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ALGEBRAIC INEQUALITIES FROM GEOMETRICAL CONFIGURATIONS

By Daniel Sitaru, Claudia Năuți – Romania

Abstract: In this article we build geometrical configurations and extract algebraic inequalities from geometrical properties.

Aplication 1 : If $x, y, z \geq 0$ then:

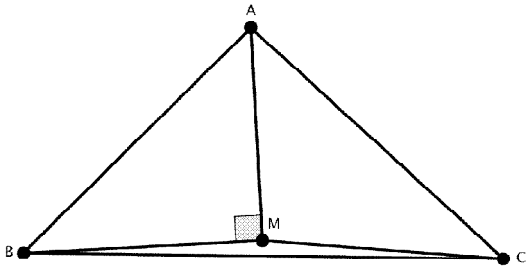
1.1. $\sqrt{x^2 + z^2 + xz\sqrt{2}} + \sqrt{y^2 + z^2 + yz\sqrt{2}} \geq \sqrt{x^2 + y^2}$

1.2 $\sqrt{x^2 + z^2 + xz\sqrt{z}} + \sqrt{x^2 + y^2} \geq \sqrt{y^2 + z^2 + yz\sqrt{2}}$

1.3 $\sqrt{y^2 + z^2 + yz\sqrt{2}} + \sqrt{x^2 + y^2} \geq \sqrt{x^2 + z^2 + xz\sqrt{2}}$

Solution: Let be ΔABC such that: $AM = x; BM = y; CM = z; \mu(\widehat{AMB}) = \frac{\pi}{2}$;

$$\mu(\widehat{AMC}) = \mu(\widehat{BMC}) = \frac{3\pi}{4}$$



By cosine theorem: $AB = \sqrt{x^2 + y^2}; BC = \sqrt{y^2 + z^2 + yz\sqrt{2}}; CA = \sqrt{x^2 + z^2 + xz\sqrt{2}}$

In ΔABC: $AB + BC > AC; BC + CA > BA; CA + AB > CB$

$$\sqrt{x^2 + z^2 + xz\sqrt{2}} + \sqrt{y^2 + z^2 + yz\sqrt{2}} \geq \sqrt{x^2 + y^2}$$

$$\sqrt{x^2 + z^2 + xz\sqrt{2}} + \sqrt{x^2 + y^2} \geq \sqrt{y^2 + z^2 + yz\sqrt{2}}$$

$$\sqrt{y^2 + z^2 + yz\sqrt{2}} + \sqrt{x^2 + y^2} \geq \sqrt{x^2 + z^2 + xz\sqrt{2}}$$

Equality holds for $x = y = z = 0$.

Aplication 2: If $x, y, z \geq 0$ then:

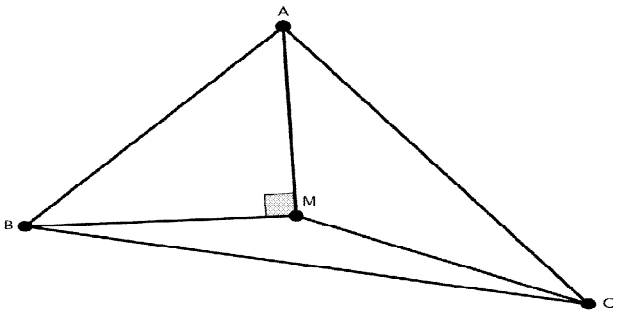
2.1. $\sqrt{x^2 + y^2} + \sqrt{x^2 + z^2 + xz} \geq \sqrt{y^2 + z^2 + yz\sqrt{3}}$

2.2. $\sqrt{x^2 + y^2} + \sqrt{y^2 + z^2 + yz\sqrt{3}} \geq \sqrt{x^2 + z^2 + xz}$

2.3. $\sqrt{x^2 + z^2 + xz} + \sqrt{y^2 + z^2 + yz\sqrt{3}} \geq \sqrt{x^2 + y^2}$

Solution: Let be ΔABC such that: $AM = x; BM = y; CM = z; \mu(\widehat{AMB}) = \frac{\pi}{2}; \mu(\widehat{BMC}) = \frac{2\pi}{3};$

$$\mu(\widehat{CMA}) = \frac{5\pi}{6}$$



By cosine theorem: $AB = \sqrt{x^2 + y^2}; BC = \sqrt{x^2 + z^2 + xz}; AC = \sqrt{y^2 + z^2 + yz\sqrt{3}}$

In ΔABC: $AB + BC > AC; BC + CA > BA; CA + AB > CB$

$$\sqrt{x^2 + y^2} + \sqrt{x^2 + z^2 + xz} \geq \sqrt{y^2 + z^2 + yz\sqrt{3}}$$

$$\sqrt{x^2 + y^2} + \sqrt{y^2 + z^2 + yz\sqrt{3}} \geq \sqrt{x^2 + z^2 + xz}$$

$$\sqrt{x^2 + z^2 + xz} + \sqrt{y^2 + z^2 + yz\sqrt{3}} \geq \sqrt{x^2 + y^2}$$

Equality holds for $x = y = z = 0$.

Aplication 3 : If $x, y, z \geq 0$ then:

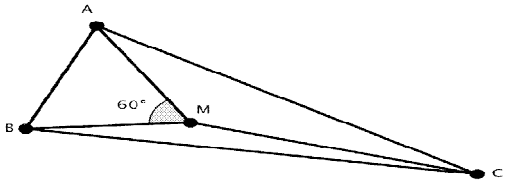
3.1. $\sqrt{y^2 + z^2 + yz\sqrt{3}} + \sqrt{x^2 + z^2 + xz\sqrt{3}} \geq \sqrt{x^2 + y^2 + xy}$

3.2 $\sqrt{y^2+z^2+yz\sqrt{3}}+\sqrt{x^2+y^2+xy}\geq\sqrt{x^2+z^2+xz\sqrt{3}}$

3.3. $\sqrt{x^2+z^2+xz\sqrt{3}}+\sqrt{x^2+y^2+xy}\geq\sqrt{y^2+z^2+yz\sqrt{3}}$

Solution: Let ΔABC such that: $AM=x; BM=y; CM=z$

$$\mu(\widehat{AMB})=60°; \mu(\widehat{BMC})=\mu(\widehat{AMC})=\frac{5\pi}{6}$$



By cosine theorem: $AB=\sqrt{x^2+y^2+xy}; BC=\sqrt{x^2+z^2+yz\sqrt{3}}; CA=\sqrt{x^2+z^2+xz\sqrt{3}}$

In ΔABC: $AB+BC>AC; BC+CA>BA; CA+AB>CB$

$$\sqrt{y^2+z^2+yz\sqrt{3}}+\sqrt{x^2+z^2+xz\sqrt{3}}\geq\sqrt{x^2+y^2+xy}$$

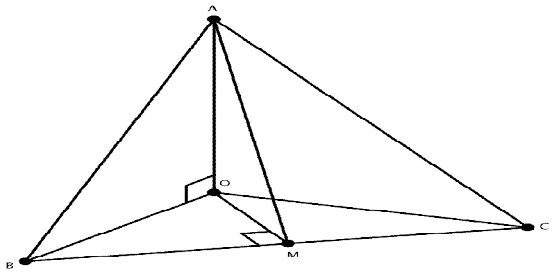
$$\sqrt{y^2+z^2+yz\sqrt{3}}+\sqrt{x^2+y^2+xy}\geq\sqrt{x^2+z^2+xz\sqrt{3}}$$

$$\sqrt{x^2+z^2+xz\sqrt{3}}+\sqrt{x^2+y^2+xy}\geq\sqrt{y^2+z^2+yz\sqrt{3}}$$

Aplication 4 : If $x,y,z\geq 0$ then:

$$\left(\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}\right)^2\geq 2\sqrt{3(x^2y^2+y^2z^2+z^2x^2)}$$

Solution:



Let be $OABC$ a tetrahedron such that: $OA \perp OB \perp OC \perp OA; OA = x; OB = y; OC = z$

$$OM = \frac{yz}{\sqrt{y^2+z^2}}; AM = \sqrt{x^2+\frac{y^2z^2}{y^2+z^2}}; AM = \frac{\sqrt{x^2y^2+y^2z^2+z^2x^2}}{\sqrt{y^2+z^2}}$$

$$S[ABC] = \frac{AM \cdot BC}{2} = \frac{1}{2} \cdot \sqrt{y^2+z^2} \cdot \frac{\sqrt{x^2y^2+y^2z^2+z^2x^2}}{\sqrt{y^2+z^2}}$$

$$S[ABC] = \frac{\sqrt{x^2y^2+y^2z^2+z^2x^2}}{2}$$

By Mitrinovic's inequality in ΔABC: $s \geq r\sqrt{3} \Rightarrow s \geq \frac{S}{s}\sqrt{3} \Rightarrow s^2 \geq S\sqrt{3}$

$$\frac{\left(\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}\right)^2}{4} \geq \frac{\sqrt{3}}{2} \cdot \sqrt{x^2y^2+y^2z^2+z^2x^2}$$

$$\left(\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}\right)^2 \geq 2\sqrt{3(x^2y^2+y^2z^2+z^2x^2)}$$

[1] Mihály Bencze; Daniel Sitaru – "**699 Olympic Mathematical Challenges**" – "Studis" Publishing House – Iași – 2017

[2] Mihály Bencze; Daniel Sitaru – "**Quantum Mathematical Power**" – "Studis" Publishing House – Iași – 2017

[3] Mihály Bencze; Daniel Sitaru – "**Olympic Mathematical Energy**" – "Studis" Publishing House – Iași – 2018

[4] George Apostolopoulos; Daniel Sitaru – "**The Olympic Mathematical Marathon**" – "Cartea Românească" Publishing House – Pitești – 2017

[5] **Romanian Mathematical Magazine – Interactive Journal – www.ssmrmh.ro**

THE BEAUTY OF INEQUALITIES

By Do Huu Duc Thinh-Vietnam

Inequality is one of the nicest and hardest major parts of Mathematics, because of its appearance in Math contests; and also it requires a solid knowledge to solve an inequality problem. But because of the elegance, Inequality attracts lots of student generations joining in solving and creating more. To maintain the passion and creativeness, many sites and forums nowadays have shared lots of topics and documents of inequality for everybody, but it's still intricate and note arranged entiringly. Maybe many people will consider that studying Inequality is boring, since its large amount of knowledges and complex presentation of that amount of knowledges, especially upper secondary high school and

higher. But I will demonstrate carefully and clearly for everyone through this document file, so everyone can realize how beautiful inequality is and its application. This file that I make is based on documents of many teachers in Vietnam and around the world, such as: Tran Quoc Anh, Nguyen Van Mau, Pham Kim Hung, Vo Quoc Ba Can, Vasile Cîrtoaje,... I spent lots of time to compile the file so if there's a mistake, I hope everybody can understand and give me feedback because my knowledge is limited. All feedbacks can be sent to the following email: thinh06032001@gmail.com or Facebook. Thank you very much!

Season 1: Some old and modern techniques of Inequalities in Math contests

In this season, I will state again some inequalities and basic techniques that are useful to find the solution of proving the inequality and solving Min-Max problems. Also I will add more lemmas with inferences and developments so everyone, especially students, can find a nice way to solve the inequality.

I. Inequalities can be proved by equivalence:

1) $a^2 + b^2 \geq \frac{(a+b)^2}{2} \geq 2ab$ ***(Basic BCS inequality for 2 numbers)***

$\rightarrow \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b} = \frac{2}{\frac{1}{a}+\frac{1}{b}}$ ***(Basic RMS-AM-GM-HM inequality for $a, b > 0$)***

$\rightarrow \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}} \leq a + b - \frac{2ab}{a+b}$ *with* $a, b \geq 0$ *such that* $a + b > 0$

2) $a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3} \geq ab + bc + ca$

$\rightarrow \begin{cases} * 3(a^2b^2 + b^2c^2 + c^2a^2) \geq (ab + bc + ca)^2 \geq 3abc(a + b + c) \\ * \frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \geq \sqrt{3(a^2 + b^2 + c^2)} \geq a + b + c \geq \sqrt{3(ab + bc + ca)} \end{cases} for\ a, b, c > 0$

3) $a^2 + b^2 + c^2 \geq 2ab + 2ac - 2bc$

4) $a^3 + b^3 + c^3 \geq 3abc\ \forall a, b, c$ *such that* $a + b + c \geq 0$ ***(special case: $a, b, c > 0$ – Cauchy's inequality for 3 non-negative numbers)***

5) $(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$ ***(Basic Bunyakovsky's inequality)***

6) $a^3 + b^3 \geq ab(a + b)\ \forall a, b$ such that $a + b \geq 0$

$\rightarrow 4(a^3 + b^3) \geq (a + b)^3 \geq 4ab(a + b)$ *for* $a + b \geq 0$

7) $2(a^2 + b^2 - ab)^2 \geq a^4 + b^4 \geq 2ab(2a^2 - 3ab + 2b^2) \forall a, b$

$\rightarrow \sqrt{a^2 - ab + b^2} \geq \frac{a^2+b^2}{a+b} = a + b - \frac{2ab}{a+b}$ *with* $a + b \neq 0$

8) $\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{8}{(a+b)^2}$ *with real numbers* a, b *such that* $ab > 0$ *(special case:* $a, b > 0$*)*

9) $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$ *with* $a, b, c > 0$ ***(Nesbitt's inequality)*** → *This is even true for real numbers* $a; b; c$ *such that* $ab + bc + ca > 0$.

10) $(a + b)(b + c)(c + a) \geq \frac{8}{9}(a + b + c)(ab + bc + ca) \geq 8abc$ *for* $a, b, c \geq 0$.

$\rightarrow \begin{cases} * abc \geq (b + c - a)(c + a - b)(a + b - c)\ for\ a, b, c\ are\ sides\ of\ a\ triangle \\ * (a + b + c)^3 \geq 27abc\ for\ a, b, c \geq 0 \end{cases}$ ($\boldsymbol{Cauchy's\ inequality\ for\ 3\ non-negative\ variables}$)

11) $\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}$ *with* $x, y > 0$ ***(Basic Bunyakovsky Cauchy-Schwarz inequality)***

** Also, from this inequality we have the chain:* $\frac{a^2}{b} + \frac{b^2}{a} \geq \sqrt{2(a^2 + b^2)} \geq a + b \geq 2\sqrt{ab}$ *for* $a, b > 0$.

12) $\begin{cases} * 2(ax + by) \geq (a + b)(x + y) \geq 2(ay + bx)\ with\ a \geq b, x \geq y \\ * 3(ax + by + cz) \geq (a + b + c)(x + y + z) \geq 3(az + by + cx)\ with\ a \geq b \geq c, x \geq y \geq z \end{cases}$ ***(Basic Chebyshev's inequality)***

13) $a^2b + b^2c + c^2a \geq ab^2 + bc^2 + ca^2$ *with* $a \geq b \geq c$

* *More general:* $a^nb + b^nc + c^na \geq ab^n + bc^n + ca^n$ *for* $a \geq b \geq c \geq 0, n \in Z^+$

14) $2(a^n + b^n) \geq (a^x + b^x)(a^y + b^y)$ *with* $a, b > 0; x, y, n \in Z^+: x + y = n$

$\rightarrow a^{m+n} + b^{m+n} \geq a^mb^n + a^nb^m$ *for* $a, b \geq 0, m, n \in Z^+$

15) $a^2 + ab + b^2 \geq \frac{3}{4}(a + b)^2; a^2 - ab + b^2 \geq \frac{1}{4}(a + b)^2 \ \forall a, b \rightarrow 3 \geq \frac{a^2-ab+b^2}{a^2+ab+b^2} \geq \frac{1}{3}$ *with real numbers* $a, b: a^2 + b^2 > 0$.

16) $\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \geq \sqrt{(a + c)^2 + (b + d)^2}$ ***(Basic Minkovsky's inequality)***

17*) $\frac{1}{(a+1)\hat{}2} + \frac{1}{(b+1)^2} \geq \frac{1}{ab+1}$ *with* $a, b \geq 0$ *(The equality happens iff* $a = b = 1$*)*

18) *Consider* $f(a; b) = \frac{1}{1+a^2} + \frac{1}{1+b^2} - \frac{2}{1+ab}$ *with* $a, b > 0$. *If* $ab \geq 1$ *then* $f(a; b) \geq 0$*; if* $ab \leq 1$ *then* $f(a; b) \leq 0$.

19) $\frac{1}{a^2-1} + \frac{1}{b^2-1} \geq \frac{2}{ab-1}$ *with* $a, b > 1$

20) $(a^2 + b^2 + c^2)(a + b + c) \geq 3.\max\{a^2b + b^2c + c^2a; ab^2 + bc^2 + ca^2\}$ *with* $a, b, c \geq 0$

21) $(a^2 + b^2 + c^2)^2 \geq (a + b + c)(a^2b + b^2c + c^2a) \ \forall a, b, c > 0$

22) $\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \geq \frac{3}{2} \geq \frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a}$ *with* $a \geq b \geq c > 0$

23*)$(a^2 + b^2 + c^2)^2 \geq 3(a^3b + b^3c + c^3a)$ ***(Vasile's inequality)***

24) $\frac{1}{1+x^2} \geq 1 - \frac{x}{2}$ *with* $\geq 0 \rightarrow \frac{1}{1+a^2} + \frac{1}{1+b^2} + \frac{1}{1+c^2} \geq \frac{3}{2} \ \forall a, b, c \geq 0: a + b + c = 3$.

25) $(a + b + c)^3 \geq a^3 + b^3 + c^3 + 24abc \ \forall a, b, c > 0$

II. Some familiar Inequalities, lemmas and techniques: (ascending by higher level)

a) For junior – early – senior:

1) $\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}} \geq \frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1a_2 \dots a_n} \geq \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$ *with* $a_i > 0 \ \forall i = 1; 2; \dots; n$ *and* $n \in Z^+, n \geq 2$

(RMS-AM-GM-HM inequality for n positive numbers)

$\rightarrow$ *If* $x_1; x_2; \dots; x_n$ *are positive real numbers that* $x_1 + x_2 \dots + x_n = 1$ *then with same condition for* a_i *we have:*

$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq a_1^{x_1} \cdot a_2^{x_2} \dots a_n^{x_n}$ ***(Weighted AM-GM inequality)***

2) *For real numbers* $a_i; b_i \ (i = 1; 2; \dots; n)$ *we have:*

$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$

(Cauchy-Schwarz inequality) $\rightarrow \frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1+a_2+\cdots+a_n)^2}{b_1+b_2+\cdots+b_n} \ \forall b_i > 0$

$\rightarrow \frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \cdots + \frac{x_{n-1}^2}{x_n} + \frac{x_n^2}{x_1} \geq \sqrt{n(x_1^2 + x_2^2 + \cdots + x_n^2)} \geq x_1 + x_2 + \cdots + x_n; \forall x_i > 0$

3) *Let* $a_1 \geq a_2 \geq \cdots \geq a_n$

$\begin{cases} * \ If \ b_1 \geq b_2 \geq \cdots \geq b_n \ then: n(a_1b_1 + a_2b_2 + \cdots + a_nb_n) \geq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) \\ * \ If \ b_1 \leq b_2 \leq \cdots \leq b_n \ then: n(a_1b_1 + a_2b_2 + \cdots + a_nb_n) \leq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) \end{cases}$

(Chebyshev's inequality)

4) *If* $x_{i.j} \geq 0 \ (i = 1; 2; \dots; n, j = 1; 2; \dots; m)$ *then we have:*

$(x_{1.1} + x_{1.2} + \cdots + x_{1.n})(x_{2.1} + x_{2.2} + \cdots + x_{2.n}) \dots (x_{m.1} + x_{m.2} + \cdots + x_{m.n}) \geq$

$\geq \left(\sqrt[m]{x_{1.1}x_{2.1} \dots x_{m.1}} + \sqrt[m]{x_{1.2}x_{2.2} \dots x_{m.2}} + \cdots + \sqrt[m]{x_{1.n}x_{2.n} \dots x_{m.n}}\right)^m$

(Hölder's inequality)

Ex: $* \ (a + b)(c + d) \geq \left(\sqrt{ac} + \sqrt{bd}\right)^2 \ \forall a, b, c, d \geq 0$

$* (a+b+c)(m+n+p)(x+y+z) \geq \left(\sqrt[3]{amx}+\sqrt[3]{bny}+\sqrt[3]{cpz}\right)^3 \ \forall a,b,c,m,n,p,x,y,z \geq 0$

$* \frac{a_1^m}{a_2^m}+\frac{a_2^m}{a_3^m}+\cdots+\frac{a_n^m}{a_1^m} \geq \frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_n}{a_1}$ *for* $a_i > 0\ (i = 1;2;\ldots;n); m \in Z^+, n \geq 2$

$\rightarrow$ $* \sqrt[n]{\frac{a_1^n+a_2^n+\cdots+a_n^n}{n}} \geq \frac{a_1+a_2+\cdots+a_n}{n}$ *with same condition* (***Power Mean Inequality***)

$* \frac{1}{a_1^2}+\frac{1}{a_2^2}+\cdots+\frac{1}{a_n^2} \geq \frac{n^3}{(a_1+a_2+\cdots a_n)^2}$ *for* $a_i > 0\ (i = 1;2;\ldots;n) \rightarrow \frac{1}{a_1^n}+\frac{1}{a_2^n}+\cdots+\frac{1}{a_n^n} \geq \frac{n^{n+1}}{(a_1+a_2+\cdots+a_n)^n}$

$\rightarrow$ *Let* a_i *and* $b_i > 0 (i = 1;2;\ldots;n)$ *and real* $p,q > 0$ *such that* $p+q = pq$. *Then we have:*

$\left(a_1^p+a_2^p+\cdots+a_n^p\right)^{\frac{1}{p}} \cdot \left(b_1^q+b_2^q+..+b_n^q\right)^{\frac{1}{q}} \geq a_1b_1+a_2b_2+\cdots+a_nb_n$ ***(General Hölder's inequality)***

5) *Let* $a_i, b_i > 0\ (i = 1;2;\ldots;n)$ *and any* $k > 1$. *Then we have:*

$\left(a_1^k+b_1^k\right)\left(a_2^k+b_2^k\right)\ldots\left(a_n^k+b_n^k\right) \geq \sqrt[k]{(a_1+a_2+\cdots+a_n)^k+(b_1+b_2+\cdots+b_n)^k}$

(Minkovsky's inequality) $\rightarrow$ *similarly for 3 variables* $a_i, b_i, c_i > 0$.

6) *For any* $x \geq -1$ *we have:* $\begin{cases} * (1+x)^r \geq 1+xr \ \ for\ r \geq 1\ and\ r \leq 0 \\ * (1+x)^r \leq 1+xr\ for\ 0 \leq r \leq 1 \end{cases}$ ***(Bernoulli's inequality)***

7) *For any positive integer* m *and* $a,b,c \geq 0$ *we have:*

$$a^m(a-b)(a-c)+b^m(b-c)(b-a)+c^m(c-a)(c-b) \geq 0$$

(Schur's inequality) $\rightarrow$ *This is also true for real* $m \geq 1$ *and equality happens iff* $a = b = c$ *or* $(a;b;c)\sim(0;k;k)$ *with* $k > 0$.

Δ *Case* $m = 1$ *– Schur deg* 3*: All forms: (*p,q,r *will be discussed in later part)*

$* a^3+b^3+c^3+3abc \geq ab(a+b)+bc(b+c)+ca(c+a) \rightarrow (a+b+c)^3+9abc \geq$

$\geq 4(a+b+c)(ab+bc+ca) \rightarrow a^2+b^2+c^2+\frac{9abc}{a+b+c} \geq 2(ab+bc+ca) \rightarrow$

$\rightarrow (a+b+c)(a^2+b^2+c^2+ab+bc+ca) \geq 3[ab(a+b)+bc(b+c)+ca(c+a)]$

$* abc \geq (b+c-a)(c+a-b)(a+b-c)$ *(Well – known result)*

$* (b-c)^2(b+c-a)+(c-a)^2(c+a-b)+(a-b)^2(a+b-c) \geq 0$

$* 3(a^3+b^3+c^3) \geq (a+b+c)[2(a^2+b^2+c^2)-ab-bc-ca]$

$* 4(a^3+b^3+c^3)+15abc \geq (a+b+c)^3$

$* \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4abc}{(a+b)(b+c)(c+a)} \geq 2$

$* \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3 \geq a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ *for* $a,b,c > 0$: $abc = 1$.

