RMM - Calculus Marathon 401 - 500

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401. Prove that:

$$csc\left(\frac{\pi}{14}\right)-4\,cos\left(\frac{2\pi}{7}\right)=2$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by Khaled Abd Almuti-Damascus-Syria

Prove that:

$$\frac{\sin\frac{7x}{2}}{\sin\frac{x}{2}}\stackrel{?}{=}2\cos 3x+2\cos x+2\cos 2x+1$$

$$\frac{\sin\frac{7x}{2}}{\sin\frac{x}{2}}=\frac{2\sin\frac{7x}{2}\cdot\cos\frac{x}{2}}{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}=\frac{2\sin\left(3x+\frac{x}{2}\right)\cdot\cos\frac{x}{2}}{\sin x}$$

$$=\frac{2\left[\sin 3x\cdot\cos\frac{x}{2}+\cos 3x\cdot\sin\frac{x}{2}\right]\cdot\cos\frac{x}{2}}{\sin x}=\frac{2\sin 3x\cdot\cos^2\frac{x}{2}+\cos 3x\cdot 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{\sin x}$$

$$=\frac{2\sin 3x\cdot\cos^2\frac{x}{2}+\cos 3x\cdot\sin x}{\sin x}=\frac{2\sin 3x\left(\frac{1+\cos x}{2}\right)+\cos 3x\cdot\sin x}{\sin x}$$

$$=\frac{\sin 3x\,(1+\cos x)+\cos 3x\cdot\sin x}{\sin x}=\frac{(3\sin x-4\sin^3 x)(1+\cos x)+\cos 3x\cdot\sin x}{\sin x}$$

$$=(3-4\sin^2 x)(1+\cos x)+\cos 3x=[3-4(1-\cos^2 x)](1+\cos x)+\cos 3x$$

$$=(4\cos^2 x-1)(1+\cos x)+4\cos^3 x-3\cos x=$$

$$=4\cos^2 x+4\cos^3 x-1-\cos x+4\cos^3 x-3\cos x=8\cos^3 x+4\cos^2 x-4\cos x-1$$

But: $\cos 3x=4\cos^3 x-3\cos x$

$4\cos^3 x=\cos 3x+3\cos x$, ***and:*** $\cos^2 x=\frac{1+\cos 2x}{2}$

$$\frac{\sin 7x}{\sin\frac{x}{2}}=2\cos 3x+6\cos x+2+2\cos 2x-4\cos x-1$$

$$\frac{\sin\frac{7x}{2}}{\sin\frac{x}{2}}=2\cos 3x+2\cos x+2\cos 2x+1\quad (*)$$

For $x=\frac{\pi}{7},\ \frac{\sin\frac{7}{2}\cdot\frac{\pi}{7}}{\sin\frac{1}{2}\cdot\frac{\pi}{7}}=2\cos\frac{3\pi}{7}+2\cos\frac{\pi}{7}+2\cos\frac{2\pi}{7}+1$

$$\frac{1}{\sin\frac{\pi}{14}} = 2\cos\frac{3\pi}{7} + 2\cos\frac{\pi}{7} + 2\cos\frac{2\pi}{7} + 1$$

$$\csc\left(\frac{\pi}{14}\right) = 2\cos\frac{3\pi}{7} + 2\cos\frac{\pi}{7} + 2\cos\frac{2\pi}{7} + 1 \quad (I)$$

Suppose, $\alpha = e^{\frac{i2\pi}{7}}$*. It is easily to prove that:*

$$S = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 + 1 = 0$$

$$\left[not: S = \alpha\cdot\frac{1-\alpha^7}{1-\alpha}, \alpha^7 = e^{i2\pi} = 1, S = 0\right]$$

and: $\alpha + \alpha^6 = 2\cos\left(\frac{2\pi}{7}\right) \quad \begin{cases} \alpha = e^{i\frac{2\pi}{7}} \\ \alpha^6 = e^{i\,12\frac{\pi}{7}} = e^{i\left(\frac{14\pi}{7}-\frac{2\pi}{7}\right)} = e^{-i\frac{2\pi}{7}} \end{cases}$

$$\alpha^2 + \alpha^5 = 2\cos\left(\frac{4\pi}{7}\right) \quad \begin{cases} \alpha^2 = e^{i4\frac{\pi}{7}} \\ \alpha^5 = e^{i10\frac{\pi}{7}} = e^{i\left(14\frac{\pi}{7}-\frac{4\pi}{7}\right)} = e^{-i\frac{4\pi}{7}} \end{cases}$$

$$\alpha^3 + \alpha^4 = 2\cos\left(\frac{6\pi}{7}\right) \quad \begin{cases} \alpha^3 = e^{i\frac{6\pi}{7}} \\ \alpha^4 = e^{i\frac{8\pi}{7}} = e^{i\left(\frac{14\pi}{7}-\frac{6\pi}{7}\right)} = e^{-i\frac{6\pi}{7}} \end{cases}$$

So: $2\cos\left(\frac{2\pi}{7}\right) + 2\cos\left(\frac{4\pi}{7}\right) + 2\cos\left(\frac{6\pi}{7}\right) = -1$

$$2\cos\left(\frac{2\pi}{7}\right) + 2\cos\left(\pi - \frac{3\pi}{7}\right) + 2\cos\left(\pi - \frac{\pi}{7}\right) + 1 = 0$$

$$2\cos\left(\frac{2\pi}{7}\right) + 1 = 2\cos\frac{3\pi}{7} + 2\cos\frac{\pi}{7} \quad (II)$$

Substituted into relation (I)

$$\csc\left(\frac{\pi}{14}\right) = 4\cos\left(\frac{2\pi}{7}\right) + 2$$

$$\csc\left(\frac{\pi}{14}\right) - 4\cos\left(\frac{2\pi}{7}\right) = 2$$

Solution 2 by Naren Bhandari-Nepal

LHS $= \frac{1}{\sin\frac{\pi}{14}} - 4\cos\frac{2\pi}{7} = \frac{1}{\cos\frac{3\pi}{7}}\left(1 - 4\cos\frac{2\pi}{7}\cdot\cos\frac{3\pi}{7}\right) \quad \sin\frac{\pi}{14} = \cos\left(\frac{\pi}{2} - \frac{3\pi}{7}\right)$

$$= \frac{1 - 4\cos\frac{2\pi}{7}\cdot\cos\frac{3\pi}{7}}{\cos\frac{3\pi}{7}} = \frac{1 - 21\left(\cos\frac{2\pi}{7}\cdot\cos\frac{3\pi}{7}\right)}{\cos\frac{3\pi}{7}}$$

Recall that: $\cos\frac{\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{3\pi}{7} = \frac{1}{8}$*. Thus: LHS* $= \frac{8\cos\frac{2\pi}{7}\cdot\cos\frac{3\pi}{7}\cdot\cos\frac{\pi}{7} - 4\left(\cos\frac{2\pi}{7}\cdot\cos\frac{3\pi}{7}\right)}{\cos\frac{3\pi}{7}} =$

$$=\frac{4\cos\frac{2\pi}{7}\cdot\cos\frac{3\pi}{7}\left(2\cos\frac{\pi}{7}-1\right)}{\cos\frac{3\pi}{7}}=4\cos\frac{2\pi}{7}\left(2\cos\frac{\pi}{7}-1\right)=$$

$$=4\left[\cos\frac{2\pi}{7}\left(2\cos\frac{\pi}{7}-1\right)\right]=4\left(\frac{1}{2}\right)=2$$

Proof identity of $\cos\frac{2\pi}{7}\left(2\cos\frac{\pi}{7}-1\right)$

$$\frac{1}{2\sin\frac{2\pi}{7}}\sin\frac{4\pi}{7}\left(2\cos\frac{\pi}{7}-1\right)$$

$$\frac{1}{2\sin 2\pi}\left(2\cos\frac{\pi}{7}\cdot\sin\frac{4\pi}{7}-\sin\frac{4\pi}{7}\right)=\frac{1}{2\sin\frac{2\pi}{7}}\left(\sin\frac{5\pi}{7}+\sin\frac{3\pi}{7}-\sin\frac{4\pi}{7}\right)=$$

$$=\frac{1}{2\sin\frac{2\pi}{7}}\left[\sin\frac{5\pi}{7}+2\cos\frac{7\pi}{2\cdot 7}\cdot\sin(-\pi)\right]=\frac{1}{2\sin\frac{2\pi}{7}}\left(\sin\frac{5\pi}{7}+2\cdot 0\right)=$$

$$=\frac{1}{2\sin\frac{2\pi}{7}}\sin\left(\pi-\frac{2\pi}{7}\right)=\frac{1}{2\sin\frac{2\pi}{7}}\cdot\sin\frac{2\pi}{7}=\frac{1}{2}$$

Solution 3 by Sagar Kumar-Patna Bihar-India

$$\left(2+4\cos\left(\frac{2\pi}{7}\right)\right)=y$$

Let $y=\sin\left(\frac{\pi}{14}\right)\left(4\cos\frac{2\pi}{7}+2\right)=\sin\left(\frac{\pi}{14}\right)\left(4-8\sin^2\frac{\pi}{7}+2\right)=$

$$=\left(2\sin\left(\frac{\pi}{14}\right)\left(3-4\sin^2\frac{\pi}{7}\right)\right)=\frac{2\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{7}\right)}{2\sin\left(\frac{\pi}{14}\right)\cos\left(\frac{\pi}{14}\right)}=\frac{\sin\left(\frac{3\pi}{7}\right)}{\cos\left(\frac{\pi}{14}\right)}=1$$

Hence $\csc\left(\frac{\pi}{14}\right)-4\cos\left(\frac{2\pi}{7}\right)=2$ ***(proved)***

Solution 4 by Ravi Prakash-New Delhi-India

Let $\theta=\frac{\pi}{14}$ ***and*** $a=\sin\left(\frac{\pi}{14}\right)=\sin\theta$

$\cos\frac{2\pi}{7}=\sin\left(\frac{\pi}{2}-\frac{2\pi}{7}\right)=\sin\left(\frac{3\pi}{14}\right)=\sin 3\theta$***. Also,*** $7\theta=\frac{\pi}{2}\Rightarrow 4\theta=\frac{\pi}{2}-3\theta\Rightarrow\sin(4\theta)=\cos 3\theta\Rightarrow$

$$\Rightarrow 4\sin\theta\cos\theta\cos 2\theta=4\cos^3\theta-3\cos\theta\Rightarrow 4\sin\theta\left(1-2\sin^2\theta\right)=4\left(1-\sin^2\theta\right)-3\Rightarrow$$

$\Rightarrow 4a\left(1-2a^2\right)=4\left(1-a^2\right)-3\Rightarrow 8a^3-4a^2-4a+1=0$***. Now,*** $\csc\theta=\frac{1}{a}=4+4a-8a^2$

$$\cos\frac{2\pi}{7}=\sin 3\theta=3a-4a^3$$

$$LHS = \csc\left(\frac{\pi}{14}\right) - 4\cos\frac{2\pi}{7} = 4 + 4a - 8a^2 - 4(3a - 4a^3) = 16a^3 - 8a^2 - 8a + 4 =$$

$$= 2(8a^3 - 4a^2 - 4a + 1) + 2 = 2(0) + 2 = 2$$

402. Prove that:

$$2 + 2\sum_{n=1}^{\infty} 2^n \cdot sin^2\left(\frac{90}{2^n}\right) \cdot tan\left(\frac{45}{2^n}\right) = \pi$$

Proposed by Radeshyam Takur-Nepal

Solution by Shafiqur Rahman-Bangladesh

$$2 + 2\sum_{n=1}^{\infty} 2^n\, sin^2\left(\frac{90^\circ}{2^n}\right) tan\left(\frac{45^\circ}{2^n}\right) = 2 + 2\sum_{n=1}^{\infty} 2^n \cdot sin^2\left(\frac{\pi}{2^{n+1}}\right) tan\left(\frac{\pi}{2^{n+2}}\right) =$$

$$= 2 + 2\sum_{n=1}^{\infty} 2^n \cdot 4\, sin^3\left(\frac{\pi}{2^{n+2}}\right) cos\left(\frac{\pi}{2^{n+2}}\right) = 2 + 2\sum_{n=1}^{\infty} 2^n \cdot sin\left(\frac{\pi}{2^{n+1}}\right)\left(1 - cos\left(\frac{\pi}{2^{n+1}}\right)\right)$$

$$= 2 + \sum_{n=1}^{\infty}\left(2^{n+1} \cdot sin\left(\frac{\pi}{2^{n+1}}\right) - 2^n \cdot sin\left(\frac{\pi}{2^n}\right)\right) = 2 + \lim_{n\to\infty} 2^{n+1} \cdot sin\left(\frac{\pi}{2^{n+1}}\right) - 2\,sin\left(\frac{\pi}{2}\right)$$

$$\therefore 2 + 2\sum_{n=1}^{\infty} 2^n\, sin^2\left(\frac{90^\circ}{2^n}\right) tan\left(\frac{45^\circ}{2^n}\right) = \pi$$

403.

$$\sin 3^\circ = \frac{1}{4}\left(\sqrt{8 - \sqrt{10 - 2\sqrt{5}} - \sqrt{3} - \sqrt{15}}\right) = \frac{1}{4}\left(\sqrt{8 - \sqrt{\sqrt{10}(\sqrt{10} - \sqrt{2})} - \sqrt{3} - \sqrt{15}}\right)$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Ahmed Salama Hegazy-Cairo-Egypt

$$L = \sin 3 = \sin(18 - 15) = \sin(18)\cos(15) - \cos(18)\sin(15)$$

Now we find $\sin(18)$, ***let*** $x = 18 \gg 5x = 90 \gg 2x + 3x = 90$

$$\therefore 2x = 90 - 3x, \therefore \sin(2x) = \sin(90 - 3x) = \cos(3x)$$

$$\therefore 2\sin x\cos x = 4\cos^3(x) - 3\cos x$$

$$\therefore 4\cos^3(x) - 2\sin x\cos x - 3\cos x = 0, \therefore \cos x\,(4\cos^2 x - 2\sin x - 3) = 0$$

$$\therefore 4(1 - \sin^2(x) - 2\sin x - 3 = 0) \gg 4\sin^2(x) + 2\sin x - 1 = 0$$

$$\therefore \sin x = \frac{\sqrt{5}-1}{4}, \therefore \sin 18 = \frac{\sqrt{5}-1}{4}, \textit{ and } \cos 18 = \sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}}$$

$$\textit{since } \sin x = \frac{1}{\sqrt{2}}\sqrt{1 - \cos 2x}, \cos x = \frac{1}{\sqrt{2}}\sqrt{1 + \cos 2x}, \textit{ put } x = 15$$

$$\therefore \sin 15 = \frac{\sqrt{3}-1}{2\sqrt{2}}, \cos 15 = \frac{1+\sqrt{3}}{2\sqrt{2}}, \textit{ by substitution we have}$$

$$\sin 3 = \frac{1}{4}\sqrt{-\sqrt{3} - \sqrt{15} + 8 - \sqrt{10 - 2\sqrt{5}}}$$

404. Prove that:

$$20\Psi_1\left(\frac{1}{3}\right) + \Psi_1\left(\frac{5}{12}\right) + \Psi_1\left(\frac{11}{12}\right) = \frac{64\pi^2}{3}, \Psi_1(x) \textbf{ – trigamma function}$$

Proposed by Vasile Mircea Popa – Romania

Solution by Zaharia Burghelea-Romania

Prove that:

$$\Omega = 20\psi_1\left(\frac{1}{3}\right) + \psi_1\left(\frac{5}{12}\right) + \psi_1\left(\frac{11}{12}\right) = \frac{64\pi^2}{3}$$

Using the duplication formula of the trigamma function, namely:

$$\psi_1(z) + \psi_1\left(z + \frac{1}{2}\right) = 4\psi_1(2z)$$

We get the following two identities:

$$\to \psi_1\left(\frac{5}{12}\right) + \psi_1\left(\frac{5}{12} + \frac{1}{2}\right) = 4\psi_1\left(\frac{5}{6}\right)$$

$$\to \psi_1\left(\frac{1}{3}\right) + \psi_1\left(\frac{1}{3} + \frac{1}{2}\right) = 4\psi_1\left(\frac{2}{3}\right)$$

Plugging those into Ω yields:

$$20\psi_1\left(\frac{1}{3}\right) + \psi_1\left(\frac{5}{12}\right) + \psi_1\left(\frac{11}{12}\right) = 20\psi_1\left(\frac{1}{3}\right) + 4\psi_1\left(\frac{5}{6}\right) =$$

$$= 4\left(4\psi_1\left(\frac{1}{3}\right) + \psi_1\left(\frac{1}{3}\right) + \psi_1\left(\frac{5}{6}\right)\right) = 4\left(4\psi_1\left(\frac{1}{3}\right) + 4\psi_1\left(\frac{2}{3}\right)\right)$$

Also, using trigamma's reflection formula:

$$\psi_1(z) + \psi_1(1-z) = \frac{\pi^2}{\sin^2(\pi z)}$$

$$\Omega = 16\left(\psi_1\left(\frac{1}{3}\right) + \psi_1\left(1-\frac{1}{3}\right)\right) = 16\left(\frac{\pi^2}{\sin^2\left(\frac{\pi}{3}\right)}\right) = \frac{64\pi^2}{3}$$

405. Find all $n \in \mathbb{N}$ such that $\Omega(n) \in \mathbb{N}$:

$$\Omega(n) = \sqrt[n]{\left(\log_n\left(\frac{n!}{(n-2)!}\right)^2\right)^2} + \log_n\left(\sqrt{\binom{2n}{3}}\right)$$

Proposed by Ajao Yinka-Nigeria

Solution by Michael Sterghiou-Greece

$$\Omega(n) = \sqrt[n]{\left(\log_n\left(\frac{n!}{(n-2)!}\right)^2\right)^2} + \log_n\left(\sqrt{\binom{2n}{3}}\right) \quad \textit{(1)}$$

$n \geq 2$ ***else (1) is not defined.***

(1) is written as: $(2\log_n[n(n-1)])^{\frac{2}{n}} + \frac{1}{2}\log_n\left[\frac{2}{3}n(n-1)(2n-1)\right]$

or $\underbrace{[2(1+\log_n(n-1))]^{\frac{2}{n}}}_{\Omega_1(n)} + \underbrace{\frac{1}{2}\left(\log_n\frac{2}{3} + 1 + \log_n(n-1) + \log_n(2n-1)\right)}_{\Omega_2(n)}$ ***(2)***

$\Omega_1(n) < [2\cdot(1+1)]^{\frac{2}{n}} = 16^{\frac{1}{n}}$ ***(as*** $\log_n(n-1) < 1$***). Also,***

$\Omega_1(n) > 1$ $\left(2^{\frac{2}{n}} > 1 \text{ and } (1+\log_n(n-1)) > 1\right)$ ***so,*** $1 < \Omega_1(n) < 16^{\frac{1}{n}}$

For $n > 9 \to 1 < \Omega_1(n) < \frac{4}{3}$

$$\Omega_2(n) > 1 \textit{ and } \left.\begin{array}{c} \log_n\frac{2}{3} < 0 \quad n \geq 2 \\ \log_n(n-1) < 1 \quad n \geq 2 \\ \log_n(2n-1) < \frac{4}{3} \quad n \geq 6 \\ 1 \leq 1 \quad n \geq 2 \end{array}\right\} \Omega_2(n) < \frac{1}{2}\left(0 + 2 + \frac{4}{3}\right) = \frac{5}{3}$$

Therefore for $n > 9: 2 < \Omega_1(n) + \Omega_2(n) = \Omega(n) < \frac{4}{3} + \frac{5}{3} = 3$

and $\Omega(n)$ *cannot be natural. By trial and error for all* $n: 2 \le n \le 9$ *we conclude that only* $\Omega(2) = 3 \in \mathbb{N}$ *[Answer:* $n = 2$*]*

406. Let $f(x) = \log\left(\frac{1}{x+\sqrt{x^2+m^2}}\right)$, then prove that:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|\lambda|(n+1)}{n^5} + \sum_{k=1}^{\infty}\frac{(m+1+k)}{(k+m)^3m^2} = \frac{\pi^2}{6}\left(\frac{\pi^2}{6} + \zeta(3)\right)$$

where

$$\lambda = \lim_{x\to 0}\frac{f^{n+2}(x)}{f^n(x)},\ f^n \text{ is n}^{\text{th}} \text{ derivatives}$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Pierre Mounir-Cairo-Egypt

$$f(x) = -\ln\left(x + \sqrt{x^2+m^2}\right) \Rightarrow f^{(1)}(x) = -\frac{1}{\sqrt{x^2+m^2}} \Rightarrow$$

$$\sqrt{x^2+m^2}f^{(1)} = -1 \Rightarrow \sqrt{x^2+m^2}f^{(2)} + \frac{xf^{(1)}}{\sqrt{x^2+m^2}} = 0 \Rightarrow$$

$$(x^2+m^2)f^{(2)} + xf^{(1)} = 0 \quad \text{(differentiating } n \text{ times)}$$

$$(x^2+m^2)f^{(n+2)} + n(2x)f^{(n+1)} + \frac{n(n-1)}{2}(2)f^{(n)} + xf^{(n+1)} + n(1)f^{(n)} = 0 \Rightarrow$$

$$\frac{f^{(n+2)}(x)}{f^{(n)}(x)} = -\frac{(2n+1)x}{x^2+m^2} \times \frac{f^{(n+1)}(x)}{f^{(n)}(x)} - \frac{n^2}{x^2+m^2} \Rightarrow$$

$$\lambda = \lim_{x\to 0}\frac{f^{(n+2)}(x)}{f^{(n)}(x)} = -\frac{n^2}{m^2} \quad (f^{(n)}(0) \text{ is defined } \forall n \in \mathbb{N})$$

Note: $f(x)$ *has infinite continuous derivatives* $\in C^{\infty}$

$$\therefore \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{n^2}{m^2} \times \frac{(n+1)}{n^5} + \sum_{k=1}^{\infty}\frac{(k+m+1)}{(k+m)^3m^2}\right]$$

$$= \sum_{m=1}^{\infty}\frac{1}{m^2}\left[\sum_{n=1}^{m}\frac{(n+1)}{n^3} + \sum_{n=m+1}^{\infty}\frac{(n+1)}{n^3}\right] \quad (n = k+m)$$

$$=\sum_{m=1}^{\infty}\frac{1}{m^2}\left[\sum_{n=1}^{\infty}\frac{(n+1)}{n^3}\right]=\sum_{m=1}^{\infty}\frac{1}{m^2}\left[\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{1}{n^3}\right]=\frac{\pi^2}{6}\left[\frac{\pi^2}{6}+\zeta(3)\right]$$

407. If $n,k\in\mathbb{N}$ then:

$$\sum_{j=0}^{n}\frac{(-1)^j}{1+j}\cdot\binom{k}{j}\binom{k-1-j}{n-j}=\frac{1}{k+1}\left((-1)^n+\binom{k}{n+1}\right)$$

Proposed by Shivam Sharma-New Delhi-India

Solution by Shafiqur Rahman-Bangladesh

$$\sum_{j=0}^{n}\frac{(-1)^j}{1+j}\binom{k}{j}\binom{k-1-j}{n-j}=\frac{k!}{n!\,(k-1-n)!}\sum_{j=0}^{n}\frac{(-1)^j}{(1+j)(k-j)}\binom{n}{j}=$$

$$=\frac{k!}{(k+1)n!\,(k-n-1)!}\sum_{j=0}^{n}\left[\frac{(-1)^j\binom{n}{j}}{1+j}+\frac{(-1)^j\binom{n}{j}}{k-j}\right]$$

$$=\frac{k!}{(k+1)n!\,(k-n-1)!}\int_0^1\left[(1-x)^n+(-1)^n x^{k-n-1}(1-x)^n\right]dx=$$

$$=\frac{k!}{(k+1)n!\,(k-n-1)!}\left[\frac{1}{n+1}+(-1)^n\frac{n!\,(k-n-1)!}{k!}\right]$$

$$\therefore\sum_{j=0}^{n}\frac{(-1)^j}{1+j}\binom{k}{j}\binom{k-1-j}{n-j}=\frac{1}{k+1}\left[\frac{k!}{(n+1)!\,(k-n-1)!}+(-1)^n\right]=$$

$$=\frac{1}{k+1}\left[(-1)^n+\binom{k}{n+1}\right]$$

408. If

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+(\pi n)^2+(n+\pi^2)+\pi^2}=\frac{\cosh(p)}{\cos(q)}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+(\pi n)^2+(n+\pi)^2+\pi^2}$$

then show that

$$p=q\sqrt{2\pi^2+3}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Kelvin Hong-Rawang-Malaysia

Let $f(z)=\frac{1}{[(\pi^2+2)z^2+2\pi z+2\pi^2]}$*, the only poles of* f *are simple, which are*

$$z_{1,2}=-\frac{\pi}{\pi^2+2}\pm\frac{\pi}{\pi^2+2}\sqrt{2\pi^2+3}i\{Im(z_1)>Im(z_2)\}$$

Note that $z_1+z_2=\frac{2\pi}{(\pi^2+2)}, z_1-z_2=\frac{i2\pi\sqrt{2\pi^2+3}}{\pi^2+2}$

$$I=\sum_{n=-\infty}^{\infty}\frac{1}{n^2+(\pi n)^2+(n+\pi)^2+\pi^2}=-\sum Res_{z=z_1,z_2}\frac{\pi\cot(\pi z)}{(\pi^2+2)z^2+2\pi z+2\pi^2}$$

$$=-\sum\left[\lim_{z\to z_1,z_2}\frac{\pi\cot(\pi z)}{2(\pi+2)z+2\pi}\right]=-\left[\frac{\pi\cot(\pi z_1)}{2\pi\sqrt{2\pi^2+3}i}-\frac{\pi\cot(\pi z_2)}{2\pi\sqrt{2\pi^2+3}i}\right]$$

$$=\frac{i}{2\sqrt{2\pi^2+3}}(\cot(\pi z_1)-\cot(\pi z_2))$$

Similarly,

$$J=\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+(\pi n)^2+(n+\pi)^2+\pi^2}=-\sum Res_{z=z_1,z_2}\frac{\pi\csc(\pi z)}{(\pi^2+2)z^2+2\pi z+2\pi^2}$$

$$=\frac{i}{2\sqrt{2\pi^2+3}}(\csc(\pi z_1)-\csc(\pi z_2))$$

After some cancellation, we have:

$$\frac{I}{J}=\frac{\cot(\pi z_1)-\cot(\pi z_2)}{\csc(\pi z_1)-\csc(\pi z_2)}=\frac{\sin[\pi(z_2-z_1)]}{\sin(\pi z_2)-\sin(\pi z_1)}$$

$$=\frac{2\sin\left[\frac{\pi(z_2-z_1)}{2}\right]\cos\left[\frac{\pi(z_2-z_1)}{2}\right]}{2\cos\left[\frac{\pi(z_1+z_2)}{2}\right]\sin\left[\frac{\pi(z_2-z_1)}{2}\right]}=\frac{\cos\left(\frac{\pi^2}{\pi^2+2}\sqrt{2\pi^2+3}i\right)}{\cos\left(\frac{\pi^2}{\pi^2+2}\right)}$$

$$=\frac{\cosh\left(\frac{\pi^2\sqrt{2\pi+3}}{\pi^2+2}\right)}{\cos\left(\frac{\pi^2}{\pi^2+2}\right)},\therefore p=\frac{\pi^2}{\pi^2+2}\sqrt{2\pi^2+3}, q=\frac{\pi^2}{\pi^2+2}$$

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$$\tan\left[\frac{\pi}{64}\right]^2\tan\left[\frac{\pi}{32}\right]^2\tan\left[\frac{\pi}{16}\right]^2\tan\left[\frac{\pi}{8}\right]^2=$$

$$=\frac{(2-\sqrt{2})\left(2-\sqrt{2+\sqrt{2}}\right)\left(2-\sqrt{2+\sqrt{2+\sqrt{2}}}\right)\left(2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)}{(2+\sqrt{2})\left(2+\sqrt{2+\sqrt{2}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)}$$

Proposed by John Horton Conway-Grenoble-France

Solution 1 by Naren Bhandari-Bajura-Nepal

From half angle form we can obtain the following recursive formula $\forall n \ge 1$ ***i.e.***

$$2\cos^2\frac{\pi}{2^n}=1+\cos\frac{\pi}{2^{n-1}}2\sin^2\frac{\pi}{2^n}=1-\cos\frac{\pi}{2^{n-1}}$$

Giving $\tan^2\frac{\pi}{2^n}=\frac{1-\cos\frac{\pi}{2^{n-1}}}{1+\cos\frac{\pi}{2^{n-1}}}$***. Setting*** $n = 3,4,5,6$ ***we obtain the*** $\tan^2\frac{\pi}{8}=\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}=\frac{2-\sqrt{2}}{2+\sqrt{2}}$

$$\tan^2\frac{\pi}{16}=\frac{1-\cos\frac{\pi}{8}}{1+\cos\frac{\pi}{8}}=\frac{1-\frac{1}{2}\left(\sqrt{2+\sqrt{2}}\right)}{1+\frac{1}{2}\left(\sqrt{2-\sqrt{2}}\right)}=\frac{2-\sqrt{2-\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}$$

Thus, we can easily deduce that $\tan^2\frac{\pi}{32}=\frac{1-\cos\frac{\pi}{16}}{1\cos\frac{\pi}{16}}=\frac{1-\frac{1}{2}\left(\sqrt{2-\sqrt{2-\sqrt{2}}}\right)}{1+\frac{1}{2}\left(\sqrt{2+\sqrt{2+\sqrt{2}}}\right)}=\frac{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$

$$\tan^2\frac{\pi}{64}=\frac{1-\cos\frac{\pi}{32}}{1+\cos\frac{\pi}{32}}=\frac{1-\frac{1}{2}\left(\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}\right)}{1+\frac{1}{2}\left(\sqrt{\sqrt{2-2+\sqrt{2+\sqrt{2}}}}\right)}=\frac{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$$

and hence $\tan^2\frac{\pi}{8}\cdot\tan^2\frac{\pi}{16}\cdot\tan^2\frac{\pi}{32}\cdot\tan^2\frac{\pi}{64}$

$$=\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\cdot\left(\frac{2-\sqrt{2-\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}\right)\cdot\left(\frac{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)\cdot\left(\frac{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}\right)$$

Solution 2 by Shafiqur Rahman-Bangladesh

$$\tan^2\left(\frac{\pi}{8}\right)\tan^2\left(\frac{\pi}{16}\right)\tan^2\left(\frac{\pi}{32}\right)\tan^2\left(\frac{\pi}{64}\right)=$$

$$=\left(\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}\right)\left(\frac{2-2\cos\frac{\pi}{8}}{2+2\cos\frac{\pi}{8}}\right)\left(\frac{2-2\cos\frac{\pi}{16}}{2+2\cos\frac{\pi}{16}}\right)\left(\frac{2-2\cos\frac{\pi}{32}}{2+2\cos\frac{\pi}{32}}\right)$$

$$=\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\left(\frac{2-\sqrt{2+2\cos\frac{\pi}{4}}}{2+\sqrt{2+2\cos\frac{\pi}{4}}}\right)\left(\frac{2-\sqrt{2+\sqrt{2+2\cos\frac{\pi}{4}}}}{2+\sqrt{2+\sqrt{2+2\cos\frac{\pi}{4}}}}\right)\left(\frac{2-\sqrt{2+\sqrt{2+\sqrt{2+2\cos\frac{\pi}{4}}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2+2\cos\frac{\pi}{4}}}}}\right)$$

$$\therefore\ \tan^2\left(\frac{\pi}{8}\right)\tan^2\left(\frac{\pi}{16}\right)\tan^2\left(\frac{\pi}{32}\right)\tan^2\left(\frac{\pi}{64}\right)=$$

$$=\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\cdot\left(\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}\right)\cdot\left(\frac{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)\cdot\left(\frac{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}\right)$$

Solution 3 by Nelson Javier Villaherrera Lopez-El Salvador

$$\tan^2\left(\frac{\pi}{8}\right)\tan^2\left(\frac{\pi}{16}\right)\tan^2\left(\frac{\pi}{32}\right)\tan^2\left(\frac{\pi}{64}\right)=\left[\frac{\sin\left(\frac{\pi}{8}\right)}{\cos\left(\frac{\pi}{8}\right)}\right]^2\left[\frac{\sin\left(\frac{\pi}{16}\right)}{\cos\left(\frac{\pi}{16}\right)}\right]^2\left[\frac{\sin\left(\frac{\pi}{32}\right)}{\cos\left(\frac{\pi}{32}\right)}\right]^2\left[\frac{\sin\left(\frac{\pi}{64}\right)}{\cos\left(\frac{\pi}{64}\right)}\right]^2$$

$$=\frac{1-\cos\left(\frac{\pi}{4}\right)}{1+\cos\left(\frac{\pi}{4}\right)}\cdot\frac{1-\cos\left(\frac{\pi}{8}\right)}{1+\cos\left(\frac{\pi}{8}\right)}\cdot\frac{1-\cos\left(\frac{\pi}{16}\right)}{1+\cos\left(\frac{\pi}{16}\right)}\cdot\frac{1-\cos\left(\frac{\pi}{32}\right)}{1+\cos\left(\frac{\pi}{32}\right)}$$

$$=\frac{\sqrt{2}-1}{\sqrt{2}+1}\cdot\frac{\sqrt{2}-\sqrt{1+\cos\left(\frac{\pi}{4}\right)}}{\sqrt{2}+\sqrt{1+\cos\left(\frac{\pi}{4}\right)}}\cdot\frac{\sqrt{2}-\sqrt{1+\cos\left(\frac{\pi}{8}\right)}}{\sqrt{2}+\sqrt{1+\cos\left(\frac{\pi}{8}\right)}}\cdot\frac{\sqrt{2}-\sqrt{1+\cos\left(\frac{\pi}{16}\right)}}{\sqrt{2}+\sqrt{1+\cos\left(\frac{\pi}{16}\right)}}$$

$$=\frac{2-\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2+2\cos\left(\frac{\pi}{4}\right)}}{2+\sqrt{2+2\cos\left(\frac{\pi}{4}\right)}}\cdot\frac{2-\sqrt{2+2\cos\left(\frac{\pi}{8}\right)}}{2+\sqrt{2+2\cos\left(\frac{\pi}{8}\right)}}\cdot\frac{2-\sqrt{2+2\cos\left(\frac{\pi}{16}\right)}}{2+\sqrt{2+2\cos\left(\frac{\pi}{16}\right)}}$$

$$=\frac{2-\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}\cdot\frac{2-\sqrt{2+\sqrt{2+2\cos\left(\frac{\pi}{4}\right)}}}{2+\sqrt{2+\sqrt{2+2\cos\left(\frac{\pi}{4}\right)}}}\cdot\frac{2-\sqrt{2+\sqrt{2+2\cos\left(\frac{\pi}{8}\right)}}}{2+\sqrt{2+\sqrt{2+2\cos\left(\frac{\pi}{8}\right)}}}$$

$$=\frac{2-\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}\cdot\frac{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\cdot\frac{2-\sqrt{2+\sqrt{2+\sqrt{2+2\cos\left(\frac{\pi}{4}\right)}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2+2\cos\left(\frac{\pi}{4}\right)}}}}$$

$$=\frac{2-\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}\cdot\frac{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\cdot\frac{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$$

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$$\frac{4}{\pi}+\int_0^1\left(\frac{\pi}{1!}-x1^7\frac{\pi^3}{3!}+x^2 2^7\frac{\pi^5}{5!}-x^3 3^7\frac{\pi^7}{7!}+x^4 4^7\frac{\pi^9}{9!}-x^5 5^7\frac{\pi^{11}}{11!}+\cdots\right)=\pi+\frac{\pi}{64}(63-7\pi^4)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Feti Sinani-Kosovo

$$\textit{We have to prove that}:$$

$$\int_0^1\sum_{n=2}^{+\infty}\frac{(-1)^{n-1}\pi^{2n-1}(n-1)^7}{(2n-1)!}x^{n-1}dx=-\frac{4}{\pi}+\frac{\pi}{64}(63-7\pi^4)$$

Since that series $\sum_{n=2}^{+\infty}\frac{(-1)^{n-1}\pi^{2n-1}(n-1)^7}{(2n-1)!}x^{n-1}$

Is uniformly convergents by Weierstrass criterion:

$$\int_0^1\sum_{n=2}^{+\infty}\frac{(-1)^{n-1}\pi^{2n-1}(n-1)^7}{(2n-1)!}x^{n-1}dx=\sum_{n=2}^{+\infty}\frac{(-1)^{n-1}\pi^{2n-1}(n-1)^7}{(2n-1)!}\int_0^1 x^{n-1}dx=\sum_{n=2}^{+\infty}\frac{(-1)^{n-1}\pi^{2n-1}(n-1)^7}{n(2n-1)!}$$

$$=\sum_{n=2}^{+\infty}\frac{(-1)^{n-1}\pi^{2n-1}}{(2n-1)!}\left(n^6-7n^5+21n^4-35n^3+35n^2-21n+7-\frac{1}{n}\right)$$

Using Taylor series of $\sin x$ *we get*

$$\sin x=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}\Rightarrow S(x)=\sin x-x=\sum_{n=2}^{+\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}\Rightarrow x\sin x-x^2=\sum_{n=2}^{+\infty}\frac{(-1)^{n-1}x^{2n}}{(2n-1)!}$$

$$\Rightarrow S_1(x)=\sin x+x\cos x-2x=2\sum_{n=2}^{+\infty}\frac{(-1)^{n-1}nx^{2n-1}}{(2n-1)!}$$

$$S_2(x)=\sin x+3x\cos x-x^2\sin x-4x=2^2\sum_{n=2}^{+\infty}\frac{(-1)^{n-1}n^2x^{2n-1}}{(2n-1)!}$$

$$S_3(x)=\sin x+7x\cos x-6x^2\sin x-x^3\cos x-8x=2^3\sum_{n=2}^{+\infty}\frac{(-1)^{n-1}n^3x^{2n-1}}{(2n-1)!}$$

$$S_4(x)=\sin x+15x\cos x-25x^2\sin x-10x^3\cos x+x^4\sin x-16x=2^4\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}n^4x^{2n-1}}{(2n-1)!}$$

$$S_5(x)=\sin x+31x\cos x-90x^2\sin x-65x^3\cos x+15x^4\sin x+x^5\cos x-32x=2^5\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}n^5x^{2n-1}}{(2n-1)!}$$

$$S_6(x)=\sin x+63x\cos x-301x^2\sin x+350x^3\cos x+140x^4\sin x+21x^5\cos x-x^6\sin x-64x$$

$$=2^6\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}n^6x^{2n-1}}{(2n-1)!};\qquad \sum_{n=2}^{+\infty}\frac{(-1)^{n-1}x^{2n-1}}{n(2n-1)!}=\frac{2}{x}\left(1-\cos x-\frac{x^2}{2}\right)$$

$$\sum_{n=2}^{+\infty}\frac{(-1)^{n-1}\pi^{2n-1}}{(2n-1)!}\left(n^6-7n^5+21n^4-35n^3+35n^2-21n+7-\frac{1}{n}\right)=$$

$$=\frac{1}{2^6}S_6(\pi)-\frac{7}{2^5}S_5(\pi)+\frac{21}{2^4}S_4(\pi)-\frac{35}{2^3}S_3(\pi)+\frac{35}{2^2}S_2(\pi)-\frac{21}{2}S_1(\pi)+7S(\pi)-\frac{4}{\pi}+\pi=$$

$$=\frac{1}{64}(-63\pi+350\pi^3-21\pi^5-64\pi)-\frac{7}{32}(-31\pi+65\pi^3-\pi^5-32\pi)+\frac{21}{16}(-15\pi+10\pi^3-16\pi)$$

$$-\frac{35}{8}(-7\pi+\pi^3-8\pi)+\frac{35}{4}(-3\pi-4\pi)-\frac{21}{2}\cdot(-\pi-2\pi)-7\pi-\frac{4}{\pi}+\pi=\frac{\pi}{64}(63-7\pi^4)-\frac{4}{\pi}$$

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411. π, e, γ **with Riemann Zeta function.**

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2n+1}\cdot\frac{4^{-n}}{n}=\ln\left(\frac{\pi}{e}\right)$$

$$\sum_{n=1}^{\infty}\left(\frac{\zeta(2n)}{n}-\frac{\zeta(2n+1)}{4^n}\right)\frac{1}{2n+1}=\gamma+\ln\left(\frac{\pi}{e}\right)$$

γ **– Euler's Gamma**

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Kamel Benaicha-Algeirs-Algerie

$$S=\sum_{n=1}^{+\infty}\frac{\zeta(2n)4^{-n}}{n(2n+1)}$$

On a: $S=\sum_{n=1}^{+\infty}\frac{\zeta(2n)}{2^{2n}}\left\{\frac{1}{n}-\frac{2}{2n+1}\right\}=\sum_{n=1}^{+\infty}\frac{\zeta(2n)}{n2^{2n}}-2\sum_{n=1}^{+\infty}\frac{\zeta(2n)}{(2n+1)2^{2n}}$ *(A)*

Nous savons que: $\sum_{k=0}^{+\infty}\zeta(2k)x^{2k}=-\frac{\pi x}{2}\cot(\pi x)$, $|x|<1$ *(B)*

Integrant cette relation de 0 *a* $\frac{1}{2}$*, on trouve que:*

$$\sum_{k=0}^{+\infty}\zeta(2k)\frac{1}{(2k+1)2^{2k+1}}=-\int_0^{\frac{1}{2}}\frac{\pi x}{2}\cot(\pi x)\,dx$$

$$=-\frac{1}{2}x\ln(\sin(x))\Big|_0^{\frac{1}{2}}+\frac{1}{2}\int_0^{\frac{1}{2}}\ln(\sin(\pi x))\,dx=\frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\ln(\sin(x))\,dx=-\frac{1}{4}\ln(2)$$

$\sum_{k=1}^{+\infty}\zeta(2k)\frac{1}{(2k+1)2^{2k}}=-\frac{1}{2}\ln(2)+\frac{1}{2}=-\frac{1}{2}\ln\left(\frac{2}{e}\right)$ *(I)* $\left(\zeta(0)=-\frac{1}{2}\right)$

Posons dans (B): $x=e^{-t}$, $\sum_{k=1}^{+\infty}\zeta(2k)e^{-2kt}=-\frac{\pi}{2}e^{-t}\cot(\pi e^{-t})+\frac{1}{2}$ *(2)*

Par integration de (2) (en t de $\ln(2)$ *a* $(+\infty)$*), on obtient:*

$-\sum_{k=1}^{+\infty}\zeta(2k)\frac{x^{2k}}{2k}\Big|_0^{\frac{1}{2}}=\left\{\frac{\pi}{2}\int\cot(\pi x)\,dx+\frac{1}{2}t\right\}\Big|_{\ln(2)}^{+\infty}$ *(avec* $t=-\ln(x)$*)*

$$=\left\{\frac{1}{2}\ln(\sin(\pi x))-\frac{1}{2}\ln(x)\right\}\Big|_0^{\frac{1}{2}}=\frac{1}{2}\ln\left(\frac{\sin(\pi x)}{x}\right)\Big|_0^{\frac{1}{2}}$$

$$-\sum_{k=1}^{+\infty}\zeta(2k)\frac{1}{2k2^{2k}}=\frac{1}{2}\{\ln(2)-\ln(\pi)\}$$

$$\sum_{k=1}^{+\infty}\zeta(2k)\frac{1}{k2^{2k}}=\ln\left(\frac{\pi}{2}\right)\quad (3)$$

D'apres (1) et (3) et la reformulation (A) de (S), on trouve que:

$$S=\ln\left(\frac{\pi}{2}\right)+\ln\left(\frac{2}{e}\right)=\ln\left(\frac{\pi}{e}\right)\therefore\sum_{n=1}^{+\infty}\frac{\zeta(2n)4^{-n}}{n(2n+1)}=\ln\left(\frac{\pi}{e}\right)\quad (II)$$

$$s=\sum_{n=1}^{+\infty}\left\{\frac{\zeta(2n)}{n}-\frac{\zeta(2n+1)}{4^n}\right\}\frac{1}{2n+1}$$

On a trouve que: $\sum_{n=1}^{+\infty}\frac{\zeta(2n)4^{-n}}{n(2n+1)}=\ln\left(\frac{\pi}{e}\right)$ ***(II)***

On a $\Psi(x+1)=-\gamma-\sum_{n=1}^{+\infty}\zeta(n+1)(-x)^n$

$$=-\gamma-\sum_{n=1}^{+\infty}\zeta(2n+1)x^{2n}+\sum_{n=0}^{+\infty}\zeta(2n+2)x^{2n+1}\quad (C)$$

Integrant la relation (C), on trouve:

$$\ln\left(\Gamma\left(\frac{1}{2}+1\right)\right)=-\frac{\gamma}{2}-\sum_{n=1}^{+\infty}\zeta(2n+1)\frac{1}{(2n+1)2^{2n+1}}+$$

$$+\sum_{n=1}^{+\infty}\zeta(2n)\frac{1}{(2n)2^{2n}}\left(entre\left(0\ et\frac{1}{2}\right)\right)$$

$$\ln\left(\frac{1}{2}\sqrt{\pi}\right)=-\frac{\pi}{2}+\frac{1}{2}\sum_{n=1}^{+\infty}\left\{\frac{\zeta(2n)}{n2^{2n}}-\frac{\zeta(2n+1)}{(2n+1)2^{2n}}\right\}\quad (4)$$

D'autre part:

$$\int_0^1\int_0^t\sum_{n=1}^{+\infty}\zeta(2n)x^{2n-1}\,dx\,dt=\sum_{n=1}^{+\infty}\frac{\zeta(2n)}{n(2n+1)}=\frac{1}{2}\ln(2)+\ln\left(\frac{\pi}{e}\right)$$

$$\sum_{n=1}^{+\infty}\frac{\zeta(2n)}{n(2n+1)}=\ln(2)+\ln\left(\frac{\pi}{e}\right)\quad (5)$$

$$\sum_{n=1}^{+\infty}\frac{\zeta(2n)}{n2^{2n}}=\ln\left(\frac{\pi}{2}\right)\quad (3)$$

D'apre les relations (3), (4), on trouve

$$\ln\left(\frac{1}{2}\right)+\frac{1}{2}\ln(\pi)=-\frac{\gamma}{2}+\frac{1}{2}\ln\left(\frac{\pi}{2}\right)-\sum_{n=1}^{+\infty}\frac{\zeta(2n+1)}{2(2n+1)2^n}$$

$$-\sum_{n=1}^{+\infty}\frac{\zeta(2n+1)}{(2n+1)4^n}=\gamma-\ln(2)\quad (6)$$

La summation de (5) et (6) donne:

$$\sum_{n=1}^{+\infty}\left\{\frac{\zeta(2n)}{n}-\frac{\zeta(2n+1)}{4^n}\right\}\frac{1}{2n+1}=\gamma+\ln\left(\frac{\pi}{e}\right)$$

412. Solve for complex numbers:

$$\left|z+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right|^2+\left|z+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right|^2+|z-1|^2-3|z|^2=z$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

Using $\omega=\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right),\omega^2=\left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)$***. The given equation reduces to***

$$\left|z-\omega^2\right|^2+|z-\omega|^2+|z-1|^2-3|z|^2=z$$

We know that $|z|^2=z\bar{z}\Rightarrow\left|z-\omega^2\right|^2=(z-\omega^2)\left(\overline{z-\omega^2}\right)=(z-\omega^2)(\bar{z}-\omega)$

$$=z\bar{z}-z\omega-\bar{z}\omega^2+\omega^3=|z|^2-z\omega-\bar{z}\omega^2+1\quad\textit{(1)}\ \{\because\omega^3=1\}$$

Here, ω ***is the cube root of unity***

$$|z-\omega|^2=(z-\omega)(\overline{z-\omega})=(z-\omega)(\bar{z}-\bar{\omega})=(z-\omega)(\bar{z}-\omega^2)=$$

$$=z\bar{z}-z\omega^2-\bar{z}\omega+\omega^3$$

$$=|z|^2-z\omega^2-\bar{z}\omega+1\quad\textit{(2)}$$

$$|z-1|^2=(z-1)(\overline{z-1})$$

$$=(z-1)(\bar{z}-1)=|z|^2-z-\bar{z}+1\quad\textit{(3)}$$

$\therefore$ ***Adding (1); (2); (3):***

$$\left|z-\omega^2\right|^2+|z-\omega|^2+|z-1|^2=3|z|^2-z(\omega+\omega^2+1)\cdot\bar{z}(\omega+\omega^2+1)+3$$

$$\{\because\omega+\omega^2+1=0\}$$

$$\Rightarrow 3|z|^2+3$$

$$\Rightarrow\left|z-\omega^2\right|^2+|z-\omega|^2+|z-1|^2=3|z|^2+3$$

$$\Rightarrow\left|z-\omega^2\right|^2+|z-\omega|^2+|z-1|^2-3|z|^2=3\quad\textit{(4)}$$

But we have:

$$\left|z-\omega^2\right|^2+|z-\omega|^2+|z-1|^2-3|z|^2=z \quad (5)$$

So, from (4) and (5) $\Rightarrow z=3$.

Solution 2 by Ravi Prakash-New Delhi-India

Let $\omega=-\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ \omega^2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

$$\left|z-\omega^2\right|^2+|z-\omega|^2+|z-1|^2-3|z|^2=z$$

$$\Rightarrow |z|^2+\left|\omega^2\right|^2-\bar{z}\omega^2-z\bar{\omega}^2+|z|^2+|\omega|^2-\bar{z}\omega-z\bar{\omega}+|z|^2+1-\bar{z}-z-3|z|^2=z$$

$$\Rightarrow 3|z|^2+3-\bar{z}(\omega^2+\omega+1)-z(\omega+\omega^2+1)-3|z|^2=z\Rightarrow z=3$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\left|z+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right|^2+\left|z+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right|^2+|z-1|^2-3|z|^2=z\Rightarrow$$

$$\Rightarrow 2\left|z+\frac{1}{2}\right|^2+2\left|\frac{\sqrt{3}}{2}i\right|^2+|z-1|^2-3|z|^2=z\Rightarrow$$

$$\Rightarrow 2\left\{|z|^2+\frac{1}{4}+\frac{\bar{z}}{2}+\frac{z}{2}\right\}+2\left(\frac{3}{4}\right)+|z|^2+1-\bar{z}-z-3|z|^2=z\Rightarrow 3=z$$

Solution 4 by Sagar Kumar-Patna Bihar-India

$$\left|z+\frac{1+i\sqrt{3}}{2}\right|^2+\left|z+\frac{1-i\sqrt{3}}{2}\right|^2+|z-1|^2-3|z|^2=z$$

Clearly LHS is real and RHS is complex $\Rightarrow z$ *must be purely real. Let* $z=x$ *and using usual notations.* $z=x+iy\to 0$

$$2\left(\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right)+(x-1)^2-3x^2=x\Rightarrow$$

$$\Rightarrow 2\left(x^2+\frac{1}{4}+x\right)+\frac{3}{2}+x^2+1-2x-3x^2=x$$

$x=3$ *(Answer)*

Solution 5 by Abdallah El Farissi-Bechar-Algerie

Let $z=x+iy$, *if* z *is a solution of the equation then* z *is real number then* $z=x$, *it follows that* $\left|x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right|^2+\left|x+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right|^2+|x-1|^2-3|x|^2=x$ *we have*

$$2\left(x+\frac{1}{2}\right)^2+\frac{3}{2}+(x-1)^2-3x^2=x\Rightarrow x=3\ \textit{then}\ z=3.$$

413. If $z\in\mathbb{C},\left|z^2-2\right|=|4z+i|$ then:

$$|z|<2\sqrt{5}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Lazaros Zachariadis-Thessaloniki-Greece

$$\left|z^2-2\right|=|4z+i|$$

$$\Rightarrow\left|z^2-2\right|^2=|4z+i|^2\Rightarrow(z^2-2)\left(\overline{z}^2-2\right)=(4z+i)(4\overline{z}-i)$$

$$\Rightarrow|z|^4-2\left(z^2+\overline{z}^2\right)+3-16|z|^2+4i(z-\overline{z})=0$$

$$\Rightarrow(x^2+y^2)^2-20x^2-12y^2-81+3=0$$

$$\Rightarrow(x^2+y^2)^2-20(x^2+y^2)=-8y^2+8y-3\to\Delta=8^2-4\cdot8\cdot3=64-96<0$$

$$\textit{so},\ -8y^2+8y-3<0\ \forall y\in\mathbb{R}$$

$$\Rightarrow|z|^4-20|z|^2<0\Rightarrow|z|^2\cdot(|z|^2-20)<0\Rightarrow|z|^2<20\Rightarrow|z|<2\sqrt{5}$$

$$z=x+yi,x,y\in\mathbb{R};\ \overline{z}=x-yi;\ z-\overline{z}=2i\cdot Imz;\ |z|^2=z\cdot\overline{z}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\textit{Let}\ z=x+iy;z^2=x^2-y^2+2ixy$$

$$\textit{Now},\left|z^2-2\right|=|4z+i|\Rightarrow\left|(x^2-y^2-2)+2ixy\right|^2=|4x+(4y+1)i|^2$$

$$\Rightarrow(x^2-y^2-2)^2+4x^2y^2=16x^2+(4y+1)^2$$

$$\Rightarrow(x^2-y^2)^2+4-4(x^2-y^2)+4x^2y=16(x^2+y^2)+8y+1$$

$$\Rightarrow(x^2+y^2)^2-20(x^2+y^2)+3=-8y^2+8y$$

$$\Rightarrow(x^2+y^2-10)^2=97-8y^2+8y$$

$$=97-8\left(\left(y-\frac{1}{2}\right)^2-\frac{1}{4}\right)=99-8\left(y-\frac{1}{2}\right)^2<100$$

$$\Rightarrow\left|x^2+y^2-10\right|<10\Rightarrow\left||z|^2-10\right|<10$$

$$\Rightarrow|z|^2-10\le\left||z|^2-10\right|<10\Rightarrow|z|^2<20\Rightarrow|z|<2\sqrt{5}$$

414. If $A \in M_2(\mathbb{Z})$ then:

$$\Omega = \det(A + A^T + A^*) + \det(-A + A^T + A^*) + \det(A - A^T + A^*) + \det(A + A^T - A^*)$$

is divisible with 12.

