gerretsen's inequality: $p^2\le 4R^2+4Rr+3r^2$ *and Euler's inequality:* $R\ge 2r$.

We obtain

$$6\sum\frac{a^4(b+c)}{r_a-r}=6\left[4p^3(R+r)+4p(-4R^2r+3Rr^2+r^3)\right]=24p\left[p^2(R+r)-4R^2r+3Rr^2+r^3\right]\le$$

$$\le 24p\left[(4R^2+4Rr+3r^2)(R+r)-4R^2r+3Rr^2+r^3\right]=12p\cdot(8R^3+8R^2r+20Rr^2+8r^3)\le$$

$$\le 12p(8R^3+4R^3+5R^3+R^3)=12p\cdot 18R^3=216pR^3=p(6R)^3.$$

The equality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

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INEQUALITY IN TRIANGLE 271
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MARIN CHIRCIU

1. In acute-angled ΔABC

$$\mathbf{2\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\tan A\tan B\tan C\geq 9\sqrt{3}.}$$

Proposed by Daniel Sitaru - Romania

Proof.

Using the known identities in triangle $\sum\cot\frac{A}{2}=\frac{p}{r}$ *and* $\prod\tan A=\frac{2pr}{p^2-(2R+r)^2}$,

the inequality we have to prove can be written: $2\cdot\frac{p}{r}+\frac{2pr}{p^2-(2R+r)^2}\geq 9\sqrt{3}$.

Using Mitrinović's inequality $p\geq 3\sqrt{3}\cdot r$ *we have* $\frac{p}{r}\geq 3\sqrt{3}$ *and* $pr\geq 3\sqrt{3}\cdot r^2$

it's enough to prove that

$$\frac{2r^2}{p^2-(2R+r)^2}\geq 1\Leftrightarrow p^2\leq 4R^2+4Rr+3r^2 \text{ (Gerretsen's inequality).}$$

The inequality holds if and only if the triangle is equilateral.

□

Remark.

The inequality can be developed:

2. In acute-angled ΔABC

$$\mathbf{n\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\tan A\tan B\tan C\geq (n+1)\cdot 3\sqrt{3},\ \text{where}\ n\geq 0.}$$

Proof.

We use the known inequalities in triangle $\sum\cot\frac{A}{2}=\frac{p}{r}$ *and* $\prod\tan A=\frac{2pr}{p^2-(2R+r)^2}$.

We have

(i) $\sum\cot\frac{A}{2}\geq 3\sqrt{3}\Leftrightarrow\frac{p}{r}\geq 3\sqrt{3}\Leftrightarrow p\geq 3\sqrt{3}\cdot r$ *(Mitrinović's inequality);*

(ii)

$\prod\tan A\geq 3\sqrt{3}\Leftrightarrow\frac{2pr}{p^2-(2R+r)^2}\geq 3\sqrt{3}$, *which follows from* $pr\geq 3\sqrt{3}\cdot r^2$ *and*

$$\frac{2r^2}{p^2-(2R+r)^2} \geq 1 \Leftrightarrow p^2 \leq 4R^2+4Rr+3r^2 \text{ (Gerretsen's inequality).}$$

From (i), (ii) *and* $n \geq 0$ *the conclusion is obtained.*

The inequality holds if and only if the triangle is equilateral.

□

3. In acute-angled ΔABC

$$n\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)+k\tan A\tan B\tan C \geq (n+k)\cdot 3\sqrt{3}, \text{ where } n \geq 0, k \geq 0.$$

Proof.

We use the known identities in triangle $\sum \cot\frac{A}{2} = \frac{p}{r}$ *and* $\prod \tan A = \frac{2pr}{p^2-(2R+r)^2}$.

We have

(i) $\sum \cot\frac{A}{2} \geq 3\sqrt{3} \Leftrightarrow \frac{p}{r} \geq 3\sqrt{3} \Leftrightarrow p \geq 3\sqrt{3}\cdot r$ *(Mitrinović's inequality);*

(ii)

$\prod \tan A \geq 3\sqrt{3} \Leftrightarrow \frac{2pr}{p^2-(2R+r)^2} \geq 3\sqrt{3}$, *which follows from* $pr \geq 3\sqrt{3}\cdot r^2$ *and*

$$\frac{2r^2}{p^2-(2R+r)^2} \geq 1 \Leftrightarrow p^2 \leq 4R^2+4Rr+3r^2 \text{ (Gerretsen's inequality)}$$

From (i), (ii) *and* $n \geq 0, k \geq 0$ *the conclusion is obtained.*

The inequality holds if and only if the triangle is equilateral or $n = k = 0$.

□

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MARIN CHIRCIU

1. In ΔABC

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\geq\frac{3}{2Rr}.$$

Proposed by Rovsen Pirguliev - Sumgait - Azerbaidian

Proof.

We have $AI=\frac{r}{\sin\frac{A}{2}}$. *We obtain* $\sum\frac{1}{AI^2}=\sum\frac{\sin^2\frac{A}{2}}{r^2}=\frac{1}{r^2}\sum\sin^2\frac{A}{2}=$

$$=\frac{1}{r^2}\cdot\frac{2R-r}{2R}=\frac{2R-r}{2Rr^2}.$$

The inequality we have to prove can be written

$\frac{2R-r}{2Rr^2}\geq\frac{3}{2Rr}\Leftrightarrow R\geq 2r$ *(Euler's inequality).*

The equality holds if and only if the triangle is equilateral.

□

Remark.

The inequality can be strengthened:

2. In ΔABC

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\geq\frac{3}{4r^2}.$$

Proposed by Marin Chirciu - Romania

Proof.

We use the identity $\sum\frac{1}{AI^2}=\frac{2R-r}{2Rr^2}$, *the inequality can be written*

$\frac{2R-r}{2Rr^2}\geq\frac{3}{4r^2}\Leftrightarrow R\geq 2r$, *obviously from Euler's inequality.*

The equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **2.** is stronger than inequality **1.***

3. In ΔABC

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\geq\frac{3}{4r^2}\geq\frac{3}{2Rr}.$$

Proof.

*See inequality **2** and Euler's inequality.*

The equality holds if and only if the triangle is equilateral.

□

Remark.

The following inequality holds:

4. In ΔABC

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\geq\frac{3}{4r^2}\geq\frac{3}{2Rr}\geq\frac{81}{4p^2}\geq\frac{3}{R^2}.$$

Proof.

*The first inequality is **2**, the second follows from Euler's inequality, the third is equivalent with $2p^2 \geq 27Rr$, which follows from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$ and Euler's inequality $R \geq 2r$ and the forth is Mitrinović's ineuality $p^2 \leq \frac{27R^2}{4}$.*

□

Now, let's find an inequality having an opposite sense.

5. In ΔABC

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\leq\frac{3R}{8r^3}.$$

Proof.

Using the identity $\sum\frac{1}{AI^2}=\frac{2R-r}{2Rr^2}$, *the inequality can be written*

$$\frac{2R-r}{2Rr^2}\leq\frac{3R}{8r^3}\Leftrightarrow 4r(2R-r)\leq 3R^2\Leftrightarrow 3R^2-8Rr+4r^2\geq 0\Leftrightarrow (R-2r)(3R-2r)\geq 0,$$

obviously from Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

The following double inequality can be written:

6. In ΔABC

$$\frac{3}{4r^2}\leq\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\leq\frac{3R}{8r^3}.$$

Proposed by Marin Chirciu - Romania

Proof.

*See inequalities **2** and **5**.*

The equality holds if and only if the triangle is equilateral.

□

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