$* (a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2) \geq (a^2b^2+b^2c^2+c^2a^2)(ab+bc+ca)$

Δ *Case* $m = 2$ *– Schur deg* 4*: All forms:*

$* a^4+b^4+c^4+abc(a+b+c) \geq ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2) \rightarrow$

$\rightarrow a^4+b^4+c^4+2abc(a+b+c) \geq (a^2+b^2+c^2)(ab+bc+ca)$

$* 2(ab+bc+ca)-(a^2+b^2+c^2) \leq \frac{6abc(a+b+c)}{a^2+b^2+c^2+ab+bc+ca} \leq \frac{9abc}{a+b+c}$

$* [(b-c)(b+c-a)]^2+[(c-a)(c+a-b)]^2+[(a-b)(a+b-c)]^2 \geq 0$

$\rightarrow$ *Let* $a,b,c,x,y,z \geq 0$. *Then we have:*

$x(a-b)(a-c)+y(b-c)(b-a)+z(c-a)(c-b) \geq 0$ *iff* $a \geq b \geq c$ *and:*

$* x \geq y^{\vee}\ z \geq y$ $\qquad$ $* ax+cz \geq by$

$* x+z \geq y$ $\qquad$ $* \sqrt{x}+\sqrt{z} \geq \sqrt{y}$

$* ax \geq by^{\vee}\ \ cz \geq by$

(General Vornicu – Schur inequality)

8) $\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\geq\frac{9}{4(ab+bc+ca)}$ *for* $a,b,c\geq 0$*, no 2 of which are* 0*. (Iran 96 inequality)*

9) $(a^2+b^2+c^2)^2\geq 3(a^3b+b^3c+c^3a)$ ***(Vasile's inequality)*** → *The equality happens iff* $a=b=c$ *and also for* $(a;b;c)=\left(k\cdot\sin^2\frac{4\pi}{7};k\cdot\sin^2\frac{2\pi}{7};k\cdot\sin^2\frac{\pi}{7}\right)$ *or any cyclic permutation.*

10) *Let* a_i *and* b_i $(i=1;2;\dots;n)$ *such that:*

$*\ a_1\geq a_2\geq\cdots\geq a_n\geq 0, b_1\geq b_2\geq\cdots\geq b_n\geq 0$

$*\ a_1\geq b_1; a_1+a_2\geq b_1+b_2;\dots;a_1+a_2+\cdots+a_{n-1}\geq b_1+b_2+\cdots+b_{n-1}$

$*\ a_1+a_2+\cdots+a_{n-1}+a_n=b_1+b_2+\cdots+b_{n-1}+b_n$

For $x_1\geq 0$ *we have:* $\sum_{sym}x_{t_1}^{a_1}x_{t_2}^{a_2}\dots x_{t_n}^{a_n}\geq\sum_{sym}x_{t_1}^{b_1}x_{t_2}^{b_2}\dots x_{t_n}^{b_n}$*, where* $(t_1;t_2;\dots;t_n)$ *are all the permutations of* $(1;2;\dots;n)$

(Muirhead's inequality)

E.g: $a^3+b^3\geq ab(a+b)\to a^3b^0+b^3a^0\geq a^2b^1+a^1b^2$

$a^2+b^2+c^2\geq ab+bc+ca\to a^2b^0+b^2c^0+c^2a^0\geq a^1b^2+b^1c^1+c^1a^1$

$a^4+b^4+c^4\geq abc(a+b+c)\to a^4b^0c^0+b^4c^0a^0+c^4a^0b^0\geq$

$\geq a^2b^1c^1+b^2c^1a^1+c^2a^1b^1$

b) For senior and higher classes: (it's very hard to express the real form of these inequalities so I will try my best.)

1) $*$ *If* $\begin{bmatrix}a_1\geq a_2\geq\cdots\geq a_n;b_1\geq b_2\geq\cdots\geq b_n\\ a_1\leq a_2\leq\cdots\leq a_n;b_1\leq b_2\leq\cdots\leq b_n\end{bmatrix}$ *and* $(k_1;k_2;\dots;k_n)$ *is an arbitrary permutation of* $(1;2;\dots;n)$ *then:* $a_1b_1+a_2b_2+\cdots+a_nb_n\geq a_1b_{k_1}+a_2b_{k_2}+\cdots+a_nb_{k_n}$.

$*$ *If* $a_1\geq a_2\geq\cdots\geq a_n$ *and* $b_1\leq b_2\leq\cdots\leq b_n$ *then:*

$a_1b_1+a_2b_2+\cdots+a_nb_n\leq a_1b_{k_1}+a_2b_{k_2}+\cdots+a_nb_{k_n}$;

$n(a_1b_1+a_2b_2+\cdots+a_nb_n)\leq(a_1+a_2+\cdots+a_n)(b_1+b_2+\cdots+b_n)$

(Rearrangement inequality)

2) * *Convex function: If* $a,b\geq 0$ *such that* $a+b=1$ *then* $f(x)$ *is called a convex function on* $I(a;b)\subset R$ *iff* $\forall x_1;x_2\in I$ *we have:* $f(ax_1+bx_2)\leq af(x_1)+bf(x_2)$

* *Concave function: If* $a,b\geq 0$ *such that* $a+b=1$ *then* $f(x)$ *is called a concave function on* $I(a;b)\subset R$ *iff* $\forall x_1;x_2\in I$ *we have:* $f(ax_1+bx_2)\geq af(x_1)+bf(x_2)$

* *If* $f(x)$ *is a convex function on interval* $I\subset R$ *then for any* $x_i\in I$ $(i=1;2;\dots;n)$ *we have:* $f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)\leq\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$ ***(Classic Jensen's inequality)***

* *If* $f(x)$ *is a convex function on interval* $I\subset R$ *then for any* $x_1\in I$ $(i=1;2;\dots;n)$ *and* $p_i>0$ *we have:*

$$\frac{p_1f(x_1)+p_2f(x_2)+\cdots+p_nf(x_n)}{p_1+p_2+\cdots+p_n}\geq f\left(\frac{p_1x_1+p_2x_2+\cdots+p_nx_n}{p_1+p_2+\cdots+p_n}\right)$$

And if $f(x)$ *is a concave function then the inequality is reversed.* ***(General Jensen's inequality)***

→ $*$ *If* $f(x)$ *is a convex and continuous function on interval* $I\subset R$ *then for any* $x_i\in I$ $(i=1;2;\dots;n)$ *and* $p_i\in(0,1)$ *such that* $p_1+p_2+\cdots+p_n=1$ *we have:* $p_1f(x_1)+p_2f(x_2)+\cdots+p_nf(x_n)\geq f(p_1x_1+p_2x_2+\cdots+p_nx_n)$*. And if* $f(x)$ *is a concave function then the inequality is reversed.*

* *The classic inequality is a special case from the general one with* $p_1=p_2=\cdots=p_n$

* *Let* a_i *and* b_i $(i=1;2;..;n)\in I$ $(I\subset R)$ *such that:*

_ $a_1\geq a_2\geq\cdots\geq a_n;b_1\geq b_2\geq\cdots\geq b_n$

_ $a_1 \geq b_1; a_1 + a_2 \geq b_1 + b_2; \ldots; a_1 + a_2 + \cdots + a_{n-1} \geq b_1 + b_2 + \cdots + b_{n-1}$
_ $a_1 + a_2 + \cdots + a_{n-1} + a_n = b_1 + b_2 + \cdots + b_{n-1} + b_n$
If $f(x)$ is a convex function on I then we have:
$f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(b_1) + f(b_2) + \cdots + f(b_n)$ ***(Karamata's inequality)***
** If $f(x)$ is a convex function on $I \subset R$ then for $a_i \in I$ $(i = 1; 2; \ldots; n)$ we have:*

$$f(a_1) + f(a_2) + \cdots + f(a_n) + n(n-2)f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \geq$$

$\geq (n-1)[f(b_1) + f(b_2) + \cdots + f(b_n)]$ *where* $b_i + \frac{a_i}{n-1} = \frac{a_1+a_2+\cdots+a_n}{n-1}$ $(i = 1; 2; \ldots; n)$
(Popoviciu's inequality)
3) *Define* $p = a + b + c, q = ab + bc + ca, r = abc$ *with* a, b, c *are any real numbers. If* $k = \sqrt{p^2 - 3q}$ *then we have:* $\frac{p^3-3pk^2-2k^3}{27} \leq r \leq \frac{p^3-3pk^2+2k^3}{27}$
→ *The minimum and maximum happens iff 2 of 3 variables* $a; b; c$ *are equal.*
4) ** Let* $f(a; b; c)$ *be a symmetric polynomial of degree 3 with* $a, b, c \geq 0$*. Then:*
$f(a; b; c) \geq 0 \Leftrightarrow f(1; 1; 1); f(1; 1; 0); f(1; 0; 0) \geq 0$ ***(SD3 theorem)***
** Let* $f(a; b; c)$ *be a cyclic homogeneous polynomial of degree 3 with* $a, b, c \geq 0$*. Then:*
$f(a; b; c) \geq 0 \Leftrightarrow f(1; 1; 1) \geq 0; f(a; b; 0) \geq 0$ ***(CD3 theorem)***
→ *Let* $f_n(a; b; c)$ *be a cyclic homogeneous polynomial of degree* n $(n = 3; 4; 5)$ *with* $a, b, c \geq 0$*. Then* $f_n(a; b; c) \geq 0 \Leftrightarrow f_n(a; 1; 1) \geq 0$ *and* $f_n(0; b; c) \geq 0$*.*
5) ***(S.O.S technique)*** *Define* $S = S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2$*, where* $S_a; S_b; S_c$ *are functions with variables* $a; b; c$*. Then* $S \geq 0$ *iff:*
* $S_a; S_b; S_c \geq 0$
* $a \geq b \geq c; S_b \geq 0; S_b + S_a \geq 0; S_b + S_c \geq 0$
* $a \geq b \geq c; S_a \geq 0; S_c \geq 0; S_a + 2S_b \geq 0; S_c + 2S_b \geq 0$
* $a \geq b \geq c; S_b \geq 0; S_c \geq 0; a^2S_b + b^2S_a \geq 0$
* $S_a + S_b + S_c \geq 0; S_aS_b + S_bS_c + S_cS_a \geq 0$
→ *Consider* $f(a; b; c) = P(a-b)^2 + Q(a-c)(b-c) \geq 0$ *(*)*
** If* $f(a; b; c)$ *is symmetric then to prove (*) is true, we assume that* $a \geq b \geq c$ *or* $c = \min\{a; b; c\}$ *or* $c = \max\{a; b; c\}$ *and prove that* $P, Q \geq 0$*.*
** If* $f(a; b; c)$ *is cyclic then to prove (*) is true, we assume that* $c = \min\{a; b; c\}$ *or* $c = \max\{a; b; c\}$ *and prove that* $P, Q \geq 0$*.* ***(S.S technique)***
c) Some identities:
(*) Some useful identities in inequality that can be proved by S.O.S, S.S technique:
1) $a^2 + b^2 - 2ab = (a-b)^2; \frac{a}{b} + \frac{b}{a} - 2 = \frac{(a-b)^2}{ab}$

$$\rightarrow \sqrt{n(a_1^2 + a_2^2 + \cdots + a_n^2)} - (a_1 + a_2 + \cdots + a_n) = \frac{\sum_{1 \leq i < j \leq n}(a_i - a_j)^2}{\sqrt{n(a_1^2 + a_2^2 + \cdots + a_n^2)} + (a_1 + a_2 + \cdots + a_n)}$$

For $n = 2$: $\sqrt{2(a^2 + b^2)} - (a + b) = \frac{(a-b)^2}{\sqrt{2(a^2+b^2)}+a+b}$

For $n = 3$: $\sqrt{3(a^2 + b^2 + c^2)} - (a + b + c) = \frac{(a-b)^2+(b-c)^2+(c-a)^2}{\sqrt{3(a^2+b^2+c^2)}+(a+b+c)} = 2\frac{(a-b)^2+(a-c)(b-c)}{\sqrt{3(a^2+b^2+c^2)}+(a+b+c)}$

$$\rightarrow (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 9 = \frac{(a-b)^2}{ab} + \frac{(b-c)^2}{bc} + \frac{(c-a)^2}{ca} = \frac{2(a-b)^2}{ab} + \left(\frac{1}{ac} + \frac{1}{bc}\right)(a-c)(b-c)$$

2) $(a + b + c)^2 - 3(ab + bc + ca) = a^2 + b^2 + c^2 - (ab + bc + ca) =$

$$= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] = (a-b)^2 + (a-c)(b-c)$$

3) $a^3 + b^3 - ab(a + b) = (a + b)(a - b)^2$

$$\rightarrow \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}=\frac{(b-c)^2}{2(a+b)(a+c)}+\frac{(c-a)^2}{2(b+c)(b+a)}+\frac{(a-b)^2}{2(c+a)(c+b)}=$$
$$=\frac{(a-b)^2}{(c+a)(c+b)}+\frac{(a+b+2c)(a-c)(b-c)}{2(a+b)(b+c)(c+a)}$$
$$\rightarrow 3(a^3+b^3+c^3)-(a+b+c)(a^2+b^2+c^2)=(a+b)(a-b)^2+(b+c)(b-c)^2+(c+a)(c-a)^2$$
$$=2(a+b)(a-b)^2+(a+b+2c)(a-c)(b-c)$$

4) $(a+b+c)(ab+bc+ca)-9abc=(a+b)(b+c)(c+a)-8abc=$
$$=a(b-c)^2+b(c-a)^2+c(a-b)^2=2c(a-b)^2+(a+b)(a-c)(b-c)$$

5) $a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]=$
$$=(a+b+c)[(a-b)^2+(a-c)(b-c)]$$

6) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3=\frac{1}{6abc}[(a-b)^2(3c+a-b)+(b-c)^2(3a+b-c)+(c-a)^2(3b+c-a)]=$
$$=\frac{(a-b)^2}{ab}+\frac{(a-c)(b-c)}{bc}$$

7) $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-(a+b+c)=\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}=\left(\frac{1}{a}+\frac{1}{b}\right)(a-b)^2+\frac{b+c}{ac}(a-c)(b-c)$

8) $a^3+b^3+c^3+3abc-ab(a+b)-bc(b+c)-ca(c+a)=$
$$=\frac{1}{2}[(b+c-a)(b-c)^2+(c+a-b)(c-a)^2+(a+b-c)(a-b)^2]$$
$$=(a+b-c)(a-b)^2+c(a-c)(b-c)$$

9) $\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}-(a+b+c)=$
$$=\frac{1}{2abc}\{(b-c)^2[(b+c)^2+a^2]+(c-a)^2[(c+a)^2+b^2]+(a-b)^2[(a+b)^2+c^2]\}$$
$$=\frac{(a+b)^2+c^2}{abc}(a-b)^2+\left[\frac{1}{c}+\frac{(a+c)(b+c)}{abc}\right](a-c)(b-c)$$

10) $\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}-(a^2+b^2+c^2)=\left(\frac{a}{b}+\frac{1}{2}\right)(a-b)^2+\left(\frac{b}{c}+\frac{1}{2}\right)(b-c)^2+\left(\frac{c}{a}+\frac{1}{2}\right)(c-a)^2$

11) $a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2-a(b^3+c^3)-b(c^3+a^3)-c(a^3+b^3)=$
$$=\frac{1}{2}[(a^2+b^2)(a-b)^2+(b^2+c^2)(b-c)^2+(c^2+a^2)(c-a)^2]$$
$$=(a^2+b^2)(a-b)^2+(a^2+ab+b^2+c^2)(a-c)(b-c)$$

(**) More identities: Actually the first nine identities are rare, so just consider the identity 10 onwards:

1) $*\frac{a+bc}{b-c}\cdot\frac{b+ca}{c-a}+\frac{b+ca}{c-a}\cdot\frac{c+ab}{a-b}+\frac{c+ab}{a-b}\cdot\frac{a+bc}{b-c}=a+b+c-1$ *with* $a\neq b\neq c$

$*\frac{a-bc}{b-c}\cdot\frac{b-ca}{c-a}+\frac{b-ca}{c-a}\cdot\frac{c-ab}{a-b}+\frac{c-ab}{a-b}\cdot\frac{a-bc}{b-c}=-a-b-c-1$ *with* $a\neq b\neq c$

2) $*\frac{a-b}{a+b}+\frac{b-c}{b+c}+\frac{c-a}{c+a}=\frac{(a-b)(b-c)(a-c)}{(a+b)(b+c)(a+c)}$

$*\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}=\frac{a(b-c)^2+b(c-a)^2+c(a-b)^2}{(a-b)(b-c)(a-c)}$ *with* $a\neq b\neq c$

3) $*\frac{a+b}{a-b}\cdot\frac{b+c}{b-c}+\frac{b+c}{b-c}\cdot\frac{c+a}{c-a}+\frac{c+a}{c-a}\cdot\frac{a+b}{a-b}=-1$ *with* $a\neq b\neq c$

$*\frac{a-b}{a+b}\cdot\frac{b-c}{b+c}+\frac{b-c}{b+c}\cdot\frac{c-a}{c+a}+\frac{c-a}{c+a}\cdot\frac{a-b}{a+b}=\frac{-[a(b-c)^2+b(c-a)^2+c(a-b)^2]}{(a+b)(b+c)(c+a)}$

4) $*\frac{a^2+bc}{b+c}\cdot\frac{b^2+ca}{c+a}+\frac{b^2+ca}{c+a}\cdot\frac{c^2+ab}{a+b}+\frac{c^2+ab}{a+b}\cdot\frac{a^2+bc}{b+c}=a^2+b^2+c^2$

$*\frac{a^2-bc}{b-c}\cdot\frac{b^2-ca}{c-a}+\frac{b^2-ca}{c-a}\cdot\frac{c^2-ab}{a-b}+\frac{c^2-ab}{a-b}\cdot\frac{a^2-bc}{b-c}=-(a+b+c)^2$ *with* $a\neq b\neq c$

5) $(a^2-bc)(b+c)+(b^2-ca)(c+a)+(c^2-ab)(a+b)=0$

$(a^2+bc)(b-c)+(b^2+ca)(c-a)+(c^2+ab)(a-b)=-2(a-b)(b-c)(c-a)$

6) $\frac{(b+c)^2}{(a-b)(a-c)}+\frac{(c+a)^2}{(b-c)(b-a)}+\frac{(a+b)^2}{(c-a)(c-b)}=1$ *with* $a\neq b\neq c$

7) $\frac{1-ab}{a-b}\cdot\frac{1-bc}{b-c}+\frac{1-bc}{b-c}\cdot\frac{1-ca}{c-a}+\frac{1-ca}{c-a}\cdot\frac{1-ab}{a-b}=-1$ *with* $a\neq b\neq c$

8) $*\frac{a^2+bc}{b-c}\cdot\frac{b^2+ca}{c-a}+\frac{b^2+ca}{c-a}\cdot\frac{c^2+ab}{a-b}+\frac{c^2+ab}{a-b}\cdot\frac{a^2+bc}{b-c}=a^2+b^2+c^2$ *with* $a\neq b\neq c$

$*\frac{a^2-bc}{b+c}\cdot\frac{b^2-ca}{c+a}+\frac{b^2-ca}{c+a}\cdot\frac{c^2-ab}{a+b}+\frac{c^2-ab}{a+b}\cdot\frac{a^2-bc}{b+c}=-(a+b+c)[ab(a-b)^2+bc(b-c)^2+ca(c-a)^2]$

9) $\frac{a^2+bc}{a^2-bc}\cdot\frac{b^2+ca}{b^2-ca}+\frac{b^2+ca}{b^2-ca}\cdot\frac{c^2+ab}{c^2-ab}+\frac{c^2+ab}{c^2-ab}\cdot\frac{a^2+bc}{a^2-bc}=-1$

10) $(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$

11) $2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$

12) $(ab^n+bc^n+ca^n)-(a^nb+b^nc+c^na)=(a-b)(b-c)(c-a)\sum_{sym}a^pb^qc^r$ *with* $p,q,r\in N$ *and* $n\geq 2$ *such that* $p+q+r=n-2$ *(?)*

E.g: for $n=2$: $(ab^2+bc^2+ca^2)-(a^2b+b^2c+c^2a)=(a-b)(b-c)(c-a)$

$n=3$: $(ab^3+bc^3+ca^3)-(a^3b+b^3c+c^3a)=(a-b)(b-c)(c-a)(a+b+c)$

$n=4$: $(ab^3+bc^3+ca^3)-(a^3b+b^3c+c^3a)=(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$

13) $a^n(a-b)(a-c)+b^n(b-c)(b-a)+c^n(c-a)(c-b)=$

$=\frac{1}{2}[(a^n+b^n-c^n)(a-b)^2+(b^n+c^n-a^n)(b-c)^2+(c^n+a^n-b^n)(c-a)^2]$

14) *If* $a,b,c\neq 0$ *such that* $abc=1$ *then:* $\frac{1}{ab+b+1}+\frac{1}{bc+c+1}+\frac{1}{ca+a+1}=1$

15) $a^2+b^2+c^2+abc=4\Leftrightarrow 2a+bc=\sqrt{(4-a^2)(4-b^2)}$, *etc* $\Leftrightarrow$

$$\Leftrightarrow\frac{2a+bc}{(2+b)(2+c)}+\frac{2b+ca}{(2+c)(2+a)}+\frac{2c+ab}{(2+a)(2+b)}=$$

$$=\frac{(2-b)(2-c)}{2a+bc}+\frac{(2-c)(2-a)}{2b+ca}+\frac{(2-a)(2-b)}{2c+ab}=\frac{a}{2a+bc}+\frac{b}{2b+ca}+\frac{c}{2c+ab}=$$

$$=\frac{bc}{2a+bc}+\frac{ca}{2b+ca}+\frac{ab}{2c+ab}=1\Leftrightarrow\frac{1}{2a+bc}+\frac{1}{2b+ca}+\frac{1}{2c+ab}=\frac{1}{a+b+c-2}=$$

$$=\frac{a+b+c+2}{2(ab+bc+ca)-abc}\Leftrightarrow(a+b+c-2)^2=(2-a)(2-b)(2-c)$$

→ *From the identity, there exists* $x,y,z>0$ *such that:* $a=2\sqrt{\frac{xy}{(z+x)(z+y)}}$; $b=2\sqrt{\frac{yz}{(x+y)(x+z)}}$;

$c=2\sqrt{\frac{xz}{(y+z)(y+x)}}$. *And there exists triangle* ABC *such that:* $a=2\cos A$; $b=2\cos B$;

$$c=2\cos C$$

16) $a^2+b^2+c^2+2abc=1\Leftrightarrow a+bc=\sqrt{(1-a^2)(1-b^2)}$, *etc* $\Leftrightarrow$

$$\Leftrightarrow\frac{a+bc}{(1+b)(1+c)}+\frac{b+ca}{(1+c)(1+a)}+\frac{c+ab}{(1+a)(1+b)}=$$

$$=\frac{(1-b)(1-c)}{a+bc}+\frac{(1-c)(1-a)}{b+ca}+\frac{(1-a)(1-b)}{c+ab}=1\Leftrightarrow$$

$$\Leftrightarrow\frac{a}{a+bc}+\frac{b}{b+ca}+\frac{c}{c+ab}=2\Leftrightarrow\frac{bc}{a+bc}+\frac{ca}{b+ca}+\frac{ab}{c+ab}=1\Leftrightarrow$$

$$\Leftrightarrow\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}=\frac{2}{a+b+c-1}=\frac{a+b+c+1}{ab+bc+ca-abc}\Leftrightarrow$$

$$\Leftrightarrow(a+b+c-1)^2=2(1-a)(1-b)(1-c)$$

→ *If we substitute* $a\to\frac{a}{2}$; $b\to\frac{b}{2}$; $c\to\frac{c}{2}$ *we will get identity 15, so:*

From the identity, there exists $x,y,z>0$ *such that:* $a=\sqrt{\frac{xy}{(z+x)(z+y)}}$; $b=\sqrt{\frac{yz}{(x+y)(x+z)}}$;

$c = \sqrt{\frac{zx}{(y+z)(y+x)}}$. *And there exists triangle* ABC *such that:* $a = \cos A$; $b = \cos B$; $c = \cos C$.