Proposed by Marian Ursărescu-Romania

Solution by Ravi Prakash-New Delhi-India

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, A^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, a, b, c, d \in \mathbb{Z}$

$$A + A^T + A^* = \begin{pmatrix} 2a + d & c \\ b & a + 2d \end{pmatrix} = B_1 \quad \textit{(say)}$$

$$-A + A^T + A^* = \begin{pmatrix} d & c - 2b \\ b - 2c & a \end{pmatrix} = B_2 \quad \textit{(say)}$$

$$A - A^T + A^* = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = B_2 \quad \textit{(say)}$$

$$A + A^T - A^* = \begin{pmatrix} 2a - d & c + 2b \\ 2c + b & 2d - a \end{pmatrix} = B_4 \quad \textit{(say)}$$

$$\therefore \det(B_1) + \det(B_2) + \det(B_3) + \det(B_4)$$

$$= (2a + d)(a + 2d) - bc + ad - (b - 2c)(c - 2b) + ad - bc$$

$$+(2a - d)(2d - a) - (c + 2b)(2c + b)$$

$$= 2a^2 + 5ad + 2d^2 - bc + ad - (5bc - 2c^2 - 2b^2) + ad - bc$$

$$+5ad - 2d^2 - 2a^2 - (2c^2 + 5bc + 2b^2)$$

$$= 12(ad - bc) \textit{ which is divisible by } 12.$$

415. GENERALIZATION FOR A DAN RADU SECLEMAN'S INEQUALITY

If $A, B \in M_n(\mathbb{R}), n \ge 2, p \ge 1, n, p \in \mathbb{N}, A^{2p+1} + B^{2p} = I_n, A^{4p+1} = A^{2p}$

then:

$$\det(I_n + A^{2p} + B^{2p}) \ge 0$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

$$A^{2p+1} + B^{2p} = I_n | \cdot A^{2p} \Rightarrow A^{4p+1} + B^{2p} \cdot A^{2p} = A^{2p} \Rightarrow$$

$$A^{2p} + B^{2p}A^{2p} = A^{2p} \Rightarrow B^{2p}A^{2p} = O_n \quad (1)$$

$$A^{2p} | A^{2p+1} + B^{2p} = I_n \Rightarrow A^{4p+1} + A^{2p}B^{2p} = A^{2p} \Rightarrow A^{2p}B^{2p} = O_n \quad (2)$$

From (1)+(2) we must show:

$$\det(I_n + A^{2p} + B^{2p} + A^{2p}\cdot B^{2p}) \ge 0 \Leftrightarrow$$

$$\det[(I_n + A^{2p})(I_n + B^{2p})] \ge 0 \Leftrightarrow \det(I_n + A^{2p})\cdot\det(I_n + B^{2p}) \ge 0 \quad \textbf{(3)}$$

But $\det(I + A^{2p}) = \det(I_n^2 - i^2A^{2p}) =$

$$= \det[(I_n + iAP)(I_n - iAP)] = \det\left[(I_n + iA^p)\overline{(I_n + \iota A^p)}\right] \ge 0 \quad \textbf{(4)}$$

Similarly: $\det(I_n + B^{2p}) \ge 0$ ***(5)***

From (4)+(5)$\Rightarrow \det(I_n + A^{2p})\,(I_n + B^{2p}) \ge 0 \Rightarrow$ ***(3) its true.***

416. If $A, B \in M_2(\mathbb{C})$, $\det A \ne 0$, $\det B \ne 0$ then:

$$\det(A\det B + B\det A) + \det\left(\frac{A}{\det A} + \frac{B}{\det B}\right) = \det(A + B)\left(\det(AB) + \frac{1}{\det(AB)}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ ***and*** $\alpha, \beta \in \mathbb{C}$

$$\alpha A + \beta B = \begin{pmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ \alpha c_1 + \beta c_2 & \alpha d_1 + \beta d_2 \end{pmatrix}$$

$$\det(\alpha A + \beta B) = (\alpha a_1 + \beta a_2)(\alpha d_1 + \beta d_2) - (\alpha b_1 + \beta b_2)(\alpha c_1 + \beta c_2)$$

$$= \alpha^2(a_1d_1 - b_1c_1) + \alpha\beta(a_2d_1 + a_1d_2 - b_1c_2 - b_2c_1) + \beta^2(a_2d_2 - b_2c_2)$$

Let $\det(A) = a = a_1d_1 - b_1c_1$; $\det(B) = b = a_2d_2 - b_2c_2$

$$\therefore \det(bA + aB) = b^2a + ab(a_2d_1 + a_1d_2 - b_1c_2 - b_2c_1) + a^2b$$

and $\det\left(\frac{1}{a}A + \frac{1}{b}B\right) = \frac{1}{a^2}(a) + \frac{1}{ab}(a_2d_1 + a_1d_2 - b_1c_2 - b_2c_1) + \frac{1}{b^2}(b)$

$$= \frac{1}{a} + \frac{1}{b} + \frac{1}{ab}(a_2d_1 + a_1d_2 - b_1c_2 - b_2c_1)$$

Thus, $\det(\det(A)B + \det(B)A) + \det\left(\frac{1}{\det(A)}A + \frac{1}{\det(B)}B\right)$

$$= a^2b + ab^2 + \frac{1}{a} + \frac{1}{b} + \left(ab + \frac{1}{ab}\right)(a_2d_1 + a_1d_2 - b_1c_2 - b_2c_1) \quad \textbf{(1)}$$

Also, $\det(A + B) = 1^2a + 1^2b + (a_2d_1 + a_1d_2 - b_1c_2 - b_2c_1) =$

$$= a + b + (a_2d_1 + a_1d_2 - b_1c_2 - b_2c_1)$$

$$\left[\frac{1}{\det(AB)}+\det(AB)\right]\det(A+B)=$$

$$=\left(\frac{1}{ab}+ab\right)[a+b+(a_2d_1+a_1d_2-b_1c_2-b_2c_1)]$$

$$=a^2b+ab^2+\frac{1}{a}+\frac{1}{b}+\left(ab+\frac{1}{ab}\right)[a_2d_1+a_1d_2-b_1c_2-b_2c_1]\quad \textbf{(2)}$$

From (1), (2): $\det(\det(A)B+\det(B)A)+\det\left(\frac{1}{\det(A)}A+\frac{1}{\det(B)}B\right)=$

$$=\left(\det(AB)+\frac{1}{\det(AB)}\right)\det(A+B)$$

Solution 2 by Marian Ursărescu-Romania

First we prove this: Theorem (by Vasile Pop and Ovidiu Furdui)

If $A,B\in M_2(\mathbb{C})$ ***and*** $x,y\in\mathbb{C}$ ***then:***

$$\det(xA+yB)=x^2\det A+y^2\det B+xy[\det(A+B)-\det A-\det B]$$

Demonstration: we use a determinant formula: If $A,B\in M_2(\mathbb{C})\wedge x\in\mathbb{C}$ ***then:***

$$\det(A+xB)=\det A+(\det(A+B)-\det A-\det B)x+(\det B)x^2$$

For our theorem if $x=0\Rightarrow$ ***then its trivial.***

If $x\neq 0\Rightarrow\det(xA+yB)=\det\left[x\left(A+\frac{y}{x}B\right)\right]=$

$$=x^2\det\left(A+\frac{y}{x}B\right)=x^2[\det A+(\det(A+B)-\det A-\det B)]\frac{y}{x}+\det B\frac{y^2}{x^2}$$

$$=\det A\,x^2+(\det(A+B)-\det A-\det B)xy+\det B\,y^2\quad \textit{(done)}$$

Now for our problem: Let $x=\det B$, $y=\det A\Rightarrow$

$$\det(A\det B+B\det A)=(\det B)^2+\det A+(\det A)^2\cdot\det B+$$

$$+\det(AB)\,(\det(A+B)-\det A-\det B)\quad \textbf{(1)}$$

Let $x=\frac{1}{\det A}, y=\frac{1}{\det B}\Rightarrow$

$$\det\left(\frac{A}{\det A}+\frac{B}{\det B}\right)=\frac{1}{\det A}+\frac{1}{\det B}+\frac{1}{\det(AB)}(\det(A+B)-\det A-\det B)\quad \textbf{(2)}$$

From (1) + (2) $\Rightarrow\det(A\det B+B\det A)+\det\left(\frac{A}{\det A}+\frac{B}{\det B}\right)=$

$$=\det A\,(\det B)^2+(\det A)^2\det B+\det(AB)\det(A+B)-$$

$$-(\det A)^2\cdot\det B-\det A\cdot(\det B)^2+\frac{1}{\det A}+\frac{1}{\det B}+\frac{\det(A+B)}{\det AB}-\frac{1}{\det B}-\frac{1}{\det A}=$$

$$= \det(AB)\cdot\det(A+B) + \frac{\det(A+B)}{\det AB} = \det(A+B)\left(\det(AB) + \frac{1}{\det(AB)}\right)$$

(Done)

417.

$$\Omega = \begin{vmatrix} \sin^2 x & \sin^2 y\cdot\cos^2 x & \cos^2 y\cdot\cos^2 x \\ \cos^2 y\cdot\cos^2 x & \sin^2 x & \sin^2 y\cdot\cos^2 x \\ \sin^2 y\cdot\cos^2 x & \cos^2 y\cdot\cos^2 x & \sin^2 x \end{vmatrix}, x, y \in \mathbb{R}$$

Prove that: $|\Omega| \le 1$.

Proposed by Daniel Sitaru – Romania

Solution 1 by Le Van-Vietnam

Let $a = (\sin x)^2; b = (\cos x\cos y)^2; c = (\cos x\sin y)^2$

Note that $a+b+c=1$ ***and*** $ab+bc+ca>0$

$$\Omega = \det(a\ c\ b\ |b\ a\ c\ |c\ b\ a) = a^3+b^3+c^3-3abc = a^2+b^2+c^2-ab-bc-ca \ge 0$$

$$\Omega = (a+b+c)^2 - 3(ab+bc+ca) = 1-3(ab+bc+ca) \le 1$$

$0 \le \Omega \le 1$. ***Q.E.D.***

Solution 2 by Ravi Prakash-New Delhi-India

$$\Omega = \begin{vmatrix} \sin^2 x & \sin^2 y\cos^2 x & \cos^2 y\cos^2 x \\ \cos^2 y\cos^2 x & \sin^2 x & \sin^2 y\cos^2 x \\ \sin^2 y\cos^2 x & \cos^2 y\cos^2 x & \sin^2 x \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$, ***we get*** $\Omega = \begin{vmatrix} 1 & 1 & 1 \\ \cos^2 y\cos^2 x & \sin^2 x & \sin^2 y\cos^2 x \\ \sin^2 y\cos^2 x & \cos^2 y\cos^2 x & \sin^2 x \end{vmatrix}$

Using $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, ***we get***

$$\Omega = \begin{vmatrix} 1 & 0 & 0 \\ \cos^2 y\cos^2 x & \sin^2 x - \cos^2 y\cos^2 x & \cos^2 x(\sin^2 y - \cos^2 y) \\ \sin^2 y\cos^2 x & (\cos^2 y - \sin^2 y)\cos^2 x & \sin^2 x - \sin^2 y\cos^2 x \end{vmatrix} =$$

$$= (\sin^2 x - \cos^2 y\cos^2 x)(\sin^2 x - \sin^2 y\cos^2 x) + \cos^4 x(\sin^2 y - \cos^2 y)^2$$

$$= \sin^4 x - \sin^2 x\cos^2 x(\cos^2 y + \sin^2 y) + \cos^4 x\sin^2 y\cos^2 y +$$

$$+ \cos^4 x(\cos^4 y + \sin^4 y - 2\sin^2 y\cos^2 y)$$

$$= \sin^4 x - \sin^2 x\cos^2 x - 3\cos^4 x\sin^2 y\cos^2 y + \cos^4 x(\cos^2 y + \sin^2 y)^2$$

$$= \sin^4 x + \cos^4 x - \sin^2 x\cos^2 x - 3\cos^4 x\sin^2 y\cos^2 y$$

$$= (\sin^2 x + \cos^2 x)^2 - 3\sin^2 x\cos^2 x - 3\cos^4 x\sin^2 y\cos^2 y$$

$$= 1 - 3\sin^2 x\cos^2 x - 3\cos^4 x\sin^2 y\cos^2 y \le 1$$

Also, $3\sin^2 x\cos^2 x + 3\cos^4 x\sin^2 y\cos^2 y = \frac{3}{4}\sin^2 2x + \frac{3}{4}\cos^4 x\sin^2 2y \le$

$$\le \frac{3}{4} + \frac{3}{4} = \frac{3}{2} \Rightarrow 1 - \frac{3}{2} \le 1 - 3\sin^2 x\cos^2 x - 3\cos^4 x\sin^2 y\cos^2 y \le 1$$

$$\Rightarrow -\frac{1}{2} \le \Omega \le 1 \Rightarrow |\Omega| \le 1$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\Omega = \sin^2 x(\sin^4 x - \cos^4 x\sin^2 y\cos^2 y) -$$

$$-\sin^2 y\cos^2 x(\sin^2 x\cos^2 x\cos^2 y - \cos^4 x\sin^4 y) +$$

$$+\cos^2 x\cos^2 y(\cos^4 x\cos^4 y - \sin^2 x\cos^2 x\sin^2 y) =$$

$$\overset{(1)}{=} \sin^6 x + \cos^6 x\sin^6 y + \cos^6 x\cos^6 y - 3\sin^2 x\cos^4 x\sin^2 y\cos^2 y =$$

$$= a^3 + b^3 + c^3 - 3abc(a = \sin^2 x, b = \cos^2 x\sin^2 y, c = \cos^2 x\cos^2 y)$$

$$= \frac{1}{2}(a + b + c)\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$$

$$= \frac{1}{2}(\sin^2 x + \cos^2 x\sin^2 y + \cos^2 x\cos^2 y)\{(a-b)^2 + (b-c)^2 + (c-a)^2\} \ge 0$$

$\therefore \Omega \overset{(a)}{\ge} 0$. ***Also,*** $\because 3\sin^2 x\cos^4 x\sin^2 y\cos^2 y \ge 0, \therefore$ ***by (1),***

$$\Omega \le \sin^6 x + \cos^6 x\sin^6 y + \cos^6 x\cos^6 y = \sin^6 x + \cos^6 x(\sin^6 y + \cos^6 y) \le$$

$$\le \sin^6 x + \cos^6 x \ (\because \sin^6 y + \cos^6 y \le \sin^2 y + \cos^2 y = 1 \ \& \cos^6 x \ge 0)$$

$$\le \sin^2 x + \cos^2 x = 1 \therefore \Omega \overset{(1)}{\le} 1$$

(a), (b) $\Rightarrow 0 \le \Omega \le 1 \Rightarrow -1 < \Omega \le 1 \Rightarrow |\Omega| \le 1$ ***(Proved)***

418. If $A, B, C \in M_n(\mathbb{R}), AB = BA, AC = CA, BC = CB, n \in \mathbb{N}, n \ge 2$ then:

$$\det(A^2 - 6AB + 10B^2 + 16BC + 10C^2 - 6AC) \ge 0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\det(A^2-6AB+10B^2+16Bc+10C^2-6AC)=$$
$$=\det[(A+(-3+i)B+(-3-i)C)(A+(-3-i)B+(-3+i)C]$$
$$=\det\left((A+(-3+i)B+(-3-i)C)\left(\overline{A+(-3+i)B(-3-i)C}\right)\right.$$
$$=\det(A+(-3+i)B+(-3-i)C)\,\overline{\det(A+(-3+\imath)B+(-3-\imath)C)}$$
$$=|\det(A+(-3+i)B+(-3-i)C)|^2\geq 0$$

Solution 2 by Marian Ursărescu-Romania

We make a generalization:

Lemma 1: Let $P\in R[x], p(x)=x^2+ax+b, \Delta=b^2-4b<0$. *Then* $\forall A,B\in M_n(\mathbb{R})$ *the following statement is true:* $\det[(A+x_1B+x_2C)(A+x_2B+x_1C)]\geq 0, x_1,x_2$ *being the roots of* p

Demonstration: $If\ \Delta<0\Rightarrow x_1,x_2\in\mathbb{C}, x_2=x_1$ *and using* $\det(x\cdot\overline{x})\geq 0$, $\forall x\in M_n(\mathbb{R})\Rightarrow$

$$\det[(A+x_1B+x_2C)(A+x_2B+x_1C)]=\det[(A+x_1B+x_2C)(\overline{A+x_1B+x_2C})]\geq 0$$

Lemma 2. If $AB=BA, AC=CA, BC=CB$ *then the conclusion of this theorem can be written this way:* $\det[A^2+b(B^2+C^2)-a(AB+AC)+(a^2-2b)BC]\geq 0$

Demonstration: $\det[(A+x_1B+x_2C)(A+x_2B+x_1C)]=$

$$=\det\left[A^2+x_1x_2(B^2+C^2)+(x_1+x_2)(AB+AC)+\left(x_1^2+x_2^2\right)BC\right]=$$
$$=\det[A^2+b(B^2+C^2)-a(AB+AC)+(a^2-2b)BC]\geq 0$$

(we used $AB=BA, AC=CA, BC=CB$ *and Viéte relations)*

Now, in our case: $a=6, b=10$. *Done.*

419. If $A,B\in M_2(\mathbb{R}), AB=BA, \det A=\alpha>0, \det(A+i\alpha B)=0$ then find:

$$\Omega=\det(A^2-\alpha AB+\alpha^2B^2)$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Serban George Florin-Romania

$$x^2-\alpha x+\alpha^2=0, \Delta=\alpha^2-4\alpha^2=-3\alpha^2<0$$
$$x_1=\frac{\alpha+\alpha\sqrt{3}i}{2}, x_2=\frac{\alpha-\alpha\sqrt{3}i}{2}$$

$$\det(A+Bx)=x^2\det B+u\cdot x+\det A=x^2\det B+ux+\alpha$$

$$\det(A+i\alpha B)=(i\alpha)^2\det B+ui\alpha+\alpha=\left(-\alpha^2\det B+\alpha\right)+ui\alpha=0\Rightarrow u\alpha=0\Rightarrow u=0$$

$$\Rightarrow -\alpha^2\det B+\alpha=0\Rightarrow \alpha\cdot\det B=1$$

$$\Omega=\det(A^2-\alpha AB+\alpha^2B^2)=\det(A-x_1B)\cdot\det(A-x_2B)$$

$$\Omega=\left(x_1^2\det B+\alpha\right)\cdot\left(x_2^2\det B+\alpha\right)=(x_1x_2)^2\cdot\det{}^2B+\alpha^2+\alpha\det B\left(x_1^2+x_2^2\right)=$$

$$=\alpha^4\cdot\det{}^2B+\alpha^2+1(-\alpha^2)=\alpha^2\cdot(\alpha\det B)^2=\alpha^2\cdot 1^2=\alpha^2$$

Vieté relationships $x_1+x_2=\alpha, x_1x_2=\alpha^2, x_1^2+x_2^2=S^2-2P=\alpha^2-2\alpha^2=-\alpha^2$

Solution 2 by Ravi Prakash-New Delhi-India

If $AB=BA, \det(A)=\alpha>0, \det(A+\alpha iB)=0$***, find*** $\det(A^2-\alpha AB+\alpha^2B^2)$***. As***

$\det(A)>0, A^{-1}$ ***exists. Let*** $C=A^{-1}B$***.***

Now, $\det(A+\alpha iB)=0$ ***(1)***

$$\Rightarrow \det\left[\alpha iA\left(-\frac{i}{\alpha}I+A^{-1}B\right)\right]=0\Rightarrow \det(\alpha iA)\det\left(C-\frac{i}{\alpha}I\right)=0 \quad \textbf{(2)}$$

$$\Rightarrow \det\left(C-\frac{i}{\alpha}I\right)=0$$

$$[\because \det(i\alpha A)=-\alpha^2(\alpha)\neq 0]$$

Characteristic equation of C ***is***

$$t^2-tr(C)t+\det(C)=0 \quad \textbf{(3)}$$

In view of (2), $\frac{i}{\alpha}$ ***satisfies (3)*** $\Rightarrow -\frac{1}{\alpha^2}-\frac{i}{\alpha}tr(C)+\det(C)=0$

$\Rightarrow \det(C)=\frac{1}{\alpha^2}$ ***and*** $tr(C)=0$

As $\det(C)\neq 0$***, we get*** $\det(A^{-1}B)\neq 0\Rightarrow \det(B)\neq 0$ ***and*** $\det(B)=\frac{1}{\alpha}\Rightarrow$

$\Rightarrow \det(B)\neq 0$ ***and*** $\det(B)=\frac{1}{\alpha}\Rightarrow B^{-1}$ ***exists. Let*** $D=AB^{-1}$***. From (1):***

$$\det[(D+i\alpha)B]=0\Rightarrow \det(D+i\alpha)\det(B)=0\Rightarrow \det(D+i\alpha)=0 \quad \textbf{(4)}$$

Characteristic equation of D ***is*** $t^2-\left(tr(D)\right)t+\det(D)=0$ ***(5)***

In view of (4) $-i\alpha$ ***satisfies (4)***

$$\therefore -\alpha^2+tr(D)(i\alpha)+\det(D)=0\Rightarrow \det(D)=\alpha^2, tr(D)=0$$

$\therefore$ ***characteristic equation (5) becomes*** $t^2+\alpha^2=0$***.***

Now, $A^2-\alpha AB+\alpha^2B^2=(A^2B^{-2}-\alpha AB^{-1}+\alpha^2I)B^2=(D^2-\alpha D+\alpha^2I)B^2=$

$$= (0 - \alpha D)B^2 = -\alpha AB^{-1}B^2 = -\alpha AB$$

$$\det(A^2 - \alpha AB + \alpha^2 B^2) = (-\alpha)^2 \det(AB) = \alpha^2 \det(A)\det(B) = \alpha^2$$

420. Find $A, B \in M_2(\mathbb{R})$ such that:

$$\det A < 0, \det(A - B) > 0, \det(A + B) < 0, \det(2A + B) > 0$$

Proposed by Marian Ursărescu – Romania

Solution by Andrew Okukura-Japan&Romania

We will use the following formula:

$\det(A + xB) = ax^2 + bx + c$, *when:* $a = \det B$, $b = tr(AB^*)$, $c = \det A$

We will note $p(x) = \det(A + xB)$. *Because* p *is a polygon of second degree, it's obvious that it can be at most two changes in the value of* $sgn\,(p(x))$. *But:*

$p(-1) > 0, p(0) < 0, p\left(\frac{1}{2}\right) > 0, p(1) < 0 \Rightarrow 3$ *changes of sign. That means there are no matrices with the properties in the hypothesis.*

Observation:

$$\det(2A + B) = 4\det\left(A + \frac{1}{2}B\right) = 4p\left(\frac{1}{2}\right) > 0 \Rightarrow p\left(\frac{1}{2}\right) > 0$$

421. $A \in M_3(\mathbb{R}), \det(A^2 + 2A + 2I_3) = \det(A + I_3) = 0$

Find:

$$\Omega = \det A$$

Proposed by Marian Ursărescu-Romania

Solution by Djeeraj Badera-India

$A \in M_3(\mathbb{R})$ *then characteristics polynomial has highest degree* 3

$\therefore$ *We have to find a polynomial with their eigen values*

$\therefore \det(A^2 + 2A + 2I_3) = 0 \therefore$ *then polynomial is* $x^2 + 2x + 2 = 0$

It has two different eigen values $(-1 + i)$ *and* $(-1 - i)$

[by solving quadratic equation]

Here $|A+I|=0 \therefore$ ***one eigen value of*** A ***is*** $-1 \therefore$ ***characteristic polynomial is***

$$=(x+1)(x^2+2x+2)=x^3+2x^2+2x+x^2+2x+2=x^3+3x^2+4x+2$$

$$\therefore \text{ then } \det(A)=\text{product of eigen value}=-2$$

422. Solve for natural numbers:

$$(x+y)^{x^n+y^n}=(x+1)^{x^n}\cdot(y+1)^{y^n}, n\in\mathbb{N}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

Solution 1 by Michael Sterghiou-Greece

$$(x+y)^{x^n+y^n}=(x+1)^{x^n}\cdot(y+1)^{y^n}, n\in\mathbb{N} \quad (1)$$

$$(1)\to\left(\frac{x+y}{x+1}\right)^{x^n}=\left(\frac{y+1}{x+y}\right)^{y_n} \quad (2)$$

$$1)\ y>1\wedge x=1\ \left(\frac{y+1}{2}\right)^{n}>1\wedge\left(\frac{y+1}{x+y}\right)^{y^n}=1 \text{ contradiction}$$

$$2)\ y>1\ \wedge x>1 \text{ LHS of (2)}>1\wedge \text{RHS of (2)}<1 \gg$$

$$\text{Similar if } y=1, x>1\ \frac{2}{x+1}<1 \text{ etc} \gg$$

$$3)\ y=0\to x^{x^n}=(x+1)^{x^n} \text{ contradiction}$$

$$4)\ x=0 \text{ likewise}$$

$$\text{Therefore } x=y=1.$$

Solution 2 by Serban George Florin-Romania

$$\text{If } y=0\Rightarrow x^{x^n}=(x+1)^{x^n} \text{ false}\Rightarrow y\neq 0 \text{ and } x\neq 0$$

$$\Rightarrow x^n\geq 1, y^n\geq 1, x^n+y^n\geq 2$$

$$\Rightarrow \left.\begin{matrix}(x+1)|x+y\\(x+1)|x+1\end{matrix}\right\}\Rightarrow (x+1)|y-1\overset{(if\ y\geq 2)}{\Rightarrow} x+1\leq y-1, x\leq y-2$$

$$\Rightarrow \left.\begin{matrix}y+1|x+y\\y+1|y+1\end{matrix}\right\}\Rightarrow y+1|x-1\overset{(if\ x\geq 2)}{\Rightarrow} y+1\leq x-1, x\geq y+z$$

$$\Rightarrow x\leq y-2 \text{ and } x\geq y+2 \text{ false.}$$

$$\Rightarrow x\geq 2 \text{ or } y\geq 2 \text{ false}\Rightarrow x=1 \text{ or } y=1$$

$$\text{If } x=1\Rightarrow (y+1)^{y^n+1}=2(y+1)^{y^n}, (y+1)^{y^n}(y+1-2)=0$$

$$(y+1)^{y^n}\cdot(y-1)=0, y+1\neq 0\Rightarrow (y+1)^{y^n}\neq 0\Rightarrow y=1$$

If $y = 1 \Rightarrow (x+1)^{x^n+1} = 2(x+1)^{x^n}, (x+1)^{x^n}(x+1-2) = 0$

$$(x+1)^{x^n}(x-1) = 0 \Rightarrow x+1 \neq 0 \Rightarrow (x+1)^{x^n} \neq 0$$

$$\Rightarrow x-1 = 0 \Rightarrow x = 1 \Rightarrow x = y = 1$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$(x+y)^{x^n+y^n} \overset{(1)}{=} (x+1)^{x^n}(y+1)^{y^n}$$

(1)$\Leftrightarrow (x^n+y^n)\ln(x+y) = x^n\ln(x+1) + y^n\ln(y+1)$

$$\Leftrightarrow x^n \ln\left(\frac{x+y}{x+1}\right) + y^n \ln\left(\frac{x+y}{y+1}\right) \overset{(1)}{=} 0$$

$$\because x \geq 1 \therefore x+y \geq y+1 \Rightarrow \frac{x+y}{y+1} \geq 1$$

$$\Rightarrow \ln\left(\frac{x+y}{y+1}\right) \geq 0 \Rightarrow y^n \ln\left(\frac{x+y}{y+1}\right) \overset{(i)}{\geq} 0 \ (\because y^n \geq 1)$$

Also, $\because y \geq 1 \therefore x+y \geq x+1 \Rightarrow \frac{x+y}{x+1} \geq 1$

$$\Rightarrow \ln\left(\frac{x+y}{x+1}\right) \geq 0 \Rightarrow x^n \ln\left(\frac{x+y}{x+1}\right) \overset{(ii)}{\geq} 0 \ (\because x^n \geq 1)$$

(i)+(ii)$\Rightarrow$ ***LHS of (1)***≥ 0***, equality if*** $x = y = 1$

and $\because$ ***LHS*** $= 0 \therefore x = y = 1$ ***(Answer)***

423. **Solve in $\mathbb{R}$:**

$$log_2(2^{cos\,x}+1) + log_3(3^{cos\,x}+2) + log_4(4^{cos\,x}+3) = \sqrt[3]{27-cos\,x}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Amit Dutta-Jamshedpur-India

AM-GM

$$2^{\cos x}+1 \geq 2\cdot 2^{\left(\frac{\cos x}{2}\right)} \Rightarrow \log_2(2^{\cos x}+1) \geq 1 + \left(\frac{\cos x}{2}\right)$$

$3^{\cos x}+2 = 3^{\cos x}+1+1 \geq 3\cdot 3^{\left(\frac{\cos x}{3}\right)} \Rightarrow$ ***Taking log***

$$\Rightarrow \log_3(3^{\cos x}+2) \geq 1 + \left(\frac{\cos x}{3}\right)$$

$$4^{\cos x}+3 = 4^{\cos x}+1+1+1 \geq 4\cdot 4^{\frac{\cos x}{4}} \Rightarrow \log_4(4^{\cos x}+3) \geq 1 + \left(\frac{\cos x}{4}\right)$$

So, the original equation becomes:

$$\left(1+\frac{\cos x}{2}\right)+\left(1+\frac{\cos x}{3}\right)+\left(1+\frac{\cos x}{4}\right)\le\sqrt[3]{27-\cos x}$$

$$\Rightarrow 3+\frac{\cos x}{2}+\frac{\cos x}{3}+\frac{\cos x}{3}\le\sqrt[3]{27-\cos x}$$

$$\Rightarrow 3+\left(\frac{13}{12}\right)\cos x\le\sqrt[3]{27-\cos x}\quad (1)$$

Now, Case I

If $\cos x>0$, ***(1)*** $\Rightarrow 3+\left(\frac{13}{12}\right)\cos x\le\sqrt[3]{27-\cos x}$

$LHS>3, RHS<3$

From (1), we can see that this case is false

Case II $(\cos x<0)$

$$\because \log_2(2^{\cos x}+1)+\log_3(3^{\cos x}+2)+\log_4(4^{\cos x}+3)=\sqrt[3]{27-\cos x}$$

when $(\cos x<0)$

$$2^{\cos x}<2^0<1\Rightarrow 2^{\cos x}+1<2\Rightarrow \log_2(2^{\cos x}+1)<1$$

$$3^{\cos x}<3^0<1\Rightarrow 3^{\cos x}+2<3\Rightarrow \log_3(3^{\cos x}+2)<1$$

$$4^{\cos x}<4^0<1\Rightarrow 4^{\cos x}+3<4\Rightarrow \log_4(4^{\cos x}+3)<1$$

$$\Rightarrow \log_2(2^{\cos x}+1)+\log_3(3^{\cos x}+2)+\log_4(4^{\cos x}+3)<3$$

But $\sqrt[3]{27-\cos x}>3$ ***so,*** $LHS<3, RHS>3$***. So, this case is also false***

Case III when $\cos x=0$

$$\log_2(2^{\cos x}+1)+\log_3(3^{\cos x}+2)+\log_4(4^{\cos x}+3)=\sqrt[3]{27-\cos x}$$

when $\cos x=0$***; LHS = RHS =*** 3

$\therefore$ ***The case possible is*** $\cos x=0\Rightarrow x=(2n+1)\frac{\pi}{2}, n\in$ ***Integer***

Solution 2 by Ravi Prakash-New Delhi-India

If $0<\cos x<1$,

$$2^{\cos x}+1>2\Rightarrow \log_2(2^{\cos x}+1)>1$$

$$3^{\cos x}+2>3\Rightarrow \log_3(3^{\cos x}+2)>1$$

$$4^{\cos x}+3>4\Rightarrow \log_4(4^{\cos x}+3)>1$$

$$\Rightarrow \log_2(2^{\cos x}+1)+\log_3(3^{\cos x}+2)+\log_4(4^{\cos x}+3)>3$$

and $(27-\cos x)^{\frac{1}{3}}<3$***. Similarly, if*** $-1<\cos x<0$***, then***

$LHS < 3$ and $RHS > 3$. Thus, only possible solution is $\cos x = 0 \Rightarrow x = (2n +$

$1\pi 2, n\in\mathbb{Z}$

Solution 3 by Soumava Chakraborty-Kolkata-India

Case 1 $\cos x \geq 0$

$\because y = \log x$ ***&*** $z = a^t$ ***are increasing*** $f^n s, \therefore \cos x \geq 0 \Rightarrow$

$\Rightarrow LHS \geq \log_2 2 + \log_3 3 + \log_4 4 = 3$ ***&*** $RHS \leq \sqrt[3]{27} = 3$ ***&***

$\because LHS = RHS \therefore LHS = RHS = 3$, ***which occurs when*** $\cos x = 0$

Case 2 $\cos x \leq 0$. ***We have*** $LHS \leq 3$ ***&*** $RHS \geq 3$ ***&*** $\because LHS = RHS, \therefore LHS = RHS = 3$, ***which occurs at*** $\cos x = 0$ $\therefore$ ***given equality occurs at*** $\cos x = 0 \Rightarrow x = 2n\pi \pm \frac{\pi}{2} (n \in \mathbb{Z})$

(Answer)

424. Find $x, y, z \geq 0$ such that:

$$\frac{2x^2+4}{z^2+2y+3} + \frac{2y^2+4}{x^2+2z+3} + \frac{2z^2+4}{y^2+2x+3} = 3$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

$$2y \leq y^2 + 1 \Rightarrow z^2 + 2y + 3 \leq z^2 + y^2 + 4 \Rightarrow$$

$$E = \sum \frac{2x^2+4}{z^2+y^2+4} = 3 \Rightarrow E = \sum \frac{x^2+2}{y^2+2+z^2+2} = \frac{3}{2} \quad \textit{(1)}$$

Let $x^2 + 2 = a, y^2 + 2 = b, z^2 + 2 = c \Rightarrow$ ***(1) becomes***

$$\sum \frac{a}{b+c} = \frac{3}{2} \quad \textit{(2)}$$

$$\textit{But} \sum \frac{a}{b+c} \geq \frac{3}{2} \quad \textit{(3)}$$

From (2)+(3)$\Rightarrow a = b = c \Rightarrow x^2 = y^2 = z^2 \Rightarrow x = y = z = 1.$

425. Solve for real numbers:

$$\frac{1}{x+1} + \frac{1}{x(x+2)} + \frac{1}{x(x+1)(x+3)} + \cdots + \frac{1}{x(x+1)\cdot\ldots\cdot(x+99)(x+101)} =$$

$$=\frac{1}{3}-\frac{1}{x(x+1)(x+2)\cdot\ldots\cdot(x+100)(x+101)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michael Sterghiou-Greece

(1) Constraints all denominators $\neq 0$

(1) is written $LHS+\frac{1}{\prod_{101}}=\frac{1}{3}$ *(2) where* $\prod_k\ =\prod_{k=1}^{101}x(x+1)\ldots(x+n)$

(2) $\frac{x(x+2)\ldots(x+101)}{\prod_{101}}+\frac{(x+1)(x+3)\ldots(x+101)}{\prod_{101}}+\cdots+\frac{x+100}{\prod_{101}}+\frac{1}{\prod_{101}}=\frac{1}{3}\Leftrightarrow$

$$\Leftrightarrow\frac{x(x+2)\ldots(x+101)}{\prod_{101}}+\cdots+\frac{x+101}{\prod_{101}}=\frac{1}{3}\Leftrightarrow$$

$$\Leftrightarrow\frac{x(x+2)\ldots(x+100)}{\prod_{100}}+\cdots+\frac{x+100}{\prod_{100}}+\frac{1}{\prod_{100}}=\frac{1}{3}$$

after we have eliminated $x+101\neq 0$*. We observe that the process continues like this and in every step we eliminate the last term* $x+k$*. After* $101-k$ *steps we will have:* $\frac{x(x+2)\ldots(x+k)}{\prod_k}+\cdots+\frac{x+k-1}{\prod_k}+\frac{1}{\prod_k}=\frac{1}{3}$ *and will eliminate* $x+k$*.*

The last steps for $k=99$ *will be* $\frac{x(x+2)}{\prod_2}+\frac{x+1}{\prod_2}+\frac{1}{\prod_2}=\frac{1}{3}\Leftrightarrow\frac{x+2}{x+2}\cdot\left[\frac{x}{\prod_1}+\frac{1}{\prod_1}\right]=\frac{1}{3}\leftrightarrow$

$\overset{\prod_1=\frac{1}{x(x+1)}}{\leftrightarrow}\frac{x+1}{x(x+1)}=\frac{1}{3}\leftrightarrow\frac{1}{x}=\frac{1}{3}\leftrightarrow x=3$*. This is the only solution and it is accepted.*

Solution 2 by Ravi Prakash-New Delhi-India

Consider $S=\frac{1}{x+1}+\frac{1}{x(x+2)}+\frac{1}{x(x+1)(x+3)}+\cdots+\frac{1}{x(x+1)(x+2)\ldots(x+99)(x+101)}+$

$+\frac{1}{x(x+1)(x+2)\ldots(x+100)(x+101)}$*. Combining last two terms*

$$S=\frac{1}{x+1}+\frac{1}{x(x+2)}+\frac{1}{x(x+1)(x+3)}+\cdots+\frac{1}{x(x+1)\ldots(x+98)(x+100)}+$$

$+\frac{x+100+1}{x(x+1)(x+2)\ldots(x+100)(x+101)}$*. Simplify the last term and combine it with last but one term* $S=\frac{1}{x+1}+\frac{1}{x(x+2)}+\frac{1}{x(x+1)(x+3)}+\cdots+\frac{x+99+1}{x(x+1)(x+2)\ldots(x+100)}$

Continuing in this way we get: $S=\frac{1}{x+1}+\frac{1}{x(x+1)}=\frac{1}{x}$*. Thus,* $S=\frac{1}{x}=\frac{1}{3}\Rightarrow x=3$*.*

Solution 3 by Sagar Kumar-Kolkata-India

$$\overset{LHS}{=}\sum_{r=1}^{101}\frac{(x-1)!\,(x+r-1)}{(x+r)!}\Rightarrow (x-1)!\sum_{r=1}^{101}\left(\frac{1}{(x+r-1)!}-\frac{1}{(x+r)!}\right)\Rightarrow$$

$$\Rightarrow \frac{1}{x}-\frac{(x-1)!}{(x+101)!}\quad (1)$$

$$\overset{RHS}{=}\frac{1}{3}-\frac{(x-1)!}{(x+101)!}\quad (2)$$

From (1) and (2) $x=3$ is the only solution.

426. Solve for real numbers:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ e^x & e^{-x} & e & e^{-1} \\ e^{2x} & e^{-2x} & e^2 & e^{-2} \\ e^{4x} & e^{-4x} & e^4 & e^{-4} \end{vmatrix}\cdot\begin{vmatrix} 1 & 1 & 1 & 1 \\ e^x & e^{-x} & e & e^{-1} \\ e^{3x} & e^{-3x} & e^3 & e^{-3} \\ e^{4x} & e^{-4x} & e^4 & e^{-4} \end{vmatrix}=0$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

Let $a=e^x, b=e$. ***Put*** $\Delta_1=\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & \frac{1}{a} & b & \frac{1}{b} \\ a^2 & \frac{1}{a^2} & b^2 & \frac{1}{b^2} \\ a^4 & \frac{1}{a^4} & b^4 & \frac{1}{b^4} \end{vmatrix}=\frac{1}{a^4b^4}\Delta_2$

$$\Delta_3=\begin{vmatrix} 1 & a^4 & 1 & b^4 \\ a & a^3 & b & b^3 \\ a^2 & a^2 & b^2 & b^2 \\ a^4 & 1 & b^4 & 1 \end{vmatrix}$$

$$C_1\to C_1-C_2, C_3\to C_3-C_4$$

$\Delta_2=(1-a^2)(1-b^2)\Delta_3$ ***where*** $\Delta_4=\begin{vmatrix} 1+a^2 & a^4 & 1+b^2 & b^4 \\ a & a^3 & b & b^3 \\ 0 & a^2 & 0 & b^2 \\ -(1+a^2) & 1 & -(1+b^2) & 1 \end{vmatrix}$

Expand along R_3

$$\Delta_4=-a^2\begin{vmatrix} 1+a^2 & 1+b^2 & b^4 \\ a & b & b^3 \\ -(1+a^2) & -(1+b^2) & 1 \end{vmatrix}-b^2\begin{vmatrix} 1+a^2 & a^4 & 1+b^2 \\ a & a^3 & b \\ -(1+a^2) & 1 & -(1+b^2) \end{vmatrix}$$

$$R_3 \to R_3 + R_1$$

$$\Delta_4 = -a^2\begin{vmatrix}1+a^2 & 1+b^2 & b^4\\ a & b & b^3\\ 0 & 0 & 1+b^4\end{vmatrix} - b^2\begin{vmatrix}1+a^2 & a^4 & 1+b^2\\ a & a^3 & b\\ 0 & 1+a^4 & 0\end{vmatrix} =$$

$$= -a^2(1+b^4)[(1+a^2)b-(1+b^2)a] + b^2(1+a^4)[(1+a^2)b-(1+b^2)a]$$

$$= [(b-a)-ab(b-a)][b^2-a^2-a^2b^2(b^2-a^2)] =$$

$$= (b-a)(1-ab)(b^2-a^2)(1-a^2b^2) = (b-a)^2(b+a)(1-ab)^2(1+ab)$$

Thus, $\Delta_1 = \frac{(1+a)}{(ab)^4}(1-b^2)(a+b)(1+ab)(1-a)(b-a)^2(1-ab)$

Next, put $\Delta_2 = \begin{vmatrix}1 & 1 & 1 & 1\\ 0 & \frac{1}{a} & b & \frac{1}{b}\\ a^3 & \frac{1}{a^3} & b^3 & \frac{1}{b^3}\\ a^4 & \frac{1}{a^4} & b^4 & \frac{1}{b^4}\end{vmatrix} = \frac{1}{a^4b^4}\Delta_5$ *where*

$$\Delta_5 = \begin{vmatrix}1 & a^4 & 1 & b^4\\ a & a^3 & b & b^3\\ a^3 & a & b^3 & b\\ a^4 & 1 & b^4 & 1\end{vmatrix}$$

Use $C_1 \to C_1 - C_2, C_3 \to C_3 - C_4$

$\Delta_5 = (1-a^2)(1-b^2)\Delta_6$ *where*

$$\Delta_6 = \begin{vmatrix}1+a^2 & a^4 & 1+b^2 & b^4\\ a & a^3 & b & b^3\\ -a & a & -b & b\\ -(1+a^2) & 1 & -(1+b^2) & 1\end{vmatrix}$$

$$R_4 \to R_4 + R_1, R_3 \to R_3 + R_2$$

$$\Delta_6 = \begin{vmatrix}1+a^2 & a^4 & 1+b^2 & b^4\\ a & a^3 & b & b^3\\ 0 & a+a^3 & 0 & b+b^3\\ 0 & 1+a^4 & 0 & 1+b^4\end{vmatrix} = (1+a^2)\begin{vmatrix}a^3 & b & b^3\\ a+a^3 & 0 & b+b^3\\ 1+a^4 & 0 & 1+b^4\end{vmatrix} -$$

$$-a\begin{vmatrix}a^4 & 1+b^2 & b^4\\ a+a^3 & 0 & b+b^3\\ 1+a^4 & 0 & 1+b^4\end{vmatrix} = -(1+a^2)b[(1+a^3)(1+b^4)-(1+a^4)(b+b^3)] +$$

$$+a(1+b^2)[(a+a^3)(1+b^4)-(1+a^4)(b+b^3)] =$$

$$= (a-b)(1-ab)[(a-b)(1-a^3b^3)+(1-ab)(a^3-b^3)]$$

$$= (a-b)^2(1-ab)^2[1+ab+a^2b^2+a^2+b^2+ab]$$

Thus, $\Delta = \frac{1}{(ab)^8}(1-a^2)^2(1-b^2)^2(b-a)^4(1-ab)^4(a+b)(1+ab)$

$$(1+2ab+a^2b^2+a^2+b^2)$$

As $a, b > 0, b \neq 1$

$$\Delta = 0 \Leftrightarrow a^2 - 1 \text{ or } b = a \text{ or } ab = 1 \Leftrightarrow e^x = 1 \text{ or } e^x = e, e^{x+1} = 1 \Leftrightarrow$$

$$\Leftrightarrow x = 0, x = 1, x = -1.$$

427. Solve for real numbers:

$$\frac{\left(e^{\pi x^{2018}}+1\right)\left(e^{2\pi x^{2018}}+1\right)\left(e^{4\pi x^{2018}}+1\right)\left(e^{8\pi x^{2018}+1}\right)\dots\left(e^{2^n\pi x^{2018}}+1\right)}{\left(\pi^{\frac{2e}{x}}+1\right)\left(\pi^{\frac{4e}{x}}+1\right)\left(\pi^{\frac{8e}{x}}+1\right)\left(\pi^{\frac{16e}{x}}+1\right)\dots\left(\pi^{\frac{2^{n+1}e}{x}}+1\right)} = \frac{e^{2^{n+1}\pi x^{2018}-1}}{\pi^{\frac{2^{n+2}e}{x}}-1}$$

Proposed by Jhoaw Carlos-La Paz-Bolivia

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{Put } e^{\pi x^{2018}} = t, \pi^{\frac{2e}{x}} = u$$

Numerator of LHS

$$= (t+1)(t^2+1)(t^4+1)\dots\left(t^{2^n}+1\right)$$

$$= \frac{1}{t-1}(t^2-1)(t^2+1)(t^4+1)\dots\left(t^{2^n}+1\right)$$

$$= \cdots$$

$$= \frac{1}{t-1}\left(t^{2^{n+1}}-1\right)$$

Denominator of RHS

$$= (u+1)(u^2+1)\dots\left(u^{2^n}+1\right) = \frac{u^{2^{n+1}}-1}{u-1}$$

$$\therefore LHS = \frac{u-1}{t-1}\cdot\frac{t^{2^{n+1}}-1}{u^{2^{n+1}}-1} \quad (1)$$

$$\text{Also, RHS} = \frac{t^{2^{n+1}}-1}{u^{2^{n+1}}-1} \quad (2)$$

From (1), (2), we get

$$u-1 = t-1 \Rightarrow u = t$$

$$\Rightarrow e^{\pi x^{2018}} = \pi^{\frac{2e}{x}} \Rightarrow \pi x^{2018} = \frac{2e}{x}\ln\pi \Rightarrow x^{2019} = \frac{2e\ln\pi}{\pi} \Rightarrow x = \left(\frac{2e\ln\pi}{\pi}\right)^{\frac{1}{2019}}$$

Solution 2 by Naren Bhandari-Nepal

Given that: $\prod_{k=0}^{n}\frac{e^{x^{2018\cdot 2^k\pi}}+1}{\pi^{\frac{2^{k+1}e}{x}}+1} = \frac{e^{x^{2018\cdot 2^{n+1}\pi}}-1}{\pi^{\frac{2^{n+2}e}{x}}-1}$

Now, multiplying and dividing by $e^{x^{2018\pi}}-1$ *and* $\pi^{\frac{2e}{x}}-1$ *as follows we obtain*

$$\frac{\pi^{\frac{2e}{x}}-1}{e^{x^{2018\pi}}-1}\prod_{k=0}^{n}\left(\frac{e^{x^{2018\pi}}-1}{\pi^{\frac{2e}{x}}-1}\cdot\frac{e^{x^{2018}\cdot 2^k\pi}+1}{\pi^{\frac{2^{k+1}e}{x}}+1}\right) = \frac{\pi^{\frac{2e}{x}}-1}{e^{x^{2018\pi}}-1}\left(\frac{e^{x^{2018\cdot 2^{n+1}}}-1}{\pi^{\frac{2^{n+2}e}{x}}-1}\right)$$

Equating the equality we have then:

$$\frac{\pi^{\frac{2e}{x}}-1}{e^{x^{2018\pi}}-1} = 1 \Rightarrow \pi^{\frac{2e}{x}}-1 = e^{x^{2018\pi}}-1 \Rightarrow \pi^{\frac{2e}{x}} = e^{x^{2018}\pi}$$

Taking logarithm both sides it follows

$$x^{2019} = \frac{2e}{\pi}\log\pi \Rightarrow x = \sqrt[2019]{\frac{2e}{\pi}\log\pi}$$

428. Solve for real numbers:

$$\cos^{12}x + 4\cos^8 x\sin 2x + 2\sin^2 2x\,(3\cos^4 x - 4) + 4\sin^3 2x - 3\cos x + 19 = 0$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

Solution by Ravi Prakash-New Delhi-India

$$\cos^{12}x + 4\cos^8 x\sin 2x + (3\cos^4 x - 4)(2\sin^2 2x) + 4\sin^3 2x - 3\cos x + 19 = 0$$

$$\Rightarrow \cos^{12}x + 8\cos^9 x\sin x + 24\cos^6 x\sin^2 x + 32\cos^3 x\sin^3 x - 32\cos^2 x\sin^2 x -$$

$$-3\cos x + 19 = 0 \Rightarrow$$

$$\Rightarrow (\cos^3 x + 2\sin x)^4 - 16\sin^4 x - 32\cos^2 x\sin^2 x - 3\cos x + 19 = 0$$

$$\Rightarrow (\cos^3 x + 2\sin x)^4 - 16(\sin^2 x + \cos^2 x)^2 + 16\cos^4 x - 3\cos x + 19 = 0$$

$$\Rightarrow (\cos^3 x + 2\sin x)^4 + 16\cos^4 x = 3(\cos x - 1)$$

LHS ≥ 0 *and* $RHS \leq 0$*. Equality when* $LHS = 0, RHS = 0$

$$(\cos^3 x + 2\sin x)^4 + 16\cos^4 x = 0, \cos x - 1 = 0$$

$\Rightarrow \cos^3 x + 2\sin x = 0$, $\cos x = 0$ *and* $\cos x = 1$. *Thus, no solution.*

429. $A \in M_2(\mathbb{R})$, $\det A = tr\, A = 1$. **Solve for real numbers:**

$$\det(A^4 + I_2) + 10\det(A^2 + I_2) + x = 4\det(A^3 + I_2) + 16\det(A + I_2)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $tr\,(A) = a + d = 1$, $\det(A) = ad - bc = 1$

Ch. eqn. of A *is:* $(a - t)(d - t) - bc = 0$

$t^2 - (a + d)t + ad - bc = 0$ *or* $t^2 - t + 1 = 0$

A *satisfies* $t^2 - t + 1 = 0 \Rightarrow A^2 - A + I = 0 \Rightarrow A^2 = A - I \Rightarrow A^3 = A^2 - A = -I$

$$A^4 = -A$$

Thus, $\det(A^4 + I_2) + 10\det(A^2 + I_2) = \det(-A + I_2) + 10\det(A)$

$= \det(-A^2) + 10\det(A) = \det(A^2) + 10\det(A) = (\det A)^2 + 10\det A = 11$

$$\det(A^3 + I_2) = \det(-I_2 + I_2) = 0$$

$$\det(A + I_2) = \begin{vmatrix} a + 1 & b \\ dc & d + 1 \end{vmatrix} = ad + (a + d) + 1$$

$$-bc = 3$$

$$= 3$$

Thus, $11 + x = 0 + (16)(3) \Rightarrow x = 37$.