⊕ *Also if we let* $a^2 = yz; b^2 = zx; c^2 = xy$ *with* $x, y, z > 0$ *then* $abc = xyz$, *so we have 2 identities 17-18:*

17) $xy + yz + zx + xyz = 4 \Leftrightarrow \frac{1}{x+2} + \frac{1}{y+2} + \frac{1}{z+2} = \frac{x}{x+2} + \frac{y}{y+2} + \frac{z}{z+2} = 1 \Leftrightarrow$

$$\Leftrightarrow \frac{\sqrt{x}}{2+x} + \frac{\sqrt{y}}{2+y} + \frac{\sqrt{z}}{2+z} = \frac{\sqrt{xyz}}{\sqrt{xy} + \sqrt{yz} + \sqrt{zx} - 2}$$

18) $xy + yz + zx + 2xyz = 1 \Leftrightarrow \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 2 \Leftrightarrow \frac{x}{x+1} + \frac{y}{y+1} + \frac{z}{z+1} = 1 \Leftrightarrow$

$$\Leftrightarrow \frac{\sqrt{x}}{1+x} + \frac{\sqrt{y}}{1+y} + \frac{\sqrt{z}}{1+z} = \frac{2\sqrt{xyz}}{\sqrt{xy} + \sqrt{yz} + \sqrt{zx} - 1}$$

→ *From identity 17, there also exists,* $m, n, p > 0$ *such that* $x = \frac{2m}{n+p}; y = \frac{2n}{p+m}; z = \frac{2p}{m+n}$, *similarly for identity 18.*

19) $x(x^2 + xy + y^2) + y(y^2 + yz + z^2) + z(z^2 + zx + x^2) = y(x^2 + xy + y^2) +$

$$+z(y^2 + yz + z^2) + x(z^2 + zx + x^2) = (x + y + z)(x^2 + y^2 + z^2)$$

$$z(x^2 + xy + y^2) + x(y^2 + yz + z^2) + y(z^2 + zx + x^2) = (x + y + z)(xy + yz + zx)$$

20) $(x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2) = p^2 + pq + q^2$, *where*

$$p = x^2y + y^2z + z^2x; q = xy^2 + yz^2 + zx^2$$

$$= 3(x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2) + [(x-y)(y-z)(z-x)]^2$$

$$= \frac{3}{4}[xy(x+y) + yz(y+z) + zx(z+x)]^2 + \frac{1}{4}[(x-y)(y-z)(z-x)]^2$$

$$= \frac{1}{2}[(x+y+z)^2(x^2y^2 + y^2z^2 + z^2x^2) + (xy + yz + zx)^2(x^2 + y^2 + z^2)]$$

$$\rightarrow \begin{cases} *\ (xy + yz + zx)^2(x^2 + y^2 + z^2) = (x^2 + 2yz)(y^2 + 2zx)(z^2 + 2xy) + [(x-y)(y-z)(z-x)]^2 \\ *\ (x + y + z)^2(x^2y^2 + y^2z^2 + z^2x^2) = (2x^2 + yz)(2y^2 + zx)(2z^2 + xy) + [(x-y)(y-z)(z-x)]^2 \end{cases}$$

21) $2(x^2 + y^2)(y^2 + z^2)(z^2 + x^2) = [xy(x+y) + yz(y+z) + zx(z+x) - 2xyz]^2 +$

$$+[(x-y)(y-z)(z-x)]^2$$

22) $2[x^2(y-z)^4 + y^2(z-x)^4 + z^2(x-y)^4] = [x(y-z)^2 + y(z-x)^2 + z(x-y)^2]^2 +$

$$+[(x-y)(y-z)(z-x)]^2$$

23) $\frac{a^2}{a^2+bc} + \frac{b^2}{b^2+ca} + \frac{c^2}{c^2+ab} = \frac{3}{2} \Leftrightarrow \left(\frac{ab+bc+ca}{a+b+c}\right)^3 = abc \Leftrightarrow a^2 = bc$ *or* $b^2 = ca$ *or* $c^2 = ab$

d) Useful lemmas: In above parts, I've showed some of it. In this part I will state more lemmas, maybe a lot but worth it :)

• *Inequalities with condition about* p, q, r *(denote* $p = a + b + c; q = ab + bc + ca;$ $r = abc$*) – part 1: In this part, I will state lemma with familiar conditions, about the „unusual" conditions, I will show in later seasons.*

d. 1) If $a, b, c > 0$ ***such that*** $abc = 1$ ***then:***

1) $a^m + b^m + c^m \geq a^n + b^n + c^n$ $(m, n \in Z^+; m > n)$

2) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c; \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \rightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{3}{2}(a + b + c - 1) \rightarrow$

$$\rightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 3 \geq a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

3) $\frac{1}{a^{2k}+a^k+1} + \frac{1}{b^{2k}+b^k+1} + \frac{1}{c^{2k}+c^k+1} \geq 1$ *with* $k \in Z^+$

4) $\frac{1}{a^k+b^k+1} + \frac{1}{b^k+c^k+1} + \frac{1}{c^k+a^k+1} \leq 1$ *with* $k \in Z^+$

5) $*\frac{1}{a(b+1)}+\frac{1}{b(c+1)}+\frac{1}{c(a+1)}\geq\frac{3}{2}$

$*\frac{1}{a(a+b)}+\frac{1}{b(b+c)}+\frac{1}{c(c+a)}\geq\frac{3}{2}\rightarrow\frac{1}{a(ma+nb)}+\frac{1}{b(mb+nc)}+\frac{1}{c(mc+na)}\geq\frac{3}{m+n}$ *with* $m,n>0$ *(*)*

6) $\frac{1}{a^2+2b^2+3}+\frac{1}{b^2+2c^2+3}+\frac{1}{c^2+2a^2+3}\leq\frac{1}{2}$

7) $\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca}\leq 1$

8) *Let* $p=a+b+c; q=ab+bc+ca$ *then:*

$$*\ p^2+3\geq 4q\rightarrow\begin{cases}p+\frac{3}{q}\geq 4\\ p+\frac{3}{q}\geq 4\cdot\frac{q}{p}\end{cases}\qquad *\ pq\geq 5p-6$$

9) $\frac{a}{b+c+1}+\frac{b}{c+a+1}+\frac{c}{a+b+1}\geq 1\rightarrow\frac{a}{b+c+1}+\frac{b}{c+a+1}+\frac{c}{a+b+1}\geq\frac{a}{a+2}+\frac{b}{b+2}+\frac{c}{c+2}\geq 1\geq$

$$\geq\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\rightarrow\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1}\geq 1\geq\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}$$

$$\left(Extra:\frac{1}{3a+1}+\frac{1}{3b+1}+\frac{1}{3c+1}\geq\frac{3}{4}\geq\frac{1}{a+3}+\frac{1}{b+3}+\frac{1}{c+3}\right)$$

10) $(a^2+1)(b^2+1)(c^2+1)\geq(a+b)(b+c)(c+a)\geq(a+1)(b+1)(c+1)\geq 8$

11) $\sqrt{(n^2-1)a^2+1}+\sqrt{(n^2-1)b^2+1}+\sqrt{(n^2-1)c^2+1}\leq n(a+b+c)$ *with*

$$a+b+c\geq ab+bc+ca$$

$\rightarrow$ *Special case:* $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ *and without condition* $abc=1$.

12) $\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2}+\frac{1}{(c+1)^2}\geq\frac{3}{4}$

13) $(a+b+c)^5\geq 81(a^2+b^2+c^2)$

d. 2) ***If*** $\boldsymbol{a,b,c}>0$ ***such that*** $\boldsymbol{a+b+c=3}$ ***then:***

1) $a^2+b^2+c^2\geq 3\geq ab+bc+ca$

2) $a^m+b^m+c^m\geq 3$ *for* $m\in Z^+$

3) $\frac{a^m}{b^n}+\frac{b^m}{c^n}+\frac{c^m}{a^n}\geq 3$ *for* $m,n\in Z^+$ *such that* $m\geq n$

4) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq a^2+b^2+c^2$

5) $a^{n-1}+b^{n-1}+c^{n-1}\leq\frac{a^n+b^n}{a+b}+\frac{b^n+c^n}{b+c}+\frac{c^n+a^n}{c+a}\leq a^n+b^n+c^n$ *with all* $n\in Z^+$

6) $\frac{1}{a^2+b^2+n}+\frac{1}{b^2+c^2+n}+\frac{1}{c^2+a^2+n}\leq\frac{3}{n+2}$ *with all* $n\geq 2$

7) $\frac{a^m}{b^n+c^n}+\frac{b^m}{c^n+a^n}+\frac{c^m}{a^n+b^n}\geq\frac{3}{2}$ *with* $m,n\in Z^+$ *such that* $m>n$

8) $abc(a^2+b^2+c^2)\leq 3\rightarrow(abc)^n\cdot(a^2+b^2+c^2)\leq 3$ *with* $n\in Z^+\rightarrow\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq$

$$\geq\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\geq a^2+b^2+c^2$$

9) $\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}\geq 1$

10) $a^2b+b^2c+c^2a+abc\leq 4$

11) $a^2+b^2+c^2\geq a^2b+b^2c+c^2a$

12*) $\frac{a}{b^2+pc}+\frac{b}{c^2+pa}+\frac{c}{a^2+pb}\geq\frac{3}{1+p}$ *with* $p\geq 1$

d.3) If $\boldsymbol{a,b,c}>0$: $ab+bc+ca=3$ ***then:***

1) $a+b+c\geq 3abc\rightarrow(a+b+c)^5\geq 243abc$

2) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq a+b+c$

3) $\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\geq\frac{3}{2}\rightarrow\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2}+\frac{1}{(c+1)^2}\geq\frac{3}{4}$

4) $a^2b + b^2c + c^2a \le a^2 + b^2 + c^2$

5) $\frac{1}{(a+b)^2+m} + \frac{1}{(b+c)^2+m} + \frac{1}{(c+a)^2+m} \le \frac{3}{4+m}$ *with* $m \ge 2$

$\rightarrow \frac{1}{(xa+yb)^2+m} + \frac{1}{(xb+yc)^2+m} + \frac{1}{(xc+ya)^2+m} \le \frac{3}{(x+y)^2+m}$ *with* $x, y > 0$ *and* $m \ge 2(x^2 - xy + y^2)$ *(?)*

6) $\frac{1}{a^2+b^2+k} + \frac{1}{b^2+c^2+k} + \frac{1}{c^2+a^2+k} \le \frac{3}{2+k}$ *with* $k \ge 1$

7) $\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \le 1$

8*) $\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \le 1$

d.4) If $a, b, c > 0$ ***such that*** $a^2 + b^2 + c^2 = 3$ ***then:***

1) $a + b + c \ge ab + bc + ca \rightarrow \sqrt[n]{a} + \sqrt[n]{b} + \sqrt[n]{c} \ge ab + bc + ca$ *with* $n \in Z^+$

2) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \frac{9}{a+b+c} \rightarrow (a + b + c)^3 \ge 9(ab + bc + ca)$

3) $\frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} \le 1; \frac{a}{b+1} + \frac{b}{c+1} + \frac{c}{a+1} \ge \frac{3}{2}$

4) $a^3b + b^3c + c^3a \le 3$

5) $a^2b + b^2c + c^2a \le 2 + abc \rightarrow$ *In case* $a, b, c \ge 0$: *the equality happens if* $\begin{cases} a = b = c = 1 \\ a = \sqrt{2}; b = 1; c = 0 \text{ or any cyclic permutation} \end{cases}$

d.5) If $a, b, c > 0$ ***such that*** $a + b + c = ab + bc + ca$ ***then:***

1) $\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \le 1$

2) $a + b + c + abc \ge 4 \rightarrow \begin{cases} * \; a + b + c + 1 \ge 4abc \\ * \; (a + 1)(b + 1)(c + 1) \ge 8 \end{cases}$

3) $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \le \frac{3}{2}$

4) $(a + b)(b + c)(c + a) \ge 8$

5) $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge a^2 + b^2 + c^2 \rightarrow \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + \frac{3n}{a^2+b^2+c^2} \ge 3 + n$ *with* $n \le 3$

6) $a + b + c \ge abc + 2$

d. 6) If $a, b, c > 0$ ***such that*** $ab + bc + ca + abc = 4$ ***then:***

1) $a + b + c \ge ab + bc + ca \ge 3 \rightarrow \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{a+b+c}{2}$

$$\rightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge a + b + c$$

2) $\frac{a+b+c}{2} \ge \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \ge \frac{3}{2} \rightarrow \frac{a+b+c}{2} \ge \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}$

3) $\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \le 1$

4) $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \le 3$

5) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge a + b + c$

6) $a^2b + b^2c + c^2a \le a^2 + b^2 + c^2$

d.7) If $a, b, c > 0$ ***such that*** $a^2 + b^2 + c^2 + abc = 4$ ***then:***

1) $ab + bc + ca \le a + b + c \le 3 \rightarrow ab + bc + ca \le abc + 2 \le a + b + c \le 3$

2) $\frac{3}{2} \le \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \le \frac{9}{2(ab+bc+ca)}$

3) $a(a^2 + b^2) + b(c^2 + a^2) + c(a^2 + b^2) \le 6$

4) $\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \ge 3 \rightarrow \frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \ge a^2 + b^2 + c^2$

5) $a + bc \le a + \left(\frac{b+c}{2}\right)^2 \le 2$*; etc*

6) $a^2+b^2+c^2+ab+bc+ca \geq 2(a+b+c) \rightarrow a+b+c \geq \sqrt{a}+\sqrt{b}+\sqrt{c} \geq ab+bc+ca$

***d*.8) If** $\boldsymbol{a,b,c>0}$ **such that** $\boldsymbol{a^2+b^2+c^2=2(ab+bc+ca)}$ **then:** $\frac{a+b+c}{3} \geq \sqrt[3]{2abc}$
$\rightarrow$ ***the equality happens if*** $(a;b;c)\sim(k;k;4k)$ ***with*** $k>0$.

***d*.9) If** $\boldsymbol{a,b,c>0}$ **such that** $\boldsymbol{ab+bc+ca=abc+2}$ **then:**
1) *Assume that* $(b-1)(c-1) \geq 0$ *then:* $c+ab \geq 2 \rightarrow a^2+b^2+c^2+abc \geq 4$
2) $\max\{ab;bc;ca\} \geq 1; \max\{a;b;c\} \geq 1$

• ***Inequalities with classic condition (like*** $\boldsymbol{a,b,c>0,\ldots}$***) – part 1:***

***d*.10)** *If* $a,b,c>0$ *then:* $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a+b+c}{\sqrt[3]{abc}}$

***d*.11)** *If* $a_1;a_2;\ldots;a_n>0$ *then:* $\frac{1}{1+a_1^n}+\frac{1}{1+a_2^n}+\cdots+\frac{1}{1+a_n^n} \geq \frac{n}{1+a_1a_2\ldots a_n}$

***d*.12)** *If* $a,b,c \geq 0$*, no 2 of which are* 0 *then:* $\frac{a^n+b^n}{a+b}+\frac{b^n+c^n}{b+c}+\frac{c^n+a^n}{c+a} \leq 3\cdot\frac{a^n+b^n+c^n}{a+b+c}$ *with* $\forall n \in Z^+$

***d*.13)** *If* $a,b,c>0$ *then:* $\frac{1}{2}+\frac{a^2+b^2+c^2}{ab+bc+ca} \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{a^2+b^2+c^2}{2(ab+bc+ca)}+1$

***d*.14)** *If* $a,b,c,x,y,z>0$ *then:*

$$a(y+z)+b(z+x)+c(x+y) \geq 2\sqrt{(ab+bc+ca)(xy+yz+zx)}$$

***d*.15)** *If* $a,b,c,d \geq 0$ *then:*
$a^4+b^4+c^4+d^4+2abcd \geq a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2$ *(Turkevich's inequality)→ The equality happens iff* $a=b=c=d$ *and* $a=b=c=k>0, d=0$ *or any cyclic permutation.*

***d*.16)** *If* $a,b,c>0$ *then:* $\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b} \geq 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$

***d*.17)** *If* a,b,c *are sides of a triangle then:*

$$\sqrt{3(a+b+c)} \geq \sqrt{a}+\sqrt{b}+\sqrt{c} \geq \sqrt{b+c-a}+\sqrt{c+a-b}+\sqrt{a+b-c}$$

***d*.18,19)** *If* $a,b,c>0$ *then:*
$* \frac{a^3}{b^2+bc+c^2}+\frac{b^3}{c^2+ca+a^2}+\frac{c^3}{a^2+ab+b^2} \geq \frac{a^2+b^2+c^2}{a+b+c} \geq \frac{a+b+c}{3}$
$* \frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2} \geq a+b+c \geq \frac{3(ab+bc+ca)}{a+b+c}$

***d*.20)** *If* $a,b,c>0$ *then:*

$$\frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2} \geq \frac{a+b+c}{ab+bc+ca} \geq \frac{a}{a^2+2bc}+\frac{b}{b^2+2ca}+\frac{c}{c^2+2ab}$$

***d*.21)** *If* $a,b,c>0$ *then:* $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$

***d*.22)** *If* $a,b,x,y,z>0$ *then:* $\frac{x}{ay+bz}+\frac{y}{az+bx}+\frac{z}{ax+by} \geq \frac{3}{a+b}$

***d*.23*)** *If* $x \geq 0$ *then:* $e^x \geq 1+x+\frac{x^2}{2} \geq 1+x \rightarrow e^x \geq \left(1+\frac{x}{n}\right)^n$

***d*.24)** *If* $a_i>0\ \forall i=1;2;\ldots;n$ *then:* $\sum_{i=1}^{n}\frac{a_i^3}{a_i^2+a_i\cdot a_{i+1}+a_{i+1}^2} \geq \frac{a_1+a_2+\cdots+a_n}{3}$ $(a_{n+1}=a_1)$

***d*.25)** *If* $a,b,c>0$ *then:* $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{2ab}{(a+b)^2} \geq 2$

***d*.26)** *If* $a,b,c>0$ *then:*
$a^2+b^2+c^2+2abc+1 \geq 2(ab+bc+ca) \rightarrow a^2+b^2+c^2+abc+5 \geq 3(a+b+c)$

***d*.27*)** *If* a,b,c *are sides of a triangle then:* $a^2b(a-b)+b^2c(b-c)+c^2a(c-a) \geq 0$ *(IMO 1983)*

***d*.28)** *If* $a,b,c>0$ *and* $m \in Z^+$ *then:* $\frac{a^m}{b^{m-1}}+\frac{b^m}{c^{m-1}}+\frac{c^m}{a^{m-1}} \geq \frac{a^{m-1}}{b^{m-2}}+\frac{b^{m-1}}{c^{m-2}}+\frac{c^{m-1}}{a^{m-2}} \geq \cdots \geq$

$$\geq \frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\geq \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a+b+c$$

***d*.29)** *If* $a,b,c>0$ *then:* $\frac{a}{a+\sqrt{(a+b)(a+c)}}+\frac{b}{b+\sqrt{(b+c)(b+a)}}+\frac{c}{c+\sqrt{(c+a)(c+b)}}\leq 1$

***d*.30)** *If* $a,b,c>0$ *then:* $\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{b+c}}+\sqrt{\frac{c}{c+a}}\leq \frac{3}{\sqrt{2}}$

***d*.31)** *If* a,b,c *are sides of a triangle then:* $a^4+b^4+c^4\leq 2(a^2b^2+b^2c^2+c^2a^2)$ *(the equality happens iff* a,b,c *are sides of a degenerate triangle.)*

***d*.32*)** *If* $a,b,c>0$ *and* $k\geq 0$ *then:* $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq \frac{ka+b}{ka+c}+\frac{kb+c}{kb+a}+\frac{kc+a}{kc+b}$

***d*.33)** *If* $a,b,c>0$ *then:* $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{8abc}{(a+b)(b+c)(c+a)}\geq 4$

***d*.34)** *If* $a,b,c\geq 0$*, no* 2 *of which are* 0 *then:* $\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{8abc}{(a+b)(b+c)(c+a)}\geq 2$ *(**Jack Garfunkel's inequality**)*

• Inequalities with classic condition – part 2: In this part I will show some inequalities about variables p,q,r *(denote* $p=a+b+c; q=ab+bc+ca; r=abc$*), even Schur's inequality.*

Firstly, we have some identities about p,q,r***:***

1) $a^2+b^2+c^2=p^2-2q$
2) $a^2b^2+b^2c^2+c^2a^2=q^2-2pr$
3) $(a+b)(b+c)(c+a)=pq-r\rightarrow ab(a+b)+bc(b+c)+ca(c+a)=pq-3r$
4) $(a+b)(a+c)+(b+c)(b+a)+(c+a)(c+b)=p^2q$
5) $a^3+b^3+c^3=p^3-3pq+3r$
6) $a^3b^3+b^3c^3+c^3a^3=q^3-3pqr+3r^2$
7) $a^4+b^4+c^4=p^4-4p^2q+2q^2+4pr$
8) $ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)=p^2q-2q^2-pr$
9) *Denote* $S=-4p^3r+p^2q^2+18pqr-4q^3-27r^2$*, then:*

* $[(a-b)(b-c)(c-a)]^2=S$

* $a^2b+b^2c+c^2a=\frac{pq-3r-\sqrt{S}}{2}$ *if* $(a-b)(b-c)(c-a)<0; \frac{pq-3r+\sqrt{S}}{2}$ *if* $(a-b)(b-c)(c-a)\geq 0$

* $a^3b+b^3c+c^3a=\frac{p^2q-2q^2-pr-p\sqrt{S}}{2}$ *if* $(a-b)(b-c)(c-a)<0; \frac{p^2q-2q^2-pr+p\sqrt{S}}{2}$ *if* $(a-b)(b-c)(c-a)\geq 0$

10) $a^2b^2(a^2+b^2)+b^2c^2(b^2+c^2)+c^2a^2(c^2+a^2)=-2p^3r+p^2q^2+4pqr-2q^3-3r^2$
11) $ab(a^4+b^4)+bc(b^4+c^4)+ca(c^4+a^4)=p^4q-p^3r-4p^2q^2+7pqr+2q^3-3r^2$
12) $a^6+b^6+c^6=p^6-6p^4q+6p^3r+9p^2q^2-12pqr-2q^3+3r^2$

• Some inequalities about the relation of p,q,r***:***

1) $p^2\geq 3q; q^2\geq 3pr$
2) $pq\geq 9r$

$$\rightarrow p^3\geq \frac{27}{8}(pq-r)\geq 3pq\geq \frac{9q^2}{p}\geq 27r$$

3) $p^3+9r\geq 4pq\rightarrow r\geq \frac{p(4q-p^2)}{9}\rightarrow r\geq \max\left\{0;\frac{p(4q-p^2)}{9}\right\}$
4) $2p^3+9r\geq 7pq$
5) $p^2q+3pr\geq 4q^2$
6) $p^4+4q^2+6pr\geq 5p^2q$
7) $r\geq \frac{(4q-p^2)(p^2-q)}{6p}\rightarrow r\geq \max\left\{0;\frac{(4q-p^2)(p^2-q)}{6p}\right\}$

8) $r \le \frac{p(5q-p^2)}{18}; r \le \frac{p^4-7p^2q+13q^2}{9p}$. *Combining inequality* 3, 6, 8 *and we get:*

$$\min\left\{\frac{p(5q-p^2)}{18}; \frac{p^4-7p^2q+13q^2}{9p}\right\} \ge r \ge \max\left\{0; \frac{p(4q-p^2)}{9}; \frac{(4q-p^2)(p^2-q)}{6p}\right\}$$

9*) $\frac{9pq-2p^3-2k\sqrt{k}}{27} \le r \le \frac{9pq-2p^3+2k\sqrt{k}}{27}$, *with* $k = p^2 - 3q$. *This result comes from solving the inequality* $S = -4p^3r + p^2q^2 + 18pqr - 4q^3 - 27r^2 \ge 0$ *with variable* r. → *From this result, by AM-GM we have:*

$$* 27r \ge 9pq - 2p^3 - 2k\sqrt{k} = 9pq - 2p^3 - \frac{2(p^2-3q)(p^2-2q)\sqrt{p^2(p^2-3q)}}{p(p^2-2q)}$$

$$\ge \frac{p^2(9q-2p^2)(p^2-2q) - (p^2-3q)[(p^2-2q)^2 + p^2(p^2-3q)]}{p(p^2-2q)} =$$

$$= \frac{(p^2-2q)(-3p^4+14p^2q-6q^2) - p^2(p^2-3q)^2}{p(p^2-2q)} = \frac{(4q-p^2)(4p^2-10p^2q+3q^2)}{p(p^2-2q)}$$

$$* 27r \le 9pq - 2p^3 + 2k\sqrt{k} = 9pq - 2p^3 + \frac{2(p^2-3q)\left(p^2-\frac{3}{2}q\right)\sqrt{p^2(p^2-3q)}}{p\left(p^2-\frac{3}{2}q\right)}$$

$$\le \frac{p(9pq-2p^3)(2p^2-3q) + 2(p^2-3q)\left[\left(p^2-\frac{3}{2}q\right)^2 + p^2(p^2-3q)\right]}{p(2p^2-3q)} =$$

$$= \frac{(2p^2-3q)[2(9pq-2p^3) + (p^2-3q)(2p^2-3q)] + 4p^2(p^2-3q)^2}{2p(2p^2-3q)}$$

$$= \frac{(2p^2-3q)(-2p^4+9p^2q+9q^2) + 4p^2(p^4-6p^2q+9q^2)}{2p(2p^2-3q)} = \frac{27q^2(p^2-q)}{2p(2p^2-3q)}$$

So we have the chain: $\frac{q^2}{3p} \ge \frac{q^2(p^2-q)}{2p(2p^2-3q)} \ge r \ge \frac{(4q-p^2)(4p^2-10p^2q+3q^2)}{27p(p^2-2q)}$, *more interesting, the third inequality is stronger than Schur deg* 3 *and* 4, *since:*

$\frac{(4q-p^2)(4p^4-10p^2q+3q^2)}{27p(p^2-2q)} - \frac{p(4q-p^2)}{9} = \frac{(4q-p^2)(p^2-3q)(p^2-q)}{27p(p^2-2q)} \ge 0$ *in case* $4q \ge p^2$

$$\frac{(4q-p^2)(4p^2-10p^2q+3q^2)}{27p(p^2-2q)} - \frac{(4q-p^2)(p^2-q)}{6p} = \frac{(4q-p^2)^2(p^2-3q)}{54p(p^2-2q)} \ge 0$$

Hence we can also conclude that: $r \ge \max\left\{0; \frac{(4p-p^2)(4p^4-10p^2q+3q^2)}{27p(p^2-2q)}\right\}$

Further more, we can write inequality 9 *as:* $\frac{p^3-3pk-2k\sqrt{k}}{27} \le r \le \frac{p^3-3pk+2k\sqrt{k}}{27}$

$$\Leftrightarrow \frac{\left(p+\sqrt{k}\right)^2\left(p-2\sqrt{k}\right)}{27} \le r \le \frac{\left(p-\sqrt{k}\right)^2\left(p+2\sqrt{k}\right)}{27}$$

For my thinking, the p, q, r *technique is one of the nicest and hardest techniques to use, due to long calculations, time loss and it requires carefulness in calculation. But the nice thing to say is its benefit in solving hard inequality about symmetric and cyclic expressions. There is a similar technique to* p, q, r, *that is* u, v, w *technique (can be found and search in inequality sites and forums)*

ELEMENTE DE GEOMETRIA TRIUNGHIULUI IN COORDONATE BARICENTRICE

By Neculai Stanciu-Romania

Articolul vine uşor în completarea programei şcolare din liceu şi are scopul de a pune în evidenţã noi metode de rezolvare a problemelor de geometrie şi de a lãrgi orizontul

matematic al elevilor.În cele ce urmează, voi enunţa şase teoreme importante şi voi demonstra numeroase aplicaţii ale acestor teoreme referitoare la unele egalităţi şi inegalităţi în triunghi.

Teorema 1.Se consideră un triunghi fix ABC şi notăm BC=a,CA=b,AB=c,S=Aria(ABC). Atunci pentru orice $M \in E_2$ (unde, E_2 este planul euclidian)există şi este unic tripletul ordonat $(x,y,z) \in R^3$,x+y+z=1 astfel încât $x\overrightarrow{MA}+y\overrightarrow{MB}+z\overrightarrow{MC}=\vec{0}$
şi reciproc,pentru orice triplet ordonat $(x,y,z) \in R^3$,x+y+z=1 există şi este unic un punct $M \in E_2$ astfel încât $x\overrightarrow{MA}+y\overrightarrow{MB}+z\overrightarrow{MC}=\vec{0}$ şi în acest caz vom spune că punctul M are coordonatele baricentrice (x,y,z) în raport cu triunghiul ABC şi vom nota M(x,y,z).Pentru orice $X \in E_2$ avem $x\overrightarrow{XA}+y\overrightarrow{XB}+z\overrightarrow{XC}=\overrightarrow{XM}$ (demonstraţie în [1]pag.66).

Exemple de coordonate baricentrice pentru câteva puncte remarcabile într-un triunghi:

1.1.A(1,0,0),B(0,1,0),C(0,0,1);

1.2.$G\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)-$ *centrul de greutate*;

1.3.$I\left(\frac{a}{2p},\frac{b}{2p},\frac{c}{2p}\right)-$ *centrul cercului inscris* $C(I,r)$;

1.4.$I_a\left(\frac{a}{2(p-a)},\frac{b}{2(p-a)},\frac{c}{2(p-a)}\right)-$ *centrul cercului exinscris* $C(I_a,r_a)$;

1.5.$N\left(\frac{p-a}{p},\frac{p-b}{p},\frac{p-c}{p}\right)-$ *punctul lui Nagel*;

1.6.$\Gamma\left(\frac{(p-b)(p-c)}{r(4R+r)},\frac{(p-c)(p-a)}{r(4R+r)},\frac{(p-a)(p-b)}{r(4R+r)}\right)-$ *punctul lui Gergone*;

1.7.$H\left(ctgBctgc,ctgCctgA,ctgActgB\right)-$ *ortocentrul*;

1.8.$O(\frac{R^2\sin 2A}{2S},\frac{R^2\sin 2B}{2S},\frac{R^2\sin 2C}{2S})-$ *centrul cercului circumscris* $C(O,R)$;

1.9.$L(\frac{a^2}{a^2+b^2+c^2},\frac{b^2}{a^2+b^2+c^2},\frac{c^2}{a^2+b^2+c^2})-$ *punctul lui Lemoine.*

Teorema 2. Puterea punctuluiM(x,y,z) $\in E_2$ faţă de cercul C(O,R),circumscris triunghiului ABC este dat de relaţia:

$$p_c(M)=-(yza^2+zxb^2+xyc^2);OM^2=R^2-p_c(M).$$

(demonstraţie în [1] pag.68).

Aplicaţii ale teoremei 2.

2.1.$p_c(G)=-\frac{a^2+b^2+c^2}{9},OG^2=R^2-\frac{a^2+b^2+c^2}{9}$ *si*,

$a^2+b^2+c^2\le 9R^2$;

Demonstraţie:se aplică teorema2 şi 1.2.

2.2.$p_c(I)=-2Rr,OI^2=R^2-2Rr$, *si*, $R\ge 2r$;

Demonstraţie:rezultă imediat din teorema2 şi 1.3.

2.3.$p_c(I_a)=-2Rr_a,OI_a{}^2=R^2+2Rr_a$;

Demonstraţie:folosim teorema2 şi 1.4.

2.4. $p_c(N) = -4r(R-r), ON = R-2r;$

Demonstraţie:utilizãm teorema 2 şi 1.5.

2.5. $p_c(H) = -8R^2 \cos A \cos B \cos C, OH^2 = R^2(1-8\cos A \cos B \cos C),$

$si, \cos A \cos B \cos C \le \frac{1}{8}, iar$

$OH^2 = 9R^2 - (a^2+b^2+c^2), deoarece, \overrightarrow{OH} = 3\overrightarrow{OG}.$

Demonstraţie:se aplicã teorema 2 şi 1.7.

2.6. $p_c(\Gamma) = -r(R+r)\left(\frac{2p}{4R+r}\right);$

Demonstraţie:se foloseşte teorema 2 şi 1.6.

2.7. $p_c(L) = -3\left(\frac{abc}{a^2+b^2+c^2}\right)^2, OL^2 = R^2 - 3\left(\frac{abc}{a^2+b^2+c^2}\right)^2, si,$

$a^2+b^2+c^2 \ge 4S\sqrt{3};$

Demonstraţie:rezultã imediat din teorema 2 şi 1.9.

Teorema 3.a)Pentru M(x,y,z) $\in E_2$ existã relaţia:

$xMA^2 + yMB^2 + zMC^2 = -p_c(M)$

b)Pentru orice X$\in E_2$ existã relaţia:

$xXA^2 + yXB^2 + zXC^2 = XM^2 - p_c(M), si$

$xXA^2 + yXB^2 + zXC^2 \ge yza^2 + zxb^2 + xyc^2.$

(demonstraţie în [1] pag.69)

Aplicaţii ale teoremei 3 :

3.1. $M \equiv G \Rightarrow XA^2 + XB^2 + XC^2 = 3XG^2 + \frac{a^2+b^2+c^2}{3} \forall X \in E_2;$

$GA^2 + GB^2 + GC^2 = \frac{a^2+b^2+c^2}{3};$

$XA^2 + XB^2 + XC^2 \ge \frac{a^2+b^2+c^2}{3} \forall X \in E_2,$

cu egalitate pentru $X \equiv G.$

3.2. $M \equiv I \Rightarrow aXA^2 + bXB^2 + cXC^2 = 2pXI^2 + abc, \forall X \in E_2;$

$aIA^2 + bIB^2 + cIC^2 = abc; aXA^2 + bXB^2 + cXC^2 \ge abc, \forall X \in E_2,$

cu egalitate pentru $X \equiv I.$

3.3. $M \equiv I_a \Rightarrow bXB^2 + cXC^2 + abc = aXA^2 + 2(p-a)XI_a^2, \forall X \in E_2;$

$X \equiv I_a \Rightarrow bI_aB^2 + cI_aC^2 + abc = aI_aA^2; bXB^2 + cXC^2 + abc \ge aXA^2$

cu egalitate pentru $X \equiv I_a.$

3.4. $Daca M(x,y,z) \in E_2, atunci \forall X \in E_2 avem:$

$yza^2 + zxb^2 + xyc^2 \le xXA^2 + yXB^2 + zXC^2 \le XM^2 + R^2, cu\ egalitate\ in\ s\tan ga, pentru X \equiv M$

si egalitate in dreapta, pentru $M \equiv O.$ *De exemplu, pentru* $M \equiv I \Rightarrow$

$abc \le aXA^2 + bXB^2 + cXC^2 \le 2p(XI^2 + R^2).$

3.5.$Daca, X \equiv M \Rightarrow xMA^2 + yMB^2 + zMC^2 =$

$= yza^2 + zxb^2 + xyc^2 = R^2 - OM^2, si\ xMA^2 + yMB^2 + zMC^2 \le R^2, cu\ egalitate\ pentru M \equiv O;$

3.6.$Daca, X \equiv A \Rightarrow MA^2 = p_c(M) + zb^2 + yc^2, de\ unde, rezulta:$

$MA^2 + MB^2 + MC^2 = 3p_c(M) + (y+z)a^2 + (z+x)b^2 + (x+y)c^2.$

Teorema 4.Dacã

$M_k(x_k, y_k, z_k) \in E_2, k = \overline{1,2}$,atunci distanţa între punctele M_1,M_2 este datã de relaţia:

$M_1M_2{}^2 = -\left[(y_1 - y_2)(z_1 - z_2)a^2 + (z_1 - z_2)(x_1 - x_2)b^2 + (x_1 - x_2)(y_1 - y_2)c^2\right]$

(demonstraţie în [1] pag.70).

Aplicaţii ale teoremei 4: Utilizând coordonatele baricentrice (vezi exemplele date)şi teorema 4 obţinem urmãtoarele relaţii:

4.1.$OG^2 = R^2 - \frac{a^2+b^2+c^2}{9}$;4.2.$OH^2 = R^2(1 - 8\cos A\cos B\cos C) =$

$9R^2 - (a^2+b^2+c^2) \Rightarrow a^2+b^2+c^2 = 8R^2(1+\cos A\cos B\cos C), de\ unde\ rezulta,$

$a^2+b^2+c^2 = 8R^2 \Leftrightarrow \Delta ABC este, dreptunghic$

4.3.$OI^2 = R^2 - 2Rr; HN = 2OI;$ 4.4.$NI^2 = 9GI^2 = p^2 + 5r^2 - 16Rr \Rightarrow$

$p^2 + 5r^2 \ge 16Rr;$ 4.5. $HI^2 = 4R(R+r) + 3r^2 - p^2 \le 4R(R+r) + 3r^2;$

4.6.$\Gamma I^2 = r^2\left[1 - \left(\frac{p\sqrt{3}}{4R+r}\right)^2\right] \Rightarrow p\sqrt{3} \le 4R + r;$ 4.7.$LI^2 = \frac{4S^2R(R+r)}{p^2 - r^2 - 4Rr};$

4.8.$G\Gamma^2 = \frac{4}{9(4R+r)^2}\left[p^2(4R^2+8Rr-5r^2) - r(4R+r)^3\right] \Rightarrow (2R-r)(2R+5r)p^2 \ge r(4R+r)^3;$

4.9.$H\Gamma^2 = 4R^2\left[1 - \frac{2p^2(2R-r)}{R(4R+r)^2}\right] \Rightarrow 2p^2(2R-r) \le R(4R+r)^2.$

Teorema 5. Dacã $M_k(x_k, y_k, z_k) \in E_2, k = \overline{1,2}$,atunci avem:

$$\overrightarrow{OM_1}\overrightarrow{OM_2} = R^2 - \frac{1}{2}\left[(y_1z_2 + y_2z_1)a^2 + (z_1x_2 + z_2x_1)b^2 + (x_1y_2 + x_2y_1)c^2\right]$$

(demonstraţie în [1] pag.71.).**Aplicaţii ale teoremei 5:** Folosind aceastã teoremã obţinem egalitãţi şi inegalitãţi importante printre care cele ce urmeazã:

5.1.$\overrightarrow{OG}\overrightarrow{OM} = R^2 - \frac{1}{6}\left[(y+z)a^2 + (z+x)b^2 + (x+y)c^2\right]$

5.1.1.$\overrightarrow{OG}\overrightarrow{OA} = R^2 - \frac{1}{6}(b^2+c^2); OG \perp OA \Leftrightarrow b^2+c^2 = 6R^2, si, b^2+c^2 \le 6R^2$

$\Leftrightarrow m(AOG) \le 90^0$

5.1.2.$\overrightarrow{OG}\overrightarrow{OI} = R^2 - \frac{1}{6}(p^2 + r^2 - 2Rr), si, 2R(3R+r) \ge p^2 + r^2 \Leftrightarrow m(IOG) \le 90^0, iar,$

$2R(3R+r) = p^2 + r^2 \Leftrightarrow a = b = c, sau, OI \perp OG;$

5.1.3.$\overrightarrow{OG}\overrightarrow{ON} = (R+r)^2 - \frac{p^2}{3}, si,\ p \le \sqrt{3}(R+r) \Leftrightarrow m(NOG) \le 90^0, iar,$

$p = \sqrt{3}(R+r) \Leftrightarrow a = b = c, sau, ON \perp OG;$

$5.2.\overrightarrow{OI}\overrightarrow{OM} = R(R+r) - \frac{1}{2}(xbc + yca + zab);$

$5.2.1.\overrightarrow{OI}\overrightarrow{ON} = R^2 + 5Rr - \frac{1}{2}(p^2 + r^2), si, 2R(R+5r) \geq p^2 + r^2$

$\Leftrightarrow m(ION) \leq 90^0, iar,$

$2R(R+5r) = p^2 + r^2 \Leftrightarrow a = b = c, sau, OI \perp ON;$

$5.2.2.\overrightarrow{OI}\overrightarrow{OA} = R^2 + Rr - \frac{1}{2}bc, si, aR \geq (b+c)r \Leftrightarrow m(AOI) \leq 90^0, iar,$

$aR = (b+c)r \Leftrightarrow a = b = c, sau, OA \perp OI.$

Teorema6. Dacă

$M_k(x_k, y_k, z_k) \in E_2, k = \overline{1,2}, avem:$

$$a)\overrightarrow{IM_1}\overrightarrow{IM_2} = -4Rr + \frac{1}{2}\left[(z_1+z_2)ab + (x_1+x_2)bc + (y_1+y_2)ca\right] -$$

$$-\frac{1}{2}\left[(y_1z_2 + y_2z_1)a^2 + (z_1x_2 + z_2x_1)b^2 + (x_1y_2 + x_2y_1)c^2\right];$$

(demonstraţie în [1],pag.73)

$$b)\overrightarrow{GM_1}\overrightarrow{GM_2} = \frac{2}{9}(a^2+b^2+c^2) - \frac{1}{6}\left[(x_1+x_2)a^2 + (y_1+y_2)b^2 + (z_1+z_2)c^2\right] -$$

$$-\frac{1}{2}\left[(y_1z_2 + y_2z_1)a^2 + (z_1x_2 + z_2x_1)b^2 + (x_1y_2 + x_2y_1)c^2\right];$$

(demonstraţie în [1],pag.74)

$$c)\overrightarrow{MM_1}\overrightarrow{MM_2} = p_c(M) + \frac{1}{2}\left\{\left[y(z_1+z_2) + z(y_1+y_2) - (y_1z_2 + y_2z_1)\right]a^2 + ...\right\}.$$

(demonstraţie în [1],pag.75)

Observaţie. Cu ajutorul acestor relaţii remarcabile se pot determina ,în particular,produse scalare,distanţe,egalităţi şi inegalităţi utilizând puncte din mulţimea:

$\{A, B, C, O, G, H, I, I_a, I_b, I_c, N, \Gamma, L\}$

asociatã unui triunghi ABC.**(Exerciţiu!).**

Mai fac observaţia cã particularizări şi unele extinderi ale coordonatelor baricentrice sunt abordate în lucrările [2] , [3] şi articolele din [4] şi [5] .Un fapt care motivează studiul coordonatelor baricentrice este legătura acestora cu calculul vectorial recent (relativ) introdus în programele şcolare IX-XII .

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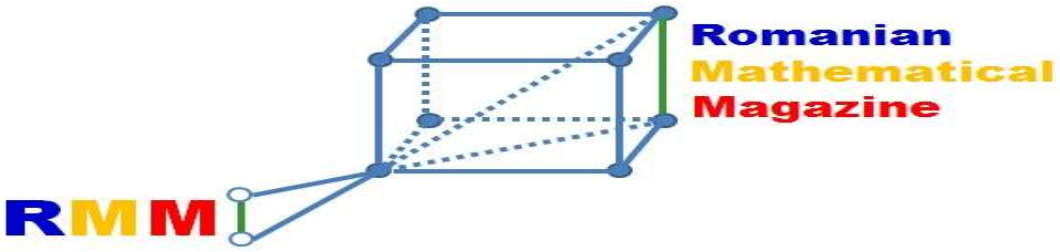

FEW APPLICATIONS OF COCEA'S INEQUALITY

By Daniel Sitaru, Claudia Nănuți-Romania

Abstract: In this paper are developed a few applications of a 1993's inequality of Romanian mathematician C. Cocea.

Keywords: circumcentre, centroid, orthocentre, incentre, Nagel, Lemoine

Cocea's inequality

Let P be a point in (ABC), where A, B, C necollinears. If $PA = x; PB = y; PC = z$ then:

$$ayz + bxz + cxy \geq abc \quad (*)$$

Proof:

If P is the origin of the system of rectangular axis lets denote a, b, c the affixes of A, B respectively C. $A(a); B(b); C(c); a, b, c$ different in pairs

$$1 = |1| = \left|\frac{ab}{(c-a)(c-b)} + \frac{bc}{(a-b)(a-c)} + \frac{ca}{(b-a)(b-c)}\right| \leq$$

$$\leq \left|\frac{ab}{(c-a)(c-b)}\right| + \left|\frac{bc}{(a-b)(a-c)}\right| + \left|\frac{ca}{(b-a)(b-c)}\right| =$$

$$= \frac{|a|\cdot|b|}{|c-a|\cdot|c-b|} + \frac{|b|\cdot|c|}{|a-b|\cdot|a-c|} + \frac{|c|\cdot|a|}{|b-a|\cdot|b-c|}$$

$$|b-c|\cdot|b|\cdot|c| + |a-c|\cdot|a|\cdot|c| + |a-b|\cdot|a|\cdot|b| \geq |b-c|\cdot|c-a|\cdot|a-b|$$

$$ayz + bxz + cxy \geq abc$$

Application 1

In ΔABC the following relationship holds:

$$R \geq 2r \text{ (Euler's inequality)}$$

Solution:

Let $P = 0$ (circumcenter) in (*)

$$PA = PB = PC = x = y = z = R$$

$$a\cdot R\cdot R + b\cdot R\cdot R + c\cdot R\cdot R \geq abc$$

$$R^2(a+b+c) \geq 4RS$$

$$R \cdot 2s \geq 4S \Rightarrow 2sR \geq 4rs \Rightarrow R \geq 2r$$

Application 2

In $\Delta\, ABC$ the following relationship holds:

$$4(am_bm_c + bm_cm_a + cm_am_b) \geq 9abc$$

Solution:

Let $P = G$ (centroid) in (*)

$$PA = \frac{2}{3}m_a; PB = \frac{2}{3}m_c; PC = \frac{2}{3}m_a$$

$$a \cdot \frac{2}{3}m_b \cdot \frac{2}{3}m_c + b \cdot \frac{2}{3}m_c \cdot \frac{2}{3}m_a + c \cdot \frac{2}{3}m_a \cdot \frac{2}{3}m_b \geq abc$$

$$4(am_bm_c + bm_cm_a + cm_am_b) \geq 9abc$$

Application 3

In ΔABC the following relationship holds:

$$\frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}} + \frac{\sin B}{\sin\frac{C}{2}\sin\frac{A}{2}} + \frac{\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}} \geq \frac{2s}{r}$$

Solution:

Let $P = I$ (incentre) in (*)

$$PA = \frac{r}{\sin\frac{A}{2}}; PB = \frac{r}{\sin\frac{C}{2}}; PC = \frac{r}{\sin\frac{A}{2}}$$

$$a \cdot \frac{r}{\sin\frac{B}{2}} \cdot \frac{r}{\sin\frac{C}{2}} + b \cdot \frac{r}{\sin\frac{C}{2}} \cdot \frac{r}{\sin\frac{A}{2}} + c \cdot \frac{r}{\sin\frac{A}{2}} \cdot \frac{r}{\sin\frac{B}{2}} \geq abc$$

$$\frac{a}{\sin\frac{B}{2}\sin\frac{C}{2}} + \frac{b}{\sin\frac{C}{2}\sin\frac{A}{2}} + \frac{c}{\sin\frac{A}{2}\sin\frac{B}{2}} \geq \frac{abc}{r^2}$$

$$\frac{2R\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}} + \frac{2R\sin B}{\sin\frac{C}{2}\sin\frac{A}{2}} + \frac{2R\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}} \geq \frac{4Rrs}{r^2}$$

$$\frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}} + \frac{\sin B}{\sin\frac{C}{2}\sin\frac{A}{2}} + \frac{\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}} \geq \frac{4Rs}{r \cdot 2R} = \frac{2s}{r}$$

Application 4

In $\Delta\, ABC$ the following relationship holds:

$$\sin A\,|\cos A\cos C| + \sin B\,|\cos C\cos A| + \sin C\,|\cos A\cos B| \geq \frac{rs}{2R^2}$$

Solution:

Let $P = H$ (orthocenter) in (*)

$$PA = 2R|\cos A|; PB = 2R|\cos B|; PC = 2R|\cos C|$$

$$a \cdot 2R|\cos B| \cdot 2R|\cos C| + b \cdot 2R|\cos C| \cdot 2R|\cos A| + c \cdot 2R|\cos A| \cdot 2R|\cos B| \geq abc$$

$$a|\cos B \cos C| + b|\cos C \cos A| + c|\cos A \cos B| \geq \frac{abc}{4R^2}$$

$$2R \sin A \cdot |\cos B \cos C| + 2R \sin B\, |\cos C \cos A| + 2R \sin C\, |\cos A \cos B| \geq \frac{4Rrs}{4R^2} = \frac{rs}{R}$$

$$\sin A \cdot |\cos B \cos C| + \sin B\, |\cos C \cos A| + \sin C\, |\cos A \cos B| \geq \frac{rs}{2R^2}$$

Applicaton 5

In $\Delta\, ABC$ the following relationship holds:

$$\sum_{cyc(a,b,c)} a\sqrt{((b-c)^2 + 4r^2)((c-a)^2 + 4r^2)} \geq abc$$

Solution:

Let $P = N$ (Nagel's point) in (*)

$$PA = \sqrt{(b-c)^2 + 4r^2}; PB = \sqrt{(c-a)^2 + 4r^2}; PC = \sqrt{(a-b)^2 + 4r^2}$$

$$\sum_{cyc(a,b,c)} a \cdot PB \cdot PC \geq abc$$

$$\sum_{cyc(a,b,c)} a\sqrt{((b-c)^2 + 4r^2)((c-a)^2 + 4r^2)} \geq abc$$

Application 6

In $\Delta\, ABC$ the following relationship holds:

$$a^2 m_b m_c + b^2 m_c m_a + c^2 m_a m_b \geq \frac{(a^2 + b^2 + c^2)^2}{4}$$

Solution:

Let $P = K$ (Lemoine's point) in (*)

$$PA = \frac{2bcm_a}{a^2 + b^2 + c^2}; PB = \frac{2cam_b}{a^2 + b^2 + c^2}; PC = \frac{2abm_c}{a^2 + b^2 + c^2}$$

$$\sum_{cyc(a,b,c)} \left(a \cdot \frac{2cam_b}{a^2 + b^2 + c^2} \cdot \frac{2cbm_c}{a^2 + b^2 + c^2}\right) \geq abc$$

$$4abc \sum_{cyc(a,b,c)} \frac{a^2 m_b m_c}{(a^2 + b^2 + c^2)^2} \geq abc$$

$$a^2 m_b m_c + b^2 m_c m_a + c^2 m_a m_c \geq \frac{(a^2 + b^2 + c^2)^2}{4}$$

In all applications equality holds if $a = b = c$ (equilateral triangle)

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ABOUT PROBLEM UP.147.
ROMANIAN MATHEMATICAL MAGAZINE-AUTUMN EDITION-2018

By Marin Chirciu-Romania

1) In ΔABC

$$\frac{r_a^2}{tg\frac{B}{2}tg\frac{C}{2}}+\frac{r_b^2}{tg\frac{C}{2}tg\frac{A}{2}}+\frac{r_c^2}{tg\frac{A}{2}tg\frac{B}{2}}\geq\frac{9\left(a^2+b^2+c^2\right)}{4}.$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

Soluție:

Demonstrăm rezultatul ajutător:

Lemă.