Solution 2 by Sagar Kumar-Kolkata-India

Let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. *Applying Cayley Hamilton* $\Rightarrow (\alpha - \gamma)(\delta - \lambda) - \beta\gamma = 0$

$$\lambda^2 - (\gamma - \alpha)\lambda + \alpha\delta - \beta\gamma = 0$$

But given $\alpha + \gamma = \alpha\delta - \beta\gamma = 1$. *Hence characteristic equation is* $\lambda^2 - \lambda + 1 = 0$

$$A^2 - A + I = 0 \Rightarrow A^3 = A^2 - A = A - I - A = -I$$

$$A^4 = -A$$

$$\det(A^4 + I_2) + 10\det(A^2 + I_2) + x = 4\det(A^3 + I_2) + 16\det(A + I_2) \Rightarrow$$

$$\Rightarrow \det(I_2 - A) + 10\det(A) + x = 4\det(0) + 16\det(A + I_2) \Rightarrow$$

$$\Rightarrow \begin{vmatrix} 1-\alpha & -\beta \\ \gamma & 1-\gamma \end{vmatrix} + 10 + x = 16\begin{vmatrix} 1+\alpha & \beta \\ \gamma & 1+\delta \end{vmatrix}$$

$$\Rightarrow 1-(\alpha+\delta)+\alpha\delta+10+x = 16(1+\alpha+\delta+\alpha\delta-\gamma\beta)-\beta\gamma$$

$$\Rightarrow 10+x+\alpha\delta = 16(2+1)-\beta\gamma$$

$$x = 48-11 = 37 \quad \textit{(Answer)}$$

Solution 3 by Shafiqur Rahman-Bangladesh

Characteristic eq² of 2 by 2 matrix A *is* $\lambda^2 - \lambda Tr(A) + \det(A) = 0 \Rightarrow \lambda^2 - \lambda + 1 = 0$

$\therefore A^2 - A + 1 = 0 \Rightarrow A^3 + I = 0$ *and Characteristic roots are* $\lambda_1 = -\omega, \lambda_1 = -\omega^2$*.*

$$\det(A^n + I_2) = (\lambda_1^n + 1)(\lambda_2^n + 1) = (1+(-\omega)^n)(1+(-\omega^2)^n) \ \& \ \det(A^3 + I_2) = 0$$

Now, $\det(A^4 + I_2) + 10\det(A^2 + I_2) + x = 4\det(A^3 + I_2) + 16\det(A + I_2)$

$$\Rightarrow (1+\omega^4)(1+\omega^8) + 10(1+\omega^2) + x = 4 \times 0 + 16(1-\omega)(1-\omega^2) \Rightarrow$$

$$\Rightarrow 1 + 10 + x = 16 \times 3 \therefore x = 37$$

Solution 4 by Marian Ursărescu-Romania

$$pA(x) = x^2 - tr\,Ax + \det A = x^2 - x + 1, \textit{ with} \begin{cases} \lambda_1 + \lambda_2 = 1 \\ \lambda_1\lambda_2 = 1 \end{cases}$$

$$\det(A + I_2) = (\lambda_1 + 1)(\lambda_2 + 1) = \lambda_1\lambda_2 + \lambda_1 + \lambda_2 + 1 = 3 \ \textbf{\textit{(1)}}$$

$$\det(A^2 + I_2) = (\lambda_1^2 + 1)(\lambda_2^2 + 1) = (\lambda_1\lambda_2)^2 + \lambda_1^2 + \lambda_2^2 + 1 = 2 + (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2$$

$$= 2 + 1 - 2 = 1 \quad \textbf{\textit{(2)}}$$

$$\det(A^3 + I_2) = (\lambda_1^3 + 1)(\lambda_2^3 + 1) = (\lambda_1\lambda_2)^2 + \lambda_1^3 + \lambda_2^3 + 1 = 2 + (\lambda_1 + \lambda_2)(\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2)$$

$$= 2 + \lambda_1^2 + 2\lambda_1\lambda_2 + \lambda_2^2 - 3\lambda_1\lambda_2 = 2 + (\lambda_1 + \lambda_2)^2 - 3\lambda_1\lambda_2 = 2 + 1 - 3 = 0 \ \textbf{\textit{(3)}}$$

$$\det(A^4 + I_2) = (\lambda_1^4 + 1)(\lambda_2^4 + 1) = (\lambda_1\lambda_2)^4 + \lambda_1^4 + \lambda_2^4 + 1 =$$

$$= 2 + \lambda_1^4 + \lambda_2^4 = 2 + \lambda_1^4 + \lambda_2^4 + 2\lambda_1^2 + \lambda_2^2 - 2\lambda_1^2\lambda_2^2 =$$

$$= 2 + (\lambda_1^2 + \lambda_2^2)^2 - 2 = ((\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2) = 1 \quad \textbf{\textit{(4)}}$$

From (1)+(2)+(3)+(4)$\Rightarrow x + 1 + 10 = 4\cdot 0 + 16\cdot 3 \Rightarrow x + 11 = 48 \Rightarrow x = 37$

430. Solve for real numbers:

$$\frac{1}{(x+1)^3} + \frac{1}{(x+1)^2} + \frac{1}{x+1} = 156 + \log_5(x+1)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Rovsen Pirguliyev-Sumgait-Azerbaijan

Denote $x+1=t$*, then*

$$\frac{1}{t^3}+\frac{1}{t^2}+\frac{1}{t}=156+\log_5 t \quad (1)$$

domain the equation (1) $t>0$*:* $f(x)=\frac{1}{t^3}+\frac{1}{t^2}+\frac{1}{t}\downarrow$ *in* $(0;+\infty)$

$g(x)=156+\log_5 t\uparrow$ *in* $(0;+\infty)$ *and has at most one root*

$$t=\frac{1}{5}\Rightarrow x+1=\frac{1}{5}\Rightarrow x=-\frac{4}{5}$$

Solution 2 by Sagar Kumar-Kolkata-India

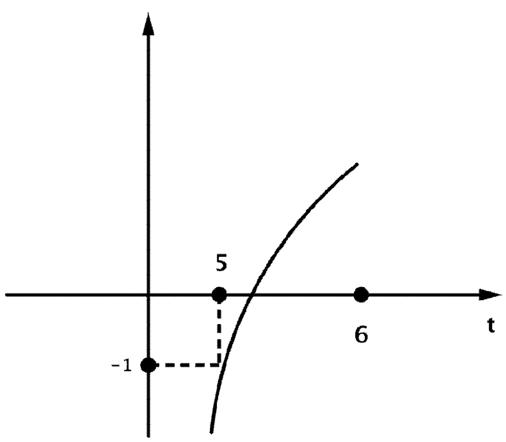
Put $\frac{1}{x+1}=t$*;* $(t^3+t^2+t)-156=-\log_5 t$

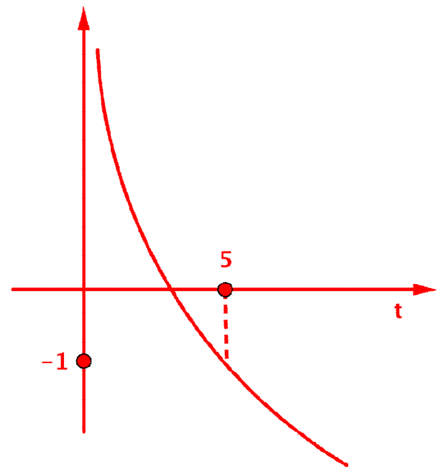
Let $f(t)=t^3+t^2+t-156$*;* $f'(t)=3t^2+2t+1$

$\Delta<0\ \ f'(t)>0\ \forall t\in\mathbb{R}\Rightarrow f(t)$ *is increasing and* $-\log_5(t)=g(t)\quad t>0$

$g'(t)=-\frac{1}{t\ln 5}\quad g'(t)<0;\forall t>0$. *Hence decreasing. Hence possible number of solution is* 1. $f(0)=-156$, $f(5)=-1$, $f(6)=102$

$f(5)f(6)<0$ *Hence one root between* 5 *and* 6





$g(t) = -\log_5(t)$. *Clearly at* $t = 5$

$g(5) = -1$

Hence $t = 5$ *is the only solution* $\frac{1}{x+1} = 5; x = \frac{1}{5} - 1$

$x = -\frac{4}{5}$ *(Answer)*

431. **Solve for real numbers:**

$$\left(x + \sqrt{x^2+1}\right)\left(x - [x] + \sqrt{(x-[x])^2+1}\right) = 1, [*] - \textit{great integer function}$$

Proposed by Dan Nedeianu-Romania

Solution by Tran Hong-Vietnam

$$\left(x + \sqrt{x^2+1}\right)\left(x - [x] + \sqrt{(x-[x])^2+1}\right) = 1 \quad (*)$$

If $x > 0$ *then*

$$x + \sqrt{x^2+1} > 1; \{x\} = x - [x] \geq 0 \Rightarrow \{x\} + \sqrt{\{x\}^2+1} > 1$$

$\Rightarrow$ *LHS (*)* $> 1 \Rightarrow$ *no solution.*

If $x \leq 0$ *then (*) becomes*

$$\{x\} + \sqrt{\{x\}^2+1} = -x + \sqrt{(-x)^2+1}$$

Let $f(u) = u + \sqrt{u^2+1}$ *with* $u \geq 0$

$$\Rightarrow f'(u) = 1 + \frac{u}{\sqrt{u^2+1}} > 0 (\forall u \geq 0) \Rightarrow f \nearrow [0, +\infty)$$

$$\Rightarrow f(\{x\}) = f(-x) \Leftrightarrow \{x\} = -x$$

$$\Leftrightarrow x - [x] = -x \Leftrightarrow 2x = [x] \in \mathbb{Z}$$

More, $0 \le \{x\} < 1 \Rightarrow 0 \le -x < 1 \Leftrightarrow -1 < x \le 0$

$$\Leftrightarrow -2 < 2x \le 0 \Leftrightarrow -2 < [x] \le 0 \Leftrightarrow [x] = -1 \text{ or } [x] = 0$$

$$\Leftrightarrow x = -\frac{1}{2} \text{ or } x = 0.$$

Answer: $x = -\frac{1}{2}$ ***or*** $x = 0.$

432. Solve for real numbers:

$$\frac{\left|\cos x \cdot \cos\frac{x}{2}\right|}{\sqrt{(2-\cos^2 x)\left(2-\cos^2\frac{x}{2}\right)}} = \frac{\cos^2 x + \cos^2\frac{x}{2}}{2+\sin^2 x + \sin^2\frac{x}{2}}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

Solution by Hoang Le Nhat Tung-Hanoi-Vietnam

$$\frac{\left|\cos x \cdot \cos\frac{x}{2}\right|}{\sqrt{(2-\cos^2 x)\left(2-\cos^2\frac{x}{2}\right)}} = \frac{\cos^2 x + \cos^2\frac{x}{2}}{2+\sin^2 x + \sin^2\frac{x}{2}}$$

$$\Leftrightarrow \frac{\left|\cos x \cdot \cos\frac{x}{2}\right|}{\sqrt{4-2\left(\cos^2 x+\cos^2\frac{x}{2}\right)+\cos^2 x\cos^2\frac{x}{2}}} = \frac{\cos^2 x+\cos^2\frac{x}{2}}{4-\left(\cos^2 x+\cos^2\frac{x}{2}\right)} \quad \textbf{(1)}$$

- Let $\begin{cases} \cos x \cdot \cos\frac{x}{2} = a \\ \cos^2 x + \cos^2\frac{x}{2} = b \end{cases}$; ***(1)*** $\leftrightarrow \frac{|a|}{\sqrt{4-2b+a^2}} = \frac{b}{4-b}$

$$\Rightarrow \frac{a^2}{4-2b+a^2} = \frac{b^2}{(4-b)^2} \leftrightarrow a^2(4-b)^2 = b^2(4-2b+a^2)$$

$$\leftrightarrow a^2b^2 - 8a^2b + 16a^2 = a^2b^2 - 2b^3 + 4b^2 \leftrightarrow -4a^2b + 8a^2 = -b^3 + 2b^2$$

$$\leftrightarrow b^3 - 2b^2 - 4a^2b + 8a^2 = 0 \leftrightarrow b^2(b-2) - 4a^2(b-2) = 0 \leftrightarrow (b-2)\left(b^2-4a^2\right) = 0$$

$$\leftrightarrow \left\{\begin{matrix} b-2=0 \\ b^2-4a^2=0 \end{matrix}\right. \rightarrow \begin{cases} b = 2 \\ b = 2a \\ b = -2a \end{cases}$$

$$b = 2 \rightarrow \cos^2 x + \cos^2\frac{x}{2} = 2 \rightarrow \cos^2 x = \cos^2\frac{x}{2} = 1 \Rightarrow x = 2k+1\ (k \in \mathbb{Z})$$

$$b=2a\to\cos^2 x+\cos^2\frac{x}{2}=2\cos x\cos\frac{x}{2}\leftrightarrow\left(\cos x-\cos\frac{x}{2}\right)^2=0\leftrightarrow\cos x=\cos\frac{x}{2}$$

$$\leftrightarrow -2\sin\frac{3x}{4}\cdot\sin\frac{x}{4}=0\leftrightarrow\begin{cases}\sin\frac{3x}{4}=0\\ \sin\frac{x}{4}=0\end{cases}\leftrightarrow\begin{cases}\frac{3x}{4}=k+1\\ \frac{x}{4}=k+1\end{cases}\leftrightarrow\begin{cases}x=\frac{4k+1}{3}\\ x=4k\pi\end{cases}$$

$$b=-2a\leftrightarrow\cos^2 x+\cos^2\frac{x}{2}=-2\cos x\cdot\cos\frac{x}{2}\leftrightarrow\left(\cos x+\cos\frac{x}{2}\right)^2=0$$

$$\leftrightarrow\cos x+\cos\frac{x}{2}=0\leftrightarrow 2\cos\frac{3x}{4}\cdot\cos\frac{x}{4}=0\leftrightarrow\begin{cases}\cos\frac{3x}{4}=0\\ \cos\frac{x}{4}=0\end{cases}\leftrightarrow\begin{cases}\frac{3x}{4}=\frac{\pi}{2}+k2\pi\\ \frac{x}{4}=\frac{\pi}{2}+k2\pi\end{cases}$$

$$\leftrightarrow\begin{cases}x=\frac{2\pi}{3}+\frac{8k\pi}{3}\\ x=2\pi+8k\pi\end{cases}(k\in\mathbb{Z})$$

433. Solve for real numbers:

$$(1+\sin x)\cdot(\sin x)^{\cos x}+(1+\cos x)\cdot(\cos x)^{\sin x}=1+\sin x+\cos x$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

Solution 1 by Lazaros Zachariadis-Thessaloniki-Greece

$$\underbrace{1+\sin x+\cos x}_{LHS}=\underbrace{(1+\sin x)(\sin x)^{\cos x}+(1+\cos x)(\cos x)^{\sin x}}_{RHS}$$

$$RHS=(1+\sin x)\big(1+(\sin x-1)\big)^{\cos x}+(1+\cos x)\big(1+(\cos x-1)\big)^{\sin x}$$

$$\overset{Bernoulli}{\le}(1+\sin x)(1+\cos x\cdot\sin x-\cos x)+(1+\cos x)(1+\cos x\sin x-\sin x)$$

$$=1+\sin x-\cos^3 x+1+\cos x-\sin^3 x$$

$$=(1+\sin x+\cos x)-(\cos^3 x+\sin^3 x)+1$$

$$=LHS-(\cos^3 x+\sin^3 x)+1$$

So, $RHS=LHS$ if-f $\cos^3 x+\sin^3 x=1$

$$x=2k\pi,k\in\mathbb{Z}\vee x=2k\pi+\frac{\pi}{2},k\in\mathbb{Z}$$

Solution 2 by Ravi Prakash-New Delhi-India

Thus, $(1+\sin x)(\sin x)^{\cos x}+(1+\cos x)(\cos x)^{\sin x}=1+\sin x+\cos x$

Has now solution in $\left(0,\frac{\pi}{2}\right)$

It follows that the only solution in $[0,2\pi]$ ***are*** $0,\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi$***, i.e. multiples of*** $\frac{\pi}{2}$.

Hence general solution is $\frac{n\pi}{2}, n\in\mathbb{Z}$.

434. Solve for real numbers:

$$2^x\cdot 3^{\frac{1}{x}}+3^x\cdot 2^{\frac{1}{x}}=\sqrt{6}\left(\sqrt{2}+\sqrt{3}\right)\left(5-\sqrt{6}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

Equation $\Leftrightarrow 2^x\cdot 3^{\frac{1}{x}}+3^x\cdot 2^{\frac{1}{x}}=4\sqrt{3}+9\sqrt{2}$. ***If*** $x<0\Rightarrow 2^x\cdot 3^{\frac{1}{x}}<1$ ***and*** $3^x\cdot 2^{\frac{1}{x}}<1\Rightarrow$

$2^x\cdot 3^{\frac{1}{x}}+3^x\cdot 2^{\frac{1}{x}}<2\Rightarrow$ ***equation can't have negative solutions***

Let $x>0; x=\frac{1}{2}$ ***and*** $x=2$ ***are solutions for this equation. We've proved that this are its only solutions. Let*** $p:(0,+\infty)\to\mathbb{R}; p(x)=a^x b^{\frac{1}{x}}, a,b>1$

We show that p ***is strictly increasing for*** $\left(\sqrt{\log_a b},+\infty\right)$

and strictly decreasing for $\left(0,\sqrt{\log_a b}\right)$ ***(1)***

p ***strictly increasing for*** $\left(\sqrt{\log_a b},+\infty\right)\Leftrightarrow \forall x_1,x_2>\sqrt{\log_a b}$

Such that $x_1<x_2\Rightarrow p(x_1)<p(x_2)\Leftrightarrow a^{x_1}b^{\frac{1}{x_1}}<a^{x_2}b^{\frac{1}{x_2}}\Leftrightarrow b^{\frac{x_2-x_1}{x_1x_2}}<a^{x_2-x_1}\Leftrightarrow$

$b<a^{x_1x_2}$ ***(because*** $a,b>1$ ***and*** $x_1<x_2$***)*** $\Leftrightarrow$

$\log_a b<x_1x_2$***, relation which is true because*** $x_1,x_2>\sqrt{\log_a b}$

Similarly, for $\left(0,\sqrt{\log_a b}\right)$

Let $p_1(x)=2^x\cdot 3^{\frac{1}{x}}$ ***and*** $p_2(x)=3^x\cdot 2^{\frac{1}{x}}$

For (1)$\Rightarrow p_1$ ***it is increasing for*** $\left(\sqrt{\log_2 3},+\infty\right)$ ***and strictly decreasing for*** $\left(0,\sqrt{\log_2 3}\right)$ ***(2)***

For (2)⇒ p_2 it is strictly increasing for $\left(\sqrt{\log_3 2}, +\infty\right)$ and strictly decreasing for $\left(0, \sqrt{\log_3 2}\right)$. Because $\log_3 2 < \log_2 3 \Rightarrow p_1(x) + p_2(x)$ it is strictly decreasing for $\left(0, \sqrt{\log_3 2}\right) \Rightarrow$ for this interval the equation

$p_1(x) + p_2(x) = 4\sqrt{3} + 9\sqrt{2}$ has a unique solution $x = \frac{1}{2}$.

$p_1(x) + p_2(x)$ it is strictly increasing for $\left(\sqrt{\log_2 3}, +\infty\right) \Rightarrow$ for this interval the equation $p_1(x) + p_2(x) = 4\sqrt{3} + 9\sqrt{2}$ has a unique solution $x = 2$.

For internal $\left(\sqrt{\log_3 2}, \sqrt{\log_2 3}\right)$, $p_1(x) + p_2(x) < 4\sqrt{3} + 9\sqrt{2} \Rightarrow$ the only solutions are

$$x = \frac{1}{2}, x = 2$$

Solution 2 by Michael Sterghiou-Greece

$$2^x \cdot 3^{\frac{1}{x}} + 3^x \cdot 2^{\frac{1}{x}} = \sqrt{6}\left(\sqrt{2} + \sqrt{3}\right)\left(5 - \sqrt{6}\right) \quad (1)$$

RHS of (1) $= 9\sqrt{2} + 4\sqrt{3}$. Consider the function $f(x) = 2^x \cdot 3^{\frac{1}{x}} + 3^x 2^{\frac{1}{x}}$ over $(0, +\infty)$. For $x < 0$ (1) does not have any solution as LHS $< 2 <$ RHS.

$$f''(x) = \frac{1}{x^4}\begin{bmatrix} 2^x 3^{\frac{1}{x}}(x^4 \log^2 2 - 2x^2 \log 2 \log 3 + 2x \log 3 + \log^2 3) + \\ 2^{\frac{1}{x}} 3^x (x^4 \log^2 3 - 2x^2 \log 2 \log 3 + 2x \log 2 + \log^2 2) \end{bmatrix} \quad (2)$$

If $x < 1$ then $-2x^2 \log 2 \log 3 + 2x \log 3 + \log^2 3 > 0$ (from the trinomial, its max is > 0 and $f(0_+), f(1) > 0$). Likewise, for $x \geq 1$ $y^2 \log^2 3 - 2y \log 2 \log 3 + \log^2 2 > 0$ with $y = x^2$ (trinomial). In a similar manner we can show that the 2nd term of (2) is > 0, hence $f''(x) > 0$ and $f'(x) \uparrow$. $f'(x) = -\frac{2^{\frac{1}{x}} 3^x \log 2}{x^2} + 2^{\frac{1}{x}} \cdot 3^x \log 3 - \frac{2^x 3^{\frac{1}{x}} \log 3}{x^2} + 2^x 3^{\frac{1}{x}} \log 2$

Observe that $f'(1) = 0$, therefore $x = 1$ is a min of $f(x)$ on $(0, +\infty)$. Equal to $f(1) = 12$. But RHS > 12. So, there are only two solutions one < 1 and the other > 1 and by inspection, we see that they are $x = 2$ and by symmetry $x = \frac{1}{2}$. Done!

435. Find all continuous functions: $f, g, h: \mathbb{R} \to \mathbb{R}$ such that:

$$f\left(\frac{x+y}{2}\right)=\frac{g(x)+h(y)}{2}, \forall x,y\in\mathbb{R}$$

Proposed by Marian Ursarescu-Romania

Solution by Chris Kyriazis-Athens-Greece

Let's set $y=0$: $f\left(\frac{x}{2}\right)=\frac{g(x)+h(0)}{2}\Rightarrow g(x)=2f\left(\frac{x}{2}\right)=h(0)$ *(1)*

Set $x=0$: $f\left(\frac{y}{2}\right)=\frac{g(0)+h(y)}{2}\Rightarrow h(y)=2f\left(\frac{y}{2}\right)-g(0)$ *(2)*

Using (1), (2) we have: $f\left(\frac{x+y}{2}\right)=f\left(\frac{x}{2}\right)+f\left(\frac{y}{2}\right)-\frac{h(0)+g(0)}{2}$ *or*

$f(a+b)=f(a)+f(b)-\frac{h(0)+g(0)}{2}$ *where* $a:\frac{x}{2}, b:\frac{y}{2}, a,b\in\mathbb{R}$*. Now let's set*

$k(a)=f(a)-\frac{h(0)+g(0)}{2}$*. Then* $k(a+b)=k(a)+k(b), \forall a,b\in\mathbb{R}$*. So* k *is a Cauchy*

function and continuos. So $k(x)=cx, c\in\mathbb{R}\Rightarrow f(x)=cx-\frac{h(0)+g(0)}{2}, \forall x\in\mathbb{R}$ *and*

$$g(x)=cx-h(0)-\frac{h(0)+g(0)}{2}\Rightarrow$$

$$\Rightarrow g(x)=cx-\frac{3h(0)+g(0)}{2}, h(x)=cx-\frac{3g(0)-h(0)}{2};$$

and similarly these functions satisfy the equation.

436. **Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that:**

$$f(x)+f(y)+x^2y+xy^2=f(x+y), \forall x,y\in\mathbb{R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$f(x)+f(y)=f(x+y)-xy(x+y)\Rightarrow f(x)-\frac{x^3}{3}+f(y)-\frac{y^3}{3}=f(x+y)=$$

$$=\frac{x^3}{3}-\frac{y^3}{3}-xy(x+y)\Rightarrow f(x)-\frac{x^3}{3}+f(y)-\frac{y^3}{3}=f(x+y)-\frac{1}{3}(x+y)^3 \quad (1)$$

Now, let $g(x)=f(x)-\frac{x^3}{3}$, g *continuous (2)*

From (1)+(2) $\Rightarrow g(x)+g(y)=g(x+y)\Rightarrow g(x)=ax, a\in\mathbb{R}$ *(3) (from Cauchy*

equation). From (2)+(3) $\Rightarrow f(x)-\frac{x^3}{3}=ax\Rightarrow f(x)=\frac{x^3}{3}+ax$

Solution 2 by Shafiqur Rahman-Bangladesh

$$f(x+y)=f(x)+f(y)+xy(x+y)=f(x)+f(y)+\frac{1}{3}[(x+y)^3-(x^3+y^3)]\Rightarrow$$

$$\Rightarrow\left(f(x+y)-\frac{1}{3}(x+y)^3\right)=\left(f(x)-\frac{1}{3}x^3\right)+\left(f(y)-\frac{1}{3}y^3\right)$$

Let $g(x)=f(x)-\frac{1}{3}x^3 \therefore g(x+y)=g(x)+g(y)\Rightarrow g(x)=kx\Rightarrow f(x)-\frac{1}{3}x^3=kx$

$\therefore f(x)=\frac{1}{3}x^3+kx$ where $k\in\mathbb{R}$.

Solution 3 by Yen Tung Chung-Taichung-Taiwan

$$f(x)+f(y)+x^2y+xy^2=f(x+y)\Rightarrow f(x+y)-f(x)=f(y)+x^2y+xy^2$$

Since $f(0+0)-f(0)=f(0)\Rightarrow f(0)=0$ and

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(h)+x^2h+xh^2}{h}$$

$$=\lim_{h\to 0}\left(\frac{f(h)}{h}+x^2+xh\right)=f'(0)+x^2$$

Let $f'(0)=A$, we have $f'(x)=x^2+A$ so, $f(x)=\int_0^x(x^2+A)\,dx=\frac{1}{3}x^3+Ax, A\in\mathbb{R}$

Solution 4 by Rovsen Pirguliyev-Sumgait-Azerbaijan

Let $g(x)=f(x)-\frac{x^3}{3}$, then we have $g(x+y)=g(x)+g(y)\Rightarrow g(x)=kx, \forall k\in\mathbb{R}$.

Then $\Rightarrow kx=f(x)-\frac{x^3}{3}\Rightarrow f(x)=\frac{x^3}{3}+kx$, where $k\in\mathbb{R}$.

437. Find all function $f:\mathbb{R}\to\mathbb{R}$ satisfying:

$$f(x+ny^2)\geq(y+1)^n f(x), \forall x,y\in\mathbb{R}, 1\leq n\in\mathbb{N}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tran Hong-Vietnam

Set $x:=x-n, y=1\Rightarrow f(x)\geq 0, \forall x\in\mathbb{R}$ (*)

Let $y=\frac{1}{n}\Rightarrow f\left(x+\frac{1}{n}\right)\geq\left(1+\frac{1}{n}\right)^n f(x), \forall x\in\mathbb{R}$ (1)

Set: $x:=x+\frac{1}{n}\Rightarrow f\left(x+\frac{2}{n}\right)\geq\left(1+\frac{1}{n}\right)^n f\left(x+\frac{1}{n}\right), \forall x\in\mathbb{R}$ (2)

$$\overset{(1),(2)}{\Rightarrow} f\left(x+\frac{2}{n}\right) \geq \left(1+\frac{1}{n}\right)^{2n} f(x), \forall x \in \mathbb{R}$$

By induction we have: $f\left(x+\frac{k}{n}\right) \geq \left(1+\frac{1}{n}\right)^{kn} f(x), \forall x \in \mathbb{R}, k \in \mathbb{N}$

Let $k = n \Rightarrow f(x+1) \geq \left(1+\frac{1}{n}\right)^{n^2} f(x), \forall x \in \mathbb{R}$ ***(3)***

Suppose exists $x_0 \in \mathbb{R}$ ***such that*** $f(x_0) \neq 0 \Rightarrow f(x_0) > 0$ ***(because (*)).***

From (3) we let n ***from to*** ∞

$$f(x_0+1) \geq \lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^{n^2} f(x_0)$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^{n}\right]^{n} f(x_0) = \lim_{n\to\infty} e^n = +\infty$$

But $f(t_0+1)$ ***is real number*** $\Rightarrow$ ***contradiction.***

$$\Rightarrow f(x) = 0, \forall x \in \mathbb{R}.$$

438. Find all functions $f: \mathbb{R} \to \mathbb{R}$ continuous in $x = 0$ such that:

$$f(2018x) = f(2019)x + x^2$$

Proposed by Nguyen Van Canh-Vietnam

Solution 1 by Andrew Okukura-Romania

We will define the continuous in $x = 0$ ***funciton*** $g(x) = f(x) + \frac{1}{4037}x^2, g: \mathbb{R} \to \mathbb{R}$

We have:

$$f(2018x) = g(2018x) - \frac{2018^2}{4037}x^2$$

$$f(2019x) = g(2019x) - \frac{2019^2}{4037}x^2$$

By replacing in the hypothesis of the relationship, we have:

$$g(2018x) - \frac{2018^2}{4037}x^2 = g(2019x) - \frac{2019^2x^2}{4037} + x^2 \Rightarrow g(2018x) = g(2019x)$$

$$2018x = y \Rightarrow 2019x = \frac{2019}{2018}y$$

$$\Rightarrow g(x)=g\left(\frac{2019}{2018}y\right)\Rightarrow g\left(\frac{2018}{2019}y\right)=g(y)\Rightarrow g\left(\left(\frac{2018}{2019}\right)^2y\right)=g\left(\frac{2018}{2019}y\right)\Rightarrow$$

$$\dots\Rightarrow g\left(\left(\frac{2018}{2019}\right)^ny\right)=g\left(\left(\frac{2018}{2019}\right)^{n-1}y\right)\Rightarrow g\left(\left(\frac{2018}{2019}\right)^ny\right)=g(y),\forall y\in\mathbb{R}$$

But g is continuous in 0, so far $n\to\infty$ $g\left(\left(\frac{2018}{2019}\right)^ny\right)=g(0)$, because

$\left(\frac{2018}{2019}\right)^n+0$ *and for any $y\in\mathbb{R}$; $y\left(\frac{2018}{2019}\right)^n\to 0$. That means $g(y)=g(0)$, and as such*

$$f(x)=-\frac{1}{4037}x^2+f(0)=-\frac{1}{4037}x^2+C$$

Solution 2 by Marian Ursărescu-Romania

More general: $1<a<b\Rightarrow f(ax)=f(bx)+x^2$, *let* $bx=t\Rightarrow x=\frac{t}{b}\Rightarrow$

$$f\left(\frac{a}{b}t\right)=f(t)+\frac{1}{b^2}t^2,\ \text{now}\ \frac{a}{b}=\alpha_1,\alpha\in(0,1)\Rightarrow$$

$$\left.\begin{aligned}f(\alpha t)-f(t)&=\frac{1}{b^2}t^2\\ f(\alpha^2t)-f(\alpha t)&=\frac{1}{b^2}\alpha^2t^2\\ &\vdots\\ f(\alpha^nt)-f(\alpha^{n-1}t)&=\frac{1}{b^2}\alpha^{2(n-1)}t^2\end{aligned}\right\}\Rightarrow$$

$$f(\alpha^4t)-f(t)=\frac{1}{b^2}t^2\left(1+\alpha^2+\dots+\alpha^{2(n-1)}\right)\Rightarrow$$

$$\lim_{n\to\infty}f(\alpha^nt)-f(t)=\lim_{n\to\infty}\frac{1}{b^2}t^2\frac{\alpha^{2n}-1}{\alpha^2-1}\Rightarrow$$

$$f\left(\lim_{n\to\infty}\alpha^nt\right)-f(t)=\frac{1}{b^2}t^2\frac{1}{1-\alpha^2}\Rightarrow$$

$$f(0)-f(t)=\frac{1}{b^2}\frac{t^2}{1-\frac{a^2}{b^2}}\Rightarrow f(0)-f(t)=\frac{t^2}{b^2-a^2}$$

Let $f(0)=c\Rightarrow f(t)=c-\frac{t^2}{(b-a)(b+a)}$

In our case $a=2018,b=2019$

$$f(x)=c-\frac{x^2}{4037}$$

Solution 3 by Remus Florin Stanca-Romania

$f\left(\frac{2018}{2019}x\right) = f(x) + \frac{x^2}{2019^2}$; ***let*** $\frac{2018}{2019} = \alpha, \alpha < 1$ ***so:***

$$f(\alpha x) = f(x) + x^2 \cdot \left(\frac{\alpha}{2018}\right)^2 \succ f(x) = (\alpha x) - x^2\left(\frac{\alpha}{2018}\right)^2 \quad \textit{(1)}$$

$$\Rightarrow f(x) = f(\alpha^2 x) - (\alpha x)^2\left(\frac{\alpha}{2018}\right)^2 - x^2\left(\frac{\alpha}{2018}\right)^2$$

We prove by using the Mathematical induction that:

$$f(x) = f(\alpha^n x) - \frac{x^2}{2018^2}\alpha^2((\alpha^2)^0 + \dots + (\alpha^2)^{n-1})\ \forall n \in \mathbb{N}^*:$$

1. we prove that $P(1)$: ***"***$f(x) = f(\alpha x) - \frac{x^2}{2018^2}\alpha^2(\alpha^2)^0$***" is true:***

$$\Leftrightarrow f(x) = f(\alpha x) - x^2\left(\frac{\alpha}{2018}\right)^2 \Leftrightarrow \textit{(1)} \succ \textit{it is true}$$

2. we suppose that $P(n)$: ***"***$f(x) = f(\alpha^n x) - \frac{x^2}{2018^2}\alpha^2((\alpha^2)^0 + \dots + (\alpha^2)^{n-1})$***" is true***

3. we prove that $P(n+1)$: ***"***$f(x) = f(\alpha^{n+1}x) - \left(\frac{x}{2018}\right)^2\alpha^2((\alpha^2)^0 + \dots + (\alpha^2)^n)$***" is true***

by using $P(n)$***:*** $f(x) = f(\alpha^n x) - \frac{x^2}{2018^2}\alpha^2((\alpha^2)^0 + \dots (\alpha^2)^{n-1})(a)$

$$\overset{(1)}{\succ} f(\alpha^n x) = f(\alpha^{n+1}x) - (\alpha^n x)^2\frac{\alpha^2}{2018^2}$$

$$\overset{(a)}{\Rightarrow} f(x) = f(\alpha^{n+1}x) - \frac{x^2\alpha^2}{2018^2}((\alpha^2)^0 + \dots + (\alpha^2)^{n-1} + (\alpha^2)^n) \Rightarrow$$

$\succ P(n+1)$ ***is true*** $\succ$ ***proved. So,*** $f(x) = f(\alpha^n x) - \frac{x}{2018^2}\alpha^2 \cdot \frac{(\alpha^2)^n - 1}{\alpha^2 - 1} \Rightarrow$

$$\Rightarrow \lim_{n\to\infty} f(x) = \lim_{n\to\infty}\left(f(\alpha^n x) - \frac{x^2}{2018^2}\alpha^2 \cdot \frac{(\alpha^n)^2 - 1}{\alpha^2 - 1}\right),$$

$\alpha < 1$ ***and*** f ***is continuous in the point*** $x = 0$

$$\Rightarrow f(x) = f(0) - \frac{x^2}{2018^2}\alpha^2\frac{1}{1-\alpha^2} = f(0) - \frac{x^2}{2019^2 - 2018^2} = f(0) - \frac{x^2}{4037}$$

$$\Rightarrow f(x) = f(0) - \frac{x^2}{4037}$$

439. Find all ROLLE functions $f: [0,1] \to \mathbb{R}$ such that:

$$\begin{cases} f(0) = f(1) = \dfrac{2019}{2018} \\ 2017f'(x) + 2018f(x) \le 2019, \forall x \in (0,1) \end{cases}$$

Proposed by Nguyen Van Canh-Vietnam

Solution 1 by Tran Hong-Vietnam

$$2017f'(x) + 2018f(x) \le 2019$$

$$\Leftrightarrow f'(x) + \frac{2018}{2017}f(x) \le \frac{2019}{2017}$$

$$\textit{Let } g(x) = e^{\frac{2018}{2017}x}\left[f(x) - \frac{2019}{2018}\right]$$

$$\Rightarrow g'(x) = e^{\frac{2018}{2017}x}\left[\frac{2018}{2017}\left\{f(x) - \frac{2019}{2018}\right\} + f'(x)\right] \le 0 \ \ \forall x \in (0,1);$$

$$\Rightarrow g(x) \searrow \textit{on } (0,1)$$

$$\textit{More: } g(0) = g(1) = 0 \Rightarrow g(x) = 0 \ \ \forall x \in (0,1)$$

$$\Rightarrow f(x) = \frac{2019}{2018}, \forall x \in [0,1]$$

Solution 2 by Ravi Prakash-New Delhi-India

$$2017f'(x) + 2018f(x) \le 2019$$

$$\Rightarrow f'(x) + \frac{2018}{2017}f(x) \le \frac{2019}{2017}$$

$$\textit{Multiplying both sides by } e^{\frac{2018x}{2017}} \textit{ to obtain:}$$

$$\frac{d}{dx}\left[e^{\frac{2018x}{2017}}f(x)\right] \le \frac{2019}{2017}e^{\frac{2018x}{2017}}$$

$$\Rightarrow \frac{d}{dx}\left[e^{\frac{2018x}{2017}}\left(f(x) - \frac{2019}{2018}\right)\right] \le 0$$

$$\Rightarrow F(x) = e^{\frac{2018x}{2017}}\left(f(x) - \frac{2019}{2018}\right) \textit{ decreases on } [0,1]$$

$$\textit{But } F(0) = F(1) = 0$$

$$\therefore F(x) \textit{ must be constant on } [0,1]$$

$$\Rightarrow F(x) = F(0) = 0 \Rightarrow f(x) = \frac{2019}{2018}, \forall x \in [0,1]$$

440. If

$$\int_0^1\left(\int_0^t \frac{ln(x)}{1-x^5}dx\right)dt=\frac{A\Psi_1\left(\frac{1}{5}\right)+B\Psi_1\left(\frac{2}{5}\right)}{C}$$

Then prove that

$$25(A-B)+2C=0$$

where $\Psi_n(x)$ is Poly – Gamma function.

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Shafiqur Rahman-Bangladesh

$$\int_0^1\left(\int_0^t \frac{\ln x}{1-x^5}dx\right)dt=\left[t\int_0^t \frac{\ln x}{1-x^5}dx\right]_0^1-\int_0^1 \frac{t\ln t}{1-t^5}dt=\int_0^1\frac{(1-t)\ln t}{1-t^5}dt\left[t\to t^{\frac{1}{5}}\right]=$$

$$=\frac{1}{25}\left(\int_0^1\frac{t^{\frac{1}{5}-1}\ln t}{1-t}dt-\int_0^1\frac{t^{\frac{2}{5}}\ln t}{1-t}dt\right)$$

$$\therefore \int_0^1\left(\int_0^t \frac{\ln x}{1-x^5}dx\right)dt=\frac{-\Psi_1\left(\frac{1}{5}\right)+\Psi_1\left(\frac{2}{5}\right)}{25}=\frac{A\Psi_1\left(\frac{2}{5}\right)+B\Psi_1\left(\frac{1}{5}\right)}{C}$$

$$\therefore 25(A-B)+2C=-25\times 2+2\times 50=0$$

Solution 2 by Khalef Ruhemi-Jarash-Jordan

$$I:=\int_0^1\int_0^t\frac{\ln(x)\,dx}{1-x^5}dt=\int_0^1\int_x^1\frac{\ln(x)}{1-x^5}dt\,dx=\int_0^1\frac{(1-x)\ln(x)\,dx}{1-x^5}$$

But $\frac{1}{1-x^5}=\sum_{n=0}^{\infty}x^{5n}, |x|<1$

$$\therefore \frac{1-x}{1-x^5}=\sum_{n=0}^{\infty}(1-x)x^{5n}=\sum_{n=0}^{\infty}x^{5n}-x^{5n+1}\ \therefore I=\sum_{n=0}^{\infty}\int_0^1\left(x^{5n}-x^{5n-1}\right)\ln(x)\,dx$$

But $\int_0^1 x^p\ln(x)\,dx=\frac{-1}{(1+p)^2}$

$$\therefore I=\sum_{n=0}^{\infty}\frac{1}{(5n+2)^2}-\frac{1}{(5n+1)^2}=\sum_{n=1}^{\infty}\frac{1}{(5n-3)^2}-\frac{1}{(5n-4)^2}=$$

$$=\left(\frac{1}{25}\right)\left[\sum_{n=1}^{\infty}\frac{1}{\left(n-\frac{3}{5}\right)^2}-\sum_{n=1}^{\infty}\frac{1}{\left(n-\frac{4}{5}\right)^2}\right]=\left(\frac{1}{25}\right)\left[\Psi_1\left(\frac{2}{5}\right)-\Psi_1\left(\frac{1}{5}\right)\right]=\frac{\Psi\left(\frac{2}{5}\right)-\Psi_1\left(\frac{1}{5}\right)}{25}$$

$$\therefore A=-1,B=+1,C=25$$

$$\therefore 25(A-B)+2C=(25)(-1-1)+2(25)=-50+50=0$$

Solution 3 by Shivam Sharma-New Delhi-India

Applying $I\cdot B\cdot P$, *we get,* $I=\int_0^1\frac{(1-t)\ln(t)}{1-t^5}dt$. *Let,* $t^5=y\Rightarrow t=y^{\frac{1}{5}}$

$$dt=\frac{1}{5}y^{\frac{1}{5}-1}dy\Rightarrow\frac{1}{25}\left[\int_0^1\frac{\ln(y)\,y^{\frac{1}{5}-1}}{1-y}dy-\int_0^1\frac{\ln(y)\,y^{\frac{2}{5}-1}}{1-y}dy\right]$$

$$\Rightarrow\frac{1}{25}\left[\sum_{n=0}^{\infty}\left(\int_0^1 y^{n-\frac{1}{5}}\ln(y)\,dy-\int_0^1 y^{n-\frac{3}{5}}\ln(y)\,dy\right)\right]$$

$$\Rightarrow\frac{1}{25}\left[\sum_{n=0}^{\infty}\left(\frac{y^{n+\frac{1}{5}}\ln(y)}{\left(n+\frac{1}{5}\right)}-\frac{y^{n+\frac{1}{5}}}{\left(n+\frac{1}{5}\right)^2}\right)-\sum_{n=0}^{\infty}\left(\frac{y^{n+\frac{2}{5}}\ln(y)}{\left(n+\frac{2}{5}\right)}-\frac{y^{n+\frac{2}{5}}}{\left(n+\frac{2}{5}\right)^2}\right)\right]_0^1\Rightarrow$$

$$\Rightarrow\frac{1}{25}\left[\sum_{n=0}^{\infty}\left(\frac{1}{\left(n+\frac{2}{5}\right)^2}-\frac{1}{\left(n+\frac{1}{5}\right)^2}\right)\right]\Rightarrow\frac{1}{25}\left[\Psi_1\left(\frac{2}{5}\right)-\Psi\left(\frac{1}{5}\right)\right]$$

$$\textit{(OR) } I=\frac{1}{25}\left[\Psi\left(\frac{2}{5}\right)-\Psi\left(\frac{1}{5}\right)\right]$$

Now, $A=-1,B=+1,C=25$. *Now,*

$25(A-B)+2C=25(-1-1)+2(25)$ ***(OR)*** $25(A-B)+2C=0$ ***(proved)***

441. **Find:**

$$\Omega=\int_0^{\infty}\frac{1}{(1+x^2)(3-\cos x)}dx$$

Proposed by Zaharia Burghelea-Romania

Solution by Kartick Chandra Betal-India

$$\frac{1-a^2}{1-2a\cos x+a^2}=2\sum_{k=0}^{\alpha}a^k\cos kx+I$$

$$\therefore\int_0^{\alpha}\frac{dx}{(1+x^2)(3-\cos x)}=\frac{I}{3}\int_0^{\alpha}\frac{dx}{(1+x^2)\left(1-\frac{\cos x}{3}\right)}=\frac{I}{3}\int_0^{\alpha}\frac{dx}{(1+x^2)\left(1-\frac{2a\cos x}{1+a^2}\right)}$$

$$\frac{2a}{1+a^2}=\frac{1}{3}\ \textbf{or}\ a^2-6a+1=0$$

$$a=\frac{6-\sqrt{36-4}}{2}=3-2\sqrt{2}$$

$$=\frac{(1+a^2)}{3}\int_0^{\alpha}\frac{dx}{(1+x^2)(1+a^2-2a\cos x)}=\frac{1+a^2}{3}\cdot\frac{1}{1-a^2}\int_0^{\alpha}\left[2\sum_{k=0}^{\alpha}a^k\cos kx+I\right]\frac{dx}{1+x^2}$$

$$=\frac{2(1+a^2)}{3(1-a^2)}\sum_{k=0}^{\alpha}a^k\int_0^{\alpha}\frac{\cos kx}{1+x^2}dx+\frac{1+a^2}{3(1-a^2)}\int_0^{\alpha}\frac{1}{1+x^2}dx=$$

$$=\frac{2(1+a^2)}{3(1-a^2)}\sum_{k=0}^{\alpha}\frac{\pi}{2}a^ke^{-k}+\frac{1+a^2}{3(1-a^2)}\frac{\pi}{2}=\frac{2(1+a^2)}{3(1-a^2)}\cdot\frac{\pi}{2\left(1-\frac{a}{e}\right)}+\frac{\pi}{6}\left(\frac{1+a^2}{1-a^2}\right)=$$

$$=\frac{(1+a^2)}{(1-a^2)}\cdot\frac{\pi}{6}\left[\frac{2}{1-\frac{a}{e}}+I\right]=\frac{\pi}{6}\cdot\frac{(1+a^2)}{(1-a^2)}\left[\frac{2+1-\frac{a}{e}}{1-\frac{a}{e}}\right]=\frac{\pi}{6}\cdot\frac{(1+a^2)}{(1-a^2)}\left[\frac{3e-a}{e-a}\right]=$$

$$=\frac{\pi}{6}\left[\frac{1+(3-2\sqrt{2})^2}{1-(3-2\sqrt{2})^2}\right]\left[\frac{3e-3+2\sqrt{2}}{e-3+2\sqrt{2}}\right]=\frac{\pi}{6}\left[\frac{18-12\sqrt{2}}{12\sqrt{2}-16}\right]\left(\frac{3e+2\sqrt{2}-2}{2\sqrt{2}+e-3}\right)=$$

$$=\frac{\pi}{12}\cdot\frac{(9-6\sqrt{2})}{(3\sqrt{2}-4)}\left(\frac{3e+2\sqrt{2}-2}{2\sqrt{2}+e-3}\right)$$

442. Find:

$$\Omega=\int\left(\sum_{n=1}^{\infty}3^n\sinh^3\frac{x}{3^n}\right)dx$$

Proposed by Daniel Sitaru – Romania

Solution by Shafiqur Rahman-Bangladesh

$$\int\left(\sum_{n=1}^{\infty}3^n\sinh^3\left(\frac{x}{3^n}\right)\right)dx=\frac{1}{4}\int\left(\sum_{n=1}^{\infty}3^n\sinh\left(\frac{x}{3^{n-1}}\right)-3^{n+1}\sinh\left(\frac{x}{3^n}\right)\right)dx=$$

$$=\frac{1}{4}\int(3\sinh(x)-3x)\,dx$$

$$\therefore\int\left(\sum_{n=1}^{\infty}3^n\sinh^3\left(\frac{x}{3^n}\right)\right)dx=\frac{3}{4}\left(\cosh(x)-\frac{x^2}{2}\right)+C$$

443. Find:

$$\Omega=\int\frac{2x^4+5x^3+6x^2+6x+12}{(x^2+2x+2)\sqrt{x^2+2x+2}}dx,\ x\in\mathbb{R}$$

Proposed by Abdul Mukhtar-Nigeria

Solution by Ravi Prakash-New Delhi-India

$$2x^2+x$$

$$x^2+2x+2\overline{\big)\,2x^4+5x^3+6x^2+6x+12}$$

$$2x^4+4x^3+4x^2$$

$$\ominus\quad\ominus\quad\ominus$$

$$x^3+2x^2+6x+12$$

$$x^3+2x^2+2x$$

$$\ominus\quad\ominus\quad\ominus$$

$$4x+12$$

$$\therefore I=\int\frac{2x^2+x}{\sqrt{x^2+2x+2}}dx+\int\frac{4x+12}{(x^2+2x+2)^{\frac{3}{2}}}=I_1+I_2$$

Where

$$I_1=\int\frac{2x^2+4x+4-3x-4}{\sqrt{x^2+2x+2}}dx=2\int\sqrt{x^2+2x+2}\,dx-\int\frac{3x+4}{\sqrt{x^2+2x+2}}dx$$

Put $x+1=t$

$$I_1 = 2\int\sqrt{t^2+1}\,dt - \int\frac{3(t-1)+4}{\sqrt{t^2+1}}dt = 2\left[\frac{1}{2}t\sqrt{t^2+1}+\frac{1}{2}\ln\left(t+\sqrt{t^2+1}\right)\right] -$$

$$-3\int\frac{t}{\sqrt{t^2+1}} - \int\frac{dt}{\sqrt{t^2+1}}$$

$$= t\sqrt{t^2+1} + \ln\left(t+\sqrt{t^2+1}\right) - 3\sqrt{t^2+1} - \ln\left(t+\sqrt{t^2+1}\right) =$$

$$= (t-3)\sqrt{t^2+1} = (x-2)\sqrt{x^2+2x+2}$$

$$I_2 = \int\frac{2x+2}{(x^2+2x+2)^{\frac{3}{2}}}dx + 4\int\frac{dx}{(x^2+2x+2)^{\frac{3}{2}}} = -\frac{2}{\sqrt{x^2+2x+2}} + 4I_3$$

where $I_3 = \int\frac{dx}{[(x+1)^2+1]^{\frac{3}{2}}}$. *Put* $x+1 = \tan\theta$. $I_3 = \int\frac{\sec^2\theta d\theta}{(\tan^2\theta+1)^{\frac{3}{2}}} = \int\cos\theta\,d\theta = \sin\theta =$

$$= \frac{\tan\theta}{\sec\theta} = \frac{\tan\theta}{\sqrt{\tan^2\theta+1}} = \frac{x+1}{\sqrt{x^2+2x+1}}$$

Thus, $I = (x-2)\sqrt{x^2+2x+2} + \frac{4+8x}{\sqrt{x^2+2x+2}} + C$

444. Find:

$$\Omega = \int\frac{tanh(x)}{1+e^{3x}}dx, x\in\mathbb{R}$$

Proposed by Ekpo Samuel-Nigeria

Solution by Ravi Prakash-New Delhi-India

$$\Omega = \int\frac{\tanh(x)}{1+e^{3x}}dx = \int\frac{e^{2x}-1}{(e^{2x}+1)(e^{3x}+1)}dx$$

Put $e^x = t, e^x dx = dt$

$$\Omega = \int\frac{t^2-1}{(t^2+1)(t^3+1)}dt = \int\frac{t-1}{(t^2+1)(t^2-t+1)t}dt$$

$$\frac{t-1}{(t^2+1)(t^2-t+1)} \equiv \frac{A}{t} + \frac{Bt+C}{t^2+1} + \frac{Dt+E}{t^2-t+1} \Rightarrow t-1 = A(t^2+1)(t^2-t+1) +$$

$$+(Bt+c)t(t^2-t+1) + D(Dt+E)t(t^2+1)$$

Put $t=0$; $-1 = A \Rightarrow A = -1$. *Put* $t = i; i-1 = (Bi+C)i(-i) = Bi+C \Rightarrow$

$\Rightarrow B = 1, C = -1$. *Compare coefficient of* t^n; $D = A+B+D \Rightarrow D = 0$. *Put* $t = -\omega$,

($\omega \neq 1$ is cube root of unity)

$$-\omega - 1 = E(-\omega)(\omega^2 + 1) = E\omega^2 \Rightarrow \omega^2 = E\omega^2 \Rightarrow E = 1$$

Thus, $\Omega = \int \left[-\frac{1}{t} + \frac{t-1}{t^2+1} + \frac{1}{t^2-t+1}\right] dt = -\ln t + \frac{1}{2}\ln(t^2+1) - \tan^{-1}(t) +$

$$+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C = -x + \frac{1}{2}\ln(e^{2x}+1) - \tan^{-1}(e^x) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2e^x-1}{\sqrt{3}}\right) + C$$

445. If

$$\Psi(m) = \int_0^1 x^2\, ln(x)\, ln(x+m)\, dx$$

then show that

$$\int_0^\infty \frac{\Psi(m)}{1+m^2} dm = \frac{1}{9} + \frac{\pi G}{6} - \frac{G}{9} - \frac{\zeta(3)}{16} - \frac{5\pi}{27} + \frac{5\pi^2}{432} - \frac{\pi\, ln(2)}{36}$$

where G is Catalan's constant.