2) In ΔABC

$$\frac{r_a^2}{tg\frac{B}{2}tg\frac{C}{2}}+\frac{r_b^2}{tg\frac{C}{2}tg\frac{A}{2}}+\frac{r_c^2}{tg\frac{A}{2}tg\frac{B}{2}}=\frac{\left(4R+r\right)^3-12Rp^2}{r}.$$

Demonstrație.

Folosind $r_a=\frac{S}{p-a}$ și $tg\frac{B}{2}tg\frac{C}{2}=\frac{p-a}{p}$ obținem:

$$\sum \frac{r_a^2}{tg\frac{B}{2}tg\frac{C}{2}}=\sum \frac{\frac{S^2}{(p-a)^2}}{\frac{p-a}{p}}=S^2p\sum \frac{1}{(p-a)^3}=S^2p\cdot\frac{(4R+r)^3-12Rp^2}{S^3}=\frac{(4R+r)^3-12Rp^2}{r}.$$

Să trecem la rezolvarea inegalității din enunț.

Folosim **Lema** și $a^2+b^2+c^2=2\left(p^2-r^2-4Rr\right)$ inegalitatea se scrie:

$\frac{(4R+r)^3-12Rp^2}{r}\geq\frac{9}{4}\cdot 2\left(p^2-r^2-4Rr\right)\Leftrightarrow\ 2(4R+r)^3+9r^2(4R+r)\geq 3p^2(8R+3r)$, care rezultă din inegalitatea lui Gerretsen: $p^2\leq 4R^2+4Rr+3r^2$.

Rămâne să arătăm că:

$2(4R+r)^3+9r^2(4R+r)\geq 3\left(4R^2+4Rr+3r^2\right)(8R+3r)\Leftrightarrow 8R^3-9R^2r-12Rr^2-4r^3\geq 0\Leftrightarrow$

$\Leftrightarrow (R-2r)\left(8R^2+7Rr+2r^2\right)\geq 0$, evident din inegalitatea lui Euler $R\geq 2r$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.

Inegalitatea poate fi întărită:

3) In ΔABC

$$\frac{r_a^2}{tg\frac{B}{2}tg\frac{C}{2}}+\frac{r_b^2}{tg\frac{C}{2}tg\frac{A}{2}}+\frac{r_c^2}{tg\frac{A}{2}tg\frac{B}{2}}\geq\frac{27}{4}\cdot\frac{a^3+b^3+c^3}{a+b+c}.$$

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Solutie:

Folosind **Lema** și $a^3+b^3+c^3=2p\left(p^2-3r^2-6Rr\right)$ inegalitatea se scrie:

$\frac{(4R+r)^3-12Rp^2}{r}\geq\frac{27}{4}\cdot\frac{2p\left(p^2-3r^2-6Rr\right)}{2p}\Leftrightarrow 4(4R+r)^3+27r^2(6R+3r)\geq 3p^2(16R+9r)$

care rezultă din inegalitatea lui Gerretsen: $p^2\leq 4R^2+4Rr+3r^2$.

Rămâne să arătăm că:

$4(4R+r)^3+27r^2(6R+3r)\geq 3\left(4R^2+4Rr+3r^2\right)(16R+9r)\Leftrightarrow$

$\Leftrightarrow 32R^3-54R^2r-21Rr^2+2r^3\geq 0\Leftrightarrow (R-2r)\left(32R^2+10Rr-r^2\right)\geq 0$, evident din inegalitatea lui Euler $R\geq 2r$. Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.

Inegalitatea **3)** este mai tare decât inegalitatea **1)**:

4) In ΔABC

$$\frac{r_a^2}{tg\frac{B}{2}tg\frac{C}{2}}+\frac{r_b^2}{tg\frac{C}{2}tg\frac{A}{2}}+\frac{r_c^2}{tg\frac{A}{2}tg\frac{B}{2}}\geq\frac{27}{4}\cdot\frac{a^3+b^3+c^3}{a+b+c}\geq\frac{9}{4}\left(a^2+b^2+c^2\right).$$

Solutie:

Se folosește inegalitatea **3)** și $\frac{27}{4}\cdot\frac{a^3+b^3+c^3}{a+b+c}\geq\frac{9}{4}\left(a^2+b^2+c^2\right)\Leftrightarrow$

$3\left(a^3+b^3+c^3\right)\geq(a+b+c)\left(a^2+b^2+c^2\right)\ \Leftrightarrow$

$\Leftrightarrow 2\sum a^3 \geq \sum ab(a+b)$, care rezultă din $a^3+b^3 \geq ab(a+b) \Leftrightarrow (a+b)(a-b)^2 \geq 0$ și analoagele.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.

Să obținem o inegalitate de sens contrar:

5) In ΔABC

$$\frac{r_a^2}{tg\frac{B}{2}tg\frac{C}{2}}+\frac{r_b^2}{tg\frac{C}{2}tg\frac{A}{2}}+\frac{r_c^2}{tg\frac{A}{2}tg\frac{B}{2}} \leq \frac{81R}{8r}\left(9R^2-32r^2\right).$$

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Soluție:

Folosind **Lema** inegalitatea se scrie:

$\frac{(4R+r)^3-12Rp^2}{r} \leq \frac{81R}{8r}\left(9R^2-32r^2\right)$, care rezultă din inegalitatea lui Euler $r \leq \frac{R}{2}$ și inegalitatea lui Mitrinovic $p^2 \geq 27r^2$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.

Se poate scrie dubla inegalitate:

6) In ΔABC : $\frac{27}{2}\left(5Rr-4r^2\right) \leq \frac{r_a^2}{tg\frac{B}{2}tg\frac{C}{2}}+\frac{r_b^2}{tg\frac{C}{2}tg\frac{A}{2}}+\frac{r_c^2}{tg\frac{A}{2}tg\frac{B}{2}} \leq \frac{81R}{8r}\left(9R^2-32r^2\right).$

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Soluție:

Se folosesc **3)** , **5)** și inegalitatea lui Gerretsen $p^2 \geq 16Rr-5r^2$ .Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă: În același registru se pot propune:

7) In ΔABC

$$\frac{r_a^2}{ctg\frac{B}{2}ctg\frac{C}{2}}+\frac{r_b^2}{ctg\frac{C}{2}ctg\frac{A}{2}}+\frac{r_c^2}{ctg\frac{A}{2}ctg\frac{B}{2}} = r(4R+r).$$

Soluție:

Folosind $r_a = \frac{S}{p-a}$ și $ctg\frac{B}{2}ctg\frac{C}{2} = \frac{p}{p-a}$ obținem:

$$\sum \frac{r_a^2}{ctg\frac{B}{2}ctg\frac{C}{2}} = \sum \frac{\frac{S^2}{(p-a)^2}}{\frac{p}{p-a}} = \frac{S^2}{p}\sum \frac{1}{p-a} = \frac{S^2}{p}\cdot\frac{4R+r}{S} = r(4R+r).$$

8) In ΔABC :

$$9r^2 \leq \frac{r_a^2}{ctg\frac{B}{2}ctg\frac{C}{2}}+\frac{r_b^2}{ctg\frac{C}{2}ctg\frac{A}{2}}+\frac{r_c^2}{ctg\frac{A}{2}ctg\frac{B}{2}} \leq \frac{9Rr}{2}.$$

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Solutie:

Se folosește identitatea $\sum \frac{r_a^2}{ctg\frac{B}{2}ctg\frac{C}{2}} = r(4R+r)$ și inegalitatea lui Euler $R \geq 2r$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

9) In ΔABC

$$\frac{r_a^2}{tg^2\frac{A}{2}} + \frac{r_b^2}{tg^2\frac{B}{2}} + \frac{r_c^2}{tg^2\frac{C}{2}} = 3p^2.$$

Solutie:

Folosind $r_a = \frac{S}{p-a}$ și $tg^2\frac{A}{2} = \frac{(p-b)(p-c)}{p(p-a)}$ obținem:

$$\sum \frac{r_a^2}{tg^2\frac{A}{2}} = \sum \frac{\frac{S^2}{(p-a)^2}}{\frac{(p-b)(p-c)}{p(p-a)}} = S^2 p \sum \frac{1}{(p-a)(p-b)(p-c)} = r^2 p^3 \cdot \frac{3}{r^2 p} = 3p^2.$$

10) In ΔABC

$$81r^2 \leq \frac{r_a^2}{tg^2\frac{A}{2}} + \frac{r_b^2}{tg^2\frac{B}{2}} + \frac{r_c^2}{tg^2\frac{C}{2}} \leq \frac{81R^2}{4}.$$

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Solutie: Se folosește identitatea $\sum \frac{r_a^2}{tg^2\frac{A}{2}} = 3p^2$ și inegalitatea lui Mitrinovic

$$27r^2 \leq p^2 \leq \frac{27R^2}{4}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

11) In ΔABC

$$\frac{r_a^2}{ctg^2\frac{A}{2}} + \frac{r_b^2}{ctg^2\frac{B}{2}} + \frac{r_c^2}{ctg^2\frac{C}{2}} = \frac{2p^4 - 16p^2R(4R+r) + (4R+r)^4}{p^2}.$$

Solutie: Folosind $r_a = \frac{S}{p-a}$ și $ctg^2\frac{A}{2} = \frac{p(p-a)}{(p-b)(p-c)}$ obținem:

$$\sum \frac{r_a^2}{ctg^2\frac{A}{2}} = \sum \frac{\frac{S^2}{(p-a)^2}}{\frac{p(p-a)}{(p-b)(p-c)}} = \frac{S^2}{p} \sum \frac{(p-b)(p-c)}{(p-a)^3} = \frac{r^2p^2}{p} \cdot \frac{2p^4 - 16p^2R(4R+r) + (4R+r)^4}{r^2p^3} =$$

$$= \frac{2p^4 - 16p^2R(4R+r) + (4R+r)^4}{p^2}.$$

12) In ΔABC

$$\frac{r_a^2}{ctg^2\frac{A}{2}}+\frac{r_b^2}{ctg^2\frac{B}{2}}+\frac{r_c^2}{ctg^2\frac{C}{2}}\geq\frac{9Rr}{2}.$$

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Solutie:

Folosind identitatea $\frac{r_a^2}{ctg^2\frac{A}{2}}+\frac{r_b^2}{ctg^2\frac{B}{2}}+\frac{r_c^2}{ctg^2\frac{C}{2}}=\frac{2p^4-16p^2R(4R+r)+(4R+r)^4}{p^2}$

inegalitatea se scrie:

$\frac{2p^4-16p^2R(4R+r)+(4R+r)^4}{p^2}\geq\frac{9Rr}{2}\Leftrightarrow 2(4R+r)^4\geq p^2\left(128R^2+41Rr-4p^2\right)$, care

rezultă din inegalitatea Blundon-Gerretsen $16Rr-5r^2\leq p^2\leq\frac{R(4R+r)^2}{2(2R-r)}$.

Rămâne să arătăm că:

$2(4R+r)^4\geq\frac{R(4R+r)^2}{2(2R-r)}\left[128R^2+41Rr-4\left(16Rr-5r^2\right)\right]\Leftrightarrow 23R^2-44Rr-4r^2\geq 0\Leftrightarrow$

$\Leftrightarrow (R-2r)(23R+2r)\geq 0$, evident din inegalitatea lui Euler $R\geq 2r$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

ABOUT PROBLEM 3675-CRUX MATHEMATICORUM

By Marin Chirciu-Romania

1) Let a,b, and c be the sides of a triangle and let s be its semiperimeter. Let r and R denote its iradius and circumradius respectively. Prove that

$$6\leq\sum_{cyclic}\frac{b(s-b)+c(s-c)}{a(s-a)}\leq\frac{3R}{r}.$$

Proposed by Michel Bataille, Rouen, France

Solutie:

În articol voi prezenta o soluție diferita de soluția autorului apărută în Crux Mathematicorum Vol. 38 No.8 din 2012 și voi întări dubla inegalitate din enunț.Voi demonstra rezultatul ajutător:

Lema:

2) In ΔABC : $\sum_{cyclic}\frac{b(s-b)+c(s-c)}{a(s-a)}=\frac{(4R+r)^3+s^2(r-2R)}{2Rs^2}$.

Demonstratie:

$$\sum_{cyclic}\frac{b(s-b)+c(s-c)}{a(s-a)}=\frac{\sum bc(s-b)(s-c)\left[b(s-b)+c(s-c)\right]}{abc(s-a)(s-b)(s-c)}=$$

$$=\frac{2r^3\left[(4R+r)^3+s^2(r-2R)\right]}{4Rrs\cdot r^2s}=\frac{(4R+r)^3+s^2(r-2R)}{2Rs^2}\ .$$

Să trecem la rezolvarea problemei propuse.

Inegalitatea din stânga, ținând seama de **Lemă** se scrie:

$\frac{(4R+r)^3+s^2(r-2R)}{2Rs^2}\geq 6 \Leftrightarrow (4R+r)^3\geq p^2(14R-r)$,care rezultă din inegalitatea Blundon-Gerretsen: $p^2\leq\frac{R(4R+r)^2}{2(2R-r)}$. Rămâne să arătăm că:

$(4R+r)^3\geq\frac{R(4R+r)^2}{2(2R-r)}(14R-r)\Leftrightarrow 2R^2-3Rr-2r^2\geq 0 \Leftrightarrow (R-2r)(2R+r)\geq 0$, evident din inegalitatea lui Euler: $R\geq 2r$ .Egalitatea are loc dacă și numai dacă triunghiul este echilateral. Inegalitatea din dreapta, ținând seama de **Lemă** se scrie:

$\frac{(4R+r)^3+s^2(r-2R)}{2Rs^2}\leq\frac{3R}{r}\Leftrightarrow p^2(6R^2+2Rr-r^2)\geq r(4R+r)^3$,

care rezultă din inegalitatea Gerretsen: $p^2\geq 16Rr-5r^2$ .Rămâne să arătăm că:

$(16Rr-5r^2)(6R^2+2Rr-r^2)\geq r(4R+r)^3\Leftrightarrow 16R^3-23R^2r-19Rr^2+2r^3\geq 0\Leftrightarrow$

$\Leftrightarrow (R-2r)(16R^2+9Rr-r^2)\geq 0$, evident din inegalitatea lui Euler: $R\geq 2r$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă: Inegalitatea din stânga poate fi întărită:

3) In ΔABC: $\sum\limits_{cyclic}\frac{b(s-b)+c(s-c)}{a(s-a)}\geq\frac{4}{3}\left(5-\frac{r}{R}\right)$.

Soluție: Folosind **Lema** inegalitatea se scrie:

$\frac{(4R+r)^3+s^2(r-2R)}{2Rs^2}\geq\frac{4}{3}\left(5-\frac{r}{R}\right)\Leftrightarrow 3(4R+r)^3\geq p^2(46R-11r)$,

care rezultă din inegalitatea Blundon-Gerretsen: $p^2\leq\frac{R(4R+r)^2}{2(2R-r)}$.

Rămâne să arătăm că:

$3(4R+r)^3\geq\frac{R(4R+r)^2}{2(2R-r)}(46R-11r)\Leftrightarrow 2R^2-Rr-6r^2\geq 0\Leftrightarrow (R-2r)(2R+3r)\geq 0$,

evident din inegalitatea lui Euler: $R\geq 2r$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă: Inegalitatea **3)** este mai tare decât inegalitatea stângă din enunț:

4) In ΔABC: $\sum\limits_{cyclic}\frac{b(s-b)+c(s-c)}{a(s-a)}\geq\frac{4}{3}\left(5-\frac{r}{R}\right)\geq 6$.

Soluție: Vezi inegalitatea **3)** și $\frac{4}{3}\left(5-\frac{r}{R}\right)\geq 6\Leftrightarrow R\geq 2r$ (inegalitatea lui Euler).Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă: Inegalitatea **3)** poate fi și ea întărită:

5) In ΔABC: $\sum_{cyclic}\frac{b(s-b)+c(s-c)}{a(s-a)}\geq 7-\frac{3}{2}\cdot\frac{r}{R}-\left(\frac{r}{R}\right)^2$.

Solutie:

Folosind **Lema** inegalitatea se scrie:

$\frac{(4R+r)^3+s^2(r-2R)}{2Rs^2}\geq 7-\frac{3}{2}\cdot\frac{r}{R}-\left(\frac{r}{R}\right)^2 \Leftrightarrow R(4R+r)^3\geq 2p^2(8R^2-2Rr-r^2)\Leftrightarrow$,

$\Leftrightarrow p^2\leq\frac{R(4R+r)^2}{2(2R-r)}$ (inegalitatea Blundon-Gerretsen).Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.

Inegalitatea **5)** este mai tare decât inegalitatea **3)**:

6) In ΔABC: $\sum_{cyclic}\frac{b(s-b)+c(s-c)}{a(s-a)}\geq 7-\frac{3}{2}\cdot\frac{r}{R}-\left(\frac{r}{R}\right)^2\geq\frac{4}{3}\left(5-\frac{r}{R}\right)\geq 6$.

Solutie: Vezi inegalitatea **5)** , inegalitatea **4)** și $7-\frac{3}{2}\cdot\frac{r}{R}-\left(\frac{r}{R}\right)^2\geq\frac{4}{3}\left(5-\frac{r}{R}\right)$,

adevărată din inegalitatea lui Euler: $R\geq 2r$ .Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.

Să întărim și inegalitatea din dreapta:

7) In ΔABC: $\sum_{cyclic}\frac{b(s-b)+c(s-c)}{a(s-a)}\leq\frac{3}{2}+\frac{r}{R}+\frac{2R}{r}$.

Solutie:

Folosind **Lema** inegalitatea se scrie:

$\frac{(4R+r)^3+s^2(r-2R)}{2Rs^2}\leq\frac{3}{2}+\frac{r}{R}+\frac{2R}{r}\Leftrightarrow p^2(4R^2+5Rr+r^2)\geq r(4R+r)^3$,

care rezultă din inegalitatea Gerretsen: $p^2\geq 16Rr-5r^2$ .Rămâne să arătăm că:

$(16Rr-5r^2)(4R^2+5Rr+r^2)\geq r(4R+r)^3\Leftrightarrow 4R^2-7Rr-2r^2\geq 0\Leftrightarrow$

$\Leftrightarrow (R-2r)(4R+r)\geq 0$, evident din inegalitatea lui Euler: $R\geq 2r$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.

Inegalitatea **7)** este mai tare decât inegalitatea inegalitatea dreaptă din enunț:

8) In ΔABC: $\sum_{cyclic}\frac{b(s-b)+c(s-c)}{a(s-a)}\leq\frac{3}{2}+\frac{r}{R}+\frac{2R}{r}\leq\frac{3R}{r}$.

Solutie: Vezi inegalitatea **7)** și $\frac{3}{2}+\frac{r}{R}+\frac{2R}{r}\leq\frac{3R}{r}$, adevărată din inegalitatea lui Euler: $R\geq 2r$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă: Se poate scrie dubla inegalitate care întărește rezultatul din enunț:

9) In ΔABC: $7-\frac{3}{2}\cdot\frac{r}{R}-\left(\frac{r}{R}\right)^2\leq\sum_{cyclic}\frac{b(s-b)+c(s-c)}{a(s-a)}\leq\frac{3}{2}+\frac{r}{R}+\frac{2R}{r}$.

Marin Chirciu

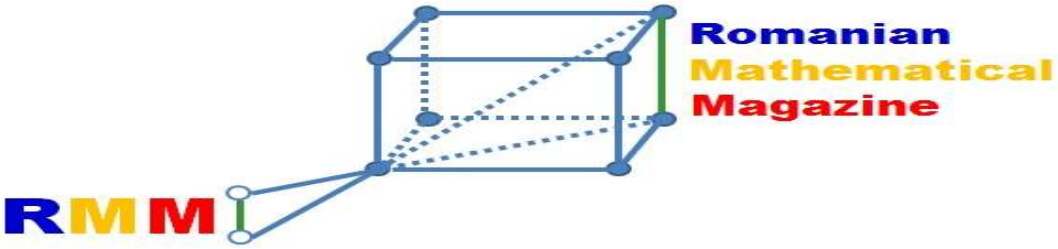
Soluție: Vezi inegalitatea **5)** și inegalitatea **7)**.Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă: Se poate scrie șirul de inegalități:

10) In ΔABC :

$$6\le\frac{4}{3}\left(5-\frac{r}{R}\right)\le 7-\frac{3}{2}\cdot\frac{r}{R}-\left(\frac{r}{R}\right)^2\le\sum_{cyclic}\frac{b(s-b)+c(s-c)}{a(s-a)}\le\frac{3}{2}+\frac{r}{R}+\frac{2R}{r}\le\frac{3R}{r}.$$

Soluție: Vezi inegalitatea **6)** și inegalitatea **8)**. Egalitatea are loc dacă și numai dacă triunghiul este echilateral.



ABOUT PROBLEM JP.173
ROMANIAN MATHEMATICAL MAGAZINE-SPRING EDITION 2019

By Marin Chirciu-Romania

1) Let ABC be a triangle. Prove that

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\ge\sqrt{\frac{6R}{r}}.$$

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

Soluție: Demonstrăm rezultatul ajutător:

Lemă.

2) In ΔABC :

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}=\frac{p^2+r^2+4Rr}{2rp}.$$

Demonstrație:

Folosind $\sin A=\frac{a}{2R}$, $\sum bc=p^2+r^2+4Rr$ și $abc=4Rrp$, obținem:

$$\sum\frac{1}{\sin A}=\sum\frac{2R}{a}=2R\sum\frac{1}{a}=2R\cdot\frac{\sum bc}{abc}=2R\cdot\frac{p^2+r^2+4Rr}{4Rrp}=\frac{p^2+r^2+4Rr}{2rp}.$$

Să trecem la rezolvarea problemei din enunț:

Folosind **Lema** inegalitatea de demonstrat se scrie:

$$\frac{p^2+r^2+4Rr}{2rp}\ge\sqrt{\frac{6R}{r}}\Leftrightarrow\left(\frac{p^2+r^2+4Rr}{2rp}\right)^2\ge\frac{6R}{r}\Leftrightarrow$$

$$\Leftrightarrow\frac{p^4+2p^2\left(r^2+4Rr\right)+r^2\left(4R+r\right)^2}{4r^2p^2}\ge\frac{6R}{r}\Leftrightarrow$$

$\Leftrightarrow p^2\left(p^2+2r^2-16Rr\right)+r^2\left(4R+r\right)^2\geq 0$.

Distingem cazurile:

Cazul 1). Dacă $\left(p^2+2r^2-16Rr\right)\geq 0$, inegalitatea este evidentă.

Cazul 2). Dacă $\left(p^2+2r^2-16Rr\right)<0$, inegalitatea se rescrie:

$r^2\left(4R+r\right)^2\geq p^2\left(16Rr-2r^2-p^2\right)$, care rezultă din inegalitatea Blundon-Gerretsen:

$16Rr-5r^2\leq p^2\leq\frac{R\left(4R+r\right)^2}{2\left(2R-r\right)}$. Rămâne să arătăm că :

$$r^2\left(4R+r\right)^2\geq\frac{R\left(4R+r\right)^2}{2\left(2R-r\right)}\left(16Rr-2r^2-\left(16Rr-5r^2\right)\right)\Leftrightarrow r^2\geq\frac{3Rr^2}{2\left(2R-r\right)}\Leftrightarrow 2\left(2R-r\right)\geq 3R$$

$\Leftrightarrow R\geq 2r$, (inegalitatea lui Euler).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă:Să punem în evidență o inegalitate de sens contrar:

3) In ΔABC :

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\leq\frac{R\sqrt{3}}{r}.$$

Marin Chirciu

Soluție: Folosind **Lema** inegalitatea se scrie:

$$\frac{p^2+r^2+4Rr}{2rp}\leq\frac{R\sqrt{3}}{r}\Leftrightarrow\left(\frac{p^2+r^2+4Rr}{2p}\right)^2\leq 3R^2\Leftrightarrow$$

$$\Leftrightarrow\frac{p^4+2p^2\left(r^2+4Rr\right)+r^2\left(4R+r\right)^2}{4p^2}\leq 3R^2\Leftrightarrow r^2\left(4R+r\right)^2\leq p^2\left(12R^2-8Rr-2r^2-p^2\right)\ ,$$

care rezultă din inegalitatea Gerretsen:

$\frac{r\left(4R+r\right)^2}{R+r}\leq 16Rr-5r^2\leq p^2\leq 4R^2+4Rr+3r^2$. Rămâne să arătăm că :

$$r^2\left(4R+r\right)^2\leq\frac{r\left(4R+r\right)^2}{R+r}\left(12R^2-8Rr-2r^2-\left(4R^2+4Rr+3r^2\right)\right)\Leftrightarrow$$

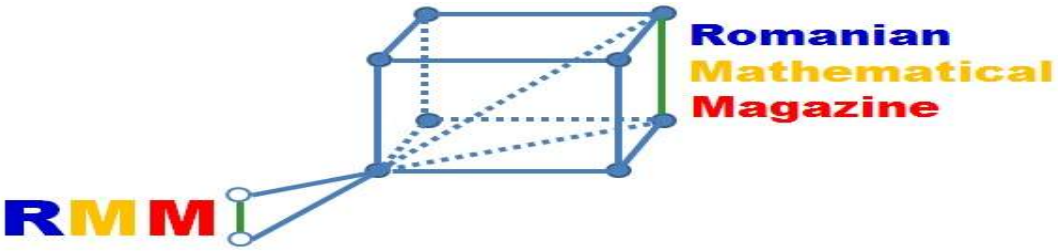
$r\left(R+r\right)\leq 8R^2-12Rr-5r^2\ \Leftrightarrow 8R^2-13Rr-6r^2\geq 0\ \Leftrightarrow\left(R-2r\right)\left(8R+3r\right)\geq 0$, evidentă din inegalitatea lui Euler $R\geq 2r$ .Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă: Se poate scrie dubla inegalitate:

4) In ΔABC :

$$\sqrt{\frac{6R}{r}}\leq\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\leq\frac{R\sqrt{3}}{r}.$$

Soluție: Vezi inegalitățile **2)** și **3)**. Egalitatea are loc dacă și numai dacă triunghiul este echilateral.



ABOUT INEQUALITY IN TRIANGLE-884
ROMANIAN MATHEMATICAL MAGAZINE

By Marin Chirciu-Romania

1) In ΔABC

$$\frac{m_a}{bc}+\frac{m_b}{ca}+\frac{m_c}{ab}\geq\frac{3}{2R}\ .$$

Daniel Sitaru

Solutie.

Folosind $bc=2Rh_a$ și $m_a\geq h_a$ obținem $\sum\frac{m_a}{bc}=\sum\frac{m_a}{2Rh_a}=\frac{1}{2R}\sum\frac{m_a}{h_a}\geq\frac{1}{2R}\cdot 3=\frac{3}{2R}$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă: Să punem în evidență o inegalitate de sens contrar.

2) In ΔABC

$$\frac{m_a}{bc}+\frac{m_b}{ca}+\frac{m_c}{ab}\leq\frac{3}{4r}\ .$$

Marin Chirciu

Solutie.

Folosind $bc=2Rh_a$ obținem

$\sum\frac{m_a}{bc}=\sum\frac{m_a}{2Rh_a}=\frac{1}{2R}\sum\frac{m_a}{h_a}$, (1).

Tripletele (m_a,m_b,m_c) și $\left(\frac{1}{h_a},\frac{1}{h_b},\frac{1}{h_c},\right)$ sunt invers ordonate. Cu inegalitatea lui Cebâșev obținem:

$\sum\frac{m_a}{h_a}\leq\frac{1}{3}\sum m_a\sum\frac{1}{h_a}\leq\frac{1}{3}\cdot\frac{9R}{2}\cdot\frac{1}{r}=\frac{3R}{2r}$, (2).

Din (1) și (2) rezultă concluzia.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.

Se poate scrie dubla inegalitate:

3) In ΔABC

$$\frac{3}{2R}\leq\frac{m_a}{bc}+\frac{m_b}{ca}+\frac{m_c}{ab}\leq\frac{3}{4r}\ .$$

Daniel Sitaru,Marin Chirciu

Solutie: Vezi inegalitățile **1)** și **2)**.Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă: Dubla inegalitate poate fi întărită:

4) In ΔABC

$$\frac{1}{2R}\left(\frac{7}{2}-\frac{r}{R}\right)\le\frac{m_a}{bc}+\frac{m_b}{ca}+\frac{m_c}{ab}\le\frac{1}{3r}\left(2+\frac{r}{2R}\right).$$

Solutie: Inegalitatea din stânga : $\frac{m_a}{bc}+\frac{m_b}{ca}+\frac{m_c}{ab}\ge\frac{1}{2R}\left(\frac{7}{2}-\frac{r}{R}\right)$ rezultă din:

$\sum\frac{m_a}{bc}=\sum\frac{m_a}{2Rh_a}=\frac{1}{2R}\sum\frac{m_a}{h_a}$, (1).

Cu inegalitatea lui Tereshin $m_a\ge\frac{b^2+c^2}{4R}$, $h_a=\frac{bc}{2R}$, $\sum\frac{b^2+c^2}{bc}=\frac{p^2+r^2-2Rr}{2Rr}$ și inegalitatea lui Gerretsen: $p^2\ge16Rr-5r^2$ obținem:

$$\sum\frac{m_a}{h_a}\ge\sum\frac{\frac{b^2+c^2}{4R}}{\frac{bc}{2R}}=\frac{1}{2}\sum\frac{b^2+c^2}{bc}=\frac{1}{2}\cdot\frac{p^2+r^2-2Rr}{2Rr}=\frac{p^2+r^2-2Rr}{4Rr}\ge\frac{p^2+r^2-2Rr}{4Rr}\ge$$

$$\ge\frac{16Rr-5r^2+r^2-2Rr}{4Rr}=\frac{14Rr-4r^2}{4Rr}=\frac{7R-2r}{2R}=\frac{7}{2}-\frac{r}{R},\ (2).$$

Din (1) și (2) obținem $\frac{m_a}{bc}+\frac{m_b}{ca}+\frac{m_c}{ab}\ge\frac{1}{2R}\left(\frac{7}{2}-\frac{r}{R}\right)$.

Inegalitatea din dreapta: $\frac{m_a}{bc}+\frac{m_b}{ca}+\frac{m_c}{ab}\le\frac{1}{3r}\left(2+\frac{r}{2R}\right)$. Folosind $bc=2Rh_a$ obținem

$\sum\frac{m_a}{bc}=\sum\frac{m_a}{2Rh_a}=\frac{1}{2R}\sum\frac{m_a}{h_a}$, (1). Tripletele (m_a,m_b,m_c) și $\left(\frac{1}{h_a},\frac{1}{h_b},\frac{1}{h_c},\right)$ sunt invers ordonate. Cu inegalitatea lui Cebâșev obținem:

$\sum\frac{m_a}{h_a}\le\frac{1}{3}\sum m_a\sum\frac{1}{h_a}\le\frac{1}{3}\cdot(4R+r)\cdot\frac{1}{r}=\frac{4R+r}{3r}$, (2).

Din (1) și (2) rezultă $\frac{m_a}{bc}+\frac{m_b}{ca}+\frac{m_c}{ab}\le\frac{1}{2R}\cdot\frac{4R+r}{3r}=\frac{4R+r}{6Rr}=\frac{1}{3r}\left(2+\frac{r}{2R}\right)$.

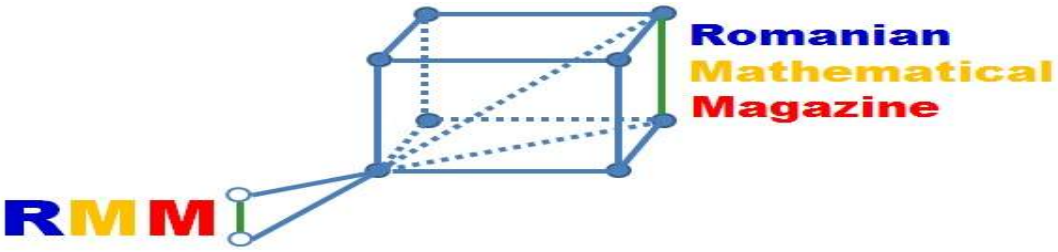
Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă: Dubla inegalitate **4)** este mai tare decât **3)**.

5) In ΔABC

$$\frac{3}{2R}\le\frac{1}{2R}\left(\frac{7}{2}-\frac{r}{R}\right)\le\frac{m_a}{bc}+\frac{m_b}{ca}+\frac{m_c}{ab}\le\frac{1}{3r}\left(2+\frac{r}{2R}\right)\le\frac{3}{4r}.$$

Soluție: Vezi 4) și inegalitatea lui Euler $R\ge2r$ .Egalitatea are loc dacă și numai dacă triunghiul este echilateral.



POLYNOMES AND MATRICES

By Marian Ursărescu-Romania

Within this article we'll try to solve problems which ask to demonstrate that the determinant of a matrice is positive.First let's remind the following properties of the numbers:

1. $\overline{z_1 + z_2 + \cdots + z_n} = \overline{z_1} + \overline{z_2} + \cdots + \overline{z_n}, \forall z_1, z_2, \ldots, z_n \in \mathbb{C}^*$
2. $\overline{z_1 \cdot z_2 \cdot \ldots \cdot z_n} = \overline{z_1} \cdot \overline{z_2} \cdot \ldots \cdot \overline{z_n}, \forall z_1, z_2, \ldots, z_n \in \mathbb{C}^*$
3. $|z|^2 = z \cdot \overline{z}, \forall\, z \in \mathbb{C}^*$

Theorem 1 : 1. $\det \overline{A} = \overline{\det A}, \forall\, A \in \mathcal{M}(\mathbb{C})$ 2. $\det\left(A \cdot \overline{A}\right) \geq 0, \forall A \in \mathcal{M}(\mathbb{C})$

Demonstration:

1. Consider $A = \left(a_{ij}\right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$ $A = \left(a_{ij}\right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$

$$\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}$$

$$\overline{\det A} = \overline{\sum_{\sigma \in S_n} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}} =$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma)\, \overline{a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}} = \sum_{\sigma \in S_n} \varepsilon(\sigma)\, \overline{a}_{1\sigma(1)} \overline{a}_{2\sigma(2)} \ldots \overline{a}_{n\sigma(n)} = \det \overline{A}$$

2. $\det\left(A \cdot \overline{A}\right) = \det A \cdot \det \overline{A} = \det A \cdot \overline{\det A} = |\det A|^2 \geq 0$

Theorem 2: Consider $P \in \mathbb{R}[x], P(x) = ax^2 + bx + c$, write $a > 0$ and $\Delta = b^2 - 4ac < 0$. Then $\det(aA^2 + bB + cI_n) \geq 0, \forall A \in \mathcal{M}_n(\mathbb{R})$.

Demonstration: If $\Delta < 0$ then P has conjugated complex roots: $x_1, x_2 \in \mathbb{C}$, with $x_2 = x_1$

$\det[(A - x_1 I_n)(A - x_2 I_n)] = \det[(A - x_1 I_n)(A - x_2 I_n)] = \det[(A - x_1 I_n)(A - x_2 I_n)] \geq 0$ (according to theorem 1)

On the other hand: $\det[(A - x_1 I_n)(A - x_2 I_n)] = \det[A^2 - (x_1 + x_2)A + x_1 x_2 I_n] =$

$$= \det\left(A^2 + \frac{b}{a}A + \frac{c}{a}I_n\right) = \frac{1}{a^n}\det(aA^2 + bB + cI_n)$$

Consequently $\det(aA^2 + bB + cI_n) \geq 0$.The generalization of theorem 2

Consider $P \in \mathbb{R}[x], P(x) = a_{2p}x^{2p} + a_{2p-1}x^{2p-1} + \cdots + a_1 x + a_0, a_{2p} > 0$ and all its roots are complexe. Then $\left(a_{2p}A^{2p} + \cdots + a_1 A + a_0 I_n\right) \geq 0, \forall A \in \mathcal{M}_n(\mathbb{R})$

Demonstration: If P has all roots complex (numbers) and $P \in \mathbb{R}[x]$ then $P(x) = a_{2p}(x - x_1)(x - x_1) \ldots (x - x_p)(x - x_p)$ and we can use the second theorem for the polynome $(x - x_k)(x - x_k), k = 1, p$

Theorem 3: Consider $P \in \mathbb{R}[x], P(x) = x^2 + ax + b, \Delta = a^2 - 4b < 0, \forall A, B \in \mathcal{M}_n(\mathbb{R})$ the following statement is true: $\det[(A + x_1B)(a + x_2B)] \geq 0, x_1, x_2$ being the roots of the polynome P.

Demonstration: If $\Delta < 0$ then $x_1, x_2 \in \mathbb{C}$ and $x_1 = x_2$

$\det[(A + x_1B)(A + x_2B)] = \det[(A + x_1B)(A + x_1B)] = \det[(A + x_1B)(A + x_1B)] \geq 0$ (according to theorem 1)

Notice that if $AB = BA$ then the conclusion of the third theorem can be written like that:

$$\det(A^2 - aAB + bB^2) \geq 0$$

Demonstration: $\det[(A + x_1B)(A + x_2B)] = \det[A^2 + (x_1 + x_2)AB + x_1x_2B^2] =$

$= \det(A^2 - aAB + bB^2) \geq 0$, (we used $AB = BA$ and Viète's relations)

Theorem 4: Consider $P \in \mathbb{R}[x], P(x) = x^2 + ax + b, \Delta = a^2 - 4b < 0, \forall A, B \in \mathcal{M}_n(\mathbb{R})$ the following statement is true: $\det[(I_n + x_1A + x_2B)(I_n + x_2A + x_1B)] \geq 0, x_1, x_2$ being the roots of P.

Demonstration: If $\Delta < 0$ then $x_1, x_2 \in \mathbb{C}$ and $x_2 = x_1$

$\det[(I_n + x_1A + x_2B)(I_n + x_2A + x_1B)] = \det[(I_n + x_1A + x_2B)(I_n + x_2A + x_1B)] =$

$= \det[(I_n + x_1A + x_1A)(I_n + x_1A + x_1B)] \geq 0$ (according to the first theorem)

If in addition, $AB = BA$ then the conclusion of the fourth theorem can be written like that:

$$\det[I_n + b(A^2 + B^2) - a(A + B) + (a^2 - 2b)AB] \geq 0$$

Demonstration: $\det[(I_n + x_1A + x_2B)(I_n + x_2A + x_1B)] =$

$$= \det[I_n + x_1x_2(A^2 + B^2) + (x_1 + x_2)(A + B) + (x_1^2 + x_2^2)AB] =$$

$= \det[I_n + b(A^2 + B^2) - a(A + B) + (a^2 + 2b)AB]$ (we used $AB = BA$ and Viète's relations)

Theorem 5: Consider $P \in \mathbb{R}[x], P(x) = x^2 + ax + b, \Delta = a^2 - 4b < 0, \forall A, B \in \mathcal{M}_n(\mathbb{R})$ the following statement is true: $\det[(A + x_1B + x_2C)(C + x_2B + x_1C)] \geq 0, x_1, x_2$ being the roots of P.

Demonstration: If $\Delta < 0$ then $x_1, x_2 \in \mathbb{C}$ and $x_2 = x_1$

$\det[(A + x_1B + x_2C)(C + x_2B + x_1C)] = \det[(A + x_1B + x_1C)(A + x_1B + x_1C)] =$

$= \det[(A + x_1B + x_1C)(C + x_1B + x_1C)] \geq 0$ (according to the first theorem)

If $AB = BA, AC = CA, BC = CB$, then the conclusion of this theorem can be written this way: $\det[A^2 + b(B^2 + C^2) - a(AB + AC) + (a^2 + 2b)BC] \geq 0$

Demonstration: $\det[(A + x_1B + x_2C)(C + x_2B + x_1C)] =$

$$= \det[A^2 + x_1x_2(B^2 + C^2) + (x_1 + x_2)(AB + AC) + (x_1^2 + x_2^2)BC] =$$

$$= \det[A^2 + b(B^2 + C^2) - a(AB + AC) + (a^2 + 2b)BC] \geq 0$$

(we used $AB = BA, AC = CA, BC = CB$ and Viète's relations). Now let's solve some problems related to these theorems.

1. Show that $\det(A^2 + I_n) \geq 0, \forall A \in \mathcal{M}_n(\mathbb{R})$

Demonstration: We use theorem 2 for $P(x) = x^2 + 1$, whose roots are $x_1 = i, x_2 = -i$

2. Show that $\det(A^2 + A + I_n) \geq 0, \forall A \in \mathcal{M}_n(\mathbb{R})$

Demonstration: We use theorem 2 for $P(x) = x^2 + x + 1$, with the roots

$$x_1 = \frac{-1 + i\sqrt{3}}{2}, x_2 = \frac{-1 - i\sqrt{3}}{2}$$

3. If $\forall A, B \in \mathcal{M}_n(\mathbb{R})$ so that $AB = BA$, then $\det(A^2 + B^2) \geq 0$

Demonstration: We use the third theorem for $P(x) = x^2 + 1$, where $x_1 + x_2 = 0, x_1x_2 = 1$

4. If $A, B \in \mathcal{M}_n(\mathbb{R})$ so that $AB = BA$, then $\det(A^2 + 2AB + 2B^2) \geq 0$
Demonstration: Using the third theorem we consider
$P(x) = x^2 + 2x + 2, x_1 + x_2 = -2, x_1x_2 = 2$

5. If $A, B \in \mathcal{M}_n(\mathbb{R}), A + B = I_n$, then $\det(A^3 + B^3) \geq 0$
Demonstration: Let's show first that $= BA$, $A + B = I_n$, we multiply right by
$B \Rightarrow AB + B^2 = B$ (1)
$A + B = I_n$, we multiply left by $B \Rightarrow BA + B^2 = B$ (2)

$$\left.\begin{matrix} If\ AB + B^2 = B\ then\ AB = B - B^2 \\ BA + B^2 = B \quad BA = B - B^2 \end{matrix}\right\} \Rightarrow AB = BA$$

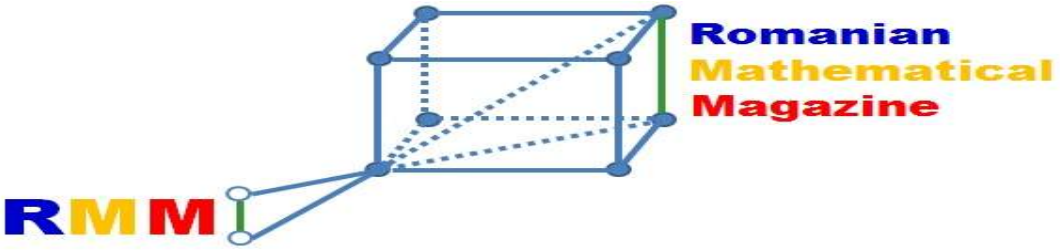
$$\det(A^3 + B^3) = \det[(A + B)(A^2 - AB + B^2)] = \det(A + B)\det(A^2 - AB + B^2) =$$
$$= \det(A^2 - AB + B^2) \geq 0 \text{ (using theorem 3 for } P(x) = x^2 - x + 1)$$

6. If $A, B, C \in \mathcal{M}_n(\mathbb{R}), AB = BA, AC = CA, BC = CA$ then
$$\det(A^2 + B^2 + C^2 - AB - AC - BC) \geq 0$$
Demonstration: We use theorem 5 for $P(x) = x^2 + x + 1$, whose roots are $x_1, x_2 \in \mathbb{C}$ with $x_1 + x_2 = -1, x_1x_2 = 1$ and $x_1^2 + x_2^2 = -1$.
7. If $A, B \in \mathcal{M}_n(\mathbb{R}), AB = BA$ then $\det(2A^2 + 2B^2 - 3AB - A - B + I_n) \geq 0$
Demonstration: We use theorem 4 for $P(x) = x^2 - x + 2$, with the roots $x_1, x_2 \in \mathbb{C}, x_1 + x_2 = 1, x_1x_2 = 2$ and $x_1x_2 = 2$ and $x_1^2 + x_2^2 = -3$.

8. If $A, B, C \in \mathcal{M}_n(\mathbb{R}), AB = BA, AC = CA, BC = CB$, then
$$\det(A^2 + (x^2 + y^2)(B^2 + C^2) + 2xA(B + C) + 2(x^2 + y^2)BC) \geq 0, \forall x, y \in \mathbb{R}$$
Demonstration: Using theorem 5 we consider $P(t) = t^2 + 2xt + x^2 + y^2, \Delta = -4y^2 \leq 0$, and the roots $x_1, x_2 \in \mathbb{C}$, $x_1 + x_2 = 2x$, $x_1x_2 = x^2 + y^2$, $x_1^2 + x_2^2 = 2(x^2 + y^2)$.
9. If $A, B \in \mathcal{M}_n(\mathbb{R}), AB = BA$, then:
$$\det(I_n + 2(a^2 + b^2)(A^2 + B^2) + 2(a + b)(A + B) + 8abAB) \geq 0, \forall a, b \in \mathbb{R}$$
Demonstration: We use theorem 4 for, $P(x) = x^2 - 2(a + b)x + 2a^2 + 2b^2$,
$\Delta = -4(a - b)^2 \leq 0, x_1, x_2 \in \mathbb{C}$, with $x_1 + x_2 = 2(a + b)$, $x_1 + x_2 = 2(a^2 + b^2)$
10. If $A \in \mathcal{M}_n(\mathbb{R})$, then $\det(I_n + A + A^2 + \cdots + A^{2k}) \geq 0, \forall k \in \mathbb{N}^*$
Demonstration: We use the generalization of theorem 2 for $P(x) = 1 + x + x^2 + \cdots + x^{2k}$ which has only complex roots

BIBLIOGRAPHY: Maths Review

PROPOSED PROBLEMS
5-CLASS-STANDARD



V.1. Prove that $\overline{1x}\left|\overline{abc}\right.$ if and only if $\overline{1x}\left|x\cdot\overline{ab}-c\right.$. ***Proposed by Marin Chirciu – Romania***

V.2. Prove that $7^{4n+3}>3^{7n+5}$, where $n\in N$. ***Proposed by Marin Chirciu – Romania***

V.3. Let be the number $A=(2n+1)(1+2+3+...+2n),\ n\in N$. Prove that $A=n(2n+1)^2,\ n\in N$. Prove that if $n=5k^2+10,\ k\in N$, then A can be written as a sum of five perfect squares, nonzero and distinct, for $k\geq 3$.

Proposed by Marin Chirciu – Romania

V.4. Find all natural numbers having maximum six digits that have the digits directly proportional with their rank from the right to the left.

Proposed by Daniel Sitaru-Romania

V.5. Prove that it exists an infinity of natural numbers (x,y,z) for which the number $13^x+13^y+13^z$ to be a perfect cube. ***Proposed by Marin Chirciu – Romania***

V.6. Find the natural numbers n such that the sum $1+2+...+n$ to be a number formed from two identical digits. ***Proposed by Titu Zvonaru, Neculai Stanciu-Romania***

V.7. Let be the natural numbers a,b,c, x,y,z which verifies the relationships $a=xy+x+y$, $b=xz+x+z$, $c=yz+y+z$. Prove that the product of the successors of the numbers a,b,c is a perfect square.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

V.8. Find all the numbers $\overline{a879b}$ divisible with 72.

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

V.9. Find all the numbers $\overline{abcde}$ for which $\overline{abcde}=4\cdot\overline{edcba}$.

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

V.10. Prove that the nonzero natural numbers a,b,c satisfy the relationship $43a^2+129b+25c=1720$, then they are prime.

Proposed by Titu Zvonaru, Neculai Stanciu-Romania

V.11. Prove that the number 194800...007171, which has in total 2016 zeros, is composed.

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

V.12. If to different letters correspond different digits, then find all the numbers $\overline{abc}$ knowing that the arithmetic means of the numbers $\overline{abc}$, $\overline{acb}$, $\overline{bac}$, $\overline{bca}$, $\overline{cab}$ and $\overline{cba}$ is $\overline{abc}$.

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

V.13. Solve the equation $x(x+1)(x+2) = \overline{abcabc}$.

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

V.14. If to different letters correspond different digits then find all the numbers $\overline{abc}$ which verify the relationship $\overline{abc} = \overline{ab} + \overline{ba} + \overline{bc} + \overline{cb} + \overline{ca} + \overline{ac}$.

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

V.15. If to different letters correspond different digits such that $\overline{adcb} + \overline{bcda} + \overline{cabd} + \overline{dbac} = \overline{abbba}$, then find $\overline{abcd}$.

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

V.16. If to different letters corresponds different digits and $\overline{cd} = 2 \cdot \overline{ab}$, then find all the perfect squares having the form $\overline{abcdab}$.

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

V.17. In how many ways can we weight 10 grams with a balance if we have 8 weights of one gram, 3 of two grams and 4 of three grams?

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

V.18. What age is Bogdan, knowing that his age is one year bigger than eight times the sum of the digits that it is formed?

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

V.19. Prove that if p is odd number and not divisible with 3 then the number $p^2 + 2015$ divides with 24.