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Khalef Ruhemi-Jarash-Jordan

$$\Psi(m) := \int_0^1 x^2 \ln(x)\ln(m+x)\, dx$$

Find $\int_0^1 \frac{\Psi(m)dm}{1+m^2}$

Let $I := \int_0^\infty \frac{\Psi(m)dm}{1+m^2} = \int_0^1 \left(x^2\ln(x)\int_0^1 \frac{\ln(m+x)}{1+m^2} dm\right) dx = I$ *(1)*

Let $F(\beta) := \int_0^\infty \frac{\ln(\beta+x)dx}{1+x^2}, \beta \geq 0 \Rightarrow F'(\beta) = \int_0^\infty \frac{dx}{(1+x^2)(\beta+x)}$

$$F'(\beta) = \left(\frac{1}{1+\beta^2}\right)\int_0^\infty \left(\frac{\beta}{1+x^2} + \frac{1}{\beta+x} - \frac{x}{1+x^2}\right) dx =$$

$$= \left(\frac{1}{1+\beta^2}\right)\left[\beta\tan^{-1}(x)\Big|_0^\infty + \frac{1}{2}\ln\left(\frac{\beta^2+x^2+2\beta x}{1+x^2}\right)\Big|_0^\infty\right] =$$

$$=\left(\frac{1}{1+\beta^2}\right)\left(\frac{\pi\beta}{2}-\ln(\beta)\right), \textit{But } f(0)=0$$

$$\therefore F(\beta)=\left(\frac{\pi}{2}\right)\int_0^\beta \frac{ydy}{1+y^2}-\int_0^\beta \frac{\ln(y)\,dy}{1+y^2}=\frac{\pi}{4}\ln(1+\beta^2)-\int_0^\beta \frac{\ln(y)\,dy}{1+y^2}$$

$$\therefore I=\int_0^1\left(x^2\ln(x)\left(\frac{\pi}{4}\ln(1+x^2)-\int_0^x \frac{\ln(y)\,dy}{1+y^2}\right)\right)dx=$$

$$=\frac{\pi}{4}\int_0^1 x^2\ln(x)\ln(1+x^2)dx-\int_0^1\int_0^x \frac{\ln(y)\cdot x^2\ln(x)}{1+y^2}dydx$$

$$=\frac{\pi}{4}\int_0^1 x^2\ln(x)\ln(1+x^2)dx-\int_0^1\left(\frac{\ln(y)}{1+y^2}\int_y^1 x^2\ln(x)\,dx\right)dy$$

$$=\frac{\pi}{4}\int_0^1 x^2\ln(x)\ln(1+x^2)dx+\int_0^1 \frac{\ln(x)}{1+x^2}\left(\frac{1}{9}+\frac{x^3}{3}\ln(x)-\frac{x^3}{9}\right)dx$$

$$\therefore I=\frac{\pi}{4}\int_0^1 x^2\ln(x)\ln(1+x^2)\,dx+\frac{1}{9}\int_0^1\frac{\ln(x)dx}{1+x^2}-\frac{1}{9}\cdot\int_0^1\frac{x^3\ln(x)dx}{1+x^2}+\frac{1}{3}\int_0^1\frac{x^3\ln^2(x)dx}{1+x^2}\quad (1)$$

Let $I=\int_0^1 x^2\ln(x)\ln(1+x^2)\,dx$***, integrating by parts***

$$I_1=\left(\frac{x^3}{3}\ln(x)-\frac{x^3}{9}\right)\ln(1+x^2)\Big|_0^1-\int_0^1\left(\frac{x^3}{3}\ln(x)-\frac{x^3}{9}\right)\frac{2x}{1+x^2}dx$$

$$\Rightarrow I_1=-\frac{1}{9}\ln(2)+\left(\frac{2}{9}\right)\int_0^1\frac{x^4dx}{1+x^2}-\frac{2}{3}\int_0^1\frac{x^4\ln(x)\,dx}{1+x^2}$$

$$I_1=-\frac{1}{9}\ln(2)+\left(\frac{2}{9}\right)\int_0^1\left(x^2-1+\frac{1}{1+x^2}\right)dx-\left(\frac{2}{3}\right)\int_0^1\left((x^2-1)\ln(x)+\frac{\ln(x)}{1+x^2}\right)dx$$

$$I_1=-\frac{1}{9}\ln(2)+\left(\frac{2}{9}\right)\left(\frac{1}{3}-1+\frac{\pi}{4}\right)-\left(\frac{2}{3}\right)\left(\frac{8}{9}\right)-\left(\frac{2}{3}\right)\int_0^1\frac{\ln(x)\,dx}{1+x^2}$$

$$\therefore I_1=-\frac{1}{9}\ln(2)-\frac{4}{27}+\frac{\pi}{18}-\frac{16}{27}+\frac{2}{3}G\therefore I_1=-\frac{20}{27}+\frac{\pi}{18}-\frac{1}{9}\ln(2)+\frac{2}{3}G$$

$$I_2 := \int_0^1 \frac{\ln(x)\,dx}{1+x^2} = -G \Rightarrow I_2 = -G$$

$$I_3 := \int_0^1 \frac{x^3\ln(x)\,dx}{1+x^2} = \int_0^1 \left(x - \frac{x}{1+x^2}\right)\ln(x)\,dx = \int_0^1 x\ln(x)\,dx - \int_0^1 \frac{x\ln(x)dx}{1+x^2}$$

$$= \frac{x^2}{2}\ln(x) - \frac{x^2}{4}\Big|_0^1 - \left(\frac{1}{2}\ln(x)\ln(1+x^2)\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{\ln(1+x^2)}{x}dx\right) =$$

$$= -\frac{1}{4} + \frac{1}{2}\int_0^1 \frac{\ln(1+x^2)dx}{x} \rightarrow \textit{let } x^2 = y \Rightarrow \frac{dx}{x} = \frac{dy}{2y}$$

$$\therefore I_3 = -\frac{1}{4} + \frac{1}{4}\int_0^1 \frac{\ln(1+x)\,dx}{x} = -\frac{1}{4} + \frac{1}{4}\left(\frac{\pi^2}{12}\right) \therefore I_3 = -\frac{1}{4} + \frac{\pi^2}{48}$$

$$\textit{Let } I_4 := \int_0^1 \frac{x^3\ln^2(x)dx}{1+x^2} = \int_0^1 x\ln^2(x)\,dx - \int_0^1 \frac{x\ln^2(x)dx}{1+x^2} =$$

$$= \frac{1}{4} - \left(\frac{1}{2}\ln^2(x)\ln(1+x^2)\Big|_0^1 - 2\int_0^1 \frac{1}{2}\frac{\ln(1+x^2)\ln(x)}{x}dx\right) =$$

$$= \frac{1}{4} + \int_0^1 \frac{\ln(1+x^2)\ln(x)}{x}dx \rightarrow \textit{let } x^2 = y \Rightarrow x = y^{\frac{1}{2}}, \frac{dx}{x} = \frac{dy}{2y}$$

$$= \frac{1}{4} + \frac{1}{4}\int_0^1 \frac{\ln(1+x)\ln(x)}{x}dx = \frac{1}{4} + \left(\frac{1}{4}\right)\left[-\ln(x)\,Li_2(-x)\Big|_0^1 + \int_0^1 +Li_2\frac{(-x)}{x}dx\right]$$

$$= \frac{1}{4} + \left(\frac{1}{4}\right)\int_0^1 Li_2\frac{(-x)}{x}dx = \frac{1}{4} + \frac{1}{4}Li_3(-1) = \frac{1}{4} - \frac{3}{16}\zeta(3)$$

$$\therefore I_4 = \frac{1}{4} - \frac{3}{16}\zeta(3) \therefore I = \left(\frac{\pi}{4}\right)\left(-\frac{20}{27} + \frac{\pi}{18} - \frac{1}{9}\ln(2) + \frac{2}{3}G\right) - \frac{1}{9}G$$

$$-\frac{1}{9}\left(-\frac{1}{4} + \frac{\pi^2}{48}\right) + \frac{1}{3}\left(\frac{1}{4} - \frac{3}{16}\zeta(3)\right)$$

$$\therefore I = -\frac{5\pi}{27} + \frac{\pi^2}{72} - \frac{\pi\ln(2)}{36} + \frac{\pi G}{6} - \frac{1}{9}G$$

$$+\frac{1}{36} - \frac{\pi^2}{(48)(9)} + \frac{1}{12} - \frac{\zeta(3)}{16}$$

$$I=\frac{1}{9}+\frac{\pi G}{6}-\frac{G}{9}-\frac{\zeta(3)}{16}-\frac{5\pi}{27}+\frac{5\pi^2}{432}-\frac{\pi\ln(2)}{36}$$

446. Find:

$$\Omega=\int_{-\beta}^{\beta}\frac{dx}{\sqrt[3]{(\beta-x)(\beta^2-x^2)}},\ \beta>0$$

Proposed by Abdul Mukhtar-Nigeria

Solution 1 by Sagar Kumar-Kolkata-India

$$\Omega=\int_{-\beta}^{\beta}\frac{dx}{(\beta+x)^{\frac{2}{3}}(\beta+x)^{\frac{1}{3}}}$$

Put $x=\beta\cos(2\theta);\ dx=-2\beta\sin(2\theta)\,d\theta$

$$\Omega=\int_{\frac{\pi}{2}}^{0}\frac{-2\beta\sin 2\theta\, d\theta}{\beta(1-\cos 2\theta)^{\frac{2}{3}}(1+\cos 2\theta)^{\frac{1}{3}}}$$

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{4\sin\theta\cos\theta d\theta}{(2)\sin^{\frac{4}{3}}\theta\cos^{\frac{2}{3}}\theta},\ \Omega=\int_0^{\frac{\pi}{2}}2\sin^{-\frac{1}{3}}\theta\cos^{\frac{1}{3}}\theta\, d\theta$$

$\Omega=\beta\left(\frac{1}{3},\frac{2}{3}\right)=\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)$. *Using reflection formula*

$$\Gamma(m)\Gamma(1-m)=\pi\csc(m\pi)\Rightarrow\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)=\pi\csc\left(\frac{2\pi}{3}\right)=\frac{2\pi}{\sqrt{3}}$$

Solution 2 by Shafiqur Rahman-Bangladesh

$$\int_{-\beta}^{\beta}\frac{dx}{\sqrt[3]{(\beta-x)(\beta^2-x^2)}}\left[\frac{\beta-x}{\beta+x}\to x\right]=\int_0^{\infty}\frac{x^{\frac{1}{3}-1}}{1+x}dx=\frac{\pi}{\sin\left(\frac{\pi}{3}\right)}$$

$$\therefore\int_{-\beta}^{\beta}\frac{dx}{\sqrt[3]{(\beta-x)(\beta^2-x^2)}}=\frac{2\pi}{\sqrt{3}}$$

447. Find:

$$\Omega = \int_1^{16} \tan^{-1}\left(\sqrt[4]{x-1}\right) dx$$

Proposed by Abdul Mukhtar-Nigerie

Solution 1 by Sagar Kumar-Kolkata-India

$$I = \int_1^{16} \tan^{-1}\left((x-1)^{\frac{1}{4}}\right) dx$$

$$\textit{Put } x-1 = t^4; dx = 4t^3 dt; I = \int_1^{(15)^{\frac{1}{4}}} 4\tan^{-1}(t)\, t^3 dt$$

$$I = 4\left(\frac{\tan^{-1}(t)(t^4)}{4}\right)\Bigg|_0^{15^{\frac{1}{4}}} - \int_0^{(15)^{\frac{1}{4}}} \frac{t^4}{t^2+1} dt$$

$$I = \tan^{-1}\left(15^{\frac{1}{4}}\right) 15 - I_1$$

$$I_1 = \int_0^{(15)^{\frac{1}{4}}} \frac{t^4-1}{t^2+1} + \frac{1}{t^2+1} dt$$

$$I_1 = \frac{t^3}{3} - t + \tan^{-1}(t)\Bigg|_0^{15^{\frac{1}{4}}} = \frac{(15)^{\frac{3}{4}}}{3} - (15)^{\frac{1}{4}} + \tan^{-1}(15)^{\frac{1}{4}}$$

$$I = 14\tan^{-1}\left(15^{\frac{1}{4}}\right) + (15)^{\frac{1}{4}} - \frac{(15)^{\frac{3}{4}}}{3}$$

Solution 2 by Serban George Florin-Romania

$$\Omega = \int_1^{16} \arctan\sqrt[4]{x-1}\, dx = \int_0^{\sqrt[4]{15}} 4t^3 \arctan t\, dt = \int_0^{\sqrt[4]{15}} (t^4)' \arctan t\, dt$$

$$x-1 = t^4, dx = 4t^3 dt$$

$$\Omega = t^4 \arctan t\Big|_0^{\sqrt[4]{15}} - \int_0^{\sqrt[4]{15}} \frac{t^4}{t^2+1} dt = 15\arctan\sqrt[4]{15} - 0 - \int_0^{\sqrt[4]{15}} \frac{t^4-1+1}{t^2+1} dt$$

$$\Omega = 15\arctan\sqrt[4]{15} - \int_0^{\sqrt[4]{15}} \left(t^2 - 1 + \frac{1}{t^2+1}\right) dt$$

$$\Omega = 15\arctan\sqrt[4]{15} - \left(\frac{t^3}{3} - t + \arctan t\right)\Bigg|_0^{\sqrt[4]{15}}$$

$$\Omega = 15\arctan\sqrt[4]{15} - \frac{\sqrt[4]{15^3}}{3} + \sqrt[4]{15}\cdot\arctan\sqrt[4]{15}$$

$$\Omega = 14\arctan\sqrt[4]{15} - \frac{\sqrt[4]{15^3}}{3} + \sqrt[4]{15}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$\Omega = \int_1^{16} \tan^{-1}\sqrt[4]{x-1}\,dx$, *let* $z^4 = x - 1 \Rightarrow 4z^3dz = dx$ *when* $x = 1, z = 0$; *when* $x = 16, z = \sqrt[4]{15}$

$$\Omega = \int_0^{\sqrt[4]{15}} 4z^3\tan^{-1}z\,dz = [z^4\tan^{-1}z]_{z=0}^{z=\sqrt[4]{15}} - \int_0^{\sqrt[4]{15}} \left[\frac{d(\tan^{-1}z)}{dz}\int 4z^3dz\right]dz$$

$$15\tan^{-1}\sqrt[4]{15} - \int_0^{\sqrt[4]{15}} \frac{z^4}{1+z^2}dz = 225\tan^{-1}\sqrt[4]{15} - \int_0^{\sqrt[4]{15}} (z^2-1)\,dz - \int_0^{\sqrt[4]{15}} \frac{dz}{1+z^2}$$

$$15\tan^{-1}\sqrt[4]{15} - \left[\frac{z^3}{3} - z\right]_{z=0}^{z=\sqrt[4]{15}} - [\tan^{-1}z]_{z=0}^{z=\sqrt[4]{15}}$$

$$= 14\tan^{-1}\sqrt[4]{15} - \frac{(15)^{\frac{3}{4}}}{3} + \sqrt[4]{15} \quad \textit{(Answer)}$$

Solution 4 by Yen Tung Chung-Taichung-Taiwan

Let $y = \sqrt[4]{x-1} \Rightarrow x = y^4 + 1, dx = 4y^3dy$

$$\int \tan^{-1}(\sqrt[4]{x-1})\,dx = \int \tan^{-1}y\cdot 4y^3dy \begin{pmatrix} u = \tan^{-1}y & dv = 4y^3dy \\ du = \dfrac{1}{1+y^2}dy & v = y^4 \end{pmatrix}$$

$$= y^4\tan^{-1}y - \int \frac{y^4}{1+y^2}dy = y^4\tan^{-1}y - \int\left(y^2 - 1 + \frac{1}{1+y^2}\right)dy =$$

$$= y^4\tan^{-1}y - \frac{1}{3}y^3 + y - \tan^{-1}y + C$$

$$= (x-2)\tan^{-1}\left(\sqrt[4]{x-1}\right) - \frac{1}{3}\left(\sqrt[4]{x-1}\right)^3 + \sqrt[4]{x-1} + C$$

So $\int_1^{16} \tan^{-1}\left(\sqrt[4]{x-1}\right) dx = \left[(x-2)\tan^{-1}\left(\sqrt[4]{x-1}\right) - \frac{1}{3}\left(\sqrt[4]{x-1}\right)^3 + \sqrt[4]{x-1}\right]\Big|_1^{16}$

$$= 14\tan^{-1}\left(\sqrt[4]{15}\right) - \frac{\left(\sqrt[4]{15}\right)^3}{3} + \sqrt[4]{15}$$

448. A note on the special Logarithmic Integral

Let's define the function $\chi(m)$ for any complex number m, $Re(m) > 0$

$$\chi(m) = \int_0^1 x\ln(x)\ln(x+m)\,dx$$

then we have the following,

$$\int_0^1 \chi\left(\frac{1}{m}\right) dm = \frac{3}{2} - \frac{\pi^2}{24} - 2\ln(2)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Togrul Ehmedov-Baku-Azerbaijan

$$I = \int_0^1 \chi\left(\frac{1}{m}\right) dm = \int_0^1\int_0^1 x\ln x\ln\left(x+\frac{1}{m}\right) dx\,dm = \int_0^1\int_0^1 x\ln x\ln(mx+1)\,dx\,dm - \int_0^1\int_0^1 x\ln x\ln(m)\,dx\,dm$$

$$I_1 = \int_0^1\int_0^1 x\ln x\ln(mx+1)\,dx\,dm = \int_0^1\int_1^{x+1} \ln x\ln(t)\,dt\,dx = \left[(x\ln x - x)\int_1^{x+1}\ln(t)\,dt\right]_0^1 - \int_0^1 (x\ln x - x)\ln(x+1)\,dx$$

$$= -\int_1^2 \ln(t)\,dt - \int_0^1 x\ln x\ln(x+1)\,dx + \int_0^1 x\ln(x+1)\,dx = 1 - 2\ln 2 - \int_0^1 x\ln x\ln(x+1)\,dx + \frac{1}{4}$$

$$= \frac{5}{4} - 2\ln 2 - \int_0^1 x\ln x\ln(x+1)\,dx$$

$$I_1 = \int_0^1 x\ln x\ln(x+1)\,dx = \int_0^1 x\ln x\sum_{k=1}^{\infty}(-1)^{k-2}\frac{x^k}{k}\,dx = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^1 x^{k+1}\ln x\,dx$$

$$=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left[\frac{x^{k+2}}{k+2}\ln x-\frac{x^{k+2}}{(k+2)^2}\right]_0^1=-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k(k+2)^2}=-\frac{1}{4}\sum_{k=1}^{\infty}(-1)^{k-1}\left[\frac{1}{k}-\frac{1}{k+2}-\frac{2}{(k+2)^2}\right]$$

$$=-\frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}+\frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k+2}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k+2)^2}=-\frac{1}{4}\ln 2+\frac{1}{4}\left(\ln 2-\frac{1}{2}\right)+\frac{1}{2}\left(\frac{\pi^2}{12}-\frac{3}{4}\right)=\frac{\pi^2}{24}-\frac{1}{2}$$

$$I_1=\frac{5}{4}-2\ln 2-\left(\frac{\pi^2}{24}-\frac{1}{2}\right)=\frac{7}{4}-2\ln 2-\frac{\pi^2}{24}$$

$$I_2=\int_0^1\int_0^1 x\ln x\ln(m)\,dx\,dm=-\int_0^1 x\ln x\,dx=\frac{1}{4}$$

$$I=\int_0^1\chi\left(\frac{1}{m}\right)dm=\int_0^1\int_0^1 x\ln x\ln\left(x+\frac{1}{m}\right)dx\,dm=\frac{7}{4}-2\ln 2-\frac{\pi^2}{24}-\frac{1}{4}=\frac{3}{2}-2\ln 2-\frac{\pi^2}{24}$$

449. Find:

$$\Omega=\int\frac{x^4e^xdx}{(x^4+4x^3+12x^2+24x+24+72e^x)^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Shafiqur Rahman-Bangladesh

$$\int\frac{x^4e^xdx}{(x^4+4x^3+12x^2+24x+24+72e^x)^2}=$$

$$=\int\frac{x^4e^{-x}dx}{(e^{-x}(x^4+4x^3+12x^2+24x+24)+72)^2}=$$

$$=-\int\frac{d(e^{-x}(x^4+4x^3+12x^2+24x+24)+72)}{(e^{-x}(x^4+4x^3+12x^2+24x+24)+72)^2}$$

$$\therefore\int\frac{x^4e^xdx}{(x^4+4x^3+12x^2+24x+24+72e^x)^2}=$$

$$=\frac{1}{e^{-x}(x^4+4x^3+12x^2+24x+24)+72}+C=$$

$$=\frac{e^x}{x^4+4x^3+12x^2+24x+24+72e^x}+C$$

Solution 2 by Shivam Sharma-New Delhi-India

$$\int \frac{(x^4e^x)dx}{e^{2x}(x^4e^{-x}+4x^3e^{-x}+12x^2e^{-x}+24xe^{-x}+24e^x+72)^2} \Rightarrow$$

$$\Rightarrow \int \frac{(x^4e^{-x})dx}{(x^4e^{-x}+4x^3e^{-x}+12x^2e^{-x}+24xe^{-x}+24e^{-x}+72)^2}$$

Let,

$$x^4e^{-x}+4x^3e^{-x}+12x^2e^{-x}+24xe^{-x}+24e^{-x}+72=u$$

$$-(x^4e^{-x})dx=dy \Rightarrow -\int \frac{du}{u^2} \Rightarrow \frac{1}{u}+C$$

(OR)

$$\Omega=\frac{e^x}{(x^4+4x^3+12x^2+24x+24+72e^x)}+C \text{ (Answer)}$$

450. Find:

$$\Omega=\int \frac{tanh(x)}{1+e^{3x}}dx, x \in \mathbb{R}$$

Proposed by Ekpo Samuel-Nigeria

Solution by Ravi Prakash-New Delhi-India

$$\Omega=\int \frac{\tanh(x)}{1+e^{3x}}dx=\int \frac{e^{2x}-1}{(e^{2x}+1)(e^{3x}+1)}dx$$

Put $e^x=t, e^xdx=dt$

$$\Omega=\int \frac{t^2-1}{(t^2+1)(t^3+1)}dt=\int \frac{t-1}{(t^2+1)(t^2-t+1)t}dt$$

$$\frac{t-1}{(t^2+1)(t^2-t+1)} \equiv \frac{A}{t}+\frac{Bt+C}{t^2+1}+\frac{Dt+E}{t^2-t+1} \Rightarrow t-1=A(t^2+1)(t^2-t+1)+$$

$$+(Bt+c)t(t^2-t+1)+D(Dt+E)t(t^2+1)$$

Put $t=0$; $-1=A \Rightarrow A=-1$. ***Put*** $t=i; i-1=(Bi+C)i(-i)=Bi+C \Rightarrow$

$\Rightarrow B=1, C=-1$. ***Compare coefficient of*** t^n; $D=A+B+D \Rightarrow D=0$. ***Put*** $t=-\omega$,

($\omega \neq 1$ ***is cube root of unity)***

$$-\omega-1=E(-\omega)(\omega^2+1)=E\omega^2 \Rightarrow \omega^2=E\omega^2 \Rightarrow E=1$$

Thus, $\Omega=\int\left[-\frac{1}{t}+\frac{t-1}{t^2+1}+\frac{1}{t^2-t+1}\right]dt=-\ln t+\frac{1}{2}\ln(t^2+1)-\tan^{-1}(t)+$

$$+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)+C=-x+\frac{1}{2}\ln(e^{2x}+1)-\tan^{-1}(e^x)+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2e^x-1}{\sqrt{3}}\right)+C$$

451. Find:

$$\Omega=\int_0^\infty\left(\frac{1}{x}\cdot\tan^{-1}\left(\frac{3x^2}{4x^4+5x^2+2}\right)\right)dx$$

Proposed by Max Wong-Hong Kong

Solution 1 by Ravi Prakash-New Delhi-India

$$\Omega=\int_0^\infty\frac{1}{x}\tan^{-1}\left(\frac{3x^2}{4x^4+5x^2+2}\right)dx=\int_0^\infty\frac{1}{x}\tan^{-1}\left[\frac{(4x^2+1)-(x^2+1)}{1+(4x^2+1)(x^2+1)}\right]dx=$$

$$=\int_0^\infty\frac{1}{x}[\tan^{-1}(4x^2+1)-\tan^{-1}(x^2+1)]\,dx$$

Let $\phi(x)=\tan^{-1}(x^2+1)$*. Note* φ *is continuous on* $[0,\infty)$

$$\therefore\ \Omega=\int_0^\infty\frac{\varphi(2x)-\varphi(x)}{x}dx\ \textit{[Frullani's Integral]}$$

$$=[\phi(0)-\varphi(\infty)]\ln\left(\frac{1}{2}\right)=\left(\frac{\pi}{4}-\frac{\pi}{2}\right)(-\ln 2)=\frac{\pi}{4}\ln(2)$$

Solution 2 by Sagar Kumar-Kolkata-India

$$I=\int_0^\infty\tan^{-1}\left(\frac{3x^2}{4x^4+5x^2+2}\right)\frac{dx}{x}$$

$$\frac{3x^2}{(1+4x^4+4x^2+x^2+1)}=\frac{3x^2}{1+(x^2+1)(4x^2+1)}$$

$$I=\int_0^\infty\frac{\tan^{-1}(4x^2+1)-\tan^{-1}(x^2+1)\,dx}{x}$$

Let $I(a)=\int_0^\infty\frac{\tan^{-1}(1+ax^2)-\tan^{-1}(1+x^2)dx}{x}$ *(1)*

$$I'(a)=\int_0^\infty\frac{x}{1+(1+ax^2)^2}dx$$

$$I'(a)=\frac{1}{2a}\int_0^\infty \frac{d(ax^2+1)}{1+(ax^2+1)^2}=\frac{\tan^{-1}(1+ax^2)}{2a}\Big|_0^\infty$$

$$I'(a)=\left(\frac{\frac{\pi}{2}-\frac{\pi}{4}}{2a}\right)=\frac{\pi}{8a}$$

$$I(a)=\frac{\pi}{8}\ln(a)+C \quad \textit{from (1)} \quad I(1)=0\Rightarrow C=0$$

$$I(4)=\frac{\pi}{4}\ln(2) \ \textit{(Answer)}$$

Solution 3 by Shafiqur Rahman-Bangladesh

$$\int_0^\infty \frac{1}{x}\tan^{-1}\frac{3x^2}{4x^4+5x^2+2}dx=\int_0^\infty \frac{1}{x}\{\tan^{-1}(4x^2+1)-\tan^{-1}(x^2+1)\}dx=$$

$$=\int_0^\infty\int_1^2 \frac{2xydy}{1+(x^2y^2+1)^2}dx=\int_1^2\int_0^\infty \frac{2xydx}{1+(x^2y^2+1)^2}dy=$$

$$=\int_1^2[\tan^{-1}(x^2y^2+1)]_0^\infty\frac{dy}{y}=\int_1^2\frac{\pi}{4}\cdot\frac{dy}{y}\therefore\int_0^\infty\frac{1}{x}\tan^{-1}\frac{3x^2}{4x^4+5x^2+2}dx=\frac{\pi}{4}\ln 2$$

Solution 4 by Shivam Sharma-New Delhi-India

$$\Rightarrow\int_0^\infty \frac{\tan^{-1}(4x^2+1)-\tan^{-1}(x^2+1)}{x}dx\Rightarrow\int_0^\infty\left(\int_1^4\frac{x\,dy}{1+(yx^2+1)^2}\right)dx\Rightarrow$$

$$\Rightarrow\int_1^4\left(\int_0^\infty\frac{x\,dx}{1+(yx^2+1)^2}\right)dy\Rightarrow\frac{\pi}{8}\int_1^4\frac{1}{y}dy$$

$$\textit{(OR)}\Rightarrow\left(\frac{\pi}{8}\right)\ln(4)\ \textit{(OR)}\ \Omega=\left(\frac{\pi}{4}\right)\ln(2)\ \textit{(Q.E.D.)}$$

452. Find:

$$\Omega=\int_0^\infty\frac{x\cdot\tan^{-1}x}{x^4+x^2+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Sagar Kumar-Kolkata-India

$$\Omega = \int_0^\infty \frac{x\tan^{-1}x}{x^4+x^2+1}dx \quad \textit{(1)}$$

$$\textit{Put } x = \frac{1}{t};\ dx = -\frac{dt}{t^2}$$

$$\Omega = \int_0^\infty \frac{x\tan^{-1}\left(\frac{1}{x}\right)dx}{x^4+x^2+1} \quad \textit{(2)}$$

$$\textit{(1)+(2)}$$

$$\Omega = \frac{\pi}{4}\int_0^\infty \frac{x\,dx}{(x^4+x^2+1)} = \frac{\pi}{8}\int_0^\infty \frac{dx}{x^2+x+1}$$

$$\Omega = \frac{\pi}{8}\int_0^\infty \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{\pi}{8}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)\frac{2}{\sqrt{3}} = \frac{\pi^2}{12\sqrt{3}}$$

453. Find:

$$\Omega = \int tan^{-1}\left(\sqrt{x+\sqrt{x^2+1}}\right)dx,\ x \in \mathbb{R}$$

Proposed by Ekpo Samuel-Nigeria

Solution 1 by Mohamed Arahman Jama-Somalia

$$\Omega = \int \tan^{-1}\left(\sqrt{x+\sqrt{x^2+1}}\right)dx =$$

$$\int \tan^{-1}\left(\sqrt{x+\sqrt{x^2+1}}\right)dx \therefore \frac{\partial dt}{\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)} = dx$$

$$\textit{Let } t = \sqrt{x+\sqrt{x^2+1}};\ t^2 = x+\sqrt{x^2+1} = \partial dt\left(\frac{\sqrt{1+x^2}}{x+\sqrt{1+x^2}}\right) = dx$$

$$(t^2-x)^2 = \left(\sqrt{x^2+1}\right)^2$$

$$t^4-2t^2x+x^2 = x^2+1$$

$$t^4-2t^2x = 1 \quad \textit{(*)}$$

$$t^4-1 = 2t^2x$$

$\therefore x=\frac{t^4-1}{2t^2}$ ***(1)***

$\therefore$ ***Note:*** $x+\sqrt{1+x^2}=t^2 \therefore \sqrt{1+x^2}=t^2-x$. ***From equality (1):*** $x=\frac{t^4-1}{2t^2}$

$$\therefore 2tdt\left[\frac{t^2-\left(\frac{t^4-1}{2t^2}\right)}{t^2}\right]=dx;\ \partial dt\left[\frac{t^4+1}{2t^3}\right]=dx$$

$$\therefore \frac{\partial}{\partial}\int \tan^{-1}(t)\cdot\left[\frac{t^4+1}{t^3}\right]dt=\int\left(t+\frac{1}{t^3}\right)\cdot\tan^{-1}(t)\,dt$$

From (*): $t^4-2t^2x-1=0$. ***Put*** $t^2=u; u^2-\partial ux-1=0$

$$u=x\pm\sqrt{1+x^2}\therefore t^2=x+\sqrt{1+x^2};\partial dt=\left(1+\frac{x}{\sqrt{x^2+1}}\right)dx$$

$$=\int\left(t+\frac{1}{t^3}\right)\cdot\tan^{-1}(t)\,dt$$

$$u=\tan^{-1}(t);dt=\left(t+\frac{1}{t^3}\right)dt$$

$$du=\frac{1}{1+t^2}dt, v=\frac{t^2}{\partial}-\frac{1}{\partial t^2}$$

$$\therefore \frac{1}{2}\left(\frac{t^2}{1}-\frac{1}{t^2}\right)\tan^{-1}(t)-\frac{1}{\partial}\left[\int\frac{t^2+1-1}{1+t^2}dt-\int\frac{1}{t^2(1+t^2)}dt\right]=$$

$$=\frac{1}{\partial}\left(t^2-\frac{1}{t^2}\right)\tan^{-1}(t)-\frac{1}{\partial}\left[\int\frac{t^2+1}{1+t^2}-\int\frac{1}{1+t^2}dt-\int\frac{1}{t^2(1+t^2)}dt\right]$$ ***(using partial decomposition)***

$$=\frac{1}{\partial}\left(t^2-\frac{1}{t^2}\right)\tan^{-1}(t)-\frac{1}{\partial}\left[t-\tan^{-1}(t)+\frac{1}{t}+\tan^{-1}(t)\right]=$$

$$=\frac{1}{\partial}\left(t^2-\frac{1}{t^2}\right)\tan^{-1}(t)-\frac{1}{2}\left[t+\frac{1}{t}\right]\therefore t=\sqrt{x+\sqrt{x^2+1}}$$

$$\frac{1}{\partial}\left(\left(x+\sqrt{x^2+1}\right)-\frac{1}{\left(x+\sqrt{x^2+1}\right)}\right)\tan^{-1}\left(\sqrt{x+\sqrt{x^2+1}}\right)-\frac{1}{2}\left[\frac{\left(\sqrt{x+\sqrt{x^2+1}}\right)^2+1}{\sqrt{x+\sqrt{x^2+1}}}\right]+C$$

$$\therefore \frac{1}{\partial}\left[\frac{\left(x+\sqrt{x^2+1}\right)^2-1}{\left(x+\sqrt{x^2+1}\right)}\right]\tan^{-1}\left(\sqrt{x+\sqrt{x^2+1}}\right)-\frac{1}{\partial}\left[\frac{x+\sqrt{x^2+1}-1}{\sqrt{x+\sqrt{x^2+1}}}\right]+C$$

Solution 2 by Serban George Florin-Romania

$$x+\sqrt{x^2+1}=y^2, \sqrt{x^2+1}=y^2-x, x^2+1=y^4-2xy^2+x^2$$

$$1=y^4-2xy^2, 2xy^2=y^4-1, x=\frac{y^4-1}{2y^2}=\frac{1}{2}(y^2-y^{-2})$$

$$dx=\frac{1}{2}(2y+2y^{-3})=(y+y^{-3})=y+\frac{1}{y^3}=\frac{y^4+1}{y^3}\cdot dy$$

$$\Omega=\int\frac{y^4+1}{y^3}\arctan y=\int(y+y^{-3})\arctan y\,dy=\int y\cdot\arctan y\,dy+$$

$$+\int y^{-3}\arctan y\,dy=\int\left(\frac{y^2}{2}\right)'\arctan y\,dy+\int\left(\frac{y^{-2}}{-2}\right)'\arctan y\,dy$$

$$\Omega=\frac{y^2\arctan y}{2}-\frac{1}{2}\int y^2\cdot\frac{1}{y^2+1}dy-\frac{y^{-2}\arctan y}{2}-\int\frac{y^{-2}}{-2}\cdot\frac{1}{y^2+1}dy$$

$$\Omega=\frac{y^2\arctan y}{2}=\frac{1}{2}\int\left(1-\frac{1}{y^2+1}\right)dy-\frac{\arctan y}{2y^2}+\frac{1}{2}\int\frac{1}{y^2(y^2+1)}dy$$

$$\Omega=\frac{y^2\arctan y}{2}-\frac{1}{2}(y-\arctan y)-\frac{\arctan y}{2y^2}+\frac{1}{2}\int\left(\frac{1}{y^2}-\frac{1}{y^2+1}\right)dy$$

$$\Omega=\frac{y^2\arctan y}{2}-\frac{y}{2}+\frac{\arctan y}{2}-\frac{\arctan y}{2y^2}+\frac{1}{2}\left(-\frac{1}{y}\right)-\frac{1}{2}\arctan y$$

$$\Omega=\frac{y^4\arctan y-y^3-\arctan y-y}{2y^2}$$

$$\Omega=\frac{\left(x+\sqrt{x^2+1}\right)^2\arctan\sqrt{x+\sqrt{x^2+1}}-\left(\sqrt{x+\sqrt{x^2+1}}\right)^3-\arctan\left(\sqrt{x+\sqrt{x^2+1}}\right)-\sqrt{x+\sqrt{x^2+1}}}{2\left(x+\sqrt{x^2+1}\right)}$$

Solution 3 by Yen Tung Chung-Taichung-Taiwan

$$\textit{Let } u=\sqrt{x+\sqrt{x^2+1}}\Rightarrow u^2-x=\sqrt{x^2+1}\Rightarrow x=\frac{1-u^4}{2u^2}$$

$$\Omega=\int\tan^{-1}\left(\sqrt{x+\sqrt{x^2+1}}\right)dx=\int\tan^{-1}u\cdot\left(\frac{1-u^2}{2u^2}\right)=\frac{1-u^4}{2u^2}\cdot\tan^{-1}u-$$

$$-\int\frac{1}{1+u^2}\cdot\frac{1-u^4}{2u^2}du=\frac{1-u^4}{2u^2}\cdot\tan^{-1}u-\int\frac{1-u^2}{2u^2}du=\frac{1-u^4}{2u^2}\cdot\tan^{-1}u-$$

$$-\frac{1}{2}\left(-\frac{1}{u}-u\right)+C=\frac{1-u^4}{2u^2}\cdot\tan^{-1}u+\frac{1}{2}\cdot\frac{1+u^2}{u}+C=$$

$$=\frac{1-\left(\sqrt{x+\sqrt{x^2+1}}\right)^4}{2\left(\sqrt{x+\sqrt{x^2+1}}\right)^2}\cdot\tan^{-1}\left(\sqrt{x+\sqrt{x^2+1}}\right)+\frac{1}{2}\cdot\frac{1+\left(\sqrt{x+\sqrt{x^2+1}}\right)^2}{\sqrt{x+\sqrt{x^2+1}}}+C$$

454. **A note on the special Logarithmic integral**

Let's define the function $\chi(m)$ for any complex number m, $Re(m)>0$

$$\chi(m)=\int_0^1 x\ln(x)\ln(x+m)\,dx$$

then we have the following,

$$\int_0^1 \chi(m)\,dm=\frac{\pi^2-16\ln(2)+4}{72}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Togrul Ehmedov-Baku-Azerbaijan

$$\int_0^1\int_0^1 x\ln x\ln(x+m)\,dx\,dm=\int_0^1\int_x^{x+1} x\ln x\ln t\,dt\,dx$$

$$=\left[\left(\frac{x^2}{2}\ln x-\frac{x^2}{4}\right)\int_x^{x+1}\ln t\,dt\right]_0^1-\int_0^1\left(\frac{x^2}{2}\ln x-\frac{x^2}{4}\right)(\ln(x+1)-\ln x)dx$$

$$=-\frac{1}{4}\int_1^2\ln t\,dt-\frac{1}{2}\int_0^1 x^2\ln x\ln(x+1)\,dx+\frac{1}{2}\int_0^1 x^2\ln^2 x\,dx+\frac{1}{4}\int_0^1 x^2\ln(x+1)\,dx-\frac{1}{4}\int_0^1 x^2\ln x\,dx$$

$$=-\frac{1}{4}(2\ln 2-1)-\frac{1}{2}\int_0^1 x^2\ln x\ln(x+1)\,dx+\frac{1}{27}+\frac{1}{4}\left(\frac{2\ln 2}{3}-\frac{5}{18}\right)+\frac{1}{36}$$

$$=-\frac{1}{3}\ln 2+\frac{53}{216}-\frac{1}{2}\int_0^1 x^2\ln x\ln(x+1)\,dx$$

$$\int_0^1 x^2\ln x\ln(x+1)\,dx=\int_0^1 x^2\ln x\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^1 x^{k+2}\ln x\,dx$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left[\frac{x^{k+3}}{k+3}\ln x - \frac{x^{k+2}}{(k+3)^2}\right]_0^1 = -\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k(k+3)^2} = -\frac{1}{9}\sum_{k=1}^{\infty}(-1)^{k-1}\left[\frac{1}{k} - \frac{1}{k+3} - \frac{3}{(k+3)^2}\right]$$

$$= -\frac{1}{9}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k} + \frac{1}{9}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k+3} + \frac{1}{3}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k+3)^2} = -\frac{1}{9}\ln 2 + \frac{1}{9}\left(\frac{5}{9} - \ln 2\right) + \frac{1}{3}\left(\frac{31}{36} - \frac{\pi^2}{12}\right)$$

$$= -\frac{2}{9}\ln 2 + \frac{41}{108} - \frac{\pi^2}{36}$$

$$\int_0^1\int_0^1 x\ln x\ln(x+m)\,dx\,dm = -\frac{1}{3}\ln 2 + \frac{53}{216} - \frac{1}{2}\int_0^1 x^2\ln x\ln(x+1)\,dx$$

$$= -\frac{1}{3}\ln 2 + \frac{53}{216} - \frac{1}{2}\left(-\frac{2}{9}\ln 2 + \frac{41}{108} - \frac{\pi^2}{36}\right) = \frac{\pi^2 - 16\ln 2 + 4}{72}$$

455. Find:

$$\Omega = \int \frac{242(x+2)^5 - (x+1)^5 - (x+3)^5}{26(x+2)^3 - (x+1)^3 - (x+3)^3}dx,\ x > 0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Put $x + 2 = t$, ***then:*** $\Omega = \int \frac{242t^5 - \{(t-1)^5 + (t+1)^5\}}{26t^3 - \{(t-1)^3 + (t+1)^3\}}dx$

$$= \int \frac{242t^5 - 2(t^5 + 10t^3 + 5t)}{26t^3 - 2(t^3 + 3t)}dx = \int \frac{240t^5 - 20t^3 - 10t}{24t^3 - 6t}dt$$

$$= \int \frac{10t(24t^4 - 2t^2 - 1)}{6t(4t^2 - 1)}dt = \frac{5}{3}\int \frac{24t^4 - 2t^2 - 1}{4t^2 - 1}dt$$

$$= \frac{5}{3}\int \frac{(4t^2 - 1)(6t^2 + 1)}{4t^2 - 1}dt = \frac{5}{3}\int (6t^2 + 1)dt = \frac{5}{3} \times \left(\frac{6}{3}t^3 + t\right) + C$$

$$= \frac{10}{3}t^3 + \frac{5}{3}t + C = \frac{10}{3}(x+2)^3 + \frac{5}{3}(x+2) + C = \frac{10}{3}x^3 + 20x^2 + \frac{125}{3}x + C$$

Solution 2 by Sagar Kumar-Patna Bihar-India

Put $x + 2 = t;\ dx = dt$

$$\Omega = \int \frac{242t^5 - (t-1)^5 - (t+1)^5}{26t^3 - (t-1)^3 - (t+1)^3}dt$$

$$(t-1)^5 + (t+1)^5 = 2(t^5 + 10t^3 + 5t)$$

$$(t-1)^3 + (t+1)^3 = 2(t^3 + 3t)$$

$$\Omega = \int \frac{240t^5 - 20t^3 - 10t}{24t^3 - 6t} dt$$

$$\Omega = \frac{10}{6} \int \frac{24t^4 - 2t^2 - 1}{4t^2 - 1} dt$$

$$\Omega = \frac{5}{3} \int \frac{6t^2(4t^2 - 1) + (4t^2 - 1)}{4t^2 - 1} dt$$

$$\Omega = \frac{5}{3}(2t^3 + t) + C$$

$$\Omega = \frac{10}{3}(x+2)^3 + \frac{5}{3}(x+2) + C$$

Solution 3 by Shivam Sharma-New Delhi-India

$$\Rightarrow \int \frac{240x^5 + 2400x^4 + 9580x^3 + 19080x^2 + 18950x + 7500}{24x^3 + 144x^2 + 282x + 180} dx$$

$$\Rightarrow \frac{10}{6} \int \frac{24x^5 + 240x^4 + 958x^3 + 1908x^2 + 1895x + 750}{4x^3 + 24x^2 + 47x + 30} dx$$

$$\Rightarrow \frac{5}{4} \int \frac{(2x+3)(2x+5)(x+2)(6x^2 + 24x + 25)}{(2x+3)(2x+5)(x+2)} dx$$

$$\Rightarrow \frac{5}{3} \int \frac{6x^2 + 24x + 25}{1} dx \Rightarrow \frac{5}{3} \int (6x^2 + 24x + 25)\, dx$$

$$\Rightarrow \frac{5}{3}\left[6\left(\frac{x^3}{3}\right) + 24\left(\frac{x^2}{2}\right) + 25x\right] + C \textbf{ (OR) } \Omega = \frac{10x^3}{3} + 20x^2 + \frac{125x}{3} + C$$

(Answer)

456. Find:

$$\Omega = \int_0^\infty \frac{\pi x - 2\log x}{\left(\frac{\pi^2}{4} + \log^2 x\right)(1 + x^2)^2} dx$$

Proposed by Khalef Ruhemi-Jerash-Jordan

Solution by Zaharia Burghelea-Romania

Substitute $x^2 = t$

$$\Omega = 2\int_0^\infty \frac{\pi\sqrt{t} - \ln t}{(\pi^2 + \ln^2 t)(1+t)^2} \frac{dt}{\sqrt{t}}$$

$$= 2\pi\int_0^\infty \frac{1}{(\pi^2+\ln^2 t)(1+t)^2}dt - 2\int_0^\infty \frac{\ln t}{(\pi^2+\ln^2 t)(1+t)^2}\frac{dt}{\sqrt{t}}$$

$$\Omega = 2\pi I_1 - 2I_2 = \frac{\pi}{6}+\frac{\pi}{12} = \frac{\pi}{4}$$

Proofs:** I_1 **is a Schroder integral, which generally is defined as:

$$\int_0^\infty \frac{1}{(\pi^2+\ln^2 x)(1+x)^n}dx = (-1)^{n-1}G_n$$

and** G_n **are Gregory coefficients that respects the following recurrence relation:

$$\frac{G_1}{n} - \frac{G_2}{n-1} + \cdots + (-1)^{n-1}\frac{G_n}{1} = \frac{1}{n+1}$$

$$I_1 = \int_0^\infty \frac{1}{(\pi^2+\ln^2 t)(1+t)^2}dt = (-1)^{2-1}G_2 = \frac{1}{12}$$

$$I_2 = \int_0^\infty \frac{\ln x}{(\pi^2+\ln^2 x)(1+x)^2}\frac{\sqrt{x}}{x}dx$$

Substitute** $\ln x = t$**, then let** $t=-y$**, add both results and simplify:

$$I_2 = \int_0^\infty \frac{x}{\pi^2+x^2}\cdot\frac{e^{\frac{x}{2}}}{(1+e^x)^2}dx = \int_0^\infty \frac{-x}{\pi^2+x^2}\cdot\frac{e^{-\frac{\pi}{2}}}{(1+e^{-x})^2}dx$$

$$2I_2 = \int_{-\infty}^\infty \frac{x}{\pi^2+x^2}\left(\frac{e^{\frac{x}{2}}}{(1+e^x)^2} - \frac{e^{-\frac{x}{2}}}{(1+e^{-x})^2}\right)dx$$

$$I_2 = -\frac{1}{4}\int_{-\infty}^\infty \frac{x}{\pi^2+x^2}\cdot\frac{\sinh\left(\frac{x}{2}\right)}{\cosh^2\left(\frac{x}{2}\right)}dx$$

Now, integrate by parts, using that: $-\frac{1}{2}\int\frac{\sinh\left(\frac{x}{2}\right)}{\cosh^2\left(\frac{x}{2}\right)} = \frac{1}{\cosh\left(\frac{x}{2}\right)} + C$

$$I_2 = \frac{1}{2}\int_{-\infty}^\infty \left(\frac{x^2-\pi^2}{(x^2+\pi^2)^2}\right)\left(\frac{1}{\cosh\left(\frac{x}{2}\right)}\right)dx$$

Using the following property of the fourier transform:

$$\int_{-\infty}^{+\infty} f(x)g(x)\,dx = \int_{-\infty}^{+\infty} (F^{-1}g)(s)(Ff)(s)ds$$

The integral simplifies to:

$$I_2 = \int_0^{\infty}\left(\sqrt{\frac{x}{2}}x(-e^{-\pi x})\right)\left(\sqrt{2\pi}\frac{1}{\cosh(\pi x)}\right)dx = -\frac{1}{\pi}\int_0^{\infty}\frac{x}{\cosh(x)}e^{-x}dx$$

The latter integral is equal to the Laplace transform in $s = 1$ ***of***

$$f(t) = \frac{t}{\cosh(t)} \to F(s) = \frac{1}{8}\left(\psi_1\left(\frac{s+1}{4}\right) - \psi_1\left(\frac{s+3}{4}\right)\right)$$

$$I_2 = F(s=1) = \frac{1}{8}\left(\psi_1\left(\frac{1}{2}\right) - \psi_1(1)\right) = \frac{1}{8}\left(\frac{\pi^2}{2} - \frac{\pi^2}{6}\right) = \frac{\pi^2}{24}$$

$$\to I_2 = \int_0^{\infty}\frac{\ln x}{(\pi^2 + \ln^2 x)(1+x)^2}\frac{\sqrt{x}}{x}dx = -\frac{\pi}{24}$$

457. Find:

$$\Omega = \int \frac{\tanh^6 x + 2\tanh^4 x}{(1+\tanh^2 x)^2}dx, x \in \mathbb{R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\Omega = \int \frac{\tanh^6 x + 2\tanh^4 x}{(1+\tanh^2 x)^2}dx = \int \frac{\tanh^6 x + 2\tanh^4 x}{(1+\tanh^2 x)^3}\cdot\frac{\operatorname{sech}^2 x}{1-\tanh^2 x}dx$$

Put $\tanh x = t$

$$\Omega = \int \frac{t^6 + 2t^4}{(t^2+1)^3}\cdot\frac{dt}{1-t^2}$$

To split into partial fraction, put $t^2 = y$

$$\frac{y^3 + 2y^2}{(y+1)^3(1-y)} = \frac{A}{y+1} + \frac{B}{(y+1)^2} + \frac{C}{(y+1)^3} + \frac{D}{1-y}$$

$$\Rightarrow y^3 + 2y^2 = A(y+1)^2(1-y) + B(y+1)(1-y) + C(1-y) + D(y+1)^3$$

Put $y = 1$

$$3=8D\Rightarrow D=\frac{3}{8}$$

Put $y=-1,\ 1=2C\Rightarrow C=\frac{1}{2}$

Comparing coefficient of y^3

$$1=-A+D\Rightarrow A=D-1=-\frac{5}{8}$$

Put $y=0$

$$0=A+B+C+D\Rightarrow B=-A-C-D=\frac{5}{8}-\frac{1}{2}-\frac{3}{8}=-\frac{1}{4}$$

Thus, $\Omega=-\frac{5}{8}\int\frac{dt}{t^2+1}-\frac{1}{4}\int\frac{dt}{(t^2+1)^2}+\frac{1}{2}\int\frac{dt}{(t^2+1)^3}+\frac{3}{8}\int\frac{dt}{1-t^2}$

Let $I_1=\int\frac{dt}{t^2+1}=\frac{t}{t^2+1}+2\int\frac{t^2}{(t^2+1)^2}dt=\frac{t}{t^2+1}+2I_1-2\int\frac{dt}{(t^2+1)^2}\Rightarrow$

$$\Rightarrow 2\int\frac{dt}{(t^2+1)^2}=\frac{t}{t^2+1}+\tan^{-1}t\Rightarrow\int\frac{dt}{(t^2+1)^2}=\frac{1}{2}\cdot\frac{t}{t^2+1}+\frac{1}{2}\tan^{-1}t$$

Next, let $I_2=\int\frac{dt}{(t^2+1)^2}=\frac{t}{(t^2+1)^2}+4\int\frac{t^2+1-1}{(t^2+1)^3}dt=\frac{t}{(t^2+1)^2}+4I_2-4\int\frac{dt}{(t^2+1)^3}\Rightarrow$

$$\Rightarrow 4\int\frac{dt}{(t^2+1)^3}=\frac{t}{(t^2+1)^2}+3I_2$$

$$\therefore\Omega=-\frac{5}{8}I_1-\frac{1}{4}I_2+\frac{1}{8}\left\{\frac{t}{(t^2+1)^2}+3I_2\right\}+\frac{3}{16}\ln\left|\frac{1+t}{1-t}\right|+C$$

$$=-\frac{5}{8}I_1+\frac{1}{8}I_2+\frac{1}{8}\frac{t}{(t^2+1)^2}+\frac{3}{16}\ln\left|\frac{1+t}{1-t}\right|+C$$

$$=-\frac{5}{8}I_1+\frac{1}{8}\left\{\frac{t}{2(t^2+1)}+\frac{1}{2}I_1\right\}+\frac{1}{8}\cdot\frac{t}{(t^2+1)^2}+\frac{3}{16}\ln\left|\frac{1+t}{1-t}\right|+C$$

$$=\frac{1}{8}\cdot\frac{t}{(t^2+1)^2}+\frac{1}{16}\cdot\frac{t}{t^2+1}-\frac{9}{16}\tan^{-1}t+\frac{3}{16}\ln\left|\frac{1+t}{1-t}\right|+C$$

where $t=\tanh x$

Solution 2 by Yen Tung Chung-Taichung-Taiwan

Let $y=\tanh x\Rightarrow x=\tanh^{-1}y,\ dx=\frac{1}{1-y^2}dy$

$$\Omega=\int\frac{\tanh^6 x+2\tanh^4 x}{(1+\tanh^2 x)^2}dx=\int\frac{y^6+2y^4}{(1+y^2)^2}\cdot\frac{1}{1-y^2}dy$$

$$= \int \left(-1 + \frac{y^4 + y^2 + 1}{(1+y^2)^2(1-y^2)}\right) dy = -y + \frac{3}{4}\int \frac{1}{1-y^2} dy + \frac{1}{4}\int \frac{1-y^2}{(1+y^2)^2} dy$$

$$= -y + \frac{3}{4}\tanh^{-1} y + \frac{1}{4}\int \frac{1}{\left(\frac{1}{y}+y\right)^2} \cdot (-1) d\left(\frac{1}{y}+y\right) = -y + \frac{3}{4}\tanh^{-1} y + \frac{1}{4} \cdot \frac{1}{\frac{1}{y}+y} + C$$

$$= -y + \frac{3}{4}\tanh^{-1} y + \frac{1}{4} \cdot \frac{y}{1+y^2} + C = \frac{3}{4}x - \tanh x + \frac{1}{4} \cdot \frac{\tanh x}{1+\tanh^2 x} + C$$

Solution 3 by Ibrahim Abdulazzez-Nigeria

$$\Omega = \int \left(\tanh^2 x - \frac{\tanh^2 x}{(1+\tanh^2 x)^2}\right) dx$$

Recall that $\tanh^2 x = 1 - \operatorname{sech}^2 x$

$$\Omega = \int (1 - \operatorname{sech}^2 x)\, dx - \int \frac{\tanh^2 x}{(1+\tanh^2 x)^2} dx$$

$$= x - \tanh x - \int \frac{\tanh^2 x}{(1+\tanh^2 x)^2} dx$$

since $\tanh x = \frac{\sinh x}{\cosh x}$***, we have***

$$= x - \tanh x - \int \frac{\sinh^2 x \cosh^2 x}{(\cosh^2 x + \sinh^2 x)^2} dx$$

But $\sinh 2x = 2\sinh x \cosh x$ ***and*** $\cosh 2x = \cosh^2 x + \sinh^2 x$

$$= x - \tanh x - \frac{1}{4}\int \tanh 2x = x - \tanh x - \frac{1}{8}(2x - \tanh 2x) + c$$

$$= \frac{3x}{4} - \tanh x + \frac{1}{8}\tanh 2x + c$$

Solution 4 by Alex Ani-Nigeria

$$\int \frac{\tanh^2 x d(\tanh x)}{(1+\tanh^2 x)^2} - \frac{1}{4}\int \tanh^2(2x)\, dx + \int \frac{\tanh^4 x\, dx}{(1+\tanh^2 x)}$$

Solving the first two integral put $\tanh x = \tan \alpha$

$$\int \sin^2(2\alpha)\, d\alpha - \frac{1}{8}(x - \tanh^2(2x))$$

$$\frac{1}{4}(2\alpha - \sin 2\alpha \cos 2\alpha) - \frac{1}{8}(x - \tanh^2(2x)) + K$$

$$\frac{1}{2}\tan^{-1}(\tanh x)-\frac{\tanh x\,(1-\tanh^2 x)}{(1+\tanh^2 x)^2}+K$$

for $I=\int\frac{\tanh^4 x dx}{(1+\tanh^2 x)}$

$$I=\int\frac{\sin^2 x\,dx}{\cos 2x}$$

$$I=\int\left(\frac{\cosh^2 x}{\cos 2x}-\frac{\operatorname{sech}^2 x}{1+\tanh^2 x}\right)dx$$

$$2I=x-\tan^{-1}(\tanh x)+k$$

$$I=\frac{1}{2}(x-\tan^{-1}(\tanh x))+k$$

compiling and simplifying.

$$\frac{3x}{8}-\frac{\tanh x\,(1-\tanh^2 x)}{(1+\tanh^2 x)^2}-\frac{1}{8}(\tanh^2 2x)+C$$

Solution 5 by Carlos Suarez-Quito-Ecuador

$$\int\frac{\tanh^6 x+2\tanh^4 x}{(1+\tanh^2 x)^2}dx$$

$$v=\tanh x;\ dv=\frac{d}{dx}(\tanh(x));\ dv=\sec^2 h(x)\,dx$$

$$\int\frac{\tanh^6 x+2\tanh^4 x}{\tan^4 x+2\tanh^2 x+1}dx$$

$$\tanh^6 x+2\tanh^4 x-\tanh^2 x+2\tanh^2 x+1-\tanh^4 x-2\tanh^4 x-\tanh^2 x-\tanh x$$

$$\int\left(\tanh^2 x-\frac{\tanh^2 x}{(1+\tanh^2 x)^2}\right)dx$$

$$\int\left(\tanh^2 x\,dx-\int\frac{\tanh^2 x}{(1+\tanh^2 x)}\right)dx$$

$$\int\tanh^2 x\,dx=x-\tanh(x)$$

$$\int\frac{\tan^2 hx}{(1+\tan^2 hx)}dx=\int\frac{\frac{\sinh^2 x}{\cosh^2 x}}{(1+v^2)}\cdot\frac{1}{(\cosh^2 x)}dx$$

$$\int\frac{\frac{\sinh^2 x}{\sinh^2 x}\cdot\cosh^2 x}{(1+v^2)}dx$$

$$v = \tanh x;\ v^2 = \tanh^2 x;\ v^2 = 1 - \sinh^2 x\,;\ \cosh^2 x = 1 + \sin^2 h\,x$$

$$v^2 = 1 - \frac{1}{\cosh^2 x};\ v^2 = \frac{\cosh^2 x - 1}{\cosh^2 x};\ v^2 = \frac{1 + \sin^2 hx - 1}{1 + \sin^2 hx}$$

$$v^2(1 + \sin^2 h(x)) = \sinh^2 x;\ v^2 + v^2 \sinh^2 x = \sinh^2 x$$

$$v^2 \sinh^2 x - \sin^2 h^2 x = -v^2;\ \sinh^2 x = \frac{v^2}{1 - v^2};\ \sinh x = \frac{v}{\sqrt{1 - v^2}}$$

$$\int \frac{\frac{v^2}{1 - v^2}}{\frac{1 + v^2}{1}} dv = \int \frac{v^2}{(1 + v^2)(1 - v^2)} dv$$

$$\int \frac{Av + B}{1 + v^2} dv + \int \frac{C}{1 + v} dv + \int \frac{D}{1 - v} dv$$

$$\int \frac{1}{2(1 + v^2)} dv + \frac{1}{4}\int \frac{dv}{v + 1} - \frac{1}{4}\int \frac{dv}{v - 1} =$$

$$= \frac{1}{4}(-\ln(1 - v) + \ln(v + 1) - 2\arctan(v)) + C$$

$$= \frac{1}{4}(\ln(\tanh x + 1) - \ln(1 - \tanh x)) - 2\arctan(\tanh x) + C$$

$$R = x - \tanh(x) + \frac{1}{4}(\ln(\tanh(x) + 1)) - \ln(1 - \tanh(x^2)) - 2\arctan(\tanh(x)) + C$$

$$R = \frac{1}{6}\Big(3(3x + \tan^{-1}(\tanh(x)) + 2\tanh(x)\,(\mathrm{sech}^2(x) - 7))\Big) + C$$

458. Find:

$$\Omega = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon < 0}} \left(\int_{-1}^{\varepsilon} \sqrt{\frac{1 + e^x}{1 - e^x}}\, dx \right)$$

Proposed by Ekpo Samuel-Nigeria

Solution 1 by Togrul Ehmedov-Baku-Azerbaijan

$$\Omega = \int_{-1}^{0} \sqrt{\frac{1 + e^x}{1 - e^x}}\, dx$$

$$\frac{1 + e^x}{1 - e^x} = t \Rightarrow e^x = \frac{t - 1}{1 + t} \Rightarrow x = \ln\left(\frac{t - 1}{1 + t}\right) \Rightarrow dx = \frac{2}{t^2 - 1} dt$$

$$\Omega = 2\int_{\frac{e+1}{e-1}}^{\infty}\frac{\sqrt{t}}{t^2-1}dt$$

$$t = z^2 \Rightarrow dt = 2z\,dz$$

$$\Omega = 4\int_{\sqrt{\frac{e+1}{e-1}}}^{\infty}\frac{z^2}{z^4-1}dz = 4\int_{\sqrt{\frac{e+1}{e-1}}}^{\infty}\frac{z^2-1}{z^4-1}dz + 4\int_{\sqrt{\frac{e+1}{e-1}}}^{\infty}\frac{1}{z^4-1}dz$$

$$= 4\int_{\sqrt{\frac{e+1}{e-1}}}^{\infty}\frac{1}{z^2+1}dz + 2\int_{\sqrt{\frac{e+1}{e-1}}}^{\infty}\frac{1}{z^2-1}dz - 2\int_{\sqrt{\frac{e+1}{e-1}}}^{\infty}\frac{1}{z^2+1}dz$$

$$= 2\left[\int_{\sqrt{\frac{e+1}{e-1}}}^{\infty}\frac{1}{z^2+1}dz + \int_{\sqrt{\frac{e+1}{e-1}}}^{\infty}\frac{1}{z^2-1}dz\right] = 2\left[\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{e+1}{e-1}} + \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right|\Bigg|_{\sqrt{\frac{e+1}{e-1}}}^{\infty}\right]$$

$$= 2\left[\tan^{-1}\sqrt{\frac{e-1}{e+1}} - \frac{1}{2}\ln\left|\frac{\sqrt{\frac{e+1}{e-1}}-1}{\sqrt{\frac{e+1}{e-1}}+1}\right|\right] = 2\tan^{-1}\sqrt{\frac{e-1}{e+1}} - \ln\left|\frac{\sqrt{e+1}-\sqrt{e-1}}{\sqrt{e+1}+\sqrt{e-1}}\right|$$

Solution 2 by Naren Bhandari-Nepal

$$N = \lim_{\epsilon\to 0\epsilon<0}\left(\int_{-1}^{\epsilon}\sqrt{\frac{1+e^x}{1-e^x}}dx\right)$$

Let us evaluate the integral $I = \left(\int_{-1}^{\epsilon}\sqrt{\frac{1+e^x}{1-e^x}}dx\right)$

Make substitution of

$$\frac{1+e^x}{1-e^x} = u \Rightarrow dx = (2e^x)^{-1}(1-e^x)^2du \Rightarrow dx = \frac{2du}{(u-1)(u+1)}$$

Since $\frac{2e^x}{1-e^x} = u-1$ ***and*** $\frac{2}{1-e^x} = u+1$ ***plugging we obtain RHS which further follows as***

$I = \int_{-1}^{\epsilon}\frac{2\sqrt{u}}{(u-1)(u+1)}du$. ***Further substitute*** $\sqrt{u} = w \Rightarrow du = 2\sqrt{u}dw = 2w\,dw$

Plugging back we have $I = \int_{-1}^{\epsilon}\frac{4w^2}{(w^2-1)(w^2+1)}dw = 2\int_{-1}^{\epsilon}\left(\frac{1}{w^2+1}+\frac{1}{w^2-1}\right)dw$

$$= \left[2\tan^{-1}\left(\sqrt{\frac{1+e^x}{1-e^x}}\right) + \ln\frac{\sqrt{1+e^x}-\sqrt{1-e^x}}{\sqrt{1+e^x}+\sqrt{1-e^x}}\right]_{-1}^{\epsilon}$$

Settings limits we have that $I = 2\tan^{-1}\left(\frac{\sqrt{(e^\epsilon+1)(e-1)}-\sqrt{(e+1)(1-e^\epsilon)}}{\sqrt{(1-e^\epsilon)(e-1)}+\sqrt{(e+1)(1+e^\epsilon)}}\right) +$

$+\ln\left[\left(\frac{\sqrt{1+e^\epsilon}-\sqrt{1-e^\epsilon}}{\sqrt{1+e^\epsilon}+\sqrt{1-e^\epsilon}}\right)\left(\frac{\sqrt{e+1}+\sqrt{e-1}}{\sqrt{e+1}-\sqrt{e-1}}\right)\right]$. ***Thus,***

$$N = \lim_{\epsilon\to 0\epsilon<0} I = 2\tan^{-1}\sqrt{\frac{e-1}{e+1}} + \ln\left[\frac{\sqrt{e+1}+\sqrt{e-1}}{\sqrt{e+1}-\sqrt{e-1}}\right]$$

459. Find:

$$\Omega = \int \frac{x\sec x\,(1+\sin x)}{x+\sin x-\cos x-1}dx$$

Proposed by Nader Al Homsi-Amman-Jordan

Solution by Ruanghaw Chaoka-Chiangrai-Thailand

$$\Omega = \int \frac{x\sec x\,(1+\sin x)}{x+\sin x-\cos x-1}dx = \int f(x)\,dx = ??$$

$$f(x) = \frac{x\sec x\,(1+\sin x)-(1+\cos x+\sin x)}{x+\sin x-\cos x-1} + \frac{1+\cos x+\sin x}{x+\sin x-\cos x-1}$$

$$= \frac{\sec x\left(x(1+\sin x)-(\cos x+\cos^2 x+\sin x\cos x)\right)}{x+\sin x-\cos x-1} + \frac{1+\cos x+\sin x}{x+\sin x-\cos x-1}$$

$$= \frac{\sec x\left(x(1+\sin x)-(\cos x+1-\sin^2 x+\sin x\cos x)\right)}{x+\sin x-\cos x-1} + \frac{1+\cos x+\sin x}{x+\sin x-\cos x-1}$$

$$= \frac{\sec x\left(x(1+\sin x)+(\sin x-\cos x-1)(1+\sin x)\right)}{x+\sin x-\cos x-1} + \frac{1+\cos x+\sin x}{x+\sin x-\cos x-1}$$

$$= \frac{\sec x\,(1+\sin x)(x+\sin x-\cos x-1)}{x+\sin x-\cos x-1} + \frac{1+\cos x+\sin x}{x+\sin x-\cos x-1}$$

$$= \sec x+\tan x + \frac{1+\cos x+\sin x}{x+\sin x-\cos x-1}$$

$$\therefore \Omega = \int\left(\sec x+\tan x+\frac{1+\cos x+\sin x}{x+\sin x-\cos x-1}\right)dx$$

$$= \ln|\tan x+\sec x| - \ln|\cos x| + \ln|x+\sin x-\cos x-1| + C$$

460.

$$\int_0^{\frac{\pi}{2}} \frac{10\tan^3 x - 19\tan^{\frac{8}{3}} x - 36\tan^2 x + 10\tan x - 19\tan^{\frac{2}{3}} x - 36}{2\tan^{\frac{8}{3}} x + 35\tan^{\frac{5}{3}} x + 108\tan^{\frac{2}{3}} x} dx$$

Proposed by Max Wong-Hong Kong

Solution by Shafiqur Rahman-Bangladesh

$$\int_0^{\frac{\pi}{2}} \frac{10\tan^3 x - 19\tan^{\frac{8}{3}} x - 36\tan^2 x + 10\tan x - 19\tan^{\frac{2}{3}} x - 36}{2\tan^{\frac{8}{3}} x + 35\tan^{\frac{5}{3}} x + 108\tan^{\frac{2}{3}} x} dx =$$

$$= \int_0^{\frac{\pi}{2}} \frac{10\tan x - 19\tan^{\frac{2}{3}} x - 36}{\tan^{\frac{2}{3}} x\,(2\tan x + 27)(\tan x + 4)} \sec^2 x\, dx =$$

$$= 18\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\, dx}{\tan^{\frac{2}{3}} x\,(2\tan x + 27)} [8\tan x \to 27\tan x] - 4\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\, dx}{\tan^{\frac{2}{3}} x\,(\tan x + 4)} -$$

$$- \int_0^{\frac{\pi}{2}} \left(\frac{1}{\tan x + 4} - \frac{2}{2\tan x + 27}\right) \sec^2 x\, dx$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\, dx}{\tan^{\frac{2}{3}} x\,(\tan x + 4)} - 4\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\, dx}{\tan^{\frac{2}{3}} x\,(\tan x + 4)} - \left[\ln \frac{\tan x + 4}{2\tan x + 27}\right]_0^{\frac{\pi}{2}} = \ln 2 + \ln\left(\frac{4}{27}\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{10\tan^3 x - 19\tan^{\frac{8}{3}} x - 36\tan^2 x + 10\tan x - 19\tan^{\frac{2}{3}} x - 36}{2\tan^{\frac{8}{3}} x + 35\tan^{\frac{5}{3}} x + 108\tan^{\frac{2}{3}} x} dx = 3\ln\left(\frac{2}{3}\right)$$

461. **Prove that:**

$$\int_0^{\frac{\pi}{2}} \left(\frac{ln\left(\frac{1 - sin(x)}{1 + sin(x)}\right)\sqrt{cos(x)}}{(1 + sin(x))\sqrt{1 - sin(x)}} \right) dx = -8$$

Proposed by Nader Al Homsi-Jordan

Solution 1 by Tran Hong-Vietnam

$$\int_0^{\frac{\pi}{2}} \frac{\ln\left(\frac{1-\sin x}{1+\sin x}\right)\sqrt{\cos x}}{(1+\sin x)\sqrt{1-\sin x}} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}\,[\ln(1-\sin x)-\ln(1+\sin x)]}{(1+\sin x)\sqrt{1-\sin x}}dx$$

$$\overset{integral\ part}{=} \left\{-\frac{2\sqrt{\cos x}}{\sqrt{1-\sin x}}\cdot\frac{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)}\left[\ln\left(\frac{1-\sin x}{1+\sin x}\right)-4\right]\right\}\Bigg|_0^{\frac{\pi}{2}} = 0-(4\cdot 2) = -8 \textit{ (Proved)}$$

Solution 2 by Tobi Joshua-Nigeria

$$y = \sin x\,;\, dy = \cos x\, dx;\, dy = \sqrt{1-y^2}dx$$

$$\int_0^1 \frac{(\ln(1-y)-\ln(1+y))(1-y^2)^{\frac{1}{4}}}{(1+y)(1-y)^{\frac{1}{2}}}\frac{dy}{\sqrt{1-y^2}} = \int_0^1 \frac{\ln\left(\frac{1-y}{1+y}\right)\left((1-y)(1+y)\right)^{\frac{1}{4}}dy}{(1+y)^{\frac{3}{2}}(1-y)'}$$

$$\int_0^1 \frac{\ln\left(\frac{1-y}{1+y}\right)(dy)}{(1+y)^{\frac{3}{4}}(1-y)^{\frac{3}{4}}} = t = \frac{1-y}{1+y}, y = \frac{1-t}{1+t}; dy = \frac{-2dt}{(1+t)^2}$$

$$2\int_0^1 \frac{\ln t\frac{dt}{(1+t)^2}}{\left(\frac{2}{1+t}\right)^{\frac{3}{4}}\left(\frac{2t}{1+t}\right)^{\frac{3}{4}}} = \frac{2}{2^2}\int_0^1 \frac{\ln t\, dt}{t^{\frac{3}{4}}} \Rightarrow \frac{2}{4}\int_0^1 \frac{\ln t}{t^{\frac{3}{4}}}dt$$

$$u = -\ln t\,, t = e^{-u}, dt = -e^{-u}du$$

$$\Rightarrow -\frac{1}{2}\int_0^\infty \frac{u\, e^{-u}du}{e^{-\frac{3u}{4}}} = -\frac{(4)^2}{2}\int_0^\infty ue^{-u}du = -\frac{16}{2}\int_0^\infty u^{2-1}\, e^{-u}du$$

$$\Rightarrow -8\int_0^\infty u^{2-1}\, e^{-u}du = -8\Gamma(1) = -8$$

Solution 3 by Kartick Chandra Betal-India

$$\int_0^{\frac{\pi}{2}} \frac{\ln\left(\frac{1-\sin x}{1+\sin x}\right)\cdot\sqrt{\cos x}}{(1+\sin x)\sqrt{1-\sin x}}dx = \int_0^1 \frac{\ln\left(\frac{1-x}{1+x}\right)\cdot(1-x^2)^{\frac{1}{4}}}{(1+\sin x)\sqrt{1-x}}\cdot\frac{dx}{\sqrt{1-x^2}} =$$

$$= \int_0^1 \frac{\ln\left(\frac{1-x}{1+x}\right)}{\sqrt{1-x}}\cdot(1-x^2)^{-\frac{3}{4}}dx = \int_1^0 \frac{\ln z\left\{1-\left(\frac{1-z}{1+z}\right)\right\}^{-\frac{3}{4}}}{\sqrt{1+\frac{1-z}{1+z}}}\cdot\frac{-2\,dz}{(1+z)^2}$$

Let $\frac{1-x}{1+x}=z$

$$=2\int_0^1 \frac{\ln z\left\{\frac{4z}{(1+z)^2}\right\}^{-\frac{3}{4}}\sqrt{1+z}}{\sqrt{2}}\cdot\frac{dz}{(1+z)^2}$$

$$=\frac{1}{2}\int_0^1 z^{-\frac{3}{4}}\ln z\,dz=\frac{1}{2}\left[\frac{z^{\frac{1}{4}}}{\frac{1}{4}}\ln z\right]_0^1-\frac{1}{2}\int_0^1\frac{z^{\frac{1}{4}}}{\frac{1}{4}}\cdot\frac{dz}{z}=-2\int_0^1 z^{\frac{1}{4}-1}\,dz=-2\left[\frac{z^{\frac{1}{4}}}{\frac{1}{4}}\right]_0^1=-8$$

462. Find:

$$\Omega=\int\frac{x^6\cdot\log x}{(3+x^7)^5}dx,\ x>0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$I=\int\frac{x^6\log x}{(3+x^7)^5}dx=-\frac{1}{4}\cdot\frac{1}{7}\cdot\frac{\log x}{(3+x^7)^4}+\frac{1}{28}\int\frac{1}{x(3+x^7)^4}dx=-\frac{1}{28}\cdot\frac{\log x}{(3+x^7)^4}+\frac{1}{28}I_1$$

where $I_1=\int\frac{1}{x(3+x^7)^4}dx=\int\frac{x^6}{x^7(3+x^7)^4}dx$. ***Put*** $3+x^7=t,\ 7x^6dx=dt\ \therefore\ I_1=\frac{1}{7}\int\frac{dt}{(t-3)t^4}$

$$=-\frac{1}{21}\int\left(1-\frac{t}{3}\right)^{-1}\frac{1}{t^4}dt=-\frac{1}{21}\left[1+\frac{t}{3}+\frac{t^2}{9}+\frac{t^3}{27}+\frac{t^4}{81}\left(1-\frac{t}{3}\right)^{-1}\right]\frac{1}{t^4}dt$$

$$=-\frac{1}{21}\int\left[\frac{1}{t^4}+\frac{1}{3t^3}+\frac{1}{9t^2}+\frac{1}{27t}+\frac{1}{81}\cdot\frac{1}{3-t}\right]dt$$

$$=\frac{1}{63}\cdot\frac{1}{t^3}+\frac{1}{126t^2}+\frac{1}{18t}-\frac{1}{567}\log(t)+\frac{1}{1701}\log|t-3|$$

Thus, $I=-\frac{1}{28}\cdot\frac{\log x}{(3+x^7)^4}+\frac{1}{28}\left[\frac{1}{63(3+x^7)^3}+\frac{1}{126(3+x^7)^2}+\frac{1}{18}\cdot\frac{1}{3+x^7}-\frac{1}{567}\log(3+x^7)+\frac{1}{243}\log|x|+c\right]$

Solution 2 by Marian Ursărescu-Romania

$$\left.\begin{array}{l}I=\int\frac{x^6\ln x}{(x^7+3)^5}dx=\begin{cases}u=\ln x\\ du=\frac{1}{x}dx\end{cases}\ dv=\frac{x^6}{(x^7+3)^5}dx\\ V=\int\frac{x^6}{(x^7+3)^5}dx=\begin{Bmatrix}x^7+3=t\\7x^6dx=dt\end{Bmatrix}=\frac{1}{7}\int\frac{dt}{t^5}=-\frac{1}{28}\cdot\frac{1}{(x^7+3)^4}\end{array}\right\}=$$

$$= u\cdot v-\int V\,du=-\frac{\ln x}{28(x^7+3)^4}+\frac{1}{28}\underbrace{\int\frac{dx}{x(x^7+3)^4}}_{I_1}$$

$$I_1=\int\frac{dx}{x(x^7+3)^4}=\int\frac{x^6}{x^7(x^7+3)^4}dx=\begin{Bmatrix}x^7+3=t\\7x^6dx=dt\end{Bmatrix}=\frac{1}{7}\int\frac{dt}{(t-3)\cdot t^4}=\frac{1}{7}I_2$$

$$I_2=\int\frac{dt}{t^4(t-3)}=\int\frac{A}{t}dt+\int\frac{B}{t^2}dt+\int\frac{C}{t^3}dt+\int\frac{D}{t^4}dt+\int\frac{E}{(t-3)}dt$$

$$\frac{1}{t^4(t-3)}=\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t^3}+\frac{D}{t^4}+\frac{E}{t-3}\,|\cdot t^4(t-3)$$

$$1=At^3(t-3)+Bt^2(t-3)+Ct(t-3)+D(t-3)+Et^4$$

$$1=At^4-3At^3+Bt^3-3Bt^2+Ct^2-3Ct+Dt-3D+Et^4$$

$$1=(A+E)t^4+(B-3A)t^3+(C-3B)t^2+(D-3C)t-3D$$

$$\Rightarrow\begin{cases}A+E=0\\B-3A=0\\C-3B=0\\D-3C=0\\-D=1\end{cases}\qquad\begin{matrix}A=\frac{1}{3}B=-\frac{1}{81},\ E=-A=\frac{1}{81}\\B=\frac{1}{3}C=-\frac{1}{27}\\C=\frac{1}{3}\ \ D=\frac{1}{9}\\D=-\frac{1}{3}\end{matrix}$$

$$I_2=\frac{1}{81}\ln t+\frac{1}{27}\cdot\frac{1}{t}+\frac{1}{81}\cdot\frac{1}{t^2}+\frac{1}{27}\cdot\frac{1}{t^3}+\frac{1}{81}\ln(t_3)$$

$$I=-\frac{\ln x}{28(x^7+3)^4}+\frac{1}{196}\cdot\left[\frac{1}{81}\ln(x^7+3)+\frac{1}{27(x^7+3)}+\frac{1}{18(x^7+3)^2}+\frac{1}{27(x^7+3)^3}+\frac{1}{81\ln x^7}\right]+C$$

Solution 3 by Yen Tung Chung Taichung-Taiwan

$$\int\frac{x^6\ln x}{(3+x^7)^5}dx=?$$

$$\int\frac{x^6\ln x}{(3+x^7)^5}dx=-\frac{1}{28}\cdot\frac{\ln x}{(3+x^7)^6}+\frac{1}{28}\int\frac{1}{x(3+x^7)^6}dx\begin{pmatrix}u=\ln x & dv=\frac{x^6}{(3+x^7)^5}dx\\du=\frac{1}{x}dx & v=-\frac{1}{28}\cdot\frac{1}{(3+x^7)^6}\end{pmatrix}$$

$$=-\frac{1}{28}\cdot\frac{\ln x}{(3+x^7)^6}+\underbrace{\frac{1}{28}\int\frac{x^{-35}}{(3x^{-7}+1)^6}\cdot x^{-8}dx}_{let\ y=3x^{-7}+1\Rightarrow x^{-7}=\frac{1}{3}(y-1)}=-\frac{1}{28}\cdot\frac{\ln x}{(3+x^7)^6}+\frac{1}{28}\int\frac{\frac{1}{3^5}(y-1)^5}{y^6}\cdot\left(-\frac{1}{21}dy\right)$$

$$=-\frac{1}{28}\cdot\frac{\ln x}{(3+x^7)^6}-\frac{1}{142884}\int\left(\frac{1}{y}-\frac{5}{y^2}+\frac{10}{y^3}-\frac{10}{y^4}+\frac{5}{y^5}-\frac{1}{y^6}\right)dy$$

$$=-\frac{1}{28}\cdot\frac{\ln x}{(3+x^7)^6}-\frac{1}{142884}\left(\ln|y|+\frac{5}{y}-\frac{5}{y^2}+\frac{10}{3y^3}-\frac{5}{4y^4}+\frac{1}{5y^5}\right)+C$$

$$=-\frac{1}{28}\cdot\frac{\ln x}{(3+x^7)^6}-\frac{1}{142884}\left(\ln|3x^{-7}+1|+\frac{5}{3x^{-7}+1}-\frac{5}{(3x^{-7}+1)^2}+\frac{10}{3(3x^{-7}+1)^3}-\frac{5}{4(3x^{-7}+1)^4}+\frac{1}{5(3x^{-7}+1)^5}\right)+C$$

Solution 4 by Kartick Chandra Betal-India

$$\Omega=\int\frac{x^6\cdot\ln x}{(x^7+3)^5}dx=\frac{1}{7}\ln x\cdot\frac{(x^7+3)^{-5+1}}{-5+1}-\frac{1}{7}\int\frac{1}{x}\cdot\frac{(x^7+3)^{-4}}{-4}dx=$$

$$=-\frac{\ln x}{28(x^7+3)^4}+\frac{1}{28}\int\frac{dx}{x(x^7+3)^4}=$$

$$=-\frac{\ln x}{28(x^7+3)^4}+\frac{1}{28}\int\left[\frac{x}{3^4}-\frac{1}{3^4(x^7+3)}-\frac{1}{3^3(x^7+3)^2}-\frac{1}{3^2(x^7+3)^3}-\frac{1}{3(x^7+3)^4}\right]x^6dx$$

$$=-\frac{\ln x}{28(x^7+3)^4}+\frac{\ln x}{2268}-\frac{\ln(x^7+3)}{15876}+\frac{1}{5292(x^7+3)}+\frac{1}{3528(x^7+3)^2}+\frac{1}{1764(x^7+3)}+C$$

Solution 5 by Orlando Irahola Ortega-Bolivia

$$\Omega=\frac{1}{\ln 10}\int\frac{x^6\ln x}{(x^7+3)^5}dx=\frac{1}{7\ln 10}\int\frac{7x^6}{(x^7+3)^5}\cdot\ln x\,dx=\frac{1}{7\ln 10}\left[-\frac{\ln x}{4(x^3+3)^4}+\frac{1}{4}\underbrace{\int\frac{dx}{x(x^7+3)^4}}_{\Delta\, I}\right]$$

$$u=\ln x\to du=\frac{dx}{x}\wedge\int dv=\int\frac{7x^6dx}{(x^7+3)^5}\Rightarrow v=-\frac{1}{4(x^7+3)^4}$$

$$I=\int\frac{dx}{x(x^7+3)^4}=\frac{1}{7}\int\frac{7x^6dx}{x^7(x^7+3)^4}=\frac{1}{7}\int\frac{7x^6dx}{(x^7+3)^5-3(x^7+3)^4}$$

$$z=x^7+3;\ dz=7x^6dx$$

$$I=\frac{1}{7}\int\frac{(1)dz}{z^4(z-3)}=\frac{1}{-21}\int\frac{(-3)dz}{z^4(z-3)}=-\frac{1}{21}\int\frac{(z-3-z)dz}{z^4(z-3)}\Rightarrow$$

$$\Rightarrow I=-\frac{1}{21}\left[\int\frac{(z-3)}{z^4(z-3)}dz-\int\frac{z\,dz}{z^4(z-3)}\right]$$

$$I=-\frac{1}{21}\left[\int z^{-4}dz-\int\frac{dz}{z^3(z-3)}\right]=-\frac{1}{21}\left[-\frac{1}{3z^3}-\left(-\frac{1}{3}\right)\int\frac{-3\,dz}{z^3(z-3)}\right]=$$

$$=\frac{1}{64z^3}-\frac{1}{63}\int\frac{(z-3-z)dz}{z^3(z-3)}$$

$$I=\frac{1}{63z^3}-\frac{1}{63}\left[\int\frac{dz}{z^3}-\int\frac{dz}{z^2(z-3)}\right]=\frac{1}{63z^3}-\frac{1}{63}\left[\int z^{-3}dz-\left(-\frac{1}{3}\right)\int\frac{-3\,dz}{z^2(z-3)}\right]$$

$$I=\frac{1}{63z^3}-\frac{1}{63}\left[-\frac{1}{2z^2}+\frac{1}{3}\int\frac{(z-3-z)dz}{z^2(z-3)}\right]=\frac{1}{63z^3}+\frac{1}{126z^2}-\frac{1}{189}\int\frac{(z-3)-z}{z^2(z-3)}dz$$

$$I=\frac{1}{63z^3}+\frac{1}{126z^2}-\frac{1}{189}\left[\int z^{-2}\,dz-\int\frac{dz}{z(z-3)}\right]=\frac{1}{63z^3}+\frac{1}{126z^2}-\frac{1}{189}\left[-\frac{1}{z}-\left(-\frac{1}{3}\right)\int\frac{-3\,dz}{z(z-3)}\right]$$

$$I=\frac{1}{63z^3}+\frac{1}{126z^2}+\frac{1}{189z}-\frac{1}{567}\int\frac{(z-3-z)}{z(z-3)}dz=\frac{1}{63z^3}+\frac{1}{126z^2}+\frac{1}{189z}-\frac{1}{567}\left[\int\frac{dz}{z}-\int\frac{dz}{z-3}\right]$$

Finally: $I=\frac{1}{63z^3}+\frac{1}{126z^2}+\frac{1}{189z}-\frac{1}{567}\ln z+\frac{1}{568}\ln(z-3)\Rightarrow-\frac{1}{567}\ln\left(\frac{z}{z-3}\right)$

$$z=x^7+3\Rightarrow I=\frac{1}{63}\left[\frac{1}{(x^7+3)^3}+\frac{1}{2(x^7+3)^2}+\frac{1}{3(x^7+3)}-\frac{1}{9}\ln\left(\frac{x^7+3}{x^7}\right)\right]$$

$$\Omega=-\frac{\ln x}{28\ln 10\,(x^7+3)^4}+\frac{1}{28\ln 10}I$$

Finally: $\Omega=-\frac{\log x}{28(x^7+3)^4}+\frac{1}{1764\ln 10}\left[\frac{1}{(x^7+3)^3}+\frac{1}{2(x^7+3)^2}+\frac{1}{3(x^7+3)}-\frac{1}{9}\ln\left(\frac{x^7+3}{x^7}\right)\right]+C$

463. Find:

$$\Omega=\int_0^\infty\left(\frac{log(x+1)}{x^4-x^2+1}\right)dx$$

Proposed by Vasile Mircea Popa – Romania

Solution by Zaharia Burghelea-Romania

We start by substituting $x=\frac{1}{t}$

$$I=\int_0^\infty\frac{t^2(\ln(1+t)-\ln t)}{t^4-t^2+1}dt$$

Considering:

$$I(a)=\int_0^\infty\frac{x^2(\ln(a+x)-\ln x)}{x^4-x^2+1}dx\to I'(a)=\int_0^\infty\frac{x^2}{(x^4-x^2+1)(a+x)}dx$$

We now calculate 4 integrals that are going to be used later:

$$I_1=\int\frac{1}{x^4-x^2+1}dx=\frac{1}{2}\int\frac{2+x^2-x^2}{x^4-x^2+1}dx=\frac{1}{2}\left(\int\frac{1+x^2}{x^4-x^2+1}dx+\int\frac{1-x^2}{x^4-x^2+1}dx\right)$$

$$=\frac{1}{2}\left(\int\frac{\frac{1}{x^2}+1}{x^2+\frac{1}{x^2}-1}dx+\int\frac{\frac{1}{x^2}-1}{x^2+\frac{1}{x^2}-1}dx\right)=\frac{1}{2}\left(\int\frac{\left(x-\frac{1}{x}\right)'}{\left(x-\frac{1}{x}\right)^2+1}dx-\int\frac{\left(x+\frac{1}{x}\right)'}{\left(x+\frac{1}{x}\right)^2-3}dx\right)$$

$$=\frac{1}{2}\arctan\left(\frac{x^2+1}{x}\right)-\frac{1}{4\sqrt{3}}\ln\left(\frac{x-\sqrt{3}x+1}{x+\sqrt{3}x+1}\right)+C$$

$$I_2=\int\frac{x}{x^4-x^2+1}dx=\int\frac{x}{\left(x^2-\frac{1}{2}\right)^2+\frac{3}{4}}dx=\frac{1}{2}\int\frac{d\left(x^2-\frac{1}{2}\right)}{\left(x^2-\frac{1}{2}\right)^2+\frac{3}{4}}=\frac{1}{\sqrt{3}}\arctan\left(\frac{2x^2-1}{\sqrt{3}}\right)+C$$

$$I_3=\int\frac{x^2}{x^4-x^2+1}dx=\frac{1}{2}\int\frac{2}{x^2+\frac{1}{x^2}-1}dx=\frac{1}{2}\left(\int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+1}dx+\int\frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-3}dx\right)=$$

$$=\frac{1}{2}\left(\int\frac{\left(x-\frac{1}{x}\right)'}{\left(x-\frac{1}{x}\right)^2+1}dx+\int\frac{\left(x+\frac{1}{x}\right)'}{\left(x+\frac{1}{x}\right)^2-3}dx\right)=\frac{1}{2}\arctan\left(\frac{x^2-1}{x}\right)+\frac{1}{4\sqrt{3}}\ln\left(\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}\right)+C$$

$$I_4=\int\frac{x^3}{x^4-x^2+1}dx=\frac{1}{4}\int\frac{4x^3-2x+2x}{x^4-x^2+1}dx=\frac{1}{4}\left(\int\frac{4x^3-2x}{x^4-x^2+1}dx+\int\frac{2x}{x^4-x^2+1}dx\right)=$$

$$=\frac{1}{4}\left(\int\frac{d(x^4-x^2+1)}{x^4-x^2+1}+\int\frac{d\left(x^2-\frac{1}{2}\right)}{\left(x^2-\frac{1}{2}\right)^2+\frac{3}{4}}\right)=\frac{1}{4}\ln(x^4-x^2+1)+\frac{1}{2\sqrt{3}}\arctan\left(\frac{2x^2-1}{\sqrt{3}}\right)+C$$

Back to the original integral, we have:

$$I'(a)=\frac{1}{a^4-a^2+1}\int_0^\infty\left(\frac{a^2}{a+x}+\frac{a^3x^2-a^2x^3-a+x}{x^4-x^2+1}\right)dx$$

Since the first integral doesn't converge we must group it with another integral that produces a logarithm term.

$$I'(a)=\frac{1}{a^4-a^2+1}\left(a^2\int_0^\infty\left(\frac{1}{a+x}-\frac{x^3}{x^4-x^2+1}\right)dx\right)+$$

$$+\frac{1}{a^4-a^2+1}\left(+a^3\int_0^\infty\frac{x^2}{x^4-x^2+1}dx-a\int_0^\infty\frac{1}{x^4-x^2+1}dx+\int_0^\infty\frac{x}{x^4-x^2+1}dx\right)$$

Using I_1, I_2, I_3 and I_4 we obtain:

$$I'(a)=\frac{1}{a^4-a^2+1}\left(-a^2\left(\ln a+\frac{\pi}{3\sqrt{3}}\right)+\frac{\pi}{2}a^3-\frac{\pi}{2}a+\frac{2\pi}{3\sqrt{3}}\right)$$

Since $I(0)=\int_0^\infty\frac{x^2(\ln(x)-\ln x)}{x^4-x^2+1}dx=0\to I=\int_0^1 I'(a)\,da$

$$I=-\int_0^1\frac{a^2\ln a}{a^4-a^2+1}da-\frac{\pi}{3\sqrt{3}}\int_0^1\frac{a^2}{a^4-a^2+1}da+\frac{\pi}{2}\int_0^1\frac{a^3-a}{a^4-a^2+1}da+\frac{2\pi}{3\sqrt{3}}\int_0^1\frac{1}{a^4-a^2+1}da$$

For the first integral put $a^2 = x$

$$\int_0^1 \frac{a^2 \ln a}{a^4 - a^2 + 1} da = \frac{1}{4}\int_0^1 \frac{\sqrt{x}\ln x}{x^2 - x + 1} dx = \frac{\pi^2\sqrt{3}}{9} - \frac{8}{3}G \quad (*)$$

Where G *is Catalan's Constant. And again using* I_1, I_2, I_3, I_4 *we get:*

$$= \frac{2}{3}G - \frac{\pi^2}{12\sqrt{3}} - \frac{\pi^2}{12\sqrt{3}} + \frac{\pi}{18}\ln(2+\sqrt{3}) - \frac{\pi^2}{12\sqrt{3}} + \frac{\pi^2}{6\sqrt{3}} + \frac{\pi}{9}\ln(2+\sqrt{3})$$

Finally: $I = \frac{\pi}{6}\ln(2+\sqrt{3}) + \frac{2}{3}G - \frac{\pi^2}{12\sqrt{3}}$

464. Find:

$$\int \left(\frac{\sqrt{1+\ln^2 x}\,(1+\ln x) + \ln^2 x}{\sqrt{1+\ln^2 x}\,(\ln x) + \ln^2 x + 1}\right) dx$$

Proposed by Nader Al Homsi-Jordan

Solution by Ravi Prakash-New Delhi-India

Put $\ln x = t \Rightarrow x = e^t$

$$\therefore I = \int e^t \cdot \frac{\sqrt{1+t^2}(1+t) + t^2}{t\sqrt{1+t^2} + (t^2+1)} dt = \int e^t \frac{\sqrt{1+t^2} + t\left(\sqrt{1+t^2} + t\right)}{\sqrt{t^2+1}\left(t + \sqrt{t^2+1}\right)} dt$$

$$= \int e^t \left[\frac{1}{t + \sqrt{t^2+1}} + \frac{t}{\sqrt{1+t^2}}\right] dt = \int e^t \left[\sqrt{t^2+1} - t + \frac{t}{\sqrt{t^2+1}}\right] dt$$

$$= \int e^t \sqrt{t^2+1}\,dt + \int e^t \frac{t}{\sqrt{t^2+1}} dt - \int te^t\,dt$$

$$= e^t\sqrt{t^2+1} - \int e^t \cdot \frac{t}{\sqrt{t^2+1}} + \int e^t \frac{t}{\sqrt{t^2+1}} - \{te^t - e^t dt\}$$

[Integrating by parts]

$$= e^t\sqrt{t^2+1} - te^t + e^t + C = e^t\left[\sqrt{t^2+1} - t + 1\right] + C$$

$$= x\left[1 - \ln x + \sqrt{1 + (\ln x)^2}\right] + C$$

465. Find:

$$\Omega = \int_0^\infty \frac{1 - \cos x}{8 - 4x\sin x + x^2(1 - \cos x)} dx$$

Proposed by Khalef Ruhemi-Jarash-Jordan

Solution by Zaharia Burghelea-Romania

$$I = \int_0^\infty \frac{1-\cos x}{8-4x\sin x + x^2(1-\cos x)}dx = \int_0^\infty \frac{2\sin^2\left(\frac{x}{2}\right)}{8-4x\sin x + 2\left(x\sin\left(\frac{x}{2}\right)^2\right)}dx$$

Setting $\frac{x}{2} = t$

$$I = \frac{1}{2}\int_0^\infty \frac{\sin^2 t}{1 - t\sin(2t) + (t\sin t)^2}dt$$

noticing that $(\cos t - t\sin t)^2 = \cos^2 t - 2t\sin t\cos t + (t\sin t)^2$

$$I = \frac{1}{2}\int_0^\infty \frac{\sin^2 t}{\sin^2 t + (\cos t - t\sin t)^2}dt = \frac{1}{4}\int_{-\infty}^\infty \frac{dx}{1+(x-\cot x)^2}$$

we have

$$\frac{\sin x}{x} = \lim_{n\to\infty}\prod_{k=1}^n\left(1-\frac{x^2}{(k\pi)^2}\right) \to \ln\left(\frac{\sin x}{x}\right) = \lim_{n\to\infty}\sum_{k=1}^n\left(1-\frac{x^2}{(k\pi)^2}\right)$$

Differentiating with respect to x *gives:*

$$\cot x - \frac{1}{x} = \lim_{n\to\infty}\sum_{k=1}^n \frac{2x}{x^2-(k\pi)^2} = \lim_{n\to\infty}\left(\frac{1}{x+\pi}+\frac{1}{x-\pi}+\cdots+\frac{1}{x+n\pi}+\frac{1}{x-n\pi}\right)$$

$$x - \cot x = \lim_{n\to\infty}\left(x - \left(\frac{1}{x}+\frac{1}{x+\pi}+\frac{1}{x-\pi}+\cdots+\frac{1}{x+n\pi}+\frac{1}{x-n\pi}\right)\right)$$

Since the above series converges as $n\to\infty$ *and by Glasser's Master Theorem*

(see: "A Remarkable Property of Definite Integrals" By M.L. Glasser) we have that:

$$\int_{-\infty}^\infty f(x-\cot x)\,dx = \int_{-\infty}^\infty f(x)\,dx$$

It follows that: $I = \frac{1}{4}\int_{-\infty}^\infty \frac{1}{1+x^2}dx = \frac{\pi}{4}$

466. Find:

$$\Omega = \int \frac{\sqrt{3+\cos 2x}}{\cos x}dx, x \in \left(0,\frac{\pi}{2}\right)$$

Proposed by Abdul Mukhtar-Nigeria

Solution 1 by Shafiqur Rahman-Bangladesh

$$\int \frac{\sqrt{3+\cos 2x}}{\cos x}dx = \sqrt{2}\int \frac{1+\cos^2 x}{\cos x\sqrt{1+\cos^2 x}}dx =$$

$$= \sqrt{2}\int \frac{\sec^2 x}{\sqrt{\tan^2 x+2}}dx + \sqrt{2}\int \frac{\cos x}{\sqrt{2-\sin^2 x}}dx$$

$$\therefore \int \frac{\sqrt{3+\cos 2x}}{\cos x}dx = \sqrt{2}\left(\sinh^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)+\sin^{-1}\left(\frac{\sin x}{\sqrt{2}}\right)\right)+C$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\Omega = \int \frac{\sqrt{3(\cos^2 x+\sin^2 x)+(\cos^2 x-\sin^2 x)}}{\cos x}dx =$$

$$= \int \frac{\sqrt{2\sin^2 x+4\cos^2 x}}{\cos x}dx = \int \frac{\sqrt{2-2\cos^2 x+4\cos^2 x}}{\cos x}dx$$

$\overset{(1)}{=} \sqrt{2}\int \frac{\sqrt{1+\cos^2 x}}{\cos x}dx$. *Let* $\cos x = t \therefore dx \overset{(1)}{=} \frac{-dt}{\sqrt{1-t^2}}$ ($\because \sin x = \sqrt{1-\cos^2 x}$ as $0 < x < \frac{\pi}{2}$)

(1), (2)$\Rightarrow \Omega \overset{(3)}{=} -\sqrt{2}\int \frac{\sqrt{1+t^2}dt}{t\sqrt{1-t^2}}$

Let $1+t^2 = z^2$ $(z>0) \therefore dt \overset{(4)}{=} \frac{z\,dz}{\sqrt{z^2-1}}$ $\left(\because t > 0 \text{ as } t = \cos x \,\&\, 0 < x < \frac{\pi}{2}\right)$

(3), (4)$\Rightarrow \Omega = -\sqrt{2}\int \frac{z^2 dz}{(z^2-1)\sqrt{2-z^2}}$ *(To be noted that* $2-z^2 = 1-\cos^2 x = \sin^2 x > 0$*)*

$$= -\sqrt{2}\int \frac{(z^2-1)dz}{(z^2-1)\sqrt{2-z^2}} - \sqrt{2}\int \frac{dz}{(z^2-1)\sqrt{2-z^2}}$$

$$= -\sqrt{2}\int \frac{dz}{\sqrt{2-z^2}} - \sqrt{2}\int \frac{dz}{(z^2-1)\sqrt{2-z^2}} \overset{(5)}{=} I + J \text{ (say)}$$

$$J = -\sqrt{2}\int \frac{(z^2-1)+(2-z^2)}{(z^2-1)\sqrt{2-z^2}}dz \overset{(6)}{=} I - \sqrt{2}\int \frac{\sqrt{2-z^2}}{z^2-1}dz$$

(5), (6)$\Rightarrow \Omega \overset{(7)}{=} 2I + I_1$ *(say)*

Let $z = \sqrt{2}\cos\theta\,; 0 < \theta < \frac{\pi}{4} \therefore \sin^2\theta = \frac{2-z^2}{2}$

$$\therefore dz = -\sqrt{2}\sin\theta\,d\theta \Rightarrow I_1 = +\sqrt{2}\int \frac{\sqrt{2}\sin\theta\cdot\sqrt{2}\sin\theta\,d\theta}{2\cos^2\theta-1}$$

$$=\sqrt{2}\int\left(\frac{1-\cos 2\theta}{\cos 2\theta}\right)d\theta=\sqrt{2}\int\sec 2\theta d\theta-\sqrt{2}\int d\theta$$

$$=\frac{1}{\sqrt{2}}\ln|\sec 2\theta+\tan 2\theta|-\sqrt{2}\theta+k=\frac{1}{\sqrt{2}}\ln\left|\frac{(\cos\theta+\sin\theta)^2}{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}\right|-\sqrt{2}\theta+k$$

$$=\frac{1}{\sqrt{2}}\ln\left|\frac{\frac{z}{\sqrt{2}}+\frac{\sqrt{2-z^2}}{\sqrt{2}}}{\frac{z}{\sqrt{2}}-\frac{\sqrt{2-z^2}}{\sqrt{2}}}\right|-\sqrt{2}\cos^{-1}\left(\frac{z}{\sqrt{2}}\right)+k$$

$$\overset{(8)}{=}\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{1+\cos^2 x}+\sin x}{\sqrt{1+\cos^2 x}-\sin x}\right)-\sqrt{2}\cos^{-1}\sqrt{\frac{1+\cos^2 x}{2}}+k$$

(7), (8) $\Rightarrow \Omega=-2\sqrt{2}\int\frac{dz}{\sqrt{2-z^2}}+\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{1+\cos^2 x}+\sin x}{\sqrt{1+\cos^2 x}-\sin x}\right)-\sqrt{2}\cos^{-1}\sqrt{\frac{1+\cos^2 x}{2}}+k$

$$=-2\sqrt{2}\sin^{-1}\left(\frac{z}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{1+\cos^2 x}+\sin x}{\sqrt{1+\cos^2 x}-\sin x}\right)-\sqrt{2}\cos^{-1}\sqrt{\frac{1+\cos^2 x}{2}}+C$$

$$=-2\sqrt{2}\sin^{-1}\sqrt{\frac{1+\cos^2 x}{2}}+\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{1+\cos^2 x}+\sin x}{\sqrt{1+\cos^2 x}-\sin x}\right)$$

$-\sqrt{2}\cos^{-1}\sqrt{\frac{1+\cos^2 x}{2}}+C$ ***(Answer)***

467. Find:

$$\Omega=\int\left(\frac{\tan x}{(1+\tan x)\sqrt{\tan x}}\right)dx, x\in\left(0,\frac{\pi}{2}\right)$$

Proposed by Abdul Mukhtar-Nigeria

Solution 1 by Ekpo Samuel-Nigeria

$$\int\frac{\tan x\sqrt{\tan x}}{(1+\tan x)\sqrt{\tan a}-\sqrt{\tan x}}dx\to\int\frac{\tan(x)\sqrt{\tan x}}{(1+\tan x+\tan x)}dx\Rightarrow\int\frac{\sqrt{\tan x}}{1+\tan x}dx$$

Let $u=\tan x, du=\sec^2 x\,dx, dx=\frac{du}{\sec^2 x}=\frac{dy}{u^2+1}$

$\to\int\frac{\sqrt{u}}{1+u}\cdot\frac{du}{u^2+1}$ | ***Let*** $a=\sqrt{u}, du=\frac{1}{2\sqrt{u}}dx\to\partial\int\frac{a^2}{(a^2+1)(a^4+1)}da$ ***decompose to***

$$\to\partial\int\left(-\frac{1}{2(a^2+1)}+\frac{1}{4(x^2+\sqrt{2}x+1)}-\frac{1}{4(-x^2+\sqrt{2}x-1)}\right)du$$

$$\to \partial\left[-\frac{1}{2}\int\frac{da}{a^2+1}+\frac{1}{4}\int\frac{da}{\left(a+\frac{\sqrt{2}}{2}\right)+\frac{2}{4}}-\frac{1}{4}\int\frac{da}{\left(a-\frac{\sqrt{2}}{2}\right)^2+\frac{2}{a}}\right]$$

$$\to 2\left[-\frac{1}{2}\int\frac{da}{a^2+1}+\frac{1}{4}\int\frac{da}{\frac{\left(2a+\sqrt{2}\right)^2+2}{4}}-\frac{1}{4}\int\frac{da}{\frac{\left(2a-\sqrt{2}\right)^2+2}{4}}\right]$$

$$\to 2\left[-\frac{1}{2}\int\frac{da}{a^2+1}+\frac{1}{4}\int\frac{4}{\left(2a+\sqrt{2}\right)^2+2}da-\frac{1}{4}\int\frac{4}{\left(2a-\sqrt{2}\right)^2+2}da\right]$$

$$=2\left[-\frac{1}{2}\tan^{-1}(a)+\frac{\sqrt{2}}{4}\tan^{-1}\left(\frac{2a+\sqrt{2}}{\sqrt{2}}\right)-\frac{\sqrt{2}}{4}\tan^{-1}\left(\frac{2a-\sqrt{2}}{\sqrt{2}}\right)\right]$$

$$=2\left[-\frac{1}{2}\tan^{-1}(a)+\frac{\sqrt{2}}{4}\tan^{-1}\left(\frac{2\sqrt{2}a+2}{2}\right)-\frac{\sqrt{2}}{4}\tan^{-1}\left(\frac{2\sqrt{2}a-2}{2}\right)\right]$$

$$=2\left(-\frac{1}{2}\tan^{-1}(a)+\frac{\sqrt{2}}{4}\tan^{-1}\left(\sqrt{2}a+1\right)-\frac{\sqrt{2}}{4}\tan^{-1}\left(\sqrt{2}a-1\right)\right)$$

Recall $a=\sqrt{u}; n=\tan x\,; a=\sqrt{-\tan x}$

$$\Rightarrow -\tan^{-1}\left(\sqrt{\tan(x)}\right)+\frac{\sqrt{2}}{2}\tan^{-1}\left(\sqrt{2}\left(\sqrt{\tan x}+1\right)\right)-\frac{\sqrt{2}}{2}\tan^{-1}\left(\sqrt{2}\sqrt{\tan x}-1\right)$$

$$-\tan^{-1}\left(\sqrt{\tan x}\right)+\frac{2\tan^{-1}\left(\sqrt{2}\left(\sqrt{\tan x}+1\right)\right)}{\sqrt{2}}-\frac{2\tan^{-1}\left(\sqrt{2}\sqrt{\tan x}-1\right)}{\sqrt{2}}$$

Solution 2 by Kelvin Hong-Malaysia

Let $u^2=\tan x$, ***then*** $2udu=\sec^2 x\,dx=(1+u^4)dx$. ***Therefore***

$$I=\int\frac{2u^2}{(1+u^2)(1+u^4)}du=\int\frac{(1+u^2)^2-(1+u^4)}{(1+u^2)(1+u^4)}du=\int\frac{1+u^2}{1+u^4}-\frac{1}{1+u^2}du$$

$$=\int\frac{1+u^2}{\left(1+\sqrt{2}u+u^2\right)\left(1-\sqrt{2}u+u^2\right)}-\frac{1}{1+u^2}du$$

$$=\int\frac{1}{2+2\sqrt{2}u+2u^2}+\frac{1}{2-2\sqrt{2}u+u^2}-\frac{1}{1+u^2}du$$

$$=\int\frac{1}{\left(\sqrt{2}u+1\right)^2+1}+\frac{1}{\left(\sqrt{2}u-1\right)^2}-\frac{1}{u^2+1}du$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}u+1)+\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}u-1)-\tan^{-1}u+C$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}u+1+\sqrt{2}u-1}{1-(\sqrt{2}u+1)(\sqrt{2}u-1)}\right)-\tan^{-1}u+C$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{2\sqrt{2}u}{2-2u^2}\right)-\tan^{-1}u+C=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}\tan x}{1-\tan x}\right)-\tan^{-1}\sqrt{\tan x}+C$$