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

V.20. How many digit has the number 2^{96} ?

Proposed by Titu Zvonaru, Neculai Stanciu-Romania

V.21. Suma a 12 numere naturale consecutive este un număr cu cifre diferite, printre care se află cifrele 1,2,3 și 4. Care este cel mai mic număr posibil dintre cele 12 numere?

Proposed by Ionică Constantin-Romania

V.22. Determinați numărul $\overline{abcd}$, știind că $4\overline{abcd} + 1026 = \overline{dcba}$

Proposed by Ionică Constantin-Romania

V.23. Scrieți numărul 164^{2019} ca sumă de patru pătrate perfecte.

Proposed by Ionică Constantin-Romania

V.24. Se consideră numărul $N = 7 + 7^2 + 7^3 + \cdots + 7^{2019}$. Arătați că $6N + 7$ este pătrat perfect.

Proposed by Ionică Constantin-Romania

V.25. Câte numere naturale de patru cifre se pot scrie ca sumă a cinci numere naturale consecutive?

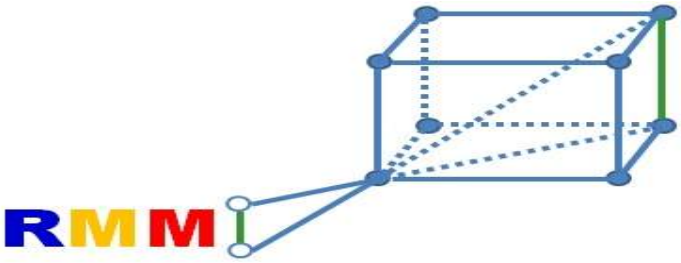
Proposed by Ionică Constantin-Romania

V.26. Să se arate că dacă $23 \cdot \overline{ab} = \overline{cd} + 369$, atunci numărul $\overline{abcd}$ se divide cu 41.

Proposed by Ionică Constantin-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

6-CLASS-STANDARD



Romanian Mathematical Magazine

VI.1. Find all the pairs (x, y) of integers which verify the relationship $x^2 + xy - y = 2018$.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VI.2. Let be a,b,c,d,e,f real numbers, strictly positive which verify the relationships (i) $a^2+b^2+c^2=16$; (ii) $d^2+e^2+f^2=36$; (iii) $ad+be+cf=24$.

Find $\frac{a+b+c}{d+e+f}=\frac{2}{3}$. ***Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania***

VI.3. If $p \geq 5$ is a prime number, then find the reminder of the number division p^2 to 12.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VI.4. If a and b are digits, then solve the equation $(\overline{ab}+\overline{ba})(\overline{ab}-\overline{ba}) = x^2$.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VI.5. Find all the pairs $(x, y) \in Z \times Q$, which verify the relationship $\frac{x^2+xy+y^2}{x+2y}=\frac{7}{5}$.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VI.6. Find all the integers a for which a^2-8a is a perfect square.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VI.7. Find the set $A=\left\{x \in Z \middle| \frac{18-9x^8}{4-3x^4} \in Z\right\}$.

Proposed by Neculai Stanciu,Iuliana Traşcă-Romania

VI.8. Let be the isosceles triangle ABC with $AB = AC$, the point D on the line BC, with C between B and D and the point E the symmetric of the point B towards the line AD. Prove that $\angle BAD \equiv \angle ECD$. ***Proposed by Titu Zvonaru,Neculai Stanciu-Romania***

VI.9. Prove that it doesn't exists a prime number p such that both numbers $p^3 + 2015$ and $p^3 + 2017$ to be prime. ***Proposed by Neculai Stanciu,Titu Zvonaru-Romania***

VI.10. Fie A= 47^{2018} , arătați că :

a) A nu se poate scrie ca sumă a două cuburi de numere întregi
b) A nu se poate scrie ca suma a trei cuburi de numere întregi.

Proposed by Cremeneanu Luiza Lorena, Prunaru Constantina-Romania

VI.11. If $a,b \in \boldsymbol{Z}$ such that $25a - 12b$ is a multiple of 11, then $2a + 3b$ is a multiple of 11.

Proposed by Marin Chirciu-Romania

VI.12. Solve in nonzero natural numbers set the following equation $\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) = 2$

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VI.13. If to different letters correspond different digits, then find all the digits a,b,c which verify the relationship $\overline{ab} \cdot \overline{ba} = \overline{cbac}$.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VI.14. From 100 pupils of a school, that have participated at baccalaureate exam, 70 passed test A, 80 test B, 75 test C and respectively 85 test D. Prove that at least 10 pupils have passed this exam.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VI.15. Prove that the number 102400...0081, which has in total 2015 zeros, is composed

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VI.16. Prove that for any natural number n $(1 \le n \le 2012)$, it exists a natural number m such that all the numbers $m \cdot 2^n + 1$ to be composed.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VI.17. Find all the triplets (x, y, z) of natural numbers with $0 < x < y < z$, which verify the relationship $x^2 + y^2 = z^2 + m$ $(m \in N^*)$.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VI.18. If $a = \frac{1}{2011}$, then compute the whole part of the number

$$S = \frac{1}{1+a} + \frac{1}{1+a+a^2} + ... + \frac{1}{1+a+a^2+...+a^{2011}}.$$

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VI.19. If $a,b,c \in R^*$ such that $a+b+c=ab+bc+ca=3$, then find:

$a^{2011}+b^{2011}+c^{2011}$.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VI.20. Let a,b,c be the nonzero real numbers such that $a+b+c=9$ and $ab+bc+ca=27$. Find $a^{2014}+b^{2014}+c^{2014}$.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VI.21. Arătați că fracția $\frac{1\cdot 2\cdot 3\cdot \ldots \cdot 2018+1}{1\cdot 2\cdot 3\cdot \ldots \cdot 2019+1}$ este ireductibilă.

Proposed by Ionică Constantin-Romania

VI.22. Fie numerele natural a și b. Arătați că dacă fracția $\frac{3a+4b}{4a+3b}$ se simplifică prin 17, atunci și fracția $\frac{8a+5b}{9a+11b}$ se simplifică cu 17. ***Proposed by Ionică Constantin-Romania***

VI.23. Arătați că nu există numere naturale x, y, z pentru care $x + y + z = 18$ și $x^2 + y^2 + z^2 = 107$. ***Proposed by Ionică Constantin-Romania***

VI.24. Se consideră numerele raționale $x = \frac{3^{59}+4}{2^{88}+3}$ și $y = \frac{5^{99}+7}{7^{74}+5}$. Arătați că $x + y > 2{,}7(3)$.

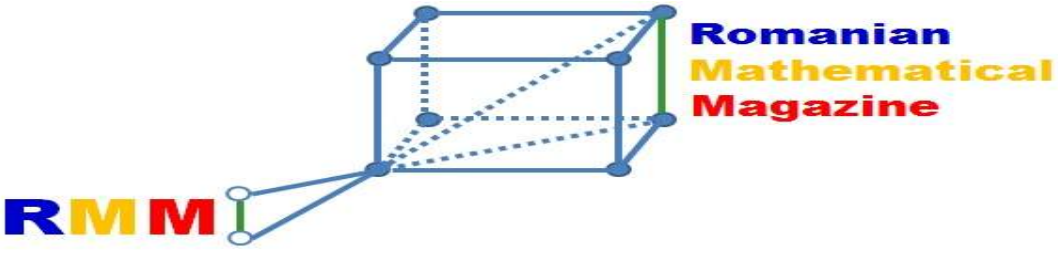
Proposed by Ionică Constantin-Romania

VI.25. Fie $a, b, c \in \mathbb{N}$ care nu sunt divizibile cu 3. Arătați că suma $a^2 + b^2 + c^2$ este divizibilă cu 3. ***Proposed by Ionică Constantin-Romania***

VI.26. Numerele $a + b$ și $a - b$ sunt direct proporționale cu 12 și 3. Determinați numerele a și b știind că $a \cdot b = 135$. ***Proposed by Ionică Constantin-Romania***

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

7-CLASS-STANDARD



VII.1. Let a, b, c be positive real numbers satisfying $a + b + c = 3$. Prove that:

$$\frac{a^2+b}{b^2+a}+\frac{b^2+c}{c^2+b}+\frac{c^2+a}{a^2+c}\geq 3$$

Proposed by Pham Quoc Sang-Vietnam

VII.2. Find all pairs (x, y) of integers satisfying the equation

$$x^4 - (y + 2)x^3 + (y - 1)x^2 + (y^2 + 2)x + y = 2.$$

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

VII.3. Find the numbers $a, b, c \in \mathbb{N}^*$ knowing that:

$$\frac{a+1}{b} \in \mathbb{N}, \frac{b+1}{c} \in \mathbb{N} \text{ and } \frac{c+1}{a} \in \mathbb{N}$$

Proposed by Gheorghe Alexe, George – Florin Şerban– Romania

VII.4. Let $ABCD$ be a trapeze where $AB \parallel CD; AB = a; CD = b; AD = c; BC = d; a > b.$ Prove that Area $[ABCD] < \frac{(a+b)(a-b+c+d)^2}{16(a-b)}$. ***Proposed by Daniel Sitaru – Romania***

VII.5. Given $a, b, c > 0$ and $a^2 + b^2 + c^2 = 6$, prove $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + a + b + c \geq 6$

Proposed by Nguyen Phuc Tang– Vietnam

VII.6. Prove that the following inequalities hold for all positive real numbers:

a. $\frac{a^3}{ab+c^2} + \frac{b^3}{bc+a^2} + \frac{c^3}{ca+b^2} \geq \frac{3}{2} \cdot \frac{a^2+b^2+c^2}{a+b+c}$

b. $\frac{1}{a(b+c)} + \frac{1}{b(c+a)} + \frac{1}{c(a+b)} \geq \frac{3}{2} \cdot \frac{a+b+c}{a^3+b^3+c^3}$

Proposed by Nguyen Viet Hung–Vietnam

VII.7. Prove that the following inequalities holds for all positive real numbers a, b, c:

a. $\frac{b}{a^2} + \frac{c}{b^2} + \frac{a}{c^2} \geq \frac{3(a+b+c)}{a^2+b^2+c^2}$

b. $\frac{b^3}{a^2} + \frac{c^3}{b^2} + \frac{a^3}{c^2} \geq \frac{3(a^2+b^2+c^2)}{a+b+c}$

Proposed by Nguyen Viet Hung-Vietnam

VII.8. Prove that for all positive real numbers a, b, c :

$$\frac{a(b^2 + c^2)}{2a^2 + bc} + \frac{b(c^2 + a^2)}{2b^2 + ca} + \frac{c(a^2 + b^2)}{2c^2 + ab} \geq \frac{6abc}{ab + bc + ca}$$

Proposed by Hung Nguyen Viet-Vietnam

VII.9. Let x, y, z > 0 be positive real numbers. Then

$$\frac{1}{x + y} + \frac{1}{y + z} + \frac{1}{z + x} \geq \frac{4\sqrt{3xyz(x + y + z)}}{(x + y)(y + z)(z + x)}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania, Martin Lukarevski-Macedonia

VII.10. Prove that for all positive real numbers a, b, c:

a. $\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geq\frac{a+b+c}{2}+\frac{(b-c)^2}{2(a+b+c)}$

b. $\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{a+b+c}{2}+\frac{(a+b-2c)^2}{2(a+b+c)}$

Proposed by Nguyen Viet Hung-Vietnam

VII.11 Prove that if $x_i\in(0,\infty); i\in\overline{1,n}; n\in\mathbb{N}; n\geq 3; x_{n+1}=x_1; x_1x_2\cdot\ldots\cdot x_n=1$, then

$$\sum_{i=1}^{n}\frac{\frac{x_i}{x_{i+1}}+\frac{x_{i+1}}{x_i}+1}{\sqrt{x_i^2+x_ix_{i+1}+x_{i+1}^2}}\geq n\sqrt{3}$$

Proposed by Daniel Sitaru– Romania

VII.12. Solve for natural numbers: $\frac{1^4}{x}+\frac{2^4}{x+1}+\frac{3^4}{x+2}+\cdots+\frac{10^4}{x+9}=3025$

Proposed by Daniel Sitaru– Romania

VII.13. We consider the right triangle ABC with the right angle in A. Let D be the contact point of altitude from A on BC and E the bisector's intersection of the angle $\angle ADC$ with AC. If $M\in(AE), N\in(DC)$ and $\{F\}=MN\cap AD$ such that $\frac{AE}{DC}=\frac{ME}{NC}=k>1$, find in function of k the rapport $\frac{BD}{AF}$.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VII.14. Prove that the number $N=\overline{\underbrace{111\ldots1}_{for\ 2010\ times}}$ is composed.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VII.15. Prove that the number $\overline{\underbrace{999\ldots9}_{for\ 2011\ times}}+\overline{1\underbrace{999\ldots9}_{for2011times}\underbrace{000\ldots0}_{for1005times}}$ is composed.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VII.16. We consider an ABC triangle with the lengths sides $AB=16, BC=34, CA=30$ and the points M respectively N on the side BC such that $BM=4, MN=12$.
Compute the measure of the angle $\angle MAN$.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VII.17. We denote "*Super – Heron*" triangle (do not confuse with "Super – Hero") a triangle that has the lengths sides consecutive natural numbers and the area also a natural number.

Recent[1](2007) was proved the existence of an infinite number of such triangles.
Confirm or infirm the existence of "Super – Heron" inscriptible quadrilaterals.
(*Indication.* Use the area formula of the incriptible quadrilateral given by the Indian mathematician *Brahamgupta* in VII century - A.H: $A=\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where a,b,c,d are the quadrilateral lengths sides, and s is its semiperimeter)

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VII.18. We consider a right-angled triangle with the perimeter P and the area A natural numbers.Prove that: the hypotenuse is a natural number if and only if P is natural even number and $\frac{P}{2}$ divides $A \Leftrightarrow P|2A$.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VII.19. Which conditions must meet the triangle's angle so this can be divided in two isosceles triangles? Find the angles of an isosceles triangle which can be divided in two isosceles triangles. ***Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania***

VII.20. Find the biggest triangle with the dimensions natural numbers whose sum is even number for which the area is also a natural number and it is equal with its perimeter.

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VII.21. Fie x, y, z numere reale astfel încât au loc simultan relațiile: $5x - 7y + 9 \geq 0$, $7y - 9z + 11 \geq 0, 9z - 5x - 20 \geq 0$. Arătați că $5x - 28y + 27z = 24$.

Proposed by Ionică Constantin-Romania

VII.22. Să se arate că într-un triunghi cu lungimile laturilor a, b, c și care are perimetrul egal cu 15 este adevărată relația $\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq 9$.

Proposed by Ionică Constantin-Romania

VII.23. Dacă a, b, c, d sunt lungimile laturilor unui patrulater și verifică relația: $\frac{a}{2019a+b+c+d} = \frac{b}{a+2019b+c+d} = \frac{c}{a+b+2019c+d} = \frac{d}{a+b+c+2019d}$, stabiliți natura patrulaterului.

Proposed by Ionică Constantin-Romania

VII.24. Se știe că lungimile diagonalelor unui romb verifică relația: $\frac{D}{d} + \frac{d}{D} = 2\sqrt{2}$. Determinați măsurile unghiurilor rombului. ***Proposed by Ionică Constantin-Romania***

VII.25. Stabiliți dacă numărul:

$a = \sqrt{1 + 2018 \cdot \sqrt{1 + 2019 \cdot \sqrt{1 + 2020 \cdot \sqrt{1 + 2021 \cdot \sqrt{1 + 2022 \cdot 2024}}}}}$ este rațional sau irațional.

Proposed by Ionică Constantin-Romania

VII.26. Arătați că numărul $a = 2019^4 - 10 \cdot 2018^2 - 20 \cdot 2018 + 15$ este pătrat perfect.

Proposed by Ionică Constantin-Romania

[1] http://www.math.twsu.edu/~richardson/heronian/heronian.html

VII.27. Să se rezolve ecuația $\frac{(a+b)^2-2a\sqrt{a}}{b}+\frac{(a+b)^2-2b\sqrt{b}}{a}=4\sqrt{ab}$, unde *a*, *b* și *c* sunt strict pozitive și $a+b=2$. ***Proposed by Carmen Chirfot-Romania***

VII.28. Să se demonstreze că $(a+b)^2\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4\sqrt{ab}+\frac{2a}{b}+\frac{2b}{a}$, unde *a*, *b* și *c* sunt numere reale strict pozitive. ***Proposed by Carmen Chirfot-Romania***

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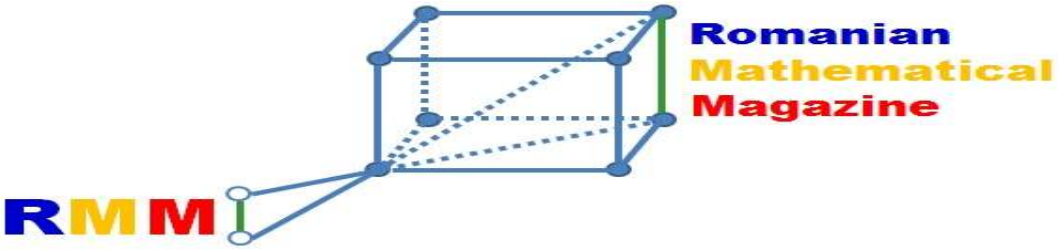
8-CLASS-STANDARD



VIII.1. Solve for integers:

$$x^{2018}+\left(\frac{x^2-x}{2}\right)^{2018}+\left(\frac{x^3-x}{6}\right)^{2018}=2018201720162015$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

VIII.2. If $A=2018^{6k+2}$, $k\in\mathbb{N}$ then:
a.Prove that A can not be written as a sum of two perfect cubes.
b.Prove that A can not be written as a sum of three perfect cubes.
Proposed by Lucian Tutescu,Daniela Iancu– Romania

VIII.3. If $a,b,c>0, \sqrt{1+a^2}+\sqrt{1+b^2}+\sqrt{1+c^2}=3\sqrt{2}$ then:

$$\frac{a}{\sqrt{1+a^2}}+\frac{b}{\sqrt{1+b^2}}+\frac{c}{\sqrt{1+c^2}}\leq\frac{3\sqrt{2}}{2}$$

Proposed by Daniel Sitaru– Romania

VIII.4. If $a,b,c>0, a+b+c=3$ then: $\sum\left(\sqrt{a(a+2b)}+\sqrt{b(b+2a)}\right)\leq 6\sqrt{3}$

Proposed by Daniel Sitaru– Romania

VIII.5. Solve in R^2 the system of equations:

$$(x+\sqrt{x^2+1})(y+\sqrt{y^2+1})=2012\ ;\ x+y=\frac{2011}{\sqrt{2012}}.$$

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

VIII.6. If $a,b,c>0, x+y+z=1$ then: $\sum(x+y)^2\geq 4\sqrt{3xyz}$

Proposed by Daniel Sitaru– Romania

VIII.7. Prove that if a, b, c > 0 then:

$$\sqrt{\frac{a}{b+c}}+2\sqrt{\frac{b}{c+a}}+4\sqrt{\frac{c}{a+b}}\leq\sqrt{7\left(\frac{a}{b+c}+\frac{2b}{c+a}+\frac{4c}{a+b}\right)}$$

Proposed by Daniel Sitaru– Romania

VIII.8. If $a,b,c>, a+b+c=3$ then:

$$\sum\frac{a^5+a-1}{a^3+a^2-1}\geq ab+bc+ca$$

Proposed by Daniel Sitaru – Romania

VIII.9. If $a,b,c>0, a^2+b^2+c^2=3$ then:

$$\frac{a^3b^2}{a^2+a+1}+\frac{b^3c^2}{b^2+b+1}+\frac{c^3a^2}{c^2+c+1}<1+ab^2+bc^2+ca^2$$

Proposed by Daniel Sitaru-Romania

VIII.10. If $x,y,z>0, x+y+z=1$ then:

$$\frac{x^2}{z}+\frac{y^2}{x}+\frac{z^2}{y}+2\left(\frac{xy}{2}+\frac{yz}{x}+\frac{zx}{y}\right)\geq 3$$

Proposed by Daniel Sitaru-Romania

VIII.11. If $x,y,z>0$ then:

$$x^2y^2z^2\left(\frac{x+y}{x^5+y^5}+\frac{y+z}{y^5+z^5}+\frac{z+x}{z^5+x^5}\right)\leq x^2+y^2+z^2$$

Proposed by Daniel Sitaru-Romania

VIII.12. If $x,y,z>0, xyz=1$ then:

$$\sum(x^2z-1+y^2z)(x^5+y^5)\geq 6$$

Proposed by Daniel Sitaru-Romania

VIII.13. If $a,b,c>0$ then:

$$\frac{ba^3}{1+a+a^2}+\frac{cb^3}{1+b+b^2}+\frac{ac^3}{1+c+c^2}<1+ab+bc+ca$$

Proposed by Daniel Sitaru – Romania

VIII.14. Solve for real numbers:

$\begin{cases}[|x|] = |[y]| \\ [|y|] = |[x]|\end{cases}$, $[*]$ - great integer function

Proposed by Daniel Sitaru – Romania

VIII.15. If $a, b, c > 0$ and $a + b + c = 3$ prove that:

$$\sum a\left(\frac{1}{b^n} + \frac{1}{c^n}\right) \geq \frac{18}{a^n + b^n + c^n}$$

where $n \geq 0$.

Proposed by Marin Chirciu– Romania

VIII.16. Let a, b, c be positive real numbers. Prove that:

$$\frac{(a^2 - ab + b^2)^2}{(a+b)^4} + \frac{(b^2 - bc + c^2)^2}{(b+c)^4} + \frac{(c^2 - ca + a^2)^2}{(c+a)^4} \geq \frac{3}{16}$$

Proposed by George Apostolopoulos-Greece

VIII.17. Prove that if $a, b, c \in \mathbb{R}$ then:

$$(2 - a - b - c + abc)^2 \leq (a^2 + 2)(b^2 + 2)(c^2 + 2)$$

Proposed by Daniel Sitaru – Romania

VIII.18. If $a, b, c > 0; n \geq 1$ then:

$$\frac{3n(a^4 + b^4 + c^4)}{(a^2 + b^2 + c^2)^2} + \frac{ab + bc + ca}{a^2 + b^2 + c^2} \geq n + 1$$

Proposed by Marin Chirciu-Romania

VIII.19. If $x, y, z > 0$ then:

$$\sqrt{\frac{13x}{6x + 7y}} + \sqrt{\frac{13y}{6y + 7z}} + \sqrt{\frac{13z}{6z + 7x}} \leq 3$$

Proposed by Marin Chirciu– Romania

VIII.20. Let a, b, c be non-negative such that a + b + c = 3. Prove that:

$$|(a - b)(b - c)(c - a)| \leq \frac{3\sqrt{3}}{2}$$

Equality occurs when?

Proposed by Nguyen Ngoc Tu-Vietnam

VIII.21. Rezolvați în $\mathbb{R}$ ecuația: $|x - 2| \cdot |x + 5| = |x + 2| \cdot |x - 5|$

Proposed by Ionică Constantin-Romania

VIII.22. Fie $x, y \in \mathbb{N}^*$ astfel încât $61|(5x + 6y)$. Arătați că $61|(x^2 + y^2)$.

Proposed by Ionică Constantin-Romania

VIII.23. Fie $x, y, z > 0$ numere reale astfel încât $xyz = 3$. Arătați că:

$$\frac{3x}{(3x^2 + y^2)(x^2 + 3z^2)} + \frac{4y}{(4y^2 + z^2)(y^2 + 4x^2)} + \frac{5z}{(5z^2 + x^2)(z^2 + 5y^2)} \leq \frac{1}{4}$$

Proposed by Ionică Constantin-Romania

VIII.24. Rezolvați ecuația:

$$\frac{x-1}{2019}+\frac{x-3}{2017}+\frac{x-5}{2015}+\frac{x-7}{2013}=\frac{x-2019}{1}+\frac{x-2017}{3}+\frac{x-2015}{5}+\frac{x-2013}{7}$$

Proposed by Ionică Constantin-Romania

VIII.25. Rezolvați în mulțimea numerelor întregi ecuația: $x^2 + xy - y = 2019$.