Solution 3 by Sagar Kumar-Patna Bihar-India

Put $\tan x=t^2;\sec^2 x\,dx=2t\,dt$

$$\Omega=\int\frac{(t^2)2tdt}{(1+t^2)(1+t^4)(t)};\ \Omega=2\int\frac{t^2dt}{(t^4+1)(t^2+1)}dt;\ \Omega=\int\frac{t^2+1}{t^4+1}-\frac{1}{t^2+1}dt$$

$$\Omega=\int\frac{d\left(t-\frac{1}{t}\right)}{\left(t-\frac{1}{t}\right)^2+2}-\int\frac{dt}{t^2+1}=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right)-\tan^{-1}(t)+C$$

Where $t=\sqrt{\tan x}$

Solution 4 by Tran Hong-Vietnam

$$\Omega=\int\frac{\tan x}{(1+\tan x)\sqrt{\tan x}}dx=$$

$$=\int\left[\left(\frac{1}{\left(\sqrt{2\tan x}+1\right)^2+1}+\frac{1}{\left(1-\sqrt{2\tan x}\right)^2+1}\right)-\frac{1}{\left(\sqrt{\tan x}\right)^2+1}\right]d(\sqrt{\tan x})$$

$$\frac{1}{\sqrt{2}}\tan^{-1}(2\sqrt{\tan x+1})-\frac{1}{\sqrt{2}}\tan^{-1}(1-\sqrt{2\tan x})-\tan^{-1}(\sqrt{\tan x})+C$$

468. **Find:**

$$\Omega=\int\frac{(\sin x-\cos x)(1+x)+3-2\cos x}{(x-\sin x)(\sin x+4+\cos x)}dx,\ x\in\left(0,\frac{\pi}{2}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Artan Ajredini-Presheva-Serbie

$$(\cos x+4+\sin x)(1-\cos x)-(x-\sin x)(\cos x-\sin x)=$$

$$=\cos x-\cos^2 x+4-4\cos x+\sin x-\sin x\cos x-$$

$$-x\cos x+x\sin x+\sin x\cos x-\sin^2 x=$$

$$=x\sin x-x\cos x-3\cos x+4-1+\sin x=$$

$$=\sin x+x\sin x-\cos x-x\cos x+3-2\cos x=$$

$$=\sin x\,(1+x)-\cos x\,(1+x)+3-2\cos x=(1+x)(\sin x-\cos x)+3-2\cos x$$

Therefore

$$\Omega=\int\frac{(\cos x+4+\sin x)(1-\cos x)-(x-\sin x)(\cos x-\sin x)}{(x-\sin x)(\sin x+4+\cos x)}dx$$

$$=\int\frac{1-\cos x}{x-\cos x}dx-\int\frac{\cos x-\sin x}{\sin x+4+\cos x}dx=$$

$$=\ln|x-\cos x|-\ln|\sin x+4+\cos x|+C=\ln\left|\frac{x-\cos x}{\sin x+4+\cos x}\right|+C$$

Solution 2 by Lahiru Samarakoon-Sri Lanka

$$\Omega=\int\frac{(\sin x-\cos x)(1+x)+3-2\cos x}{(x-\sin x)(\cos x+4+\cos x)}dx$$

$$x\in\left(0,\frac{\pi}{2}\right)$$

$$=\int\frac{\sin x+x\sin x-\cos x-x\cos x+3-2\cos x}{(x-\sin x)(\sin x+4+\cos x)}dx$$

$$=\int\frac{x(\sin x-\cos x)-\sin x\,(\sin x-\cos x)+\sin^2 x-\sin x\cos x+\sin x-\cos x+3-2\cos x}{(n-\cos x)(\sin x+\cos x+4)}dx$$

$$=\int\frac{(\sin x-\cos x)(x-\sin x)+(\sin x+\cos x+4)+\left(\overbrace{-1+\sin^2 x}^{-\cos^2 x}-\sin x\cos x-4\cos x\right)}{(x-\sin x)(\sin x+\cos x+4)}dx$$

$$=\int\frac{(\sin x-\cos x)(n-\sin x)+(\sin x+\cos x+4)(1-\cos x)}{(x-\sin x)(\sin x+\cos x+4)}dx$$

$$=\int\frac{(\sin x-\cos x)}{(\sin x+\cos x+4)}dx+\int\frac{(1-\cos x)}{x-\sin x}dx$$

$$=-\ln|\sin x+\cos x+4|+\ln|x-\sin x|+C=\ln\left|\frac{x-\sin x}{\sin x+\cos x+4}\right|+C$$

Solution 3 by Shafiqur Rahman-Bangladesh

$$\int\frac{(\sin x-\cos x)(1+x)+3-2\cos x}{(x-\sin x)(\sin x+4+\cos x)}dx=$$

$$\int\frac{(\sin x-\cos x)(x-\sin x)+(1-\cos x)(\sin x+4+\cos x)}{(x-\sin x)(\sin x+4+\cos x)}dx=$$

$$=\int\left(\frac{1-\cos x}{x-\sin x}-\frac{\cos x-\sin x}{\sin x+4+\cos x}\right)dx$$

$$\therefore\int\frac{(\sin x-\cos x)(1+x)+3-2\cos x}{(x-\sin x)(\sin x+4+\cos x)}dx=\ln\left|\frac{x-\sin x}{\sin x+4+\cos x}\right|+C$$

469. Find:

$$\Omega=\int\frac{dx}{\sqrt{\cot x}\cdot(2+\tan^2 x)^2},x\in\left(0,\frac{\pi}{2}\right)$$

Proposed by Abdul Mukhtar-Nigeria

Solution by Igor Soposki-Skopje-Macedonia

$$I=\int\frac{dx}{\sqrt{\cot x}\,(2+\tan^2 x)^2}=\int\frac{\sqrt{\tan x}}{(2+\tan^2 x)^2}dx=\int\frac{\sqrt{\tan x}}{(1+\tan^2 x)}\cdot\frac{1}{(2+\tan^2 x)^2}\cdot\frac{dx}{\cos^2 x}$$

$$=\left\{\begin{matrix}\tan x=t\\ \dfrac{dx}{\cos^2 x}=dt\end{matrix}\right\}=\int\frac{\sqrt{t}}{(1+t^2)(2+t^2)^2}dt=\left\{\begin{matrix}t=p^2\\ dt=2pdp\end{matrix}\right\}=$$

$$=\frac{1}{2}\int\frac{p^2}{(1+p^4)(2+p^4)^2}dp=\left\{\begin{matrix}\dfrac{1}{(1+m)(2+m)^2}=\dfrac{A}{1+m}+\dfrac{B}{2+m}+\dfrac{C}{(2+m)^2}\\ m=p^4\end{matrix}\right\}=A=\frac{1}{4}$$

$$\Rightarrow 1=A(2+m)^2+B(1+m)(2+m)+C(1+m)\Rightarrow B=C=-\frac{1}{4}$$

$$=\frac{1}{2}\left(\int\frac{1}{4}\frac{p^2}{(1+p^4)}dp-\int\frac{1}{4}\cdot\frac{p^2}{(2+p^4)}dp-\int\frac{1}{4}\cdot\frac{p^2}{(2+p^4)^2}dp\right)\Rightarrow$$

$$I_1=\int\frac{p^2}{1+p^4}dp=\frac{1}{2}\int\frac{2p^2+1-1}{p^4+1}dp=\frac{1}{2}\left[\int\frac{p^2+1}{p^4+1}dp-\int\frac{p^2-1}{p^4+1}dp\right]=$$

$$=\frac{1}{2}\left[\int\frac{1+\frac{1}{p^2}}{p^2+\frac{1}{p^2}}dp-\int\frac{1-\frac{1}{p^2}}{p^2+\frac{1}{p^2}}dp\right]=\frac{1}{2}\left[\int\frac{d\left(p-\frac{1}{p}\right)}{\left(p-\frac{1}{p}\right)^2+2}-\int\frac{d\left(p+\frac{1}{p}\right)}{\left(p+\frac{1}{p}\right)^2-2}\right]=$$

$$=\frac{1}{2}\left[\int\frac{dl}{l^2+2}\Big|_{l=p-\frac{1}{p}}-\int\frac{dk}{k^2-2}\Big|_{k=p+\frac{1}{p}}\right]=\frac{1}{2}\left[\frac{1}{\sqrt{2}}\arctan\frac{p}{\sqrt{2}}-\frac{1}{2\sqrt{2}}\ln\frac{p-\sqrt{2}}{p+\sqrt{2}}\right]=$$

$$=\frac{\sqrt{2}}{4}\cdot\arctan\frac{\sqrt{t}}{2}-\frac{\sqrt{2}}{8}\cdot\ln\frac{\sqrt{t}-\sqrt{2}}{\sqrt{t}+\sqrt{2}}=\frac{\sqrt{2}}{4}\arctan\frac{\sqrt{\tan x}}{2}-\frac{\sqrt{2}}{8}\ln\frac{\sqrt{\tan x}-\sqrt{2}}{\sqrt{\tan x}+\sqrt{2}}$$

$$I_2=\int\frac{p^2}{(2+p^4)}dp=\begin{Bmatrix}p=\sqrt[4]{2}\cdot z\\ dp=\sqrt[4]{2}dz\end{Bmatrix}=\int\frac{\sqrt{2}\cdot z^2}{(2+2\cdot z^4)}\cdot\sqrt[4]{2}dz=\frac{\sqrt{2}\cdot\sqrt[4]{2}}{2}\int\frac{z^2}{1+z^4}dt=$$

$$=\frac{2^{\frac{1}{2}}\cdot 2^{\frac{1}{4}}}{2}\left[\frac{\sqrt{2}}{2}\arctan\frac{z}{\sqrt{2}}-\frac{1}{2\sqrt{2}}\ln\frac{z-\sqrt{2}}{z+\sqrt{2}}\right]\cdot\frac{1}{2}=$$

$$=\left(\frac{\sqrt{2}\cdot\sqrt[4]{2}}{2}\cdot\frac{\sqrt{2}}{2}\arctan\frac{p}{\sqrt[4]{2}\cdot\sqrt{2}}-\frac{\sqrt{2}\cdot\sqrt[4]{2}}{2}\cdot\frac{1}{2\sqrt{2}}\ln\frac{p-\sqrt[4]{2}\cdot\sqrt{2}}{p+\sqrt[4]{2}\cdot\sqrt{2}}\right)\frac{1}{2}=$$

$$=\left(\frac{\sqrt[4]{2}}{2}\arctan\frac{\sqrt{t}}{\sqrt[4]{2^3}}-\frac{\sqrt[4]{2}}{4}\ln\frac{\sqrt{t}-\sqrt[4]{2^3}}{\sqrt{t}+\sqrt[4]{2^3}}\right)\frac{1}{2}=$$

$$=\left(\frac{\sqrt[4]{2}}{4}\arctan\frac{\sqrt{\tan x}}{\sqrt[4]{2^3}}-\frac{\sqrt[4]{2}}{4}\cdot\ln\frac{\sqrt{\tan x}-\sqrt[4]{2^3}}{\sqrt{\tan x}+\sqrt[4]{2^3}}\right)\cdot\frac{1}{2}$$

$$=\frac{\sqrt[4]{2}}{4}\arctan\frac{\sqrt{\tan x}}{\sqrt[4]{2^3}}-\frac{\sqrt[4]{2}}{8}\cdot\ln\frac{\sqrt{\tan x}-\sqrt[4]{2^3}}{\sqrt{\tan x}+\sqrt[4]{2^3}}$$

$$I_3=\int\frac{p^2}{(2+p^4)^2}dp=\begin{Bmatrix}p=\sqrt[4]{2}\cdot z\\ dp=\sqrt[4]{2}\cdot dt\end{Bmatrix}=\int\frac{\sqrt{2}\cdot\sqrt[4]{2}\cdot z^2}{4(1+z^4)^2}dz=\frac{\sqrt[4]{2^3}}{4}\int\frac{z^2}{(1+z^4)^2}dz=I_4$$

$$I_4=\int\frac{z^2}{(1+z^4)^2}dz=\frac{Az^3+Bz^2+Cz+D}{z^4+1}+\int\frac{az^3+bz^2+cz+p}{(z^4+1)^8}dz\Big|'$$

$$\frac{z^2}{(1+z^4)^2}=\frac{(3Az^2+2Bz+C)(z^4+1)-(Az^3+Bz^2+Cz+D)(4z^3)}{(z^4+1)^2}+\frac{az^3+bz^2+cz+p}{(z^4+1)}\Big|_{(z^4+1)^2}$$

$$z^2=3Az^6+2Bz^5+Cz^4+3Az^2+2Bz+C-4Az^6-4Bz^5-4Cz^4-4Dz^3+$$
$$+\left(az^7+bz^6+cz^5+Dz^4+az^3+bz^2+cz+d\right)=$$

$$=az^7+(-A+b)z^6+(-2B+c)z^5+(-3C+d)z^4+(-4D+a)z^3+(3A+b)z^2+(2B+C)z+c+d=0$$

$$\Rightarrow\begin{cases}a=0\\ A=b\\ 2B=c\\ 3C=d\Rightarrow C=0\\ 4D=a\Rightarrow D=0\\ 3A+b=1\Rightarrow A=\frac{1}{4}\\ 2B+c=0\Rightarrow c=0\\ c+d=0\Rightarrow d=0\end{cases}=\frac{1}{4}\cdot\frac{z^3}{z^4+1}+\frac{1}{4}\int\frac{z^2}{z^4+1}dz=\frac{1}{4}\cdot\frac{z^3}{z^4+1}+\frac{1}{4}\left[\frac{1}{2}\int\frac{z^3+1}{z^4+1}dt-\frac{1}{2}\int\frac{z^2-1}{z^4+1}dz\right]=$$

$$=\frac{1}{4}\cdot\frac{z^3}{z^4+1}+\frac{1}{8}\left[\frac{1}{\sqrt{2}}\arctan\frac{z}{\sqrt{2}}-\frac{1}{2\sqrt{2}}\arctan\frac{z-\sqrt{2}}{z+\sqrt{2}}\right]=$$

$$=\frac{1}{4}\cdot\frac{\left(\frac{p}{\sqrt{2}}\right)^3}{\left(\frac{p}{\sqrt{2}}\right)^3+1}+\frac{1}{8\sqrt{2}}\arctan\frac{\frac{p}{4\sqrt{2}}}{\sqrt{2}}-\frac{1}{16\sqrt{2}}\arctan\frac{\frac{p}{\sqrt[4]{2}}-\sqrt{2}}{\frac{p}{\sqrt[4]{2}}+\sqrt{2}}=$$

$$=\frac{1}{4}\cdot\frac{p^3}{p^3+\sqrt[4]{2^3}}+\frac{1}{8\sqrt{2}}\arctan\frac{p}{\sqrt[4]{2^3}}-\frac{1}{16\sqrt{2}}\arctan\frac{p-\sqrt[4]{2^3}}{p+\sqrt[4]{2^3}}$$

$$=\frac{1}{4}\cdot\frac{(\sqrt{t})^3}{(\sqrt{t})^3+\sqrt[4]{2}}+\frac{1}{8\sqrt{2}}\cdot\frac{\sqrt{t}}{\sqrt[4]{2^3}}-\frac{1}{16\sqrt{2}}\arctan\frac{\sqrt{t}-\sqrt[4]{2^3}}{\sqrt{t}+\sqrt[4]{2^3}}$$

$$I=\frac{1}{8}I_1-\frac{1}{8}\cdot I_2-\frac{1}{8}\cdot I_3=\frac{1}{8}\left(I_1-I_2-\frac{\sqrt[4]{2^3}}{2}\cdot I_4\right)=$$

$$=\frac{1}{8}\left[\left(\frac{\sqrt{2}}{4}\arctan\frac{\sqrt{\tan x}}{2}-\frac{\sqrt{2}}{8}\ln\frac{\sqrt{x}-\sqrt{2}}{\sqrt{\tan x}+\sqrt{2}}\right)-\left(\frac{\sqrt[4]{2}}{4}\arctan\frac{\sqrt{\tan x}}{\sqrt[4]{2^3}}-\frac{\sqrt[4]{2}}{8}\cdot\ln\frac{\sqrt{\tan x}-\sqrt[4]{2^3}}{\sqrt{\tan x}+\sqrt[4]{2^3}}\right)-\right.$$
$$\left.-\left(\frac{\sqrt[4]{2^3}}{2}\cdot\left(\frac{1}{4}\cdot\frac{(\sqrt{\tan x})^3}{(\sqrt{\tan x})^3+\sqrt[4]{2^3}}+\frac{1}{8\sqrt{2}}\arctan\frac{\sqrt{\tan x}}{\sqrt[4]{2^3}}-\frac{1}{16\sqrt{2}}\arctan\frac{\sqrt{\tan x}-\sqrt[4]{2^3}}{\sqrt{\tan x}+\sqrt[4]{2^3}}\right)\right)\right]+C$$

470. Find:

$$\Omega=\int_0^{2018}\{x\}\cdot\{x-1\}\cdot\{x-2\}\cdot\ldots\cdot\{x-2018\}dx,\{x\}=x-[x],[*]-GIF\ function$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

Solution 1 by Ravi Prakash-New Delhi-India

For $k\in\mathbb{N},\{x-k\}=x-k-[x-k]=x-k-([x]-k)=x-[x]=\{x\}$

$$\therefore\Omega=\int_0^{2018}\{x\}^{2019}dx=\int_0^{2018}(x-[x])^{2019}dx=\sum_{k=1}^{2018}\int_{k-1}^{k}(x-k+1)^{2019}dx$$

$$=\sum_{k}^{2018}\frac{1}{2020}(x-k)^{2019}]_{k-1}^{k}=\sum_{k=1}^{2018}\frac{1}{2020}(1-0)=\frac{2018}{2020}=\frac{1009}{1010}$$

Solution 2 by Sagar Kumar-Patna Bihar-India

$$I=\int_0^{2018}\prod_{k=0}^{2018}\{x-k\}dx$$

Now, $\{x-k\}=\{x\}$ ***if*** $k\in I$

$$I=\int_0^{2018}\{x\}^{2019}\,dx$$

$$I=\sum_{k=0}^{2017}\int_k^{k+1}(x-k)^{2019}\,dx=\sum_{k=0}^{2017}\frac{(x-k)^{2020}}{2020}\Big|_k^{k+1}=\frac{2018}{2020}=\frac{1009}{1010}$$

471. Find:

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_0^{\frac{\pi}{2}-\varepsilon}\left(\log\left(\frac{\sqrt{1+\cos^2 x}}{\cos x}\right)dx\right)\right)$$

Proposed by Abdul Mukhtar-Nigeria

Solution 1 by Sagar Kumar-Patna Bihar-India

$$\Omega=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(2+\tan^2 x)\,dx.\ \textit{Put}\ \tan x=t;\sec^2 x\,dx=dt$$

$$dx=\left(\frac{dt}{1+t^2}\right);\ \Omega=\frac{1}{2}\int_0^{\infty}\frac{\ln(t^2+2)}{(t^2+1)}dt.\ \textit{Let}\ \Omega(a)=\frac{1}{2}\int_0^{\infty}\frac{\ln(at^2+2)}{(t^2+1)}dt$$

$$\Omega(0)=\frac{\pi}{4}\ln 2\quad (1)$$

$$\Omega'(a)=\frac{1}{2}\int_0^{\infty}\frac{t^2}{(t^2+1)(at^2+2)}dt=\frac{1}{2}\left(\underbrace{\int_0^{\infty}\frac{dt}{(at^2+2)}}_{I_1}-\underbrace{\int_0^{\infty}\frac{dt}{(t^2+1)(at^2+2)}}_{I_2}\right)$$

$$I_1=\frac{1}{a}\int_0^{\infty}\frac{dt}{t^2+\left(\sqrt{\frac{2}{a}}\right)^2};\ I_2=\left(\frac{\pi}{2\sqrt{2}}\right)\frac{1}{\sqrt{a}}$$

$$I_2=\int_0^{\infty}\frac{dt}{(t^2+1)(at^2+2)}=\frac{1}{(2-a)}\int_0^{\infty}\frac{1}{t^2+1}-\frac{1}{t^2+\frac{2}{a}}dt$$

$$=\left(\frac{1}{2-a}\right)\left(\frac{\pi}{2}-\frac{\pi\sqrt{a}}{2\sqrt{2}}\right)=\frac{\pi}{2}\left(\frac{1}{(2-a)}-\frac{\sqrt{a}}{(2-a)\sqrt{2}}\right)$$

$$\Omega'(a)=\frac{1}{2}\left(\frac{\pi}{2\sqrt{2}(\sqrt{a})}-\frac{\pi}{2(2-a)}+\frac{\pi\sqrt{a}}{(2-a)2\sqrt{2}}\right)=\frac{\pi}{4\sqrt{2}}\left(\frac{1}{\sqrt{a}}-\frac{\sqrt{2}}{2-a}+\frac{\sqrt{a}}{a-2}\right)$$

$$\Omega(a)=\frac{\pi}{4\sqrt{2}}\left(2\sqrt{a}-\sqrt{2}\ln|a-2|+2\sqrt{a}+\sqrt{2}\ln\left|1-\frac{\sqrt{a}}{\sqrt{2}}\right|-\sqrt{2}\ln\left|\frac{\sqrt{a}}{\sqrt{2}}+1\right|\right)+C$$

$$\Omega(a)=\frac{\pi}{4\sqrt{2}}\left(2\sqrt{a}+\sqrt{2}\ln|a-2|-2\sqrt{a}-\sqrt{2}\ln\left(\frac{\sqrt{2}-\sqrt{a}}{\sqrt{2}}\right)+\sqrt{2}\left(\ln\left(\frac{\sqrt{a}+\sqrt{2}}{\sqrt{2}}\right)\right)\right)+C$$

$$\Omega(a)=\frac{\pi}{4\sqrt{2}}\left(\sqrt{2}\ln|a-2|+\sqrt{2}\ln\left|\frac{\sqrt{a}+\sqrt{2}}{\sqrt{a}-\sqrt{2}}\right|\right)+C$$

$$\Omega(0)=\frac{\pi}{4\sqrt{2}}\left(\sqrt{2}\ln 2\right)+C=\frac{\pi}{4}(\ln 2)$$

$$C=0$$

$$\Omega(1)=\frac{\pi}{4\sqrt{2}}\left(\sqrt{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right)=\frac{\pi}{2}\ln\left(\sqrt{2}+1\right)$$

Solution 2 by Togrul Ehmedov-Baku-Azerbaijan

$$\int_0^{\frac{\pi}{2}}\ln\left(\frac{\sqrt{1+\cos^2 x}}{\cos x}\right)dx=\int_0^{\frac{\pi}{2}}\ln\sqrt{1+\cos^2 x}\,dx-\int_0^{\frac{\pi}{2}}\ln(\cos x)\,dx$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(1+\cos^2 x)\,dx-\left(-\frac{\pi}{2}\ln 2\right)=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(1+\cos^2 x)\,dx+\frac{\pi}{2}\ln 2$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(2\cos^2 x+\sin^2 x)\,dx+\frac{\pi}{2}\ln 2=\frac{1}{2}\left(\pi\ln\left(\frac{\sqrt{2}+1}{2}\right)\right)+\frac{\pi}{2}\ln 2$$

$$=\frac{\pi}{2}\ln\left(\frac{\sqrt{2}+1}{2}\right)+\frac{\pi}{2}\ln 2=\frac{\pi}{2}\ln\left(\sqrt{2}+1\right)$$

Note:

$$I=\int_0^{\frac{\pi}{2}}\ln(y^2\cos^2 x+\sin^2 x)\,dx=\pi\ln\left(\frac{1+y}{2}\right)$$

Solution 3 by Zaharia Burghelea-Romania

$$I=\int_0^{\frac{\pi}{2}}\ln\left(\frac{\sqrt{1+\cos^2 x}}{\cos x}\right)dx$$

First, we rewrite the integral as: $I=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\left(\frac{1+\cos^2 x}{\cos^2 x}\right)dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(2+\tan^2 x)\,dx$

Substituting $\tan x=t$ *we get:* $I=\frac{1}{2}\int_0^{\infty}\frac{\ln(2+t^2)}{1+t^2}dt$

We now consider the following integral:

$$I(a)=\frac{1}{2}\int_0^{\infty}\frac{\ln\left(1+a(1+x^2)\right)}{1+x^2}dx\to I'(a)=\frac{1}{2}\int_0^{\infty}\frac{1+x^2}{\left(1+a(1+x^2)\right)(1+x^2)}dx$$

$$=\frac{1}{2}\int_0^{\infty}\frac{dx}{1+a+ax^2}=\frac{1}{2}\cdot\frac{1}{\sqrt{a+a^2}}\arctan\left(\sqrt{\frac{a}{a+1}}x\right)\Big|_0^{\infty}=\frac{\pi}{4}\cdot\frac{1}{\sqrt{a+a^2}}$$

Since $I(0)=0$*. We have that* $I=I(1)-I(0)=\int_0^1 I'(a)\,da$

$$I=\frac{\pi}{4}\int_0^1\frac{da}{\sqrt{a}\sqrt{1+a}}$$

With $\sqrt{a}=t$ *we get:* $I=\frac{\pi}{2}\int_0^1\frac{dt}{\sqrt{1+t^2}}=\frac{\pi}{2}\ln\left(1+\sqrt{2}\right)$

472. Prove that:

$$\int_0^1\frac{\ln^2(1+x)\left(\ln^2(1+x)+6\ln^2(1-x)\right)}{x}dx=\frac{21}{4}\zeta(5)$$

Proposed by Pedro Pantoja-Brazil

Solution by Zaharia Burghelea-Romania

Consider the following identity: $a^2(a^2+6b^2)=\frac{1}{2}\left((a+b)^4+(a-b)^4\right)-b^4$

$$I=\frac{1}{2}\int_0^1\frac{\ln^4(1-x^2)}{x}dx+\frac{1}{2}\int_0^1\frac{\ln^4\left(\frac{1-x}{1+x}\right)}{x}dx-\int_0^1\frac{\ln^4(1-x)}{x}dx$$

In the first integral let $x^2=t$ *to obtain:*

$$\int_0^1 \frac{\ln^4(1-x^2)}{x}dx = \frac{1}{2}\int_0^1 \frac{\ln^4(1-x)}{x}dx = \frac{1}{2}\int_0^1 \frac{\ln^4 x}{1-x}dx$$

For the second integral substitute: $\frac{1-x}{1+x} = t \to dx = -\frac{2}{(1+t)^2}dt$

$$\int_0^1 \frac{\ln^4\left(\frac{1-x}{1+x}\right)}{x}dx = 2\int_0^1 \frac{\ln^4 t}{1-t^2}dt \to I = \int_0^1 \frac{\ln^4 x}{1-x^2}dx - \frac{3}{4}\int_0^1 \frac{\ln^4 x}{1-x}dx$$

Recall the geometric series: $\frac{1}{1-x^z} = \sum_{n=0}^{\infty} x^{nz}, |x| < 1$

And consider the following integral: $J(k) = \int_0^1 x^k\,dx = \frac{1}{k+1}$

$$\frac{d^4}{dk^4}\big(J(k)\big) = \int_0^1 x^k \ln^4 x\,dx = (-1)^4\frac{4!}{(k+1)^5} = \frac{24}{(k+1)^5}$$

$$I = \sum_{n=0}^{\infty}\int_0^1 x^{2n}\ln^4 x\,dx - \frac{3}{4}\sum_{n=0}^{\infty}\int_0^1 x^n \ln^4 x\,dx$$

$$I = 24\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5} - 18\sum_{n=0}^{\infty}\frac{1}{(n+1)^5} = \frac{3}{4}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{2}\right)^5} - 18\sum_{n=1}^{\infty}\frac{1}{n^5}$$

$$= -\frac{3}{4}\frac{1}{4!}\psi_4\left(\frac{1}{2}\right) - 18\zeta(5) = \frac{93}{4}\zeta(5) - 18\zeta(5) = \frac{21}{4}\zeta(5)$$

Where the following identity of polygamma for $n = 4$ was used:

$$\psi_n\left(\frac{1}{2}\right) = (-1)^{n+1}n!\,(2^{n+1}-1)\zeta(n+1)$$

473. Find:

$$\Omega = \int\left(\frac{(x-1)\cos x - (x+1)\sin x}{x^2 + \sin 2x + 1}\right)dx, x \in \mathbb{R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Pierre Mounir-Cairo-Egypt

$$I = \int \frac{(x-1)\cos x - (x+1)\sin x}{x^2 + \sin(2x) + 1}dx$$

$$= \int \frac{x(\cos x - \sin x) - (\sin x + \cos x)}{x^2 + (\sin x + \cos x)^2} dx = \int \frac{\left\{\frac{x(\cos x - \sin x) - (\sin x + \cos x)}{x^2}\right\}}{1 + \frac{(\sin x + \cos x)^2}{x^2}} dx$$

$$= \int \frac{d\left(\frac{\sin x + \cos x}{x}\right)}{1 + \left(\frac{\sin x + \cos x}{x}\right)^2} = \tan^{-1}\left(\frac{\sin x + \cos x}{x}\right) + C$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

$$I = \int \frac{(x-1)\cos x - (x+1)\sin x}{x^2 + \sin 2x + 1} dx = \int \frac{x(\cos x - \sin x) - \cos x - \sin x}{(1 + \sin 2x)\left(\frac{x^2}{1 + 2\cos x \sin x} + 1\right)} dx =$$

$$= \int \frac{d\big(x(\cos x + \sin x)\big)}{(\cos x + \sin x)^2\left(\frac{x^2}{(\cos x + \sin x)^2} + 1\right)} dx$$

$$= \int \frac{d\left(\frac{x}{\cos x + \sin x}\right)}{\left(\frac{x}{\cos x + \sin x}\right)^2 + 1} dx = -\operatorname{arccotan}\frac{\cos x + \sin x}{x} + c$$

474. Find:

$$\Omega = \int_{-\infty}^{\infty} \left(\exp\left(-\frac{3x^2 + 15}{2x^2 + 18}\right) \cdot \cos\left(\frac{2x}{x^2 + 9}\right) \cdot \frac{1}{x^2 + 1}\right) dx$$

Proposed by Zaharia Burghelea-Romania

Solution by proposer

$$\Omega = \int_{-\infty}^{\infty} \exp\left(-\frac{3x^2 + 15}{2x^2 + 18}\right) \cos\left(\frac{2x}{x^2 + 9}\right) \frac{1}{x^2 + 1}$$

$$x = 3\tan t\,;\, dt = 3\sec^2 t\, dt$$

$$\Omega = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(-\frac{3}{2} \cdot \frac{9\tan^2 t + 5}{9\tan^2 t + 9}\right) \cos\left(\frac{6\tan t}{9\tan^2 t + 9}\right) \frac{3\sec^2 t}{9\tan^2 t + 1} dt$$

$$= 3e^{-\frac{3}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\exp\left(\frac{2}{3}\cos^2 t\right)\cos\left(\frac{2}{3}\sin t\cos t\right)\frac{dt}{5-4\cos(2t)}$$

$$= 6e^{-\frac{3}{2}}\exp\left(\frac{1}{3}(1+\cos(2t))\right)\cos\left(\frac{\sin(2t)}{3}\right)\frac{dt}{5-4\cos(2t)}$$

$$= 3e^{-\frac{7}{6}}\int_0^{\pi}\exp\left(\frac{\cos x}{3}+\frac{i\sin x}{3}\right)\frac{dx}{5-4\cos x}$$

$$\Omega = \frac{3}{2}e^{-\frac{7}{6}}\Re\left(\int_0^{2\pi}\exp\left(\frac{e^{ix}}{3}\right)\frac{dx}{5-2(e^{-ix}+e^{-ix})}\right)$$

Denote the inner integral as X and substitute:

$$e^{ix} = z \to dx = \frac{dz}{iz}, |z| = 1$$

$$x = \oint_{|z|=1}\frac{e^{\frac{z}{3}}}{5-2\frac{z^2+1}{z}}\frac{dz}{iz} = \frac{1}{i}\oint_{|z|=1}\frac{e^{\frac{z}{3}}}{-2z^2+5z-2}dz$$

$$-2z^2+5z-2 = -\frac{1}{2}\left((2z)^2-5(2z)+4\right) = -\frac{1}{2}(2z-4)(2z-1) = -2(z-2)\left(z-\frac{1}{2}\right)$$

In the contour $|z| = 1$ only the pole $z_2 = \frac{1}{2}$ is found, so:

$$X = \frac{1}{i}2\pi i\,Res\,(f(x), z_2),\ \textit{where}\ f(x) = \frac{e^{\frac{z}{3}}}{-2(z-2)\left(z-\frac{1}{2}\right)}$$

$$X = 2\pi\lim_{z\to z_2}(z-z_2)\frac{e^{\frac{z}{3}}}{-2(z-2)(z-z_2)} = 2\pi\frac{e^{\frac{1}{6}}}{3}$$

$$\Omega = \frac{3}{2}e^{-\frac{7}{6}}\Re\left(\frac{2}{3}\pi e^{\frac{1}{6}}\right) = \frac{\pi}{e}$$

475. Find:

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_{\varepsilon}^{\frac{\pi}{2}-\varepsilon}\left(\frac{\log(\tan x\cdot\sec x)}{\tan x+\sec x}\right)dx\right)$$

Proposed by Zaharia Burghelea-Romania

Solution by Togrul Ehmedov-Baku-Azerbaijan

$$I=\int_0^{\frac{\pi}{2}}\frac{\ln(\tan x\cdot\sec x)}{\tan x+\sec x}dx=\int_0^{\frac{\pi}{2}}\frac{\ln\frac{\sin x}{\cos^2 x}}{\frac{\sin x+1}{\cos x}}dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{\ln(\sin x)}{\sin x+1}\cos x\,dx-2\int_0^{\frac{\pi}{2}}\frac{\ln(\cos x)}{\sin x+1}\cos x\,dx$$

$$\sin x=t\Rightarrow\cos x\,dt=dt$$

$$I_1=\int_0^{\frac{\pi}{2}}\frac{\ln(\sin x)}{\sin x+1}\cos x\,dx=\int_0^1\frac{\ln(t)}{t+1}dt=-\frac{\pi^2}{12}$$

$$I_2=\int_0^{\frac{\pi}{2}}\frac{\ln(\cos x)}{\sin x+1}\cos x\,dx=\int_0^1\frac{\ln\sqrt{1-t^2}}{t+1}dt=\frac{1}{2}\int_0^1\frac{\ln(1-t^2)}{t+1}dt$$

$$=\frac{1}{2}\int_0^1\frac{\ln(1-t)}{t+1}dt+\frac{1}{2}\int_0^1\frac{\ln(1+t)}{t+1}dt=\frac{1}{2}\int_0^1\frac{\ln(1-t)}{t+1}dt+\left[\frac{1}{4}\ln^2(1+t)\right]_0^1$$

$$=\frac{1}{2}\int_0^1\frac{\ln(1-t)}{t+1}dt+\frac{1}{4}\ln^2(2)=\frac{1}{2}\left(-\frac{\pi^2}{12}+\frac{1}{2}\ln^2(2)\right)+\frac{1}{4}\ln^2(2)$$

$$=-\frac{\pi^2}{24}+\frac{1}{2}\ln^2(2)$$

$$I=-\frac{\pi^2}{12}-2\left(-\frac{\pi^2}{24}+\frac{1}{2}\ln^2(2)\right)=-\ln^2(2)$$

476. Prove:

$$\Omega=\int_{-\infty}^{\infty}\frac{\psi_0(x)\cdot\sin(\pi x)}{x}dx=-\pi\{\gamma+\log(2)\}$$

Proposed by Obidah Al Sharafy-Sana'a-Yemen

Solution by Pierre Mounir-Cairo-Egypt

$$I = PV\int_{-\infty}^{\infty}\frac{\psi(x)\sin(\pi x)}{x}dx$$

$$\because \psi(x) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+x}\right) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\frac{x}{n(n+x)}$$

$$\therefore I = -\gamma\int_{-\infty}^{\infty}\frac{\sin(\pi x)}{x}dx - PV\int_{-\infty}^{\infty}\overbrace{\sin(\pi x)\,dx}^{odd\ function} + \sum_{n=1}^{\infty}\frac{1}{n}\left(PV\int_{-\infty}^{\infty}\frac{\sin(\pi x)}{x+n}dx\right)$$

$$= -\gamma\pi - 0 + J$$

$$J = \sum_{n=1}^{\infty}\frac{1}{n}\left(PV\int_{-\infty}^{\infty}\frac{\sin(\pi t-\pi n)}{t}dt\right) \quad (t = x+n)$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}\left(\cos(\pi n)\,PV\int_{-\infty}^{\infty}\frac{\sin(\pi t)}{t}dt\right) - \sum_{n=1}^{\infty}\frac{1}{n}\left(\sin(\pi n)\,PV\int_{-\infty}^{\infty}\overbrace{\frac{\cos(\pi t)}{t}}^{odd\ function}dt\right)$$

$$= \pi\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n} = -\pi\ln 2$$

Note:

$$\ln\left|2\sin\left(\frac{x}{2}\right)\right| = -\sum_{n=1}^{\infty}\frac{\cos(nx)}{n} \Rightarrow \ln\left|2\sin\left(\frac{\pi}{2}\right)\right| = -\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$$

$$\therefore I = -\gamma\pi - \pi\ln 2 = -\pi(\gamma + \ln 2)$$

477.

$$\omega(n) = \int_{-1}^{1}\frac{(1+2x+x^2)^n(1-2x+x^2)^n}{(1-x^2)(1+2x^2+x^4)^n}dx, n \in \mathbb{N}, n \geq 2$$

Find:

$$\Omega = \lim_{n\to\infty}\frac{\omega(n+2)-\omega(n+1)}{\omega(n+1)-\omega(n)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\omega(n)=\int_{-1}^{1}\frac{(1+2x+x^2)^n(1-2x+x^2)^n}{(1-x^2)(1+2x^2+x^4)^n}dx=\int_{-1}^{1}\frac{(1+x)^{2n}(1-x)^{2n}}{(1-x^2)(1+x^2)^{2n}}dx=$$

$$=\int_{-1}^{1}\frac{(1-x^2)^{2n-1}}{(1+x^2)^{2n}}dx=2\int_{-1}^{1}\left(\frac{1-x^2}{1+x^2}\right)^{2n-1}\frac{dx}{1+x^2}$$

Put $x=\tan\theta$, $dx=\sec^2\theta$

$$\Omega=2\int_{0}^{\frac{\pi}{4}}\cos^{2n-1}(2\theta)\,d\theta$$

Put $2\theta=\varphi$,

$$\omega(n)=\int_{0}^{\frac{\pi}{2}}\cos^{2n-1}\varphi\,d\varphi=\int_{0}^{\frac{\pi}{2}}\cos\varphi\cos^{2n-2}\varphi\,d\varphi=$$

$$=[\sin\varphi\cos^{2n-2}\varphi]_0^{\frac{\pi}{2}}+(2n-2)\int_{0}^{\frac{\pi}{2}}\cos^{2n-3}\varphi\sin^2\varphi\,d\varphi$$

$$=(2n-2)\int_{0}^{\frac{\pi}{2}}\cos^{2n-3}(1-\cos^2\varphi)\,d\varphi\Rightarrow(2(n-1)+1)\varphi(n)=(2n-2)\omega(n-1)$$

$\omega(n)=\frac{2n-2}{2n-1}\omega(n-1)$. ***Now,*** $\omega(n+2)=\frac{2n+2}{2n+3}\omega(n+1)$ ***and*** $\omega(n+1)=\frac{2n}{2n+1}\omega(n)$

$$\therefore\Omega=\lim_{n\to\infty}\frac{\omega(n+2)-\omega(n+1)}{\omega(n+1)-\omega(n)}=\lim_{n\to\infty}\frac{\left(\frac{2n+2}{2n+3}-1\right)\omega(n+1)}{\left(\frac{2n}{2n+1}-1\right)\omega(n)}=\lim_{n\to\infty}\frac{2n+1}{2n+3}\cdot\frac{2n}{2n+1}=\lim_{n\to\infty}\frac{1}{1+\frac{3}{2n}}=1$$

Solution 2 by Shafiqur Rahman-Bangladesh

$$\omega(n)=\int_{-1}^{1}\frac{(1+2x+x^2)^n(1-2x+x^2)^n}{(1-x^2)(1+2x^2+x^4)^n}dx\left[x\to\tan\left(\frac{x}{2}\right)\right]=\int_{0}^{\frac{\pi}{2}}\cos^{2n-1}dx\Rightarrow$$

$$\Rightarrow\frac{\omega(n+1)}{\omega(n)}=\frac{2n}{2n+1}\therefore\Omega=\lim_{n\to\infty}\frac{\omega(n+2)-\omega(n+1)}{\omega(n+1)-\omega(n)}=\lim_{n\to\infty}\frac{\frac{\omega(n+2)}{\omega(n+1)}-1}{1-\frac{\omega(n)}{\omega(n+1)}}=$$

$$= \lim_{n\to\infty}\frac{\frac{2n+2}{2n+3}-1}{1-\frac{2n+1}{2n}} \therefore \Omega = \lim_{n\to\infty}\frac{\omega(n+2)-\omega(n+1)}{\omega(n+1)-\omega(n)} = 1$$

Solution 3 by Shivam Sharma-New Dehi-India

$$\int_{-1}^{1}\frac{(1+x)^{2n}(1-x)^{2n}}{(1-x^2)(1+x^2)^{2n}}dx$$

Let, $x = \tan(\theta) \Rightarrow 2\int_0^{\frac{\pi}{4}}\cos^{2n-1}(2\theta)\,d\theta$**. Let,** $2\theta = u \Rightarrow \int_0^{\frac{\pi}{2}}\cos^{2n-1}(2u)\,du \Rightarrow \frac{1}{2}\mathrm{B}\left(\frac{1}{2},n\right)$

$$\Rightarrow \frac{1}{2}\left[\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)}\right] \Rightarrow \frac{\sqrt{\pi}}{2}\left[\frac{(n-1)!}{\Gamma\left(n+\frac{1}{2}\right)}\right] \Rightarrow \frac{\pi}{2}\left[\frac{2^n(n-1)!}{(2n-1)!!}\right] \Rightarrow \frac{\pi}{n2^{1-2n}}\left[\frac{(n!)^2}{(2n)!}\right] \Rightarrow$$

$$\Rightarrow \frac{\pi}{n2^{1-2n}}\left[\frac{\left(\frac{n}{e}\right)^{2n}2\pi n}{\left(\frac{2n}{e}\right)^{2n}2\sqrt{\pi n}}\right] \Rightarrow \frac{\pi}{n2^{1-2n}}\left[\frac{\sqrt{\pi n}}{2^{2n}}\right]$$ **(OR)** $\omega(n) = \frac{\pi}{2}\left[\sqrt{\frac{\pi}{n}}\right]$**. Now, let,**

$$L = \lim_{n\to\infty}\left(\frac{\omega(n+2)-\omega(n+1)}{\omega(n+1)-\omega(n)}\right) \Rightarrow \lim_{n\to\infty}\left[\frac{\frac{\pi}{2}\left[\sqrt{\frac{\pi}{n+2}}-\sqrt{\frac{\pi}{n+1}}\right]}{\frac{\pi}{2}\left[\sqrt{\frac{\pi}{n+1}}-\sqrt{\frac{\pi}{n}}\right]}\right] \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{\frac{\sqrt{n+1}-\sqrt{n+2}}{\sqrt{n+2}}}{\frac{\sqrt{n}-\sqrt{n+1}}{\sqrt{n}\left(\sqrt{n+1}\right)}}\right]$$

After Rationalizing, we get: $\Rightarrow \frac{\sqrt{n}}{\sqrt{n+2}}\left[\frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{n+1}+\sqrt{n+2}}\right] \Rightarrow \lim_{n\to\infty}\frac{1}{1+\frac{2}{n}}\left[\frac{1+\sqrt{1+\frac{1}{n}}}{\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}}+1}\right] \Rightarrow \frac{2}{2}$

(OR) L=1 (Answer)

478.

$$\Omega(x) = \lim_{n\to\infty}\sum_{k=1}^{n}2^{k-1}\tan\frac{x}{2^{k-1}}\tan^2\frac{x}{2^k}$$

Prove that in acute $\Delta\, ABC$ **the following relationship holds:**

$$\Omega(A)+\Omega(B)+\Omega(C)>A\cdot B\cdot C-\pi$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Remus Florin Stanca-Romania

$$\frac{\tan\left(\frac{x}{2^{k-1}}\right)-\tan\left(\frac{x}{2^k}\right)}{1+\tan\left(\frac{x}{2^{k-1}}\right)\cdot\tan\left(\frac{x}{2^k}\right)}=\tan\left(\frac{x}{2^{k-1}}-\frac{x}{2^k}\right)=\tan\left(\frac{x}{2^k}\right);\ \textit{let}\ \tan\left(\frac{x}{2^{k-1}}\right)\tan\left(\frac{x}{2^k}\right)=\alpha$$

$$\frac{\tan\left(\frac{x}{2^{k-1}}\right)-\tan\left(\frac{x}{2^k}\right)}{1+\alpha}=\tan\left(\frac{x}{2^k}\right)\Leftrightarrow\alpha\tan\left(\frac{x}{2^k}\right)+\tan\left(\frac{x}{2^k}\right)=\tan\left(\frac{x}{2^{k-1}}\right)-\tan\left(\frac{x}{2^k}\right)$$

$$\alpha=\frac{\tan\left(\frac{x}{2^{k-1}}\right)-2\tan\left(\frac{x}{2^k}\right)}{\tan\left(\frac{x}{2^k}\right)}=\frac{\tan\left(\frac{x}{2^{k-1}}\right)}{\tan\left(\frac{x}{2^k}\right)}-2\ \textit{so we have:}\ \tan\left(\frac{x}{2^{k-1}}\right)\tan\left(\frac{x}{2^k}\right)=\frac{\tan\left(\frac{x}{2^{k-1}}\right)}{\tan\left(\frac{x}{2^k}\right)}-2$$

$$\Rightarrow\tan\left(\frac{x}{2^{k-1}}\right)\tan^2\left(\frac{x}{2^k}\right)=\tan\left(\frac{x}{2^{k-1}}\right)-2\tan\left(\frac{x}{2^k}\right)\Rightarrow 2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)\tan^2\left(\frac{x}{2^k}\right)=$$

$$=2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)-2^k\tan\left(\frac{x}{2^k}\right)$$

$$\sum_{k=1}^{n}2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)\tan^2\left(\frac{x}{2^k}\right)=\sum_{k=1}^{n}2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)-2^k\tan\left(\frac{x}{2^k}\right)$$

Let $a_k=2^k\tan\left(\frac{x}{2^k}\right),\ k=\overline{1,n}\Rightarrow\sum_{k=1}^{n}2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)\tan^2\left(\frac{x}{2^k}\right)=\sum_{k=1}^{n}a_{k-1}-a_k=$

$$=a_0-a_n=\tan(x)-2^n\tan\left(\frac{x}{2^n}\right)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)\tan^2\left(\frac{x}{2^k}\right)=\lim_{n\to\infty}\left(\tan x-2^n\tan\left(\frac{x}{2^n}\right)\right)$$

$$=\tan x-\lim_{n\to\infty}2^n\tan\left(\frac{x}{2^n}\right)=$$

$$=\tan x-\lim_{n\to\infty}\frac{\tan\left(\frac{x}{2^n}\right)}{\frac{1}{2^n}}=\tan x-\lim_{n\to\infty}\frac{\tan\left(\frac{x}{2^n}\right)}{\frac{x}{2^n}}\cdot x=\tan x-x\Rightarrow$$

$$\Rightarrow\Omega(x)=\lim_{n\to\infty}\sum_{k=1}^{n}2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)\tan^2\left(\frac{x}{2^k}\right)=\tan x-x$$

$$\Omega(A)+\Omega(B)+\Omega(C)>ABC-\pi\Leftrightarrow\tan A+\tan B+\tan C-A-B-C>ABC-\pi\Leftrightarrow$$

$$\Leftrightarrow\tan(A)+\tan(B)+\tan(C)>ABC\quad\textbf{(1)}$$

$$\frac{\tan A+\tan B}{1-\tan A\tan B}=\tan(A+B)=\tan(\pi-C)=\frac{\tan\pi-\tan C}{1+\tan\pi\tan C}=\frac{-\tan C}{1+0}=-\tan C$$

$$\Leftrightarrow \tan A+\tan B+\frac{\tan A+\tan B}{\tan A\tan B-1}>ABC\Leftrightarrow$$

$$(\tan A+\tan B)\cdot\frac{\tan A\tan B}{\tan A\tan B-1}>ABC>$$

$$-\tan(A+B)\tan A\tan B>ABC>\tan C\tan A\tan B>ABC$$

$$f(x)=\tan x-x>f(x)=\frac{1}{\cos^2 x}-1\ |\cos x|<1>\cos^2 x<1>\frac{1}{\cos^2 x}>1>$$

$$>\frac{1}{\cos^2 x}-1>0>f(x)>0>f(x)$$ ***is an increasing function***

Let $x=0>f(0)=0>f(x)>\tan x>x>\tan A\tan B\tan C>ABC$ ***(what we needed to prove)***

$$\Omega(A)+\Omega(B)+\Omega(C)>ABC-\pi$$ ***(Q.E.D.)***

Solution 2 by Shafiqur Rahman-Bangladesh

$$\Omega(x)=\lim_{n\to\infty}\sum_{k=1}^{n}2^{k-1}\tan\frac{x}{2^{k-1}}\tan^2\frac{x}{2^k}=\sum_{k=1}^{\infty}\left(2^{k-1}\tan\frac{x}{2^{k-1}}-2^k\tan\frac{x}{2^k}\right)=\tan x-x$$

Now, $\Omega(A)+\Omega(B)+\Omega(C)=\tan A+\tan B+\tan C-\pi \because \tan x=x+\frac{x^3}{3}+0(x^5)>x$

$$\therefore \Omega(A)+\Omega(B)+\Omega(C)>A\cdot B\cdot C-\pi$$

479.

$$\Omega(a,b,c)=\sum_{n=1}^{\infty}\frac{an^2+bn+c}{n!},\ a,b,c>0$$

Prove that:

$$\Omega(a,b,c)+\Omega(b,c,a)+\Omega(c,a,b)\geq 3(4e-1)\sqrt[3]{abc}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michael Stergiou-Greece

$$\Omega(a,b,c)+\Omega(b,c,a)+\Omega(c,a,b)\geq 3(4e-1)\sqrt[3]{abc}\quad \textbf{(1)}$$

$\sum_{cyc}\Omega(a,b,c)=(a+b+c)\cdot\sum_{n=1}^{\infty}\frac{n^2+n+1}{n!}$ ***(1). But*** $a+b+c\geq 3\sqrt[3]{abc}$ ***(2)***

and $\sum_{n=1}^{\infty}\frac{n^2+n+1}{n!}=4e-1$ ***because***

$$\sum_{n=0}^{\infty}\frac{1}{n!}=e\rightarrow\sum_{n=1}^{\infty}\frac{1}{n!}=e-1,\sum_{n=0}^{\infty}\frac{n}{n!}=e,\sum_{n=0}^{\infty}\frac{n^2}{n!}=2e\ (*)$$

(2)·(3)→ (1). Done.

(*) The first term for $n=0$ equals 0 so it is on to take the sum from $n=1$.

Solution 2 by Ravi Prakash-New Delhi-India

Write $an^2+bn+c=an(n-1)+(b+a)n+c,\forall n\geq 2$

Now, $\Omega(a,b,c)=\sum_{n=1}^{\infty}\frac{an^2+bn+c}{n!}=a+b+c+\sum_{n=2}^{\infty}\frac{an(n-1)+(b+a)n+c}{n!}=$

$$=a+b+c+\sum_{n=2}^{\infty}\left[\frac{a}{(n-2)!}+\frac{b+a}{(n-1)!}+\frac{c}{n!}\right]=$$

$$=a+b+c+ae+(b+a)(e-1)+c(e-2)=$$

$$=a(1+e+e-1)+b(1+e-1)+c(e-1)$$

$$\Omega(a,b,c)=2ae+be+c(e-1)$$

$$\Omega(b,c,a)=2be+ce+a(e-1)$$

$$\Omega(c,a,b)=2ce+ae+b(e-1)$$

$$\Rightarrow\Omega(a,b,c)+\Omega(b,c,a)+\Omega(c,a,b)=$$

$$=2(a+b+c)e+(a+b+c)e+(a+b+c)(e-1)=$$

$$=(a+b+c)(4e-1)\geq 3(4e-1)(abc)^{\frac{1}{3}}$$

Solution 3 by Shafiqur Rahman-Bangladesh

$$\Omega(a,b,c)=\sum_{n=1}^{\infty}\frac{an^2+bn+c}{n!}\Rightarrow\Omega(a,b,c)+\Omega(b,c,a)+\Omega(c,a,b)=$$

$$=(a+b+c)\sum_{n=1}^{\infty}\frac{n^2+n+1}{n!}=$$

$$=(a+b+c)\left(\sum_{n=1}^{\infty}\frac{1}{(n-2)!}+2\sum_{n=1}^{\infty}\frac{1}{(n-1)!}+\sum_{n=1}^{\infty}\frac{1}{n!}\right)=(a+b+c)(e+2e+e-1)\leq$$

$$\leq 3\sqrt[3]{abc}(4e-1)$$

$$\therefore\Omega(a,b,c)+\Omega(b,c,a)+\Omega(c,a,b)\leq 3(4e-1)\sqrt[3]{abc}$$

480. Find:

$$\Omega = \sum_{k=1}^{\infty}\left(\sum_{\substack{n=1\\n\neq k}}^{\infty}\frac{1}{n^2-k^2}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Khalef Ruhemi-Jarash-Jordan

$$\Omega := \sum_{j=1}^{\infty}\left(\sum_{\substack{n=1\\n\neq j}}^{\infty}\frac{1}{n^2-j^2}\right) \quad (*)$$

$$\Omega = S_1 + S_2 + S_3 + \cdots$$

$$S_1 = \sum_{n=2}^{\infty}\frac{1}{n^2-1} = \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n+1}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right) =$$

$$= \frac{1}{2}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)\right) =$$

$$= \frac{1}{2}\left(1+\frac{1}{2}\right) \Rightarrow 2S_1 = 1+\frac{1}{2} \quad (1)$$

$$S_2 = \sum_{\substack{n=1\\n\neq 2}}^{\infty}\frac{1}{n^2-4} = \frac{1}{(1)^2-4} + \sum_{n=3}^{\infty}\frac{1}{(n-2)(n+2)} =$$

$$= \frac{1}{(1)^2-(2)^2} + \sum_{n=1}^{\infty}\frac{1}{n(n+4)} = \frac{1}{(1)^2-(2)^2} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+4}$$

$$\therefore 4S_2 = \frac{(2)^2}{(1)^2-(2)^2} + \sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+1} + \sum_{n=1}^{\infty}\frac{1}{n+1}-\frac{1}{n+2} + \sum_{n=1}^{\infty}\frac{1}{n+2}-\frac{1}{n+3} +$$

$$+ \sum_{n=1}^{\infty}\frac{1}{n+3}-\frac{1}{n+4} = \frac{(2)^2}{(1)^2-(2)^2} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$\therefore 4S_2 = \frac{(2)^2}{(1)^2-(2)^2} + \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) \quad (2)$$

$$S_3 = \sum_{\substack{n=1\\n\neq 3}}^{\infty}\frac{1}{n^2-9} = \frac{1}{(1)^2-(3)^2} + \frac{1}{(2)^2-(3)^2} + \sum_{n=4}^{\infty}\frac{1}{n^2-9}$$

$$=\frac{1}{(1)^2-(3)^2}+\frac{1}{(2)^2-(3)^2}+\left(\frac{1}{6}\right)\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+6}$$

$$=\frac{1}{(1)^2-(3)^2}+\frac{1}{(2)^2-(3)^2}+\left(\frac{1}{6}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)$$

$$\therefore 6S_3=6\left(\frac{1}{(1)^2-(3)^2}+\frac{1}{(2)^2-(3)^2}\right)+\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)\quad (3)$$

$$S_4=\sum_{\substack{n=1\\ n\neq 4}}^{\infty}\frac{1}{n^2-(4)^2}=\frac{1}{(1)^2-(4)^2}+\frac{1}{(2)^2-(4)^2}+\frac{1}{(3)^2-(4)^2}+\frac{1}{8}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+8}\right)$$

$$\therefore 8S_4=8\left(\frac{1}{(1)^2-(4)^2}+\frac{1}{(2)^2-(4)^2}+\frac{1}{(3)^2-(4)^2}\right)+\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)\quad (4)$$

$$\therefore 2nS_n=(2n)\sum_{j=1}^{j=n-1}\frac{1}{j^2-n^2}+\sum_{j=1}^{j=2n}\frac{1}{j},n>1$$

$$\therefore S_n=\sum_{j=1}^{j=n-1}\frac{1}{j^2-n^2}+\frac{1}{2n}\sum_{j=1}^{j=2n}\frac{1}{j},n>1\quad (\#)$$

But $\sum_{\substack{j=1\\ j=2n}}^{j=n-1}\frac{1}{j^2-n^2}=-\sum_{j=1}^{j=n-1}\frac{1}{j(2n-j)}=-\frac{1}{2n}\cdot\sum_{j=1}^{j=n-1}\left(\frac{1}{j}+\frac{1}{2n-j}\right)$

Also, $\sum_{j=1}^{j=2n}\frac{1}{j}=\sum_{j=1}^{j=n-1}\frac{1}{j}+\sum_{j=n}^{j=2n}\frac{1}{j}=\sum_{\substack{j=1\\ j=n-1}}^{j=n-1}\frac{1}{j}+\sum_{j=0}^{j=n}\frac{1}{2n-j}=$

$$=\frac{1}{2n}+\frac{1}{n}+\sum_{j=1}^{j=n-1}\frac{1}{2n-j}+\sum_{j=1}^{j=n-1}\frac{1}{j}$$

$$\therefore S_n=-\frac{1}{2n}\cdot\sum_{j=1}^{j=n-1}\frac{1}{j}-\frac{1}{2n}\sum_{j=1}^{j=n-1}\frac{1}{2n-j}+\frac{1}{4n^2}+\frac{1}{2n^2}+\frac{1}{2n}\cdot\sum_{j=1}^{j=n-1}\frac{1}{j}+\frac{1}{2n}\sum_{j=1}^{j=n-1}\frac{1}{2n-j}$$

$$\therefore S_n=\frac{3}{4n^2},n=2,3,4\quad (\#\#)$$

$$\therefore \Omega=S_1+\sum_{n=2}^{\infty}S_n=\frac{1}{2}\left(1+\frac{1}{2}\right)+\left(\frac{3}{4}\right)\sum_{n=2}^{\infty}\frac{1}{n^2}=\left(\frac{3}{4}\right)\left[1+\sum_{n=2}^{\infty}\frac{1}{n^2}\right]=\left(\frac{3}{4}\right)\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$=\left(\frac{3}{4}\right)\zeta(2)=\left(\frac{3}{4}\right)\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{8}$$

$$\therefore\sum_{j=1}^{\infty}\left(\sum_{\substack{n=1\\ n\neq j}}^{\infty}\frac{1}{n^2-j^2}\right)=\frac{\pi^2}{8}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\text{Let } S_k=\sum_{\substack{n=1\\n\neq k}}^{\infty}\frac{1}{n^2-k^2}=\sum_{n=1}^{k-1}\frac{1}{n^2-k^2}+\sum_{n=k+1}^{\infty}\frac{1}{n^2-k^2}=\frac{1}{2k}\sum_{n=1}^{k-1}\left(\frac{1}{n-k}-\frac{1}{n+k}\right)+$$

$$+\frac{1}{2k}\sum_{n=k+1}^{\infty}\left(\frac{1}{n-k}-\frac{1}{n+k}\right)=-\frac{1}{2k}\sum_{n=1}^{k-1}\left(\frac{1}{k-n}+\frac{1}{k+n}\right)+\frac{1}{2k}\sum_{n=k+1}^{\infty}\left(\frac{1}{n-k}-\frac{1}{n+k}\right)$$

$$=-\frac{1}{2k}\left(\frac{1}{k-1}+\frac{1}{k+1}+\frac{1}{k-2}+\frac{1}{k+2}+\cdots+\frac{1}{1}+\frac{1}{2k-1}\right)+$$

$$+\frac{1}{2k}\left[\left(1-\frac{1}{2k+1}\right)+\left(\frac{1}{2}-\frac{1}{2k+2}\right)+\left(\frac{1}{3}-\frac{1}{2k+3}\right)+\cdots\right]=$$

$$=-\frac{1}{2k}\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}+\frac{1}{k+1}+\cdots+\frac{1}{2k-1}\right)+\frac{1}{2k}\left(1+\frac{1}{2}+\cdots+\frac{1}{2k}\right)=$$

$$=-\frac{1}{2k}\left(-\frac{1}{k}\right)+\frac{1}{4k^2}=\frac{3}{4k^2}$$

$$\text{Thus } \Omega=\sum_{k=1}^{\infty}\left(\sum_{\substack{n=1\\n\neq k}}^{\infty}\frac{1}{n^2-k^2}\right)=\sum_{k=1}^{\infty}S_k=\frac{3}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{3}{4}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{8}$$

481. Find:

$$\Omega=\lim_{n\to\infty}\left((n+1)\int_0^1\left(\frac{2x}{1+x^2}\right)^{n+1}dx-n\int_0^1\left(\frac{2x}{1+x^2}\right)^{n}dx\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Shafiqur Rahman-Bangladesh

$$\lim_{n\to\infty}\left((n+1)\int_0^1\left(\frac{2x}{1+x^2}\right)^{n+1}dx-n\int_0^1\left(\frac{2x}{1+x^2}\right)^{n}dx\right)[x\to\tan x]=$$

$$=\lim_{n\to\infty}\left((n+1)\left(\int_0^{\frac{\pi}{4}}2^{n+1}\sin^{n+1}x\cdot\cos^{n-1}x\,dx\right)-n\left(\int_0^{\frac{\pi}{4}}2^{n}\sin^{n}x\cdot\cos^{n-2}x\,dx\right)\right)$$

$$= \lim_{n\to\infty}\left((n+1)\left(-\frac{1}{n}+\frac{\sqrt{2\pi n}}{n+1}\right)-n\left(-\frac{1}{n-1}+\frac{\sqrt{2\pi(n-1)}}{n}\right)\right)=$$

$$= \lim_{n\to\infty}\left(\frac{1}{n(n-1)}+\sqrt{2\pi}\left(\sqrt{n}-\sqrt{n-1}\right)\right)$$

$$\therefore \lim_{n\to\infty}\left((n+1)\int_0^1\left(\frac{2x}{1+x^2}\right)^{n+1}-n\int_0^1\left(\frac{2x}{1+x^2}\right)^n dx\right)=0$$

482.

$$\omega(x)=\frac{1}{x}+\sum_{n=1}^{\infty}\frac{1}{2^n}tanh\left(\frac{x}{2^n}\right), x>0$$

Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\lim_{\substack{x\to 0\\ x>0}}\left(\frac{x^n}{\omega(x)\cdot\omega(2x)\cdot\omega(3x)\cdot\ldots\cdot\omega(nx)}\right)}$$

Proposed by Daniel Sitaru – Romania

Solution by Shafiqur Rahman-Bangladesh

$$\omega(x)=\frac{1}{x}+\sum_{n=1}^{\infty}\frac{1}{2^n}\tanh\left(\frac{x}{2^n}\right)=\frac{1}{x}+\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}\coth\left(\frac{x}{2^{n-1}}\right)-\frac{1}{2^n}\coth\left(\frac{x}{2^n}\right)\right)=\coth(x)$$

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\lim_{\substack{x\to\infty\\ x>0}}\left(\frac{x^n}{\omega(x)\cdot\omega(2x)\cdot\omega(3x)\cdot\ldots\cdot\omega(nx)}\right)}=$$

$$=\lim_{n\to\infty}\sqrt[n]{\lim_{\substack{x\to\infty\\ x>0}}\left(\frac{x^n}{\coth(x)\cdot\coth(2x)\cdot\ldots\cdot\coth(nx)}\right)}=$$

$$=\lim_{n\to\infty}\sqrt[n]{\lim_{\substack{x\to 0\\ x>0}}\left((x\tanh(x))\cdot(x\tanh(2x))\ldots(x\tanh(nx))\right)}=0$$

$$\therefore \lim_{n\to\infty}\sqrt[n]{\lim_{\substack{x\to\infty\\ x>0}}\left(\frac{x^n}{\omega(x)\cdot\omega(2x)\cdot\omega(3x)\cdot\ldots\cdot\omega(nx)}\right)}=0$$

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OR

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\lim_{\substack{x\to\infty\\x>0}}\left(x^n\omega(x)\cdot\omega(2x)\cdot\omega(3x)\cdot\ldots\cdot\omega(nx)\right)}=$$

$$=\lim_{n\to\infty}\sqrt[n]{\lim_{\substack{x\to\infty\\x>0}}\left(x^n\coth(x)\cdot\coth(2x)\cdot\ldots\cdot\coth(nx)\right)}$$

$$=\lim_{n\to\infty}\sqrt[n]{\lim_{\substack{x\to\infty\\x>0}}\left(\left(\frac{x}{\tanh(x)}\right)\cdot\left(\frac{x}{\tanh(2x)}\right)\cdots\left(\frac{x}{\tanh(nx)}\right)\right)}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{n!}\right)}=$$

$$=\lim_{n\to\infty}\left(\frac{n!}{(n+1)!}\right)=\lim_{n\to\infty}\left(\frac{1}{n+1}\right)$$

$$\therefore\lim_{n\to\infty}\sqrt[n]{\lim_{\substack{x\to\infty\\x>0}}\left(x^n\omega(x)\cdot\omega(2x)\cdot\omega(3x)\cdot\ldots\cdot\omega(nx)\right)}=0$$

483. Find:

$$\Omega=\lim_{n\to\infty}\frac{\int_0^1\left(\frac{2x}{1+x^2}\right)^{n+1}dx}{\int_0^1\left(\frac{2x}{1+x^2}\right)^{n}dx}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Shafiqur Rahman-Bangladesh

$$\lim_{n\to\infty}\frac{\int_0^1\left(\frac{2x}{1+x^2}\right)^{n+1}dx}{\int_0^1\left(\frac{2x}{1+x^2}\right)^{n}dx}[x\to\tan x]=\lim_{n\to\infty}\frac{\int_0^{\frac{\pi}{4}}2^{n+1}\sin^{n+1}x\cdot\cos^{n-1}x\,dx}{\int_0^{\frac{\pi}{4}}2^{n}\sin^{n}x\cdot\cos^{n-2}x\,dx}=$$

$$=\lim_{n\to\infty}\frac{-\frac{1}{n}+\int_0^{\frac{\pi}{2}}\sin^{n-1}x\,dx}{-\frac{1}{n-1}+\int_0^{\frac{\pi}{2}}\sin^{n-2}x\,dx}=\lim_{n\to\infty}\frac{-\frac{1}{n}+\frac{\sqrt{2\pi n}}{n+1}}{-\frac{1}{n-1}+\frac{\sqrt{2\pi(n-1)}}{n}}$$

$$\lim_{n\to\infty}\frac{\int_0^1\left(\frac{2x}{1+x^2}\right)^{n+1}dx}{\int_0^1\left(\frac{2x}{1+x^2}\right)^{n}dx}=1$$

Solution 2 by Khalef Ruhemi-Jarash-Jordan

$$\Omega=\lim_{n\to\infty}\frac{\int_0^1\left(\frac{2x}{1+x^2}\right)^{n+1}dx}{\int_0^1\left(\frac{2x}{1+x^2}\right)^{n}dx}\quad(*)$$

$$\int_0^1 \frac{x^{n+1}dx}{(1+x^2)^{n+1}} = \frac{1}{2}\int_0^1 x^n \cdot \frac{2x}{(1+x^2)^{n+1}}dx \quad \textit{(Integrate by parts)}$$

$$= \frac{1}{2}\left(\frac{x^n(1+x^2)^{-n}}{-n}\Big|_0^1 + \int_0^1 \frac{x^{n-1}dx}{(1+x^2)^n}\right) = \frac{1}{2}\left(-\frac{1}{n(2)^n} + \int_0^1 \frac{x^{n-1}dx}{(1+x^2)^n}\right)$$

$$\therefore \int_0^1 \frac{x^{n+1}dx}{(1+x^2)^{n+1}} = \frac{1}{2}\left(\frac{-1}{n(2)^n} + \int_0^1 \frac{x^{n-1}dx}{(1+x^2)^n}\right) \quad (1)$$

Then

$$\int_0^1 \frac{x^n dx}{(1+x^2)^n} = \frac{1}{2}\left(\frac{-1}{(n-1)(2)^{n-1}} + \int_0^1 \frac{x^{n-2}dx}{(1+x^2)^{n-1}}\right) \quad (2)$$

$$\textit{Let } I_n \coloneqq \int_0^1 \frac{x^{n-1}dx}{(1+x^2)^n}, \textit{ let } x = \frac{1}{y}, dx = \frac{-dy}{yz}$$

$$\therefore I_n = \int_1^\infty \frac{x^{1-n-2}}{(1+x^2)^n} x^{2n} \cdot dx = \int_1^\infty \frac{x^{n-1}dx}{(1+x^2)^n}$$

$$\therefore 2I_n = \int_1^\infty \frac{x^{n-1}dx}{(1+x^2)^n}, \textit{ let } x^2 = y \Rightarrow x = y^{\frac{1}{2}}, dx = \frac{1}{2}y^{-\frac{1}{2}}dy$$

$$\therefore 2I_n = \frac{1}{2}\int_1^\infty \frac{x^{\frac{n}{2}-1}dx}{(1+x^2)^n} = \frac{1}{2}\beta\left(\frac{n}{2},\frac{n}{2}\right) = \frac{1}{2}\cdot\frac{\Gamma^2\left(\frac{n}{2}\right)}{\Gamma(n)}$$

$$\therefore I_n = \int_0^1 \frac{x^{n-1}dx}{(1+x^2)^n} = \frac{1}{4}\cdot\frac{\Gamma^2\left(\frac{n}{2}\right)}{\Gamma(n)} \quad (3)$$

$$\textit{Then } I_{n-1} = \int_0^1 \frac{x^{n-2}dx}{(1+x^2)^{n-1}} = \frac{1}{4}\cdot\frac{\Gamma^2\left(\frac{n-1}{2}\right)}{\Gamma(n-1)} \quad (4)$$

$$\therefore \int_0^1 \frac{x^{n+1}dx}{(1+x^2)^{n+1}} = \frac{1}{2}\left(\frac{-1}{n(2)^n} + \frac{\Gamma^2\left(\frac{n}{2}\right)}{4\Gamma(n)}\right) \quad (5)$$

$$\textit{and } \int_0^1 \frac{x^n dx}{(1+x^2)^n} = \frac{1}{2}\left(\frac{-1}{(n-1)(2)^{n-1}} + \frac{\Gamma^2\left(\frac{n-1}{2}\right)}{4\Gamma(n-1)}\right) \quad (6)$$

$$\textit{But } \Omega \coloneqq 2\cdot\lim_{n\to\infty}\left(\frac{\frac{x^{n+1}dx}{(1+x^2)^{n+1}}}{\int_0^1 \frac{x^n dx}{(1+x^2)^n}}\right)$$

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$$\therefore \frac{\Omega}{2} = \lim_{n\to\infty}\left(\frac{-\frac{1}{n(2)^n} + \frac{\Gamma^2\left(\frac{n}{2}\right)}{4\Gamma(n)}}{-\frac{1}{(n-1)(2)^{n-1}} + \frac{\Gamma^2\left(\frac{n-1}{2}\right)}{4\Gamma(n-1)}}\right)$$

$$\therefore \frac{\Omega}{2} = \lim_{n\to\infty}\left(\frac{4\Gamma\left(\frac{n}{2}\right)}{4\Gamma(n)} \cdot \frac{\Gamma(n-1)}{\Gamma^2\left(\frac{n-1}{2}\right)}\right)$$

$$\therefore \frac{\Omega}{2} = \lim_{n\to\infty}\left(\frac{\Gamma(n-1)}{\Gamma(n)} \cdot \frac{\Gamma^2\left(\frac{n}{2}\right)}{\Gamma^2\left(\frac{n-1}{2}\right)}\right) = \lim_{n\to\infty}\left(\frac{\Gamma(2n-1)\Gamma^2(n)}{\Gamma(2n)\Gamma^2\left(n-\frac{1}{2}\right)}\right)$$

$$\therefore \frac{\Omega}{2} = \lim_{n\to\infty}\left(\frac{1}{(2n-1)} \cdot \left(\frac{\Gamma(n)}{\Gamma\left(n-\frac{1}{2}\right)}\right)^2\right)$$

$$\therefore \frac{\Omega}{2} = \lim_{n\to\infty}\left(\frac{1}{(2n-1)} \cdot \left(\frac{(n-1)!}{\frac{\sqrt{\pi}(2n)!}{4^n \cdot n!\left(n-\frac{1}{2}\right)}}\right)^2\right)$$

$$\frac{\Omega}{2} = \lim_{n\to\infty}\frac{1}{(2n-1)} \cdot \left(\frac{(n-1)! \cdot 4^n \cdot n! \cdot \left(n-\frac{1}{2}\right)}{\sqrt{\pi} \cdot (2n)!}\right)^2$$

$$= \lim_{n\to\infty}\left(\frac{1}{(2n-1)} \cdot \frac{\left(n-\frac{1}{2}\right)^2 \cdot 4^{2n}(n!)^2\left((n-1)!\right)^2}{\pi\left((2n)!\right)^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\left(n-\frac{1}{2}\right)^2}{\pi n^2} \cdot \frac{(4)^{2n} \cdot (n!)^4}{(2n-1)(2n!)^2}\right) = \frac{1}{\pi}\lim_{n\to\infty}\left(\frac{(4)^{2n}(n!)^4}{(2n)(2n!)^2}\right) =$$

$$= \frac{1}{2\pi}\lim_{n\to\infty}\left(\frac{2^{4n}(n!)^4}{n \cdot \left((2n)!\right)^2}\right) = \left(\frac{1}{2\pi}\right)\left(\lim_{n\to\infty}\left(\frac{2^{2n}(n!)^2}{\sqrt{n} \cdot (2n)!}\right)\right)^2$$

Using Stirling's formula $\left(n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n\right)$

Then $\Omega = \frac{1}{\pi}\left(\lim_{n\to\infty}\frac{(2\pi n)(n)^{2n} \cdot e^{-2n} \cdot 2^{2n}}{\sqrt{n} \cdot \sqrt{n} \cdot 2\sqrt{\pi}e^{-2n}2^{2n}(n)^{2n}}\right)^2$

$$\Omega = \frac{1}{\pi}\lim_{n\to\infty}\left(\sqrt{\pi}\right)^2 = \frac{1}{\pi}\cdot\pi = 1 \therefore \Omega := \lim_{n\to\infty}\left(\frac{\int_0^1\left(\frac{2x}{1+x^2}\right)^{n+1}dx}{\int_0^1\left(\frac{2x}{1+x^2}\right)^{n}dx}\right) = 1$$

484. Find:

$$\Omega = \lim_{n\to\infty}\int_0^{2018}\left|\left[nx+\frac{1}{2}\right]-nx\right|dx, [*] \text{ - great integer function}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $f_n(x) = \left|\left[nx+\frac{1}{2}\right]-nx\right|, x\in\mathbb{R}$. *Note:*

$$f_n\left(x+\frac{1}{n}\right) = \left|\left[n\left(n+\frac{1}{n}\right)+\frac{1}{2}\right]-n\left(x+\frac{1}{n}\right)\right| = \left|\left[nx+\frac{1}{2}+1\right]-nx-1\right|$$

$$= \left|\left[nx+\frac{1}{2}\right]+1-nx-1\right| = f_n(x) \therefore f_n(x) \text{ is periodic with period } \frac{1}{n}.$$

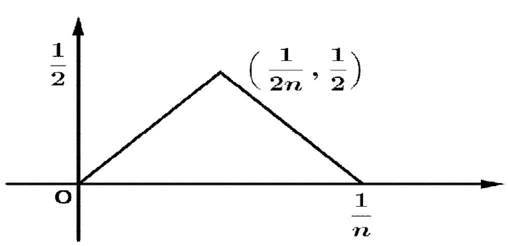
$$\Rightarrow \int_0^{2018} f_n(x)\,dx = 2018n\int_0^{\frac{1}{n}} f_n(x)\,dx$$

For $0 \le x < \frac{1}{2n}, 0 \le nx < \frac{1}{2} \Rightarrow \frac{1}{2} \le nx+\frac{1}{2} < 1 \Rightarrow \left[nx+\frac{1}{2}\right] = 0$ *and for* $\frac{1}{2n} \le x < \frac{1}{n}$

$$\frac{1}{2} \le nx < 1 \Rightarrow 1 \le nx+\frac{1}{2} < \frac{3}{2} \Rightarrow \left[nx+\frac{1}{2}\right] = 1$$

Thus, $f_n(x) = |0-nx| = nx$ *for* $0 \le x < \frac{1}{2n} = |1-nx| = 1-nx$ *for* $\frac{1}{2n} \le n < \frac{1}{n}$

$$\therefore \int_0^{\frac{1}{n}} f_n(x)\,dx = \frac{1}{2}\left(\frac{1}{n}\right)\left(\frac{1}{2}\right) = \frac{1}{4n}$$



Thus,

$$\int_0^{2018} f_n(x)dx = (2018n)\left(\frac{1}{4n}\right) = \frac{1009}{2} \Rightarrow \lim_{n\to\infty}\int_0^{2018} f_n(x)\,dx = \frac{1009}{2}$$

Solution 2 by Feti Sinani-Kosovo

$$\left[nx+\frac{1}{2}\right] = k, k\in\mathbb{Z}\,,\ k\le nx+\frac{1}{2} < k+1\,,\ \frac{k-\frac{1}{2}}{n}\le x<\frac{k+\frac{1}{2}}{n}$$

$$\int_0^{2018}\left|\left[nx+\frac{1}{2}\right]-nx\right|dx = \int_0^{\frac{1}{2n}}|-nx|\,dx + \int_{\frac{1}{2n}}^{2018-\frac{1}{2n}}\left|\left[nx+\frac{1}{2}\right]-nx\right|dx +$$

$$+\int_{2018-\frac{1}{2n}}^{2018}\left|\left[nx+\frac{1}{2}\right]-nx\right|dx = \int_0^{\frac{1}{2n}}|-nx|\ dx + \sum_{k=1}^{2018n-1}\int_{\frac{k}{n}-\frac{1}{2n}}^{\frac{k}{n}+\frac{1}{2n}}|k-nx|dx +$$

$$+\int_{2018-\frac{1}{2n}}^{2018}|2018n-nx|\,dx;\ |k-nx| = \begin{cases} k-nx, x\le\dfrac{k}{n} \\ -(k-nx), x>\dfrac{k}{n}\end{cases}$$

$$=\int_0^{\frac{1}{2n}} nx dx + \sum_{k=1}^{2018n-1}\left(\int_{\frac{k}{n}-\frac{1}{2n}}^{\frac{k}{n}}(k-nx)\,dx + \int_{\frac{k}{n}}^{\frac{k}{n}+\frac{1}{2n}}(nx-k)\,dx\right) + \int_{2018-\frac{1}{2n}}^{2018}(2018n-nx)\,dx$$

$$=\frac{1}{8n}+\sum_{k=1}^{2018n-1}\left[\left(kx-\frac{nx^2}{2}\right)\Bigg|_{\frac{k}{n}-\frac{1}{2n}}^{\frac{k}{n}} + \left(\frac{nx^2}{2}-kx\right)\Bigg|_{\frac{k}{n}}^{\frac{k}{n}+\frac{1}{2n}}\right] + \left(2018nx-\frac{nx^2}{2}\right)\Bigg|_{2018-\frac{1}{2n}}^{2018}$$

$$=\frac{1}{8n}+2\sum_{k=1}^{2018n-1}\frac{1}{8n}+\frac{1}{8n} = \frac{1}{8n}+\frac{2018n-1}{4n}+\frac{1}{8n} = \frac{2018}{4}$$

$$\therefore \Omega = \lim_{n\to\infty}\int_0^{2018}\left|\left[nx+\frac{1}{2}\right]-nx\right|dx = \frac{2018}{4}$$

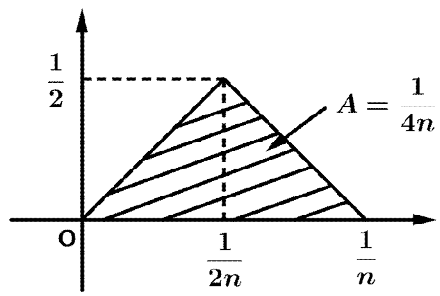
Solution 3 by Amit Shetty-Kosovo

$$\Omega = \lim_{n\to\infty}\int_0^{2018}\left|\left[nx+\frac{1}{2}\right]-nx\right|dx = \lim_{n\to\infty}\int_0^{2018}\left|nx+\frac{1}{2}-\left\{nx+\frac{1}{2}\right\}-nx\right|dx$$

$$= \lim_{n\to\infty}\int_0^{2018}\left|\frac{1}{2}-\left\{nx+\frac{1}{2}\right\}\right|dx$$

$\left|\frac{1}{2}-\left\{nx+\frac{1}{2}\right\}\right|$ *is periodic: period* $-\frac{1}{n}$

$$\therefore \Omega = \lim_{n\to\infty} 2018n\int_0^{\frac{1}{n}}\left|\frac{1}{2}-\left\{nx+\frac{1}{2}\right\}\right|dx$$



$$\therefore \Omega = \lim_{n\to\infty} 2018n\times\frac{1}{4n} = \frac{2018}{4}$$

485. **Find:**

$$\Omega = \sum_{n=0}^{\infty}\left(\frac{1+\frac{2}{1!}+\frac{3}{2!}+\cdots+\frac{n+1}{n!}}{(n+1)(n+2)}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Shafiqur Rahman-Bangladesh

$$\Omega = \sum_{n=0}^{\infty}\left(\frac{1+\frac{2}{1!}+\frac{3}{2!}+\cdots+\frac{n+1}{n!}}{(n+1)(n+2)}\right) =$$

$$= \sum_{n=0}^{\infty}\left(\frac{1+\frac{2}{1!}+\frac{3}{2!}+\cdots+\frac{n+1}{n!}}{n+1}-\frac{1+\frac{2}{1!}+\frac{3}{2!}+\cdots+\frac{n+2}{(n+1)!}}{n+2}\right)+\sum_{n=0}^{\infty}\frac{1}{(n+1)!} =$$

$$= 1+(e-1)$$

$$\therefore \Omega = \sum_{n=0}^{\infty}\left(\frac{1+\frac{2}{1!}+\frac{3}{2!}+\cdots+\frac{n+1}{n!}}{(n+1)(n+2)}\right) = e$$

Solution 2 by Khalef Ruhemi-Jarash-Jordan

$$\Omega := \sum_{n=0}^{\infty}\left(\frac{1+\frac{2}{1!}+\frac{3}{2!}+\cdots+\frac{n+1}{n!}}{(n+1)(n+2)}\right)$$

$$\therefore \Omega = \sum_{n=0}^{\infty}\left(\frac{1}{(n+1)(n+2)};\sum_{k=0}^{k=n}\frac{k+1}{k!}\right) =$$

$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)};\left(2e-\frac{1}{n!}-\frac{2e}{n!}\cdot\int_0^1 x^n\cdot e^{-x}dx\right) =$$

$$= (2e)\cdot\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}-\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)\cdot n!}-2e\int_0^1 e^{-x}\cdot\sum_{n=0}^{\infty}\frac{x^n}{(n+1)(n+2)\cdot n!}$$

But $\frac{e^x-x-1}{x^2} = \sum_{n=0}^{\infty}\frac{x^n}{(n+1)(n+2)\cdot n!}$

$$\therefore e-2 = \sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)\cdot n!}$$

$$\therefore \Omega = (2e)\cdot\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+2-e-2e\int_0^1\frac{1-e^{-x}-xe^{-x}}{x^2}dx =$$

$$= 2e+2-e-(2e)\int_0^1\frac{1-e^{-x}-xe^{-x}}{x^2}dx = e+2+(2e)\int_0^1\frac{e^{-x}+xe^{x}-1}{x^2}dx$$

$$= e+2+(2e)\left[\frac{e^{-x}+xe^{-x}-1}{x}\Big|_1^0+\int_0^1\frac{1}{x}(-xe^{-x})\,dx\right]$$

$$= e+2+(2e)\left(1-\frac{2}{e}\right)-x\int_0^1 e^{-x}\,dx = e+2+2e-4-2e\left(1-\frac{1}{e}\right) =$$

$$= 3e-2-2e+2 = e, \therefore \Omega = e$$

486. $x_n, y_n > 0, x_n \neq y_n, n \in \mathbb{N}, \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = p > 0$

Find:

$$\Omega = \lim_{n\to\infty} \frac{x_n^{y_n} - y_n^{x_n}}{\left(\frac{1}{x_n}\right)^{\frac{1}{y_n}} - \left(\frac{1}{y_n}\right)^{\frac{1}{x_n}}}$$

Proposed by Marian Ursarescu-Romania

Solution 1 by Shafiqur Rahman-Bangladesh

$$\Omega = \lim_{n\to\infty} \frac{x_n^{y_n} - y_n^{x_n}}{\left(\frac{1}{x_n}\right)^{\frac{1}{y_n}} - \left(\frac{1}{y_n}\right)^{\frac{1}{x_n}}} = \lim_{\substack{x_n\to p\\ y_n\to p}} \frac{x_n^{y_n} - y_n^{x_n}}{\left(\frac{1}{x_n}\right)^{\frac{1}{y_n}} - \left(\frac{1}{y_n}\right)^{\frac{1}{x_n}}} = \lim_{x_n\to p}\lim_{y_n\to p} \frac{x_n^{y_n} - y_n^{x_n}}{(x_n)^{-\frac{1}{y_n}} - (y_n)^{-\frac{1}{x_n}}} =$$

$$= \lim_{x_n\to p} \frac{x_n^p - p^{x_n}}{(x_n)^{-\frac{1}{p}} - p^{-\frac{1}{x_n}}} = \lim_{x_n\to p} \frac{\frac{x_n^p - p^p}{x_n - p} - p^p \cdot \frac{p^{(x_n-p)} - 1}{(x_n - p)}}{\frac{(x_n)^{-\frac{1}{p}} - p^{-\frac{1}{p}}}{x_n - p} - \frac{p^{-\frac{1}{p}}}{p x_n} \cdot \frac{p^{-\left(\frac{1}{x_n} - \frac{1}{p}\right)} - 1}{-\left(\frac{1}{x_n} - \frac{1}{p}\right)}} =$$

$$= \frac{p \cdot p^{p-1} - p^p \ln p}{-\frac{1}{p} p^{-\frac{1}{p}-1} - p^{-\frac{1}{p}-2} \ln p} = p^{p+\frac{1}{p}+p} \frac{\ln p - 1}{\ln p + 1}$$

Similarly $\lim_{\substack{x_n\to p\\ y_n\to p}} \frac{x_n^{y_n} - y_n^{x_n}}{\left(\frac{1}{x_n}\right)^{\frac{1}{y_n}} - \left(\frac{1}{y_n}\right)^{\frac{1}{x_n}}} = \lim_{y_n\to p}\lim_{x_n\to p} \frac{x_n^{y_n} - y_n^{x_n}}{(x_n)^{-\frac{1}{y_n}} - (y_n)^{-\frac{1}{x_n}}} = \lim_{y_n\to p} \frac{p^{y_n} - y_n^p}{p^{-\frac{1}{y_n}} - (y_n)^{-\frac{1}{p}}} =$

$$p^{p+\frac{1}{p}+2} \frac{\ln p - 1}{\ln p + 1}$$

$$\therefore \Omega = \lim_{n\to\infty} \frac{x_n^{y_n} - y_n^{x_n}}{\left(\frac{1}{x_n}\right)^{\frac{1}{y_n}} - \left(\frac{1}{y_n}\right)^{\frac{1}{x_n}}} = p^{p+\frac{1}{p}+2} \frac{\ln p - 1}{\ln p + 1}$$

Solution 2 by Khalef Ruhemi-Jarash-Jordan

Since $\lim_{n\to\infty}(x_n) = \lim_{n\to\infty}(y_n) = p > 0$***. Then*** $\Omega = \lim_{x\to y}\left(\frac{x^y - y^x}{\left(\frac{1}{x}\right)^{\frac{1}{y}} - \left(\frac{1}{y}\right)^{\frac{1}{x}}}\right)$***, using Lop-***

rule

$$\Omega = \lim_{x\to\infty}\left(\frac{y(x)^{y-1}-(y)^x\ln(y)}{-\frac{1}{y}\left(\frac{1}{x}\right)^{\frac{1}{y}-1}\cdot\frac{1}{x^2}+\left(\frac{1}{y}\right)^{\frac{1}{x}}\cdot\ln(y)\left(\frac{-1}{x^2}\right)}\right) = \frac{y(y)^{y-1}-(y)^y\ln(y)}{-\frac{1}{y^3}\left(\frac{1}{y}\right)^{\frac{1}{y}-1}-\frac{1}{y^2}\left(\frac{1}{y}\right)^{\frac{1}{y}}\ln(y)} =$$

$$= \frac{(y)^y\ln(y)-(y)^y}{(y)^{-2-\frac{1}{y}}+(y)^{-2-\frac{1}{y}}\ln(y)} = \frac{(y)^y(\ln(y)-1)}{(y)^{-2-\frac{1}{y}}(\ln(y)+1)} = (y)^{y+\frac{1}{y}+2}\cdot\frac{(\ln(y)-1)}{(\ln(y)+1)}$$

$$\textit{But } y := \lim_{n\to\infty}(y_n) = p$$

$$= (p)^{p+\frac{1}{p}+1}\cdot\frac{(\ln(p)-1)}{(\ln(p)+1)} \therefore \Omega = (p)^{p+\frac{1}{p}+2}\cdot\frac{(\ln(p)-1)}{(\ln(p)+1)}$$

487. Find:

$$\Omega = \lim_{k\to\infty}\left(2k\sum_{n=k+1}^{\infty}\frac{1}{(n-k)(n+k)}-\log\left(\frac{k^2}{k+1}\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Remus Florin Stanca-Romania

$$\sum_{n=k+1}^{\infty}\frac{2k}{(n-k)(n+k)} = \lim_{p\to\infty}\sum_{n=k+1}^{p}\frac{2k}{(n-k)(n+k)}$$

$$\sum_{n=k+1}^{p}\frac{2k}{(n-k)(n+k)} = \sum_{n=k+1}^{p}\frac{n+k-(n-k)}{(n-k)(n+k)} = \sum_{n=k+1}^{p}\frac{1}{n-k}-\frac{1}{n+k} = 1+\frac{1}{2}+\cdots+$$

$$+\frac{1}{p-k}-\left(\frac{1}{2k+1}+\frac{1}{2k+2}+\cdots+\frac{1}{p+k}\right) = 1+\frac{1}{2}+\cdots+\frac{1}{p-k}-\ln(p-k)+$$

$$+\ln(p-k)-\left(1+\frac{1}{2}+\cdots+\frac{1}{p+k}-\ln(p+k)+\ln(p+k)-\left(1+\frac{1}{2}+\cdots+\frac{1}{2k}\right)\right)$$

$$\Rightarrow \lim_{p\to\infty}\sum_{n=k+1}^{p}\frac{2k}{(n+k)(n-k)} = \lim_{p\to\infty}\left(\ln(p-k)-\ln(p+k)+1+\frac{1}{2}+\cdots+\frac{1}{2k}\right) =$$

$$= 1+\frac{1}{2}+\cdots+\frac{1}{2k} \succ \sum_{n=k+1}^{\infty}\frac{2k}{(n-k)(n+k)} = 1+\frac{1}{2}+\cdots+\frac{1}{2k}$$

$$\Rightarrow \Omega = \lim_{k\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{2k}-\ln\left(\frac{k^2}{k+1}\right)\right)=$$

$$=\lim_{k\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{2k}-\ln(2k)+\ln(2k)-\ln\left(\frac{k^2}{k+1}\right)\right)=$$

$$=\gamma+\lim_{k\to\infty}\ln\frac{2k^2+2k}{k^2}=\gamma+\ln(2)$$

$$\Omega=\gamma+\ln(2)$$

Solution 2 by Farid Khelili-Skikda-Algerie

$$S=\sum_{n=k+1}^{\infty}\frac{2k}{(n-k)(n+k)}=\lim_{N\to\infty}S_N$$

$$S_N=\sum_{n=k+1}^{N}\frac{2k}{(n-k)(n+k)}=\sum_{m=1}^{N-k}\frac{2k}{m(m+2k)}=\sum_{m=1}^{N-k}\left(\frac{1}{m}-\frac{1}{m+2k}\right)$$

$$S_N=\sum_{m=1}^{N-k}\left(\frac{1}{m}-\frac{1}{m+1}\right)+\sum_{m=1}^{N-k}\left(\frac{1}{m+1}-\frac{1}{m+2}\right)+\cdots+\sum_{m=1}^{N-k}\left(\frac{1}{m+2k-1}-\frac{1}{m+2k}\right)$$

$$S_N=\left(\frac{1}{1}-\frac{1}{N-k+1}\right)+\left(\frac{1}{2}-\frac{1}{N-k+2}\right)+\cdots+\left(\frac{1}{2k}-\frac{1}{N+k}\right)$$

$$S_N=\sum_{m=1}^{2k}\frac{1}{m}-\sum_{m=1}^{2k}\frac{1}{N-k+m};S=\lim_{N\to\infty}S_n=\sum_{m=1}^{2k}\frac{1}{m}$$

$$\Omega=\lim_{k\to\infty}\left(\sum_{n=k+1}^{\infty}\frac{2k}{(n-k)(n+k)}-\ln\left(\frac{k^2}{k+1}\right)\right)$$

$$\Omega=\lim_{k\to\infty}\left(\sum_{m=1}^{2k}\frac{1}{m}-\ln(2k)-\ln\left(\frac{k}{k+1}\right)+\ln(2)\right)$$

$$\Omega=\ln(2)+\lim_{k\to\infty}\left(\sum_{m=1}^{2k}\frac{1}{m}-\ln(2k)\right)=\ln(2)+\lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m}-\ln(n)\right)=\ln(2)+\gamma$$

where $\gamma=\lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m}-\ln(n)\right)$ ***is the Euler-Mascheroni constant.***

488. Find:

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$$\Omega = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_{\varepsilon}^{\frac{\pi}{2}} \left(\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x} \right) \right) dx$$

Proposed by Abdul Jesse Pratt-Nigeria

Solution 1 by Igor Soposki-Skopje-Macedonia

$$I = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_{\varepsilon}^{\frac{\pi}{2}} \arctan\left(\frac{\sqrt{1+x^2}-1}{x} \right) dx = \begin{cases} u = \arctan\left(\frac{\sqrt{1+x^2}-1}{x} \right) \\ dv = dx \Rightarrow v = x \end{cases}$$

$$du = \frac{1}{1+\left(\frac{\sqrt{1+x^2}-1}{x} \right)^2} \cdot \frac{\frac{2x}{2\sqrt{1+x^2}} \cdot x - \left(\sqrt{1+x^2}-1 \right)}{x^2} dx =$$

$$= \frac{x^2}{x^2+1+x^2-2\sqrt{1+x^2}+1} \cdot \frac{x^2 - \sqrt{1+x^2}\left(\sqrt{1+x^2}-1 \right)}{x^2 \cdot \sqrt{1+x^2}} dx =$$

$$= \frac{x^2-1-x^2+\sqrt{x^2+1}}{2\left(x^2+1-\sqrt{x^2+1} \right) \cdot \sqrt{x^2+1}} dx = \frac{\sqrt{x^2+1}-1}{2\left(x^2+1-\sqrt{x^2+1} \right)\sqrt{x^2+1}} dx$$

$$= \frac{\left(\sqrt{x^2+1} \right)^2 - 1}{2\left(x^2+1-\sqrt{x^2+1} \right)\sqrt{x^2+1} \cdot \left(\sqrt{x^2+1}+1 \right)} dx =$$

$$= \frac{x^2+1-1}{2\left(x^2+1-\sqrt{x^2+1} \right)\left(x^2+1+\sqrt{x^2+1} \right)} dx =$$

$$= \frac{x^2}{2(x^2+1)^2 - \left(\sqrt{x^2+1} \right)^2} dx = \frac{x^2}{2\left((x^2+1)^2 - (x^2+1) \right)} dx = \frac{x^2}{2\left((x^2+1)(x^2+1-1) \right)} dx =$$

$$= \frac{1}{2(x^2+1)} dx$$

$$= u \cdot v - \int v \cdot du = x \cdot \arctan\left(\frac{\sqrt{1+x^2}-1}{x} \right) - \frac{1}{2} \int \frac{x}{x^2+1} dx =$$

$$= x \cdot \arctan\left(\frac{\sqrt{1+x^2}-1}{x} \right) - \frac{1}{4} \ln(x^2+1)$$

$$I=\lim_{x\to\frac{\pi}{2}}\left[x\cdot\arctan\left(\frac{\sqrt{1+x^2-1}}{x}\right)-\frac{1}{4}\ln(x^2+1)\right]-$$

$$-\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}\{x=\varepsilon\}}\left[x\cdot\arctan\left(\frac{\sqrt{1+x^2}-1}{x}\right)-\frac{1}{4}\ln(x^2+1)\right]=$$

$$=\frac{\pi}{2}\cdot\arctan\left(\frac{\sqrt{1+\left(\frac{\pi}{2}\right)^2}-1}{\frac{\pi}{2}}\right)-\frac{1}{4}\ln\left(\left(\frac{\pi}{2}\right)^2+1\right)=$$

$$=\frac{\pi}{2}\arctan\frac{\sqrt{\pi^2+4}-1}{\pi}-\frac{1}{4}\ln\left(\frac{\pi^2+4}{4}\right)$$

$$\lim_{\varepsilon\to 0}\varepsilon\cdot\arctan\left(\frac{\sqrt{1+\varepsilon^2}-1}{\varepsilon}\right)=\lim_{\varepsilon\to 0}\varepsilon\cdot\arctan\left(\frac{\left(\sqrt{1+\varepsilon^2}-1\right)\left(\sqrt{1+\varepsilon^2}+1\right)}{\varepsilon\left(\sqrt{1+\varepsilon+1}\right)}\right)=$$

$$=\lim_{\varepsilon\to 0}\varepsilon\cdot\arctan\left(\frac{1+\varepsilon^2-1}{\varepsilon\left(\sqrt{1+\varepsilon^2}+1\right)}\right)=\lim_{\varepsilon\to 0}\varepsilon\cdot\arctan\frac{\varepsilon}{\sqrt{1+\varepsilon^2}+1}=$$

$$=0\cdot\arctan 0=0;\lim_{\varepsilon\to 0}\frac{1}{4}\ln(\varepsilon^2+1)=\frac{1}{4}\ln 1=0$$

Solution 2 by Nassim Nicholas Taleb-USA

We have the antiderivative of the integrand =

$$x\tan^{-1}\left(\frac{\sqrt{x^2+1}-1}{x}\right)-\frac{1}{4}\log(x^2+1)$$

$$\left(hint:\frac{\partial\left(x\tan^{-1}\left(\frac{\sqrt{x^2+1}-1}{x}\right)\right)}{\partial x}=\frac{x}{2x^2+2}+\tan^{-1}\left(\frac{\sqrt{x^2+1}-1}{x}\right),\frac{\partial\left(\frac{1}{4}\log(x^2+1)\right)}{\partial x}=\frac{x}{2(x^2+1)}\right)$$

$$\left[x\tan^{-1}\left(\frac{\sqrt{x^2+1}-1}{x}\right)-\frac{1}{4}\log(x^2+1)\right]_{\varepsilon}^{\frac{\pi}{2}}=$$

$$=\frac{1}{4}\left(\log\left(\frac{4(\varepsilon^2+1)}{4+\pi^2}\right)-4\varepsilon\tan^{-1}\left(\frac{\sqrt{\varepsilon^2+1}-1}{\varepsilon}\right)+2\pi\tan^{-1}\left(\frac{\sqrt{4+\pi^2}-2}{\pi}\right)\right)$$

Since $\lim_{u\to\infty}\tan^{-1}(u)=\frac{\pi}{2}$

$$\Omega=\frac{1}{4}\left(\log\left(\frac{4}{4+\pi^2}\right)+2\pi\tan^{-1}\left(\frac{\sqrt{4+\pi^2}-2}{\pi}\right)\right)$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\text{Put } x = \tan\theta\,,0 < \theta < \frac{\pi}{2}$$

$$\frac{\sqrt{1+x^2}-1}{x} = \frac{\sec\theta - 1}{\tan\theta} = \frac{1-\cos\theta}{\sin\theta} = \tan\left(\frac{\theta}{2}\right) \Rightarrow \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{\theta}{2}$$

$$\text{Let } I = \int_{\varepsilon}^{\frac{\pi}{2}} \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) dx = \int_a^b \frac{\theta}{2}\cdot \sec^2\theta\, d\theta \text{ where}$$

$$b = \tan^{-1}\left(\frac{\pi}{2}\right); a = \tan^{-1}(\varepsilon)$$

$$I = \frac{1}{2}\theta\tan\theta]_a^b - \int_a^b \tan\theta\, d\theta = \frac{1}{2}b\tan b - \frac{1}{2}a\tan a - \log(\sec\theta)]_a^b =$$

$$= \frac{1}{2}b\tan b - \frac{1}{2}a\tan a - \frac{1}{2}\log(1+\tan^2 b) + \frac{1}{2}\log(1+\tan^2 a)$$

$$\text{As } \varepsilon \to 0_+, a \to 0$$

$$\therefore \lim_{\varepsilon\to 0_+} I = \frac{\pi}{4}\tan^{-1}\left(\frac{\pi}{2}\right) - 0 - \frac{1}{2}\log\left(1+\frac{\pi^2}{4}\right)$$

Solution 4 by Kelvin Hong-Rawang-Malaysia

$$\text{Let } x = \tan\theta,$$

$$\Omega = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\frac{\pi}{2}} \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}dx = \lim_{\varepsilon\to 0}\int_{\epsilon}^{\frac{\pi}{2}} \tan^{-1}\frac{1-\frac{1}{\sqrt{1+x^2}}}{\frac{x}{\sqrt{1+x^2}}}dx$$

$$= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\tan^{-1}\frac{\pi}{2}} \tan^{-1}\frac{1-\cos\theta}{\sin\theta}\sec^2\theta\, d\theta = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\tan^{-1}\frac{\pi}{2}} \sec^2\theta\tan^{-1}\tan\frac{\theta}{2}d\theta$$

$$= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\frac{\pi}{2}} \frac{\theta}{2}\sec^2\theta\, d\theta = \lim_{\varepsilon\to 0}\left[\frac{\theta}{2}\tan\theta\Big|_{\varepsilon}^{\tan^{-1}\frac{\pi}{2}} - \frac{1}{2}\int_{\varepsilon}^{\frac{\pi}{2}}\tan\theta\, d\theta\right]$$

$$= \lim_{\varepsilon\to 0}\left[\frac{\pi}{4}\tan^{-1}\frac{\pi}{2} + \frac{1}{2}\ln\cos\left(\tan^{-1}\frac{\pi}{2}\right) - \frac{\varepsilon}{4}\tan\varepsilon - \frac{1}{2}\ln\cos\tan^{-1}\varepsilon\right]$$

$$= \frac{\pi}{4}\tan^{-1}\frac{\pi}{2} - \frac{1}{2}\ln\left(\frac{\sqrt{\pi^2+4}}{2}\right)$$

489. Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[7]{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)(\log n)^6} - \log n\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Shafiqur Rahman-Bangladesh

$$\Omega = \lim_{n\to\infty}\left(\sqrt[7]{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)(\ln n)^6} - \ln n\right) =$$

$$= \lim_{n\to\infty}\left(\sqrt[7]{\left(\ln n+\gamma+\frac{1}{2n}+O\left(\frac{1}{n^2}\right)\right)(\ln n)^6} - \ln n\right) =$$

$$= \lim_{n\to\infty}\ln n\left(\sqrt[7]{\left(1+\frac{\gamma}{\ln n}+\frac{1}{2n\ln n}+O\left(\frac{1}{n^2\ln n}\right)\right)} - 1\right)$$

$$= \lim_{n\to\infty}\ln n\left(\left(1+\frac{\gamma}{7\ln n}+\frac{1}{14n\ln n}\right)-1\right) = \lim_{n\to\infty}\left(\frac{\gamma}{7}+\frac{1}{14n}+\cdots\right)$$

$$\therefore \Omega = \lim_{n\to\infty}\left(\sqrt[7]{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)(\ln n)^6} - \ln n\right) = \frac{\gamma}{7}$$

Solution 2 by Remus Florin Stanca-Romania

$$\Omega = \lim_{n\to\infty}\ln(n)\left(\sqrt[7]{\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\log_n^e} - 1\right) =$$

$$= \lim_{n\to\infty}\frac{\sqrt[7]{\left(1+\cdots+\frac{1}{n}\right)\log_n^e} - 1}{\log_n^e} =$$

$$= \lim_{n\to\infty}\frac{\dfrac{\left(1+\cdots+\frac{1}{n}\right)\log_n^e - 1}{\left(\sqrt[7]{\left(1+\cdots+\frac{1}{n}\right)\log_n^e}\right)^0+\cdots+\left(\sqrt[7]{\left(1+\cdots+\frac{1}{n}\right)\log_n^e}\right)^6}}{\log_n^e}$$

$$\lim_{n\to\infty}\frac{1+\dots+\frac{1}{n}}{\ln(n)}\overset{Stolz\ Cesaro}{=}\lim_{n\to\infty}\frac{\frac{1}{n+1}}{\ln\frac{n+1}{n}}=\lim_{n\to\infty}\frac{\frac{1}{n}\cdot\frac{n}{n+1}}{\ln\left(1+\frac{1}{n}\right)}=1$$

$$\succ\Omega=\frac{1}{7}\lim_{n\to\infty}\ln(n)\left(\left(1+\dots+\frac{1}{n}\right)\log_n^e-1\right)=$$

$$\frac{1}{7}\lim_{n\to\infty}\left(1+\dots+\frac{1}{n}-\ln(n)\right)=\frac{\gamma}{7}\succ\Omega=\frac{\gamma}{7}$$

Solution 3 by Prem Kumar-India

As $\sum\frac{1}{n}\sim\log n+\gamma,\gamma=$ ***Euler – Mascheroni constant***

$$\Omega=\lim_{n\to\infty}\sqrt[7]{(\log n+\gamma)(\log n)^7}-\log n=\lim_{n\to\infty}\log n\left(1+\frac{\gamma}{\log n}\right)^{\frac{1}{7}}-\log n=$$

$$=\lim_{n\to\infty}\log n\left[\left(1+\frac{\gamma}{\log n}\right)^{\frac{1}{7}}-1\right]=\lim_{n\to\infty}\log n\left(1+\frac{\gamma}{7\log n}-1\right)$$

$$\therefore\Omega=\frac{\gamma}{7}$$

490.

$$\Omega_n(a)=\sum_{k=0}^{n}(k^2-a^2+1)(a+k)!,a,n\in\mathbb{N}$$

Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\Omega_n(a)-(a+1)!}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\Omega_n(a)=\sum_{k=0}^{n}(k^2-a^2+1)(a+k)!=\sum_{k=0}^{n}(k-a)(k+a)(a+k)!+(a+k)!$$

$$=\sum_{k=0}^{n}(k-a)(k+a+1-1)(a+k)!+(a+k)!=$$

$$=\sum_{k=0}^{n}(k-a)(a+k+1)!-(k-a)(a+k)!+(a+k)!=$$

$$=\sum_{k=0}^{n}(k-a)(a+k+1)!-(k-a-1)(a+k)!=$$

$$=(n-a)(n+1+a)!+(a+1)!\Rightarrow$$

$$\Omega=\lim_{n\to\infty}\sqrt[n]{(n-a)(n+1+a)!+(a+1)!-(a+1)!}=$$

$$=\lim_{n\to\infty}\sqrt[n]{(n-a)(n+1+a)!}=\lim_{n\to\infty}\frac{(n+1-a)(n+2+a)!}{(n-a)(n+1+a)!}=$$

$$=\lim_{n\to\infty}\frac{(n+1-a)(n+2+a)}{n-a}=\infty$$

Solution 2 by Remus Florin Stanca-Romania

$$\Omega_n(a)=\sum_{k=0}^{n}\big((k-a)(a+k)(a+k)!+(a+k)!\big)=$$

$$=\sum_{k=0}^{n}(k(a+k)(a+k)!-a(a+k)(a+k)!)+\sum_{k=0}^{n}(a+k)!=$$

we prove by using Mathematical induction that

$$\sum_{k=0}^{n}k\cdot k!=(n+1)!-1$$

1. We prove that $P(2): 1+2\cdot 2!=6-1$ ***is true (this fact is obvious)***

2. We suppose that $P(n): \sum_{k=1}^{n}k\cdot k!=(n+1)!-1$ ***is true.***

3. we prove that $P(n+1): \sum_{k=1}^{n+1}k\cdot k!=(n+2)!-1$ ***is true by using*** $P(n)$

$$\sum_{k=1}^{n+1}k\cdot k!=\sum_{k=1}^{n}k\cdot k!+(n+1)(n+1)!=(n+1)!-1+(n+1)!(n+1)=(n+2)!-1$$

$\Rightarrow P(n+1)$ ***is true so we prove that***

$$\sum_{k=1}^{n}k\cdot k!=(n+1)!-1$$

$$\sum_{k=0}^{n}k(a+k)(a+k)!=\sum_{k=1}^{n}k(a+k)(a+k)!$$

$$\sum_{k=1}^{n}(a+k)(a+k)! = \sum_{k=1}^{a+n} k\cdot k! - \sum_{k=1}^{a} k\cdot k! = (a+n+1)! - 1 - (a+1)! + 1 =$$

$$= (a+n+1)! - (a+1)!$$

$$(a+1)(a+1)! + (a+2)(a+2)! + (a+3)(a+3)! \dots + (a+n)(a+n)!$$

$$(a+2)(a+2)! + (a+3)(a+3)! + \dots + (a+n)(a+n)!$$

$$(a+3)(a+3)! + \dots + (a+n)(a+n)!$$

$$\vdots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots. +(a+n)(a+n)!$$

$$\sum_{k=1}^{n} k(a+k)(a+k)! = n[(a+n+1)! - (a+1)!] - \begin{bmatrix}(a+2)! - (a+1)! + \cdots \\ +(a+n)! - (a+1)!\end{bmatrix} =$$

$$n(a+n+1)! - n(a+1)! - (a+2)! - (a+3)! - \dots - (a+n)! + (n-1)(a+1)!$$

$$\sum_{k=0}^{n} a(a+k)(a+k)! = a(a+n+1)! - a\cdot a!$$

SO, $\Omega_n(a) = n(a+n+1)! - n(a+1)! - (a+2)! - \dots - (a+n)! + (n-1)(a+1)! -$

$$-a(a+n+1)! + a\cdot a! + a! + (a+1)! + \dots + (a+n)! =$$

$$= n(a+n+1)! - n(a+1)! + a! + (a+1)! + a\cdot a! + (n-1)(a+1)! - a(a+n+1)!$$

$$= n(a+n+1)! + a! + (a+1)! + a\cdot a! - a(a+n+1)! =$$

$$= 2(a+1)! + (a+n+1)!\,(n-a)$$

$$\lim_{n\to\infty} \sqrt[n]{\Omega_n(a) - (a+1)!} = \lim_{n\to\infty} e^{\frac{\ln\left((a+1)! + (a+n+1)!(n-a)\right)}{n}} =$$

$$= \lim_{n\to\infty} \frac{(a+1)! + (a+n+2)!\,(n+1-a)}{(a+1)! + (a+n+1)!\,(n-a)} =$$

$$= \lim_{n\to\infty} \frac{\frac{(a+1)!}{n(a+n+1)!} + (a+n+2)\left(1 + \frac{1}{n} - \frac{a}{n}\right)}{\frac{(a+1)!}{n(a+n+1)!} + 1 - \frac{a}{n}} = \frac{0+\infty}{0+1-0} = \infty \Rightarrow$$

$$\Omega = \infty.$$

491. Find:

$$\Omega = \lim_{p\to\infty} \sqrt[p]{\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k^2 + kp - 1}{(p + k + 1)!}\right)}$$

Proposed by Daniel Sitaru – Romania

Solution by Shafiqur Rahman-Bangladesh

$$\lim_{p\to\infty} \sqrt[p]{\lim_{k=1} \sum_{k=1}^{n} \frac{k^2 + kp - 1}{(p + k + 1)!}} = \lim_{p\to\infty} \sqrt[p]{\sum_{k}^{\infty} \frac{k}{(p + k)!} - \frac{k + 1}{(p + k + 1)!}}$$

$$= \lim_{p\to\infty} \sqrt[p]{\frac{1}{(p + 1)!}} = \lim_{p\to\infty} \frac{(p + 1)!}{(p + 2)!} = \lim_{p\to\infty} \frac{1}{p + 2}$$

$$\therefore \lim_{p\to\infty} \sqrt[p]{\lim_{n\to\infty} \sum_{k}^{n} \frac{k^2 + kp - 1}{(p + k + 1)!}} = 0$$

492.

$$\Omega(a) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{3^k} \sin^3(3^k \sin a)$$

If $a, b, c \in \left[0, \frac{\pi}{2}\right)$ then:

$$4\big(b\Omega(a) + c\Omega(b) + a\Omega(c)\big) \le 3(a^2 + b^2 + c^2)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

We have $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \Rightarrow \sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha) \Rightarrow$

$$\Rightarrow \sin^3(3^k \sin a) = \frac{1}{4}\left(3\sin(3^k \sin a) - \sin(3^{k+1} \sin a)\right)$$

$$k=1\Rightarrow \sin^3(3\sin a)=\frac{1}{4}(3\sin(3\sin a)-\sin(3^2\sin a))\Big|\frac{1}{3}$$

$$k=2\Rightarrow \sin^3(3^2\sin a)=\frac{1}{4}(3\sin(3^2\sin a)-\sin(3^3\sin a))\Big|\frac{1}{3^2}$$

$$\vdots$$

$$k=n\Rightarrow \sin^3(3^n\sin a)=\frac{1}{4}(3\sin(3^n\sin a)-\sin(3^{n+1}\sin a))\Big|\frac{1}{3^n}$$

$\Rightarrow \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{3^k}\sin^3(3^k\sin a)=\lim_{n\to\infty}\frac{1}{4}\left(\sin(3\sin a)-\frac{1}{3^n}\sin(3^{n+1}\sin a)\right)$ ***(1)***

$\left|\frac{1}{3^n}\cdot\sin(3^{n+1}\sin a)\right|\le\frac{1}{3^n}\to 0$ ***(2)*. *From (1)+(2)*** $\Rightarrow \Omega(a)=\frac{1}{4}\sin(3\sin a)$ ***(3)***

But $\sin\alpha\le\alpha,\forall\alpha\ge 0\Rightarrow \sin(3\sin a)\le 3\sin a\le 3a$ ***(4)***

From (3)+(4)$\Rightarrow \Omega(a)\le\frac{3}{4}(a)\Rightarrow$ ***inequality becomes:***

$$4\big(b\Omega(a)+c\Omega(b)+a\Omega(c)\big)\le 3(ab+ac+bc)\Rightarrow$$

We must show: $3(ab+ac+bc)\le 3(a^2+b^2+c^2)$ ***true.***

Solution 2 by Shafiqur Rahman-Bangladesh

$$4\Omega(a)=4\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{1}{3^k}\sin^3(3^k\sin a)=\lim_{n\to\infty}\sum_{k=1}^{\infty}\left(\frac{1}{3^{k-1}}\sin(3^k\sin a)-\frac{1}{3^k}\sin(3^{k+1}\sin a)\right)$$

$$=\sin(3\sin a)\le 3\sin a\le 3a\ \left[\because a\in\left[0,\frac{\pi}{2}\right)\right]$$

$\therefore 4\big(b\Omega(a)+c\Omega(b)+a\Omega(c)\big)\le 3(ab+bc+ca)\le 3(a^2+b^2+c^2)$ ***(proved)***

493.

$$\Omega_n=\sum_{k=1}^{n}\left(\int_{-\frac{1}{k}}^{\frac{1}{k}}\left((2x^8+3x^6+1)\cdot\cos^{-1}(kx)\right)dx\right)$$

Find:

$$\Omega=\lim_{n\to\infty}(\Omega_n-\pi\cdot H_n)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{Let } I_k = \int_{-\frac{1}{k}}^{\frac{1}{k}} (2x^8 + 3x^6 + 1)\cos^{-1}(kx)\,dx. \text{ Put } x = -t.$$

$$I_k = \int_{\frac{1}{k}}^{-\frac{1}{k}} (2t^8 + 3t^6 + 1)\cos^{-1}(-kt)\,(-1)dt = \int_{\frac{1}{k}}^{-\frac{1}{k}} (2x^8 + 3x^6 + 1)\,[\pi - \cos^{-1}(kx)]dx$$

$$\Rightarrow 2I_k = \pi \int_{-\frac{1}{k}}^{\frac{1}{k}} (2x^8 + 3x^6 + 1)\,dx \Rightarrow I_k = \pi \int_0^{\frac{1}{k}} (2x^8 + 3x^6 + 1)\,dx$$

$$= \pi\left(\frac{2}{9}x^9 + \frac{3}{7}x^7 + x\right)\Big]_0^{\frac{1}{k}} = \pi\left(\frac{2}{9k^9} + \frac{3}{7k^7} + \frac{1}{k}\right)$$

$$\Rightarrow \sum_{k=1}^{n} I_k = \frac{2\pi}{9}\left(\sum_{k=1}^{n}\frac{1}{k^9}\right) + \frac{3\pi}{7}\sum_{k=1}^{n}\frac{1}{k^7} + \pi H_n$$

$$\Rightarrow \Omega_n - \pi H_n = \frac{2\pi}{9}\sum_{k=1}^{n}\frac{1}{k^9} + \frac{3\pi}{7}\sum_{k=1}^{n}\frac{1}{k^7}$$

$$\lim_{n\to\infty}(\Omega_n - \pi H_n) = \frac{2\pi}{9}\zeta(9) + \frac{3\pi}{7}\zeta(7)$$

Solution 2 by Khalef Ruhemi-Jerash-Jordan

$$\Omega_n \coloneqq \sum_{k=1}^{k=n}\left(\int_{-\frac{1}{k}}^{\frac{1}{k}} (2x^8 + 3x^6 + 1)\cos^{-1}(kx)\,dx\right) \quad (*)$$

$$\text{Let } kx = y \Rightarrow dx = \frac{dy}{k}, x = \frac{y}{k}$$

$$\therefore \Omega_n = \sum_{k=1}^{k=n}\int_{-1}^{1}\left(\frac{2x^8}{k^9} + \frac{3x^6}{k^7} + \frac{1}{k}\right)\cos^{-1}(x)\,dx$$

$$\text{But } \int_{-1}^{1}\left(\frac{2x^8}{k^9} + \frac{3x^6}{k^7} + \frac{1}{k}\right)\cos^{-1}(x)\,dx = \int_{-1}^{0}\left(\frac{2x^8}{k^9} + \frac{3x^6}{k^7} + \frac{1}{k}\right)\cos^{-1}(x)\,dx + \int_{-1}^{1}\left(\frac{2x^8}{k^9} + \frac{3x^6}{k^7} + \frac{1}{k}\right)\cos^{-1}(x)\,dx$$

$$(\text{Let } x = -y; dx = -dy)$$

$$= \int_0^1\left(\frac{2x^8}{k^9} + \frac{3x^6}{k^7} + \frac{1}{k}\right)\cos^{-1}(x)\,dx + \int_0^1\left(\frac{2x^8}{k^9} + \frac{3x^6}{k^7} + \frac{1}{k}\right)\cos^{-1}(-x)\,dx$$

$$= \int_0^1 \left(\frac{2x^8}{k^9} + \frac{3x^6}{k^7} + \frac{1}{k}\right)(\cos^{-1}(x) + \cos^{-1}(-x))dx =$$

$$= \pi \cdot \int_0^1 \left(\frac{2x^8}{k^9} + \frac{3x^6}{k^7} + \frac{1}{k}\right)dx = \pi\left(\frac{2}{9k^9} + \frac{3}{7k^7} + \frac{1}{k}\right)$$

$$\therefore \Omega_n = (\pi) \cdot \sum_{k=1}^{k=n} \left(\frac{2}{9k^9} + \frac{3}{7k^7} + \frac{1}{k}\right)$$

$$\therefore \Omega_n = \left(\frac{2\pi}{9}\right)\sum_{k=1}^{k=n} \frac{1}{k^9} + \left(\frac{3\pi}{7}\right) \cdot \sum_{k=1}^{k=n} \frac{1}{k^7} + \pi \sum_{k=1}^{k=n} \frac{1}{k}$$

$$\therefore \Omega_n = \left(\frac{2\pi}{9}\right)\sum_{k=1}^{k=n} \frac{1}{k^9} + \left(\frac{3\pi}{7}\right)\sum_{k=1}^{k=n} \frac{1}{k^7} + \pi H_n$$

$$\therefore \Omega_n - \pi H_n = \left(\frac{2\pi}{9}\right)\sum_{k=1}^{k=n} \frac{1}{k^9} + \left(\frac{3\pi}{7}\right)\sum_{k=1}^{k=n} \frac{1}{k^7}$$

$$\therefore \lim_{n\to\infty}(\Omega_n - \pi H_n) = \left(\frac{2\pi}{9}\right)\zeta(9) + \left(\frac{3\pi}{7}\right)\zeta(7)$$

494. Find:

$$\Omega = \lim_{n\to\infty}\left(\int_\pi^{2\pi} \left(\frac{|\sin(nx)|}{x^2}\right)dx\right)$$

Proposed by Vasile Mircea Popa – Romania

Solution by Khalef Ruhemi-Jerash-Jordan

$$I \coloneqq \lim_{n\to\infty}\left(\int_\pi^{2\pi} \frac{|\sin(nx)|dx}{x^2}\right) \quad (*)$$

$$|\sin(nx)| = \frac{2}{\pi} - \frac{4}{\pi} \cdot \sum_{j=1}^{\infty} \frac{\cos(2jnx)}{4j^2 - 1}$$

$$\therefore I = \lim_{n\to\infty}\left(\frac{2}{\pi} \cdot \int_\pi^{2\pi} \frac{dx}{x^2} - \frac{4}{\pi} \cdot \sum_{j=1}^{\infty} \frac{1}{4j^2 - 1} \cdot \int_\pi^{2\pi} \frac{\cos(2jnx)}{x^2}dx\right)$$

But $\int_{\pi}^{2\pi}\frac{dx}{x^2}=\frac{1}{x}\Big|_{2\pi}^{\pi}=\frac{1}{\pi}-\frac{1}{2\pi}=\frac{1}{2\pi}$

And $\int_{\pi}^{2\pi}\frac{\cos(2jnx)dx}{x^2}=\left(\frac{1}{x^2}\right)\cdot\frac{\sin(2jnx)}{(2jn)}\Big|_{\pi}^{2\pi}+\frac{7}{2jn}\int_{\pi}^{2\pi}\frac{\sin(2jnx)}{x^3}dx=\frac{1}{jn}\int_{\pi}^{2\pi}\frac{\sin(2jnx)dx}{x^3}$

$$\therefore I=\left(\frac{2}{\pi}\right)\left(\frac{1}{2\pi}\right)-\left(\frac{4}{\pi}\right)\lim_{n\to\infty}\left(\sum_{j=1}^{\infty}\frac{1}{(4j^2-1)j}\cdot\int_{\pi}^{2\pi}\frac{\sin(2jnx)dx}{n\cdot x^3}\right)\quad (1)$$

Since $\int_{\pi}^{2\pi}\frac{\sin(2jnx)dx}{x^3}$ *exists for every* $j,n\in\mathbb{N}$

$$\Rightarrow\int_{\pi}^{2\pi}\frac{dx}{x^3}\geq\int_{\pi}^{2\pi}\frac{\sin(2jnx)\,dx}{x^3}\geq\int_{\pi}^{2\pi}\frac{-dx}{x^3}$$

$$\therefore\frac{1}{n}\cdot\int_{\pi}^{2\pi}\frac{dx}{x^3}\geq\frac{1}{n}\cdot\int_{\pi}^{2\pi}\frac{\sin(2jnx)dx}{x^3}\geq\frac{1}{n}\int_{\pi}^{2\pi}\frac{-dx}{x^3}$$

Since $\int_{\pi}^{2\pi}\frac{dx}{x^3}$ *exists, then*

$$\lim_{n\to\infty}\left(\frac{1}{n}\cdot\int_{\pi}^{2\pi}\frac{dx}{x^3}\right)\geq\lim_{n\to\infty}\left(\frac{1}{n}\int_{\pi}^{2\pi}\frac{\sin(2jnx)\,dx}{x^3}\right)\geq\lim_{n\to\infty}\left(-\frac{1}{n}\int_{\pi}^{2\pi}\frac{dx}{x^3}\right)$$

Then $\lim_{n\to\infty}\left(\frac{1}{n}\int_{\pi}^{2\pi}\frac{\sin(2jnx)dx}{x^3}\right)=0$

Going to (1) we obtain: $I=\frac{1}{\pi^2}-\left(\frac{4}{\pi}\right)(0)=\frac{1}{\pi^2}$

$$\therefore\lim_{n\to\infty}\left(\int_{\pi}^{2\pi}\frac{|\sin(nx)|dx}{x^2}\right)=\frac{1}{\pi^2}$$

495. Find:

$$\Omega=\sum_{n=1}^{\infty}\left(log\left(\frac{n^2+3n+2}{n^2+3n}\right)+\frac{1}{4}log\left(\frac{(n-1)^2+3(n-1)+2}{(n-1)^2+3(n-1)}\right)+\cdots+\frac{1}{n^2}log\frac{3}{2}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\Omega=\sum_{n=1}^{\infty}\left[\log\left(\frac{n^2+3n+2}{n^2+3n}\right)+\frac{1}{4}\log\left(\frac{(n-1)^2+3(n-1)+2}{(n-1)^2+3(n-1)}\right)+\cdots+\frac{1}{n^2}\log\frac{3}{2}\right]$$

$$=\frac{1}{1^2}\log\left(\frac{3}{2}\right)$$

$$+\frac{1}{2^2}\log\left(\frac{3}{2}\right)+\log\left(\frac{6}{5}\right)$$

$$+\frac{1}{3^2}\log\left(\frac{3}{2}\right)+\frac{1}{2^2}\log\left(\frac{6}{5}\right)+\log\left(\frac{10}{9}\right)$$

$$+\frac{1}{4^2}\log\left(\frac{3}{2}\right)+\frac{1}{3^2}\log\left(\frac{6}{5}\right)+\frac{1}{2^2}\log\left(\frac{10}{9}\right)+\log\left(\frac{15}{14}\right)$$

$$+\frac{1}{5^2}\log\left(\frac{3}{2}\right)+\frac{1}{4^2}\log\left(\frac{6}{5}\right)+\frac{1}{3^2}\log\left(\frac{10}{9}\right)+\frac{1}{2^2}\log\left(\frac{15}{14}\right)+\log\left(\frac{21}{20}\right)$$

$$+\cdots$$

Adding columnise, we get

$$\Omega=\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)\Omega_1=\frac{\pi^2}{6}\Omega_1$$

where

$$\Omega_1=\log\left(\frac{3}{2}\right)+\log\left(\frac{6}{5}\right)+\log\left(\frac{10}{9}\right)+\log\left(\frac{15}{14}\right)+\log\left(\frac{21}{20}\right)+\cdots$$

$$=\log\left(\frac{3}{2}\right)+\log\left(\frac{3\times 4}{2\times 5}\right)+\log\left(\frac{4\times 5}{3\times 6}\right)+\log\left(\frac{5\times 6}{4\times 7}\right)+\log\left(\frac{6\times 7}{5\times 8}\right)+\cdots$$

$$=\log\left(\frac{3}{2}\right)+\sum_{k=1}^{\infty}\log\left(\frac{(k+2)(k+3)}{(k+1)(k+4)}\right)=\log\left(\frac{3}{2}\right)+\log 2=\log 3$$

Thus, $\Omega=\frac{\pi^2}{6}\log 3$

Solution 2 by Naren Bhandari-New Delhi-India

$$\Omega=\sum_{n=1}^{\infty}\left(\log\left(\frac{n^2+3n+2}{n^2+3j}\right)+\log\frac{1}{4}\left(\frac{(n-1)^2+3(n-1)+2}{(n-1)^2+3(n-1)}\right)+\cdots+\frac{1}{n^2}\log\left(\frac{3}{2}\right)\right)$$

$$=\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{1}{j^2}\log\left(\frac{j^2+3j+2}{j^2+3j}\right)\right)=\sum_{n=1}^{\infty}\sum_{j=n}^{\infty}\frac{1}{j^2}\log\left(\frac{j^2+3j+2}{nj^2+3j}\right)=$$

$$=\sum_{j=1}^{\infty}\left(\frac{1}{j^2}\right)\left(\sum_{j=1}^{\infty}\log\left(\frac{j^2+3j+2}{j^2+3j}\right)\right)=\frac{\pi^2}{6}\left(\log\left(\prod_{j=2}^{\infty}\frac{j(j+1)}{(j-1)(j+2)}\right)\right)=$$

$$=\frac{\pi^2}{6}\log 3=\pi^2\log\sqrt[6]{3}$$

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496.

$$\Omega_n(x) = \int_1^x \left(\frac{t^n - 1}{t - 1}\right) dt, n \in \mathbb{N}, n \geq 1$$

Find:

$$\Omega = \lim_{x\to 1} \frac{\tan^{-1}(nx - n) - \Omega_n(x)}{(x-1)^2}$$

Proposed by Ahmad Albaw-Amman-Jordan

Solution 1 by Sagar Kumar-Kolkata-India

$$\Omega = \lim_{n\to 0} \frac{\tan^{-1} nh - \Omega_n(1+h)}{n^2}$$

$$\Omega_n(1+h) = \int_1^{1+h} \frac{(t)^n - 1}{t-1} dt$$

$$\Omega_n'(1+h) = \frac{(1+h)^n - 1}{h} = \frac{\left(1 + nj + \frac{n(n-1)}{2!}h^2 + \cdots\right) - 1}{h} =$$

$$= \left(n + \frac{n(n-1)}{2!}h + \cdots\right)$$

$$\Omega = \lim_{h\to 0} \frac{\frac{n}{1+n^2h^2} - \left(n + \frac{n(n-1)h}{2!} + \cdots\right)}{2h}$$

$$\Omega = \lim_{n\to 0} \frac{n(1+n^2h^2)^{-1} - \left(n + \frac{(n(n-1))}{2!}h + \cdots\right)}{2h}$$

$$\Omega = \lim_{n\to 0} \frac{(n - n^3h^2 + \cdots) - \left(n + \frac{n(n-1)}{2!}h + \cdots\right)}{2h}$$

$$\Omega = -\frac{(n(n-1))}{4}$$

Solution 2 by Remus Florin Stanca-Romania

$$\Omega_n(x) = \int_1^x t^0 + t^1 + \cdots + t^{n-1}dt = \Big|_1^x t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots + \frac{t^n}{n}dt =$$

$$= x - 1 + \frac{x^2-1}{1} + \frac{x^3-1}{3} + \cdots + \frac{x^n-1}{n}$$

$$\Omega = \lim_{x\to 1}\tan^{-1}(nx-n)\cdot\frac{1 + \frac{1-x}{\tan^{-1}(nx-n)} + \cdots + \frac{1-x^n}{n\tan^{-1}(nx-n)}}{(x-1)^2} =$$

$$= \lim_{x\to 1}\frac{n + \frac{n(1-x)}{\tan^{-1}(nx-n)} + \cdots + \frac{n(1-x^n)}{n\tan^{-1}(nx-n)}}{x-1} =$$

$$\lim_{x\to 1}\frac{\sum_{k=1}^{n}\left(1 + \frac{n(1-x^k)}{k\tan^{-1}(nx-n)}\right)}{x-1}$$

$$\lim_{x\to 1}\frac{1 + \frac{n(1-x^k)}{k\tan^{-1}(nx-n)}}{x-1} = \lim_{x\to 1} n\cdot\frac{1 + \frac{n(1-x^k)}{k\tan^{-1}(nx-n)}}{\tan^{-1}(nx-n)} =$$

$$\overset{L'H}{\underset{\frac{0}{0}}{=}} \lim_{x\to 1} n\cdot\frac{k\tan^{-1}(nx-n) + n(1-x^k)}{k(\tan^{-1}(nx-n))^2} \overset{L'H}{\underset{\frac{0}{0}}{=}}$$

$$= \lim_{x\to 1} n\cdot\frac{k\frac{1}{(nx-n)^2+1}\cdot n - nkx^{k-1}}{2k\tan^{-1}(nx-n)\frac{1}{(nx-n)^2+1}\cdot n}$$

$$= \frac{n}{2}\lim_{x\to 1}\frac{\frac{n}{(nx-n)^2+1} - nx^{k-1}}{nx-n} = \frac{1}{2}\lim_{x\to 1}\frac{\frac{1}{(nx-n)^2+1} - x^{k-1}}{x-1} =$$

$$= \frac{1}{2}\lim_{x\to 1}\frac{1 - x^{k-1}((nx-2)^2+1)}{((nx-n)^2+1)(x-1)} = \lim_{x\to 1}\frac{1 - x^{k-1}((nx-n)^2+1)}{x-1} =$$

$$\lim_{x\to 1} -\frac{1}{2}\cdot\left((k-1)x^{k-2}(nx-n)^2 + 2x^{k-1}(nx-n) + (k-1)x^{k-2}\right) = -\frac{1}{2}(k-1)$$

$$\Rightarrow \Omega = -\frac{1}{2}(0 + 1 + 2 + \cdots + n - 1) = -\frac{1}{2}\cdot\frac{(n-1)n}{2} = \frac{-(n-1)n}{4}$$

497. **Find:**

$$\Omega = \lim_{n\to\infty} n\left(1-\frac{\log\left(1+\frac{\sqrt[n]{10}}{n}\right)^{n+1}}{\log\left(1+\frac{\sqrt[n+1]{10}}{n+1}\right)^{n}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Michael Sterghiou-Greece

Ω *can be written as:* $\Omega = \lim_{n\to\infty} n\cdot\frac{\ln\left(\frac{B}{A}\right)^n-\ln A}{\ln B^n}$ *(1) where*

$B = 1+\frac{10^{\frac{1}{n+1}}}{n+1}, A = 1+\frac{10^{\frac{1}{n}}}{n}, \forall x > -1: \frac{x}{x+1} < \ln(1+x) < x$, *so*

$$\frac{\frac{10^{\frac{1}{n}}}{n}}{1+\frac{10^{\frac{1}{n}}}{n}} < \ln A < \frac{10^{\frac{1}{n}}}{n} \to \frac{10^{\frac{1}{n}}}{1+\frac{10^{\frac{1}{n}}}{n}} < \ln A^n < 10^{\frac{1}{n}} \to \lim_{n\to\infty}\ln A^n = 1$$

It holds that $\lim_{n\to\infty} 10^{\frac{1}{n}} = \lim_{n\to\infty} 10^{\frac{1}{n+1}} = 1$. *Similarly* $\lim_{n\to\infty}\ln A^n = 1$

(1) now becomes $\Omega = \lim_{n\to\infty}\left[\ln\left(\frac{B}{A}\right)^n - 1\right]$. *We will show that:*

$\frac{B}{A} = \frac{1+\frac{10^{\frac{1}{n+1}}}{n+1}}{1+\frac{10^{\frac{1}{n}}}{n}} < 1-\frac{1}{n^2}$ *(2) . Simplifications reduce (2) to:*

$$-\frac{B}{A}+1-\frac{1}{n^2} = \frac{10^{\frac{1}{n}}\cdot n^3-10^{\frac{1}{n+1}}\cdot n^3+10^{\frac{1}{n}}\cdot n^2-n^2-10^{\frac{1}{n}}\cdot n-n-10^{\frac{1}{n}}}{n^2(n+1)\left(n+10^{\frac{1}{n}}\right)} \quad (3)$$

Nominator of (3) is positive (*). Also, $\left(\frac{B}{A}\right)^{n^2} < \left(1-\frac{1}{n^2}\right)^{n^2}$ *(4),* $\frac{B}{A} < 1$. *We can also show that:* $\left(\frac{B}{A}\right)^{n^2}$ *is increasing (*),* $\left(1-\frac{1}{n^2}\right)^n \uparrow$, *and* $\left(\frac{B}{A}\right)^{n^2} - \left(1-\frac{1}{n^2}\right)^n$ *is also increasing (*).*

This together with (4) means $\lim_{n\to\infty}\left(\frac{B}{A}\right)^n = \lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^{n^2} = \frac{1}{e}$ *therefore*

$$\Omega = \ln\lim_{n\to\infty}\left(\frac{1}{e}\right) - 1 = -2$$

(*) It is quite a laborious task to show these through derivatives of the respective functions.

498. Find:

$$\Omega = \sum_{n=0}^{\infty}\left(\frac{1}{3^{n+1}}\sum_{k=1}^{n}\left(3^k \cdot tan^{-1}\left(\frac{3}{k^2-k-1}\right)\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Artan Ajredni-Presheva-Serbie

Find:

$$\Omega = \sum_{n=0}^{\infty}\left(\frac{1}{3^{n+1}}\sum_{k=1}^{n}\left(3^k \tan^{-1}\left(\frac{3}{k^2-k-1}\right)\right)\right)$$

We use the formula: $\tan^{-1}\alpha - \tan^{-1}\beta = \tan^{-1}\frac{-1-\beta}{1+\alpha\beta}$***. For*** $\alpha = k+1$ ***and*** $\beta = k-2$ ***we have*** $\tan^{-1}(k-1) - \tan^{-1}(k-2) = \tan^{-1}\left(\frac{3}{k^2-k-1}\right)$***. Therefore, for*** $n = \overline{0,m}$ ***we have***

$$(n=0)\ \frac{1}{3}\sum_{k=1}^{0}\left(3^k \tan^{-1}\left(\frac{3}{k^2-k-1}\right)\right) = \frac{1}{3}\sum_{k=0}^{0}\left(3^k \cdot \tan^{-1}\left(\frac{3}{k^2-k-1}\right)\right) - \frac{1}{3}\tan^{-1}(-3) = 0$$

$$(n=1)\frac{1}{3^7}\sum_{k=1}^{1}3^k\left(\tan^{-1}(k+1) - \tan^{-1}(k-2)\right) = \frac{1}{3^2}\left(3^1\left(\tan^{-1}2 - \tan^{-1}(-1)\right)\right) =$$

$$= \tan^{-1}2 - \tan^{-1}(-1)$$

$$(n=2)\ \frac{1}{3^2}\sum_{k=1}^{2}3^k\left(\tan^{-1}(k+1) - \tan^{-1}(k-2)\right) =$$

$$= \frac{1}{3^2}\left(3\left(\tan^{-1}(2) - \tan^{-1}(-1)\right)\right) + 3^2\left(\tan^{-1}(3) - \tan^{-1}(0)\right)$$

$$(n=3)\frac{1}{3^3}\sum_{k=1}^{3}3^k\left(\tan^{-1}(k-n) - \tan^{-1}(k-2)\right) =$$

$$= \frac{1}{3^3}\left(3\left(\tan^{-1}(2) - \tan^{-1}(-1)\right)\right) + 3^2\left(\tan^{-1}(2) - \tan^{-1}(0)\right) + 3^3\left(\tan^{-1}(4) - \tan^{-1}(1)\right)$$

$$\vdots$$

$$(n=m)\frac{1}{3^m}\left(\sum_{k=1}^{m}3^k(\tan^{-1}(k+1))-\tan^{-1}(k-2)\right)=$$

$$=\frac{1}{3^{m+1}}\left(3\left(\tan^{-1}(2)-\tan^{-1}(-1)+3^2(\tan^{-1}(9)-\tan^{-1}(0))+\cdots+3^{m+1}(\tan^{-1}(m+1)-\tan^{-1}(m-2))\right)\right)$$

Hence,

$$\sum_{n=0}^{m}\left(\frac{1}{3^{n+1}}\sum_{k=1}^{n}3^k(\tan^{-1}(k+1))-\tan^{-1}(k-2)\right)=$$

$$=\begin{bmatrix}\tan^{-1}(2)-\tan^{-1}(-1)+\tan^{-1}(3)-\tan^{-1}(0)+\tan^{-1}(4)-\tan^{-1}(1)+\tan^{-1}(5)+\\+\tan^{-1}(6)-\tan^{-1}(3)+\cdots+\tan^{-1}(m+1)-\tan^{-1}(m-2)\end{bmatrix}+$$

$$+\frac{1}{3}[\tan^{-2}(2)-\tan^{-1}(-1)+\cdots+\tan^{-1}(m)-\tan^{-1}(m-3)]+\cdots+\frac{1}{3^m}[\tan^{-1}(2)-\tan^{-1}(-1)]=$$

$$=[-\tan^{-1}(-1)-\tan^{-1}(0)-\tan^{-1}(1)+\cdots+\tan^{-1}(m-1)-\tan^{-1}(0)+\tan^{-1}(m+1)]+$$

$$+\frac{1}{3}\left[-\frac{3\pi}{4}-\frac{\pi}{4}+\tan^{-1}(m-2)+\tan^{-1}(m-1)\right]+\cdots+\frac{1}{3^m}[\tan^{-1}(2)-\tan^{-1}(-1)]=$$

$$=\left[-\pi+\tan^{-1}\left(\frac{2m}{2-m^2}\right)\right]+\frac{1}{3}\left[-\pi+\tan^{-1}\left(\frac{2m-3}{3m-m^2-1}\right)\right]+\cdots+\frac{1}{3^m}[\tan^{-1}(2)-\tan^{-1}(-1)]=$$

$$=\left[-\tan^{-1}(\tan(\pi))+\tan^{-1}\left(\frac{2m}{2-m^2}\right)\right]+\frac{1}{3}\left[-\tan^{-1}(\tan(\pi))+\tan^{-1}\left(\frac{2m-3}{3m^2-m^2-1}\right)\right]+$$

$$+\cdots+\frac{1}{3^m}[\tan^{-1}(2)-\tan^{-1}(-1)]=$$

$$=\left[-\tan^{-1}(0)+\tan^{-1}\left(\frac{2m}{2-m^2}\right)\right]+\frac{1}{3}\left[-\tan^{-1}(0)+\tan^{-1}\left(\frac{2m-3}{3m^2-m^2}\right)\right]+\cdots+\frac{1}{3^m}[\tan^{-1}(2)-\tan^{-1}(-1)]=$$

$$=\tan^{-1}\left(\frac{2m}{2m^2}\right)+\frac{1}{3}\tan^{-1}\left(\frac{2m-3}{2m-m^2-1}\right)+\cdots+\frac{1}{3^m}[\tan^{-1}(2)-\tan^{-1}(-1)]=\Omega m$$

Consequently,

$$\sum_{k=0}^{\infty}\left(\frac{1}{3^{n+1}}\sum_{k=1}^{n}\left(3^k\tan^{-1}\left(\frac{3}{k^2-k-1}\right)\right)\right)=\lim_{n\to\infty}\Omega_n=0$$

Solution 2 by Naren Bhandari-Nepal

We have

$$\Omega=\sum_{n=1}^{\infty}\left[\frac{1}{3^{n+1}}\left(\sum_{k=1}^{n}3^k\tan^{-1}\left(\frac{3}{k^2-k-1}\right)\right)\right]=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{3^k}{3^{n+1}}\tan^{-1}\left(\frac{3}{k^2-k+1}\right)\right]\quad \textbf{(1)}$$

Now we exchange n by k and vice versa where we define a new sum

$$\Omega=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{3^n}{3^{k+1}}\tan^{-1}\left(\frac{3}{n^2-n-1}\right)\right]\quad \textbf{(2)}$$

Adding sum (1) and (2) we find:

$$2\pi = \frac{1}{3}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{3^n}{3^k}\left(\tan^{-1}\left(\frac{3}{n^2-n-1}\right)+\tan^{-1}\left(\frac{3}{k^2-k-1}\right)\right)\right]$$

$$\Omega = \frac{1}{3}\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{3}{n^2-n+1}\right) = \frac{1}{3}\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{(n+1)-(n-2)}{1+(n+1)(n-2)}\right) =$$

$$= \frac{1}{3}\sum_{n=1}^{\infty}\underbrace{\left(\tan^{-1}(n+1)-\tan^{-1}(n-2)\right)}_{Telescoping\ series=0}$$

Therefore $\Omega = 0$.

Solution 3 by Ravi Prakash-New Delhi-India

$$\Omega = \sum_{n=0}^{\infty}\frac{1}{3^{n+1}}\sum_{k=1}^{n}3^k\tan^{-1}\left(\frac{3}{k^2-k-1}\right) = \frac{1}{3}(0)+\frac{1}{3^2}\left[3\tan^{-1}(-3)\right]+$$

$$+\frac{1}{3^3}\left[3\tan^{-1}(-3)+3^2\tan^{-1}\left(\frac{3}{3}\right)\right]+$$

$$+\frac{1}{3^4}\left[3\tan^{-1}(-3)+3^2\tan^{-1}\left(\frac{3}{3}\right)+3^3\tan^{-1}\left(\frac{3}{5}\right)\right]$$

$$+\cdots$$

$$+\frac{1}{3^{n+1}}\left[3\tan^{-1}(-3)+3^2\tan^{-1}\left(\frac{3}{3}\right)+\cdots+3^n\tan^{-1}\left(\frac{3}{n^2-n+1}\right)\right]$$

$$+\cdots$$

Adding columnwise

$$\Omega = \left(\sum_{r=1}^{\infty}\frac{1}{3^r}\right)\tan^{-1}(-3)+\left(\sum_{r=1}^{\infty}\frac{1}{3^r}\right)\tan^{-1}\left(\frac{2}{3}\right)+\left(\sum_{r=1}^{\infty}\frac{1}{3^r}\right)\tan^{-1}\left(\frac{3}{5}\right)+\ldots$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{3}{n^2-n-1}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{(n+1)-(n-2)}{1+(n+1)(n-2)}\right) =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left[\tan^{-1}(n+1)-\tan^{-1}(n-2)\right] = \frac{1}{2}\left[-\tan^{-1}(-1)-\tan^{-1}(0)-\tan^{-1}(1)\right] = 0$$

499.

$$\Omega(k) = \int_0^1 \frac{x^k-kx^{k-1}-x^{k-1}+1}{x^{2k+1}+2x^{k+1}+x^k+x+1}\,dx,\ k\in\mathbb{N},\ k\geq 1$$

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Find:

$$\Omega = \lim_{n\to\infty}\left(\log 2\cdot\log n - n\cdot\log\left(\frac{3}{2}\right)+\sum_{k=1}^{n}\Omega(k)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Artan Ajredni-Presheva-Serbie

We have: $\left((k+1)x^k+1\right)\left(x^k+1\right)-kx^{k-1}\left(x^{k+1}+x+1\right)=x^{2k}+2x^k-kx^{k-1}+1$

Therefore $\Omega(k)=\int_0^1\frac{\left((k+1)k^k+1\right)\left(x^k+1\right)-kx^{k-1}\left(x^{k+1}+x+1\right)}{\left(x^k+1\right)\left(x^{k+1}+x+1\right)}-\int_0^1\frac{x^{k-1}\left(x^{k+1}+x+1\right)}{\left(x^k+1\right)\left(x^{k-1}+x+1\right)}dx=$

$$=\int_0^1\frac{(k+1)x^k+1}{x^{k+1}+x+1}dx-\int_0^1\frac{kx^{k-1}}{x^k+1}dx-\frac{1}{k}\int_0^1\frac{kx^{k-1}}{x^k+1}dx=\int_0^1\frac{d\left(x^{k+1}+x+1\right)}{x^{k+1}+x+1}-$$

$$-\int_0^1\frac{d\left(x^k+1\right)}{x^k+1}-\frac{1}{k}\int_0^1\frac{d\left(x^{k+1}\right)}{x^k+1}=\ln\left(x^{k+1}+x+1\right)\Big|_0^1-\ln\left(x^k+1\right)\Big|_0^1-$$

$$-\frac{1}{k}\ln\left(x^k+1\right)\Big|_0^1=\ln 3-\ln 2-\frac{1}{k}\ln 2$$

$$\sum_{k}^{n}\Omega(k)=n\ln\left(\frac{3}{2}\right)-\ln 2\sum_{k=1}^{n}\frac{1}{k}$$

$$\Omega=\lim_{n\to\infty}\left(\ln 2\ln n-n\ln\left(\frac{3}{2}\right)+n\ln\left(\frac{3}{2}\right)-\ln 2\sum_{k}^{n}\frac{1}{k}\right)=\ln 2\lim_{n\to\infty}\left(\ln A-\sum_{k}^{n}\frac{1}{k}\right)=$$

$$=-\ln 2\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)=-\gamma\ln 2$$

Solution 2 by Yen Tung Chung-Taichung-Taiwan

$$\Omega(k)=\int_0^1\frac{x^k-kx^{k-1}-x^{k-1}+1}{x^{2k+1}+2x^{k+1}+x^k+x+1}dx=\int_0^1\frac{x^k-(k+1)x^{k-1}+1}{\left(x^k+1\right)\left(x^{k+1}+x+1\right)}dx$$

$$=\int_0^1\left(\frac{(k+1)x^k+1}{x^{k+1}+x+1}-\frac{(k+1)x^{k-1}}{x^k+1}\right)dx=\left(\ln\left|x^{k+1}+x+1\right|-\frac{k+1}{k}\ln\left|x^k+1\right|\right)\Big|_0^1$$

$$=\ln 3-\frac{k+1}{k}\ln 2$$

$$\Omega = \lim_{n\to\infty}\left(\ln 2 \ln n - n\ln\left(\frac{3}{2}\right) + \sum_{k=1}^{n}\Omega(k)\right) = \lim_{n\to\infty}\left(\ln 2 \ln n - n\ln\left(\frac{3}{2}\right) + \sum_{k=1}^{n}\left(\ln 3 - \frac{k+1}{k}\ln 2\right)\right)$$

$$= \ln 2 \times \lim_{n\to\infty}\left(\ln n + n - \sum_{k=1}^{n}\left(1 + \frac{1}{k}\right)\right) = \ln 2 \times (-\gamma) = -\gamma \ln 2$$

500.

$$x_1 = 1, x_2 = 3, x_n = x_{n-2} + 2x_{n-1}, n \ge 3$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{(-1)^{n+1}\left(x_n^2 - 2x_n x_{n-1} - x_{n-1}^2\right)}{n}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Remus Florin Stanca-Romania

For $x_n = x_{n-2} + 2x_{n-1}$ ***we solve the equation*** $t^2 - 2t - 1 = 0$

$$\Delta = 4 + 4 = 8 \Rightarrow t_{1,2} = \frac{2 \pm 2\sqrt{2}}{2} = \begin{cases} t_1 = 1 + \sqrt{2} \\ t_2 = 1 - \sqrt{2} \end{cases}$$

so $x_n = \alpha t_1^n + \beta t_2^n = \alpha\left(1 + \sqrt{2}\right)^n + \beta\left(1 - \sqrt{2}\right)^n$

$$x_1 = \alpha\left(1 + \sqrt{2}\right) + \beta\left(1 - \sqrt{2}\right) = 1| \cdot \left(1 + \sqrt{2}\right)$$

$$x_2 = \alpha\left(1 + \sqrt{2}\right)^2 + \beta\left(1 - \sqrt{2}\right)^2 = 3$$

$$\begin{cases} \alpha\left(1 + \sqrt{2}\right)^2 - \beta = 1 + \sqrt{2} \\ \alpha\left(1 + \sqrt{2}\right)^2 + \beta\left(1 - \sqrt{2}\right)^2 = 3 \end{cases}$$

———————————————— „–"

$$\beta\left(4 - 2\sqrt{2}\right) = 2 - \sqrt{2} \Rightarrow \beta = \frac{1}{2} \Rightarrow \alpha = \frac{1}{2} \Rightarrow x_n = \frac{1}{2}\left(\left(1 + \sqrt{2}\right)^n + \left(1 - \sqrt{2}\right)^n\right)$$

If we denote $1 + \sqrt{2} = p \Rightarrow 1 - \sqrt{2} = \frac{-1}{p} \Rightarrow$

$$\Omega o = \lim_{n\to\infty}(-1)^{n+1}\frac{\frac{1}{4}\left(p^{2n} + \frac{1}{p^{2n}} + 2(-1)^n\right) - \left(p^{n-1} + \frac{(-1)^{n-1}}{p^{n-1}}\right)\frac{1}{2}\left(p^n + \frac{(-1)^n}{p^n}\right) - \frac{1}{4}\left(p^{2n-2} + \frac{1}{p^{2n-2}} + 2(-1)^{n-1}\right)}{n} +$$

$$= \lim_{n\to\infty}(-1)^{n+2} \cdot \frac{1}{4}\left(p^{2n+2} + \frac{1}{p^{2n+2}} + 2(-1)^{n+1}\right) + (-1)^{n+2}\frac{1}{4}\left(p^{2n} + \frac{1}{p^{2n}} + 2(-1)^n\right) +$$

$$+(-1)^{n+3}\left(p^n+\frac{(-1)^n}{p^n}\right)\frac{1}{2}\left(p^{n+1}+\frac{(-1)^{n+1}}{p^{n+1}}\right)+$$

$$+(-1)^{n+3}\left(p^{n-1}+\frac{(-1)^{n-1}}{p^{n-1}}\right)\frac{1}{2}\left(p^n+\frac{(-1)^n}{p^n}\right)+\frac{(-1)^{n+3}}{4}\left(p^{2n}+\frac{1}{p^{2n}}+2(-1)^n\right)+$$

$$+(-1)^{n+3}\cdot\frac{1}{4}\left(p^{2n-2}+\frac{1}{p^{2n-2}}+2(-1)^{n-1}\right)=0\Rightarrow\Omega=0$$

Solution 2 by Ravi Prakash-New Delhi-India

For $n\geq 2$, let

$$a_n=x_n^2-2x_nx_{n-1}-x_{n-1}^2$$

$$a_2=3^2-2(3)(1)-1^1=2$$

$$x_n=x_{n-2}+2x_{n-1}=(x_n-x_{n-1})^2=(x_{n-2}+x_{n-1})^2$$

$$=x_n^2-2x_nx_{n-1}+x_{n-1}^2=x_{n-1}^2+2x_{n-2}x_{n-1}+x_{n-2}^2\Rightarrow$$

$$\Rightarrow x_n^2-2x_nx_{n-1}-x_{n-1}^2=-\left(x_{n-1}^2-2x_{n-2}x_{n-1}-x_{n-2}^2\right)\Rightarrow$$

$$\Rightarrow a_n=-a_{n-1}\ \forall n\geq 3\Rightarrow a_n=(-1)^{n-2}a_2=(-1)^na_2=(-1)^n(2)$$

$$\Omega=\lim_{n\to\infty}\frac{(a_n)(-1)^{n+1}}{n}=\lim_{n\to\infty}\frac{(-1)^{n+1}(-1)^n2}{n}=0$$

Solution 3 by Khaled Abd Almuti-Damascus-Syria

$$\left.\begin{matrix}x_1=1\\x_2=3\end{matrix}\right\}x_n=x_{n-2}+2x_{n-1},n\geq 3$$

Find: $\lim_{n\to\infty}\left[\frac{(-1)^{n+1}\cdot(x_n^2-2x_n\cdot x_{n-1}-x_{n-1}^2)}{n}\right]$

We can write that sequence as: $\left.\begin{matrix}x_1=1\\x_2=3\end{matrix}\right\}x_{n+1}=x_{n-1}+2x_n,x\geq 2$

Let $a,b\in\mathbb{R}$: such that $\begin{cases}a+b=2\\a\cdot b=-1\\a=1+\sqrt{2},b=1-\sqrt{2}\end{cases}$

Let: $v_n=x_{n+1}-(1+\sqrt{2})x_n$ **and** $w_n=x_{n+1}-(1-\sqrt{2})x_n$

Note: $v_{n+1}=x_{n+2}-(1+\sqrt{2})x_{n+1}=2x_{n+1}+x_n-(1+\sqrt{2})x_{n+1}$

$$v_{n+1}=(1-\sqrt{2})x_{n+1}+x_n=(1-\sqrt{2})\left[x_{n+1}+\frac{1}{1-\sqrt{2}}x_n\right]$$

$v_{n+1}=(1-\sqrt{2})(v_n)\Rightarrow(v_n)_{n\geq 1}$ **is a geometric sequence,** $q=b=1-\sqrt{2}$

In similar way, we can easily prove that $(w_n)_{n\geq}$ ***is geometric sequence such that:***

$$q = 1 + \sqrt{2}$$

$$\left.\begin{array}{r} v_n = v_1 \cdot q^{n-1} \\ v_1 = x_2 - (1+\sqrt{2})x_1 \\ v_1 = 3 - (1+\sqrt{2}) = 2 - \sqrt{2} \end{array}\right\} \Rightarrow v_n = (2-\sqrt{2}) \cdot (1-\sqrt{2})^{n-1}$$

$$w_n = w_1 \cdot q^{n-1}$$

$$w_1 = x_2 - (1-\sqrt{2})x_1$$

$$w_1 = 3 - (1-\sqrt{2}) = 2 + \sqrt{2}$$

$$w_n = (2+\sqrt{2}) \cdot (1+\sqrt{2})^{n-1}$$

$$\left.\begin{array}{r} v_n = x_{n+1} - (1+\sqrt{2})x_n \\ w_n = x_{n+1} - (1-\sqrt{2})x_n \end{array}\right\} \Rightarrow v_n - w_n = -(1+\sqrt{2})x_n + (1-\sqrt{2})x_n$$

$$v_n - w_n = -2\sqrt{2} \cdot x_n \Rightarrow x_n = \frac{w_n - v_n}{2\sqrt{2}}$$

We note: $x_n^2 - 2x_n \cdot x_{n-1} - x_{n-1}^2 = x_n^2 - 2x_n \cdot x_{n-1} + x_{n-1}^2 - x_{n-1}^2 - x_{n-1}^2$

$$= (x_n - x_{n-1})^2 - 2x_{n-1}^2 = (x_n - x_{n-1})^2 - \left(\sqrt{2} \cdot x_{n-1}\right)^2 =$$

$$= \left(x_n - x_{n-1} - \sqrt{2}x_{n-1}\right)\left(x_n - x_{n-1} + \sqrt{2}x_{n-1}\right) =$$

$$= \left(x_n - (1+\sqrt{2})x_{n-1}\right)\left(x_n - (1-\sqrt{2})x_{n-1}\right) = v_{n-1} \cdot w_{n-1} =$$

$$= \frac{(2-\sqrt{2})(1-\sqrt{2})^{n-2} \cdot (2+\sqrt{2})(1+\sqrt{2})^{n-2}}{n} = \frac{2(-1)^{n-2}}{n}$$

$$\frac{(-1)^{n+1} \cdot \left(x_n^2 - 2x_n \cdot x_{n-1} - x_{n-1}^2\right)}{n} = \frac{2(-1)^{n-1} \cdot (-1)^{n-2}}{n} = \frac{2(-1)^{2n-1}}{n} = \frac{-2}{n}$$

$$\Omega = \lim_{n\to\infty}\left[\frac{(-1)^{n+1}\left(x_n^2 - 2x_n \cdot x_{n-1} - x_{n-1}^2\right)}{n}\right] = \lim_{n\to\infty}\left(\frac{-2}{n}\right) = 0$$

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It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

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To be continued!

Daniel Sitaru