Proposed by Ionică Constantin-Romania

VIII.26. Rezolvați în mulțimea numerelor întregi ecuația:

$$7x^2 + 7xy + 8x + 8y + 2018 = 0$$

Proposed by Ionică Constantin-Romania

VIII.27. Să se demonstreze că $\frac{a^3+b^3}{ab}+\frac{b^3+c^3}{bc}+\frac{c^3+a^3}{ca}\geq 6$, unde *a*, *b* și *c* sunt strict pozitive și $a+b+c=3$. Când are loc egalitatea? ***Proposed by Carmen Chirfot-Romania***

VIII.28. Să se rezolve ecuația $\sqrt{a^2+3}+\sqrt{b^2+3}+\sqrt{c^2+3}=6$, unde *a*, *b* şi *c* sunt pozitive şi $a+b+c=3$. ***Proposed by Carmen Chirfot-Romania***

VIII.29. Să se rezolve ecuația $(a+2)\sqrt{a}+(b+2)\sqrt{b}+(c+2)\sqrt{c}+\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}=12$, unde *a*, *b* și *c* sunt strict pozitive și $a+b+c=3$. ***Proposed by Carmen Chirfot-Romania***

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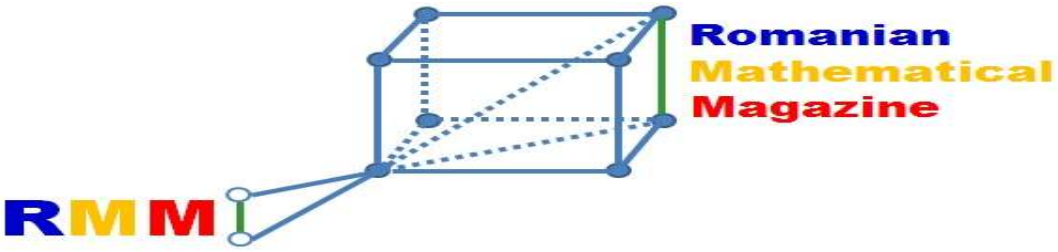
9-CLASS-STANDARD



IX.1. If $m,n\in\mathbb{N}; a,b,c>0$ then:

$$(n+1)\left(m+\frac{1}{(a+b)^{2m+2}}\right)+(m+1)\left(n+\frac{1}{(b+c)^{2m+2}}\right)+\left(m+\frac{1}{(c+a)^{m+1}}\right)\left(n+\frac{1}{(c+a)^{n+1}}\right)\geq$$
$$\geq\frac{9(m+1)(n+1)}{4(ab+bc+ca)}$$

Proposed by D.M. Bătineţu – Giurgiu, Daniel Sitaru-Romania

IX.2. If $m,n\in\mathbb{N}; x,y>0$ then in ΔABC the following relationship holds:

$$(m+x^{m+1})(n+(yab)^{m+1})+(m+(xb)^{m+1})(n+(ybc)^{n+1})+$$
$$+(m+(xca)^{m+1})(n+y^{n+1})\geq(m+1)(n+1)xy\sqrt{3}S$$

Proposed by D.M. Bătineţu – Giurgiu, Daniel Sitaru-Romania

IX.3. Let $x \in \left(0; \frac{\pi}{2}\right)$ and $n \in \mathbb{N}^*$. Prove:

$$\frac{\sin^n x}{\cos^{n+2} x} + \frac{\cos^n x}{\sin^{n+2} x} \geq 4$$

Proposed by Nguyen Van Nho-Vietnam

IX.4. In ΔABC, AA', BB', CC' - cevians, $A' \in (BC), B' \in (CA), C' \in (AB)$

$\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ – inradii, respectively circumradii in $\Delta AA'B, \Delta AA'C$
$\gamma_3, \gamma_4, \Gamma_3, \Gamma_4$ – inradii, respectively circumradii in $\Delta BB'C, \Delta BB'A$
$\gamma_5, \gamma_6, \Gamma_5, \Gamma_6$ – inradii, respectively circumradii in $\Delta CC'B, \Delta CC'B$

Prove that:

$$\Gamma_1 \cdot \Gamma_2 \cdot \Gamma_3 \cdot \Gamma_4 \cdot \Gamma_5 \cdot \Gamma_6 \geq 27 \cdot \gamma_1 \cdot \gamma_2 \cdot \gamma_3 \cdot \gamma_4 \cdot \gamma_5 \cdot \gamma_6$$

Proposed by Marian Ursărescu – Romania

IX.5. Prove that in any ΔABC the following relationship holds:

$$\min\left(\tan^2\frac{A}{2}, \tan^2\frac{B}{2}, \tan^2\frac{C}{2}\right) \leq \frac{R-r}{R+r} \leq \max\left(\tan^2\frac{A}{2}, \tan^2\frac{B}{2}, \tan^2\frac{C}{2}\right)$$

Proposed by Marian Ursărescu – Romania

IX.6. Let $a, b, c, d > 0$, with $a + b + c + d = 4$. Prove that:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + 4\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+d} + \frac{1}{d+a} + \frac{1}{a+c} + \frac{1}{b+d}\right) \geq$$
$$\geq 4 + 6\left(\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1}\right)$$

Proposed by Andrei Ștefan Mihalcea-Romania

IX.7. Let $a, b, c \in [0; +\infty) \wedge \sum_{cyc(a,b,c)} a = 1$. Prove:

$$\sum_{cyc(a,b,c)} a^4 \leq 3 + \sum_{cyc(a,b,c)} (a-1)^3$$

Proposed by Nguyen Van Nho- Vietnam

IX.8. Let $a, b, c \in (0; +\infty) \wedge a^3 + b^3 + c^3 \leq 3$. Prove:

$$\sqrt[3]{1+a^2} + \sqrt[3]{1+b^2} + \sqrt[3]{1+c^2} \leq 3\sqrt[3]{2}$$

Proposed by Nguyen Van Nho- Vietnam

IX.9. Let $a, b, c \in [0; +\infty) \wedge a + b + c = 3$. Prove:

$$a^4 + b^4 + c^4 \leq 81 + abc$$

Proposed by Nguyen Van Nho- Vietnam

IX.10. Let $a, b, c \in [0; +\infty) \wedge \sum_{cyc(a,b,c)} a = 1$. Prove:

$$\sum_{cyc(a,b,c)} \sqrt{a^2+3} \leq 2\sqrt{3} + 2$$

Proposed by Nguyen Van Nho- Vietnam

IX.11. If $a, b \in [0; +\infty)$ and $n \in \mathbb{N}^* \wedge n \geq 2$ then:

$$(ab)^{\frac{n}{2}} \leq \frac{\sum_{k=0}^{n} a^k b^{n-k}}{n+1} \leq \frac{a^n + b^n}{2}$$

Proposed by Nguyen Van Nho-Vietnam

IX.12. Prove that in any ΔABC the following inequality holds:

$$\frac{\tan^2\frac{A}{2}}{\tan^2\frac{B}{2}}+\frac{\tan^2\frac{B}{2}}{\tan^2\frac{C}{2}}+\frac{\tan^2\frac{C}{2}}{\tan^2\frac{A}{2}}\geq 3\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right)$$

Proposed by Marian Ursărescu – Romania

IX.13. Let be $\Delta ABC, I$ the circumcenter and R_A, R_B, R_C the circumradii of $\Delta IBC, \Delta IAC$ and ΔIAB. Prove that: $R_BR_C\sin A + R_AR_C\sin B + R_AR_B\sin C \leq \frac{3\sqrt{3}}{2}R^2$

Proposed by Marian Ursărescu – Romania

IX.14. Prove that in any ΔABC the following inequality holds:

$$\frac{ar_a}{br_b}+\frac{br_b}{cr_c}+\frac{cr_c}{ar_a}\geq 4-2\frac{r}{R}$$

Proposed by Marian Ursărescu – Romania

IX.15. In ΔABC the following relationship holds:

$$\frac{6r^2}{R^2}\leq\frac{h_a}{m_a+w_a}+\frac{h_b}{m_b+w_b}+\frac{h_c}{m_c+w_c}$$

Proposed by Seyran Ibrahimov-Azerbaijan

IX.16. If $x, y > 0$ then in ΔABC the following relationship holds:

$$\frac{h_a}{h_bh_c(xh_b+yh_c)}+\frac{h_b}{h_ch_a(xh_c+yh_a)}+\frac{h_c}{h_bh_b(xh_a+yh_b)}\geq\frac{9}{(x+y)s^2}$$

Proposed by D.M. Bătineţu – Giurgiu; Claudia Năuţi – Romania

IX.17. If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then in ΔABC the following relationship holds:

$$\frac{(\sin x+\tan x)^2bc}{\sin y\sin z}+\frac{(\sin y+\tan y)^2ca}{\sin z\sin x}+\frac{(\sin z+\tan z)^2ab}{\sin x\sin y}\geq 16\sqrt{3}S$$

Proposed by D.M. Bătineţu – Giurgiu; Claudia Năuţi – Romania

IX.18. In ΔABC the following relationship holds: $(a+b)h_c+(b+c)h_a+(c+a)h_b\geq 12S$

Proposed by Seyran Ibrahimov- Azerbaijan

IX.19. In ΔABC the following relationship holds:

$$2\sum m_a\cdot\sin A\geq w_a(\sin B+\sin C)+w_b(\sin A+\sin C)+w_c(\sin A+\sin B)$$

Proposed by Seyran Ibrahimov- Azerbaijan

IX.20. In ΔABC the following relationship holds: $a^4+m_a^4+b^4+m_b^4+c^4+m_c^4\geq 24S^4$

Proposed by Seyran Ibrahimov-Azerbaijan

IX.21. Prove that: $\cos ec\frac{\pi}{14}-4\cos\frac{2\pi}{7}=2$

Proposed by Vasile Mircea Popa-Romania

IX.22. Să se demonstreze egalitatea:

$$\sqrt[3]{2\left(\cos\frac{\pi}{13}+\cos\frac{5\pi}{13}\right)}-\sqrt[3]{2\left(\cos\frac{2\pi}{13}-\cos\frac{3\pi}{13}\right)}-\sqrt[3]{2\left(\cos\frac{4\pi}{13}+\cos\frac{6\pi}{13}\right)}=\sqrt[3]{7-3\sqrt[3]{13}}$$

Proposed by Vasile Mircea Popa-Romania

IX.23. In any ΔABC the following inequality holds:

$$\frac{1}{m_a^2}+\frac{1}{m_b^2}+\frac{1}{m_c^2}>\frac{8}{7R^2}$$

Proposed by Marian Ursărescu – Romania

IX.24. Let a, b, c be nonnegative real numbrs such that $a+b+c=3$. Prove:

$$\frac{1}{2a+3}+\frac{1}{2b+3}+\frac{1}{2c+3}\geq\frac{6}{abc+9}$$

Proposed by Le Khanh Sy-Vietnam

IX.25. Prove that:

$$\cos\frac{2\pi}{13}\cos\frac{3\pi}{13}=\frac{\sqrt{13}}{6}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{5}{2\sqrt{13}}\right)\right)+\frac{1}{12}$$

Proposed by Vasile Mircea Popa – Romania

IX.26. Prove that in any acute-angled triangle the following inequality holds:

$$\min(\sin B\sin C\cos A,\sin A\sin C\cos B,\sin A\sin B\sin C)\leq\frac{1}{2}\left(\left(\frac{r}{R}\right)^2+\frac{r}{R}\right)\leq$$
$$\leq\max(\sin B\sin C\cos A,\sin A\sin C\cos B,\sin A\sin B\cos C)$$

Proposed by Marian Ursărescu – Romania

IX.27. Let ABC be an acute-angled triangle and O circumcenter. We denote with R_A, R_B, R_C and r_A, r_B, r_C the inscribed circumradii in $\Delta OBC, \Delta OAC, \Delta OAB$. Prove that:

$$\frac{1}{R_A^2r_A^2}+\frac{1}{R_B^2r_B^2}+\frac{1}{R_C^2r_C^2}\geq\frac{16\left(\sqrt{3}+2\right)^2r^2}{R^6}$$

Proposed by Marian Ursărescu – Romania

IX.28. Solve for real numbers:

$$\begin{cases}\tan^{-1}y+\tan^{-1}x=z-\tan x\\ \tan^{-1}x+\tan^{-1}z=y-\tan z\\ \tan^{-1}y+\tan^{-1}z=x-\tan y\end{cases}$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

IX.29. If $x, y, z\in\mathbb{R}^+\wedge x^n+y^n+z^n=n, n\in\mathbb{N}$. Show:

$$\sqrt[n]{x^n+ry}+\sqrt[n]{y^r+nx}+\sqrt[n]{z^n+rx}\leq\sqrt[n]{3^{n-1}n\left(1+\sqrt[n]{n\cdot 3^{n-1}}\right)}$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

IX.30. Let ΔABC be acute-angled and A', B', C' the points in which the heights of the triangle intersects the circumcenter. If $S_{ABC}=S_{A'B'C'}$, then ΔABC is equilateral.

Proposed by Marian Ursărescu – Romania

IX.31. Prove that in any $\Delta\ ABC$ the following inequality holds:

$$\frac{(a+b)(a+c)}{bc}+\frac{(b+a)(b+c)}{ac}+\frac{(c+a)(c+b)}{ab}\leq 4\left(\frac{R}{r}\right)+2\left(\frac{r}{R}\right)+3$$

Proposed by Marian Ursărescu – Romania

IX.32. In any $\Delta\ ABC$ the following inequality holds:

$$\frac{m_a m_b}{r_c} + \frac{m_a m_c}{r_b} + \frac{m_b m_c}{r_a} \leq \frac{9}{4} \cdot \frac{R^2}{r}$$

Proposed by Marian Ursărescu – Romania

IX.33. Prove that in any acute-angled ΔABC the following inequality holds:

$$\max\left(a\cot\frac{A}{2}, b\cot\frac{B}{2}, c\cot\frac{C}{2}\right) \geq 7r - \frac{R}{2}$$

Proposed by Marian Ursărescu – Romania

IX.34. In ΔABC the following relationship holds:

$$2(m_a a + m_b b + m_c c) \leq (m_a + m_b + m_c)(2s - \max(a, b, c))$$

Proposed by Seyran Ibrahimov- Azerbaijan

IX.35. Let a, b, c be positive real numbers such that: $a + b + c = 3$. Prove that:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \geq 4$$

Proposed by Hoang Le Nhat Tung –Vietnam

IX.36. Let x, y, z be positive real numbers such that: $x^2 + y^2 + z^2 = 3\sqrt{xyz}$. Find the maximum value of:

$$P = \sqrt[4]{\frac{xy(x^4 + y^4)}{x^5 - x^3 + y^2 + 1}} + \sqrt[4]{\frac{yz(y^4 + z^4)}{y^5 - y^3 + z^2 + 1}} + \sqrt[4]{\frac{zx(z^4 + x^4)}{z^5 - z^3 + x^2 + 1}}$$

Proposed by Hoang Le Nhat Tung –Vietnam

IX.37. Let G be the centroid in ΔABC and $M \in BC; N \in CA; P \in AB$. Prove that in ΔABC the following relationship holds:

$$\frac{(GM + GP)a}{GM \cdot h_a} + \frac{(GP + GM)b}{GM \cdot h_b} + \frac{(GA + GB)c}{GP \cdot h_c} \geq 4\sqrt{3}$$

Proposed by D.M. Bătineţu – Giurgiu, Dan Nănuți-Romania

IX.38. Let ΔMNP be the circumcevian triangle of incentre of ΔABC and $x, y > 0$. Prove that:

$$\frac{AM}{h_b h_c(xh_b + yh_c)} + \frac{BN}{h_c h_a(xh_c + yh_a)} + \frac{CP}{h_a h_b(xh_a + yh_b)} \geq \frac{9}{(x + y)s^2}$$

Proposed by D.M. Bătineţu – Giurgiu, Claudia Nănuți-Romania

IX.39. If $m, n \in \mathbb{N}; M \in BC; N \in CA; P \in AB$ then in ΔABC the following relationship holds:

$$\left(m + \left(\frac{BN + CP}{h_a}\right)^{m+1}\right) \cdot \left(n + \left(\frac{a}{AM}\right)^{n+1}\right) + \left(m + \left(\frac{CP + AM}{h_b}\right)^{n+1}\right) \cdot$$
$$\cdot \left(n + \left(\frac{b}{BM}\right)^{m+1}\right) + \left(m + \left(\frac{AM + BN}{CP}\right)^{m+1}\right) \cdot \left(n + \left(\frac{c}{h_c}\right)^{n+1}\right) \geq$$
$$\geq 4\sqrt{3}(m + 1)(n + 1)$$

Proposed by D.M. Bătineţu – Giurgiu, Daniel Sitaru-Romania

IX.40. If $a, b, c, d \in \mathbb{N} - \{0\}, a > b > c > d$ then:

$$bd(2^a - 1)(2^c - 1) > ac(2^b - 1)(2^d - 1)$$

Proposed by Daniel Sitaru – Romania

IX.41. Let $a, b, c > 0$. Prove that:

$$\frac{b + c}{a} + \frac{a + b}{a + c} + \frac{a + c}{a + b} \geq \frac{2a + 2b + 4c}{3a + b} + \frac{2a + 4b + 2c}{3a + c}$$

Proposed by Andrei Ștefan Mihalcea-Romania

IX.42. If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then in ΔABC the following relationshp holds:

$$\frac{\sin x + \tan x}{y + z} a^2 + \frac{\sin y + \tan y}{z + x} b^2 + \frac{\sin z + \tan z}{x + y} c^2 > 4\sqrt{3}S$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

IX.43. If $m, n \in \mathbb{N}$ then in ΔABC the following relationship holds:

$$(m + (ab)^{m+1})(n + 1) + (m + b^{m+1})(n + c^{n+1}) + (m + 1)(n + (ca)^{n+1}) \geq$$
$$\geq 4\sqrt{3}(m + 1)(n + 1)S$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

IX.44. Let $a, b, c > 0$. Show that:

$$\frac{a + c}{b} + \frac{b + c}{a} \geq 2(a + b + 2c)\left(\frac{2}{a + 3b} + \frac{2}{3a + b} - \frac{1}{a + b}\right)$$

Proposed by Andrei Ștefan Mihalcea-Romania

IX.45. In ΔABC the following relationship holds:

$$\sqrt{\frac{r_a}{w_a}} + \sqrt{\frac{r_b}{w_b}} + \sqrt{\frac{r_c}{w_c}} \geq \sqrt{\frac{2R}{r}} + \sqrt{\frac{2r}{R}}$$

Proposed by Bogdan Fustei-Romania

IX.46. In ΔABC the following relationship holds:

$$\frac{R_b R_c + R_a R_c + R_a R_b}{R\sqrt{R}} \leq \sqrt{2(h_a + h_b + h_c)}$$

Proposed by Bogdan Fustei-Romania

IX.47. In ΔABC the following relationship holds:

$$\frac{\sqrt{3}}{4} \sum \frac{m_a}{\sin\frac{B}{2}\sin\frac{C}{2}} \geq \frac{(h_a + h_b + h_c)(R_a + R_b + R_c)}{(m_a + m_b + m_c)} \sqrt{\sum \frac{m_a}{h_a}}$$

Proposed by Bogdan Fustei-Romania

IX.48. In ΔABC the following relationship holds:

$\frac{R_b R_c}{R_a} + \frac{R_a R_c}{R_b} + \frac{R_a R_b}{R_c} \geq 3R$ where R_a, R_b, R_c circumradii of $\Delta BIC, \Delta AIC, \Delta AIB$

Proposed by Bogdan Fustei-Romania

IX.49. In ΔABC the following relationship holds:

$$m_a \sqrt{\frac{R_a}{AI}} + m_b \sqrt{\frac{R_b}{CI}} + m_c \sqrt{\frac{R_c}{CI}} \geq \frac{(h_a + h_b + h_c)(R_a + R_b + R_c)}{m_a + m_b + m_c}$$

where R_a, R_b, R_c circumradii $\Delta BIC, \Delta AIC, \Delta AIB$. ***Proposed by Bogdan Fustei-Romania***

IX.50. In ΔABC the following relationship holds:

$\frac{AI}{R_a} + \frac{BI}{R_b} + \frac{CI}{R_c} \geq 3$ where R_a, R_b, R_c circumradii of $\Delta BIC, \Delta AIC, \Delta AIB$

Proposed by Bogdan Fustei-Romania

IX.51. In ΔABC the following relationship holds:

$\sqrt{\frac{R_a}{AI}} + \sqrt{\frac{R_b}{BI}} + \sqrt{\frac{R_c}{CI}} \geq 3$ where R_a, R_b, R_c circumradii of $\Delta BIC, \Delta AIC, \Delta AIB$

Proposed by Bogdan Fustei-Romania

IX.52. In ΔABC the following relationship holds:

$$\frac{AI + BI + CI}{2R} \leq \sqrt{\frac{m_a}{R_a}} + \sqrt{\frac{m_b}{R_b}} + \sqrt{\frac{m_c}{R_c}}$$

Proposed by Bogdan Fustei-Romania

IX.53. In ΔABC the following relationship holds:

$$\frac{a + b + c}{2R} > \sum \frac{m_a}{\sqrt{(R_b - R)(R_c - R)}}$$

Proposed by Bogdan Fustei-Romania

IX.54. In ΔABC the following relationship holds:

$$3(2R^2 + r^2) \geq \frac{h_a^2 w_a}{m_a} + \frac{h_b^2 w_b}{m_b} + \frac{h_c^2 w_c}{m_c}$$

Proposed by Seyran Ibrahimov- Azerbaijan

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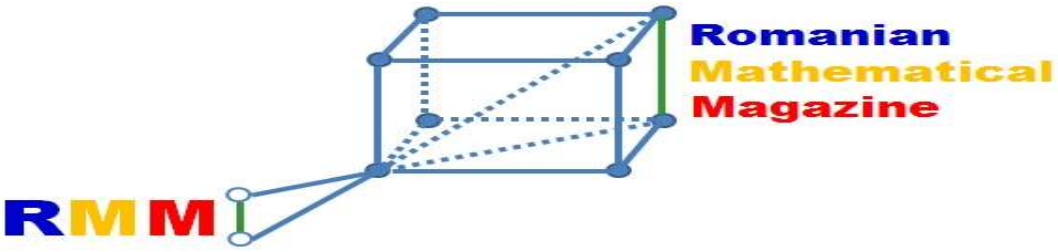
10-CLASS-STANDARD



X.1. 1. If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{(y + z)a}{xh_a} + \frac{(z + x)b}{yh_b} + \frac{(x + y)c}{zh_c} \geq 4\sqrt{3}$$

Proposed by D.M. Bătineţu – Giurgiu, Claudia Nănuți-Romania

X.2. If $x, y > 0$; G – centroid; $d_a = d(G, BC)$; $d_b = d(G, CA)$; $d_c = d(G, AB)$ then in ΔABC the following relationship holds:

$$\frac{AM}{d_b d_c(xd_b + yd_c)} + \frac{BN}{d_c d_a(xd_c + yd_a)} + \frac{CP}{d_a d_b(xd_a + yd_b)} \geq \frac{243}{(x + y)s^2}$$

Proposed by D.M. Bătineţu – Giurgiu, Daniel Sitaru-Romania

X.3. In ΔABC, G – centroid; $d_a = d(A, BC)$; $d_b = d(B, AC)$; $d_c = d(C, AB)$. Prove that:

$$\frac{d_a}{d_b d_c(xd_b + yd_c)} + \frac{d_b}{d_c d_a(xd_c + yd_a)} + \frac{d_c}{d_a d_b(xd_a + yd_b)} \geq \frac{81}{(x + y)s^2}$$

Proposed by D.M. Bătineţu – Giurgiu, Dan Nănuți-Romania

X.4. If $m, n, r, s \in \mathbb{N}$; $a, b, c > 0$ then:

$$\left((m + 1)(n + ab)^{n+1}\right) + (m + b^{m+1})(n + c^{n+1}) + \left(m + (ca)^{m+1}(n + 1)\right) \cdot$$

$$\cdot\left((r+a^{2r+2})(s+a^{s+1})+(r+b^{r+1})(s+b^{2s+2})+(s+1)(s+c^{3s+3})\right)\geq$$
$$\geq(m+1)(n+1)(r+1)(s+1)abc(a+b+c)^2$$

Proposed by D.M. Bătineţu – Giurgiu, Daniel Sitaru-Romania

X.5. In ΔABC; G – centroid; $d_a=d(G,BC);d_b=d(G,AC);d_c=d(G,AB),M\in BC$; $N\in CA;P\in AB$. Prove that:

$$\frac{GM}{d_bd_c(dx_b+yd_c)}+\frac{GN}{d_cd_a(xd_c+yd_a)}+\frac{GP}{d_ad_b(xd_a+yd_b)}\geq\frac{81}{(x+y)s^2}$$

Proposed by D.M. Bătineţu – Giurgiu, Dan Nănuți-Romania

X.6. If $n\geq 0;a,b,c>0;m,p\in\mathbb{N}$ then:

$$((n+a^{n+1})\left(p+\left(\frac{1}{2b+3\sqrt[3]{ac^2}}\right)^{p+1}\right)^{m+1}+\left((n+b^{p+1})\left(n+\left(\frac{1}{2c+3\sqrt[3]{ba^2}}\right)^{n+1}\right)\right)^{m+1}+$$
$$+\left((n+c^{n+1})\left(p+\frac{1}{2a+3\sqrt[3]{cb^2}}\right)^{p+1}\right)^{m+1}\geq\frac{3(m+1)^{m+1}(p+1)^{m+1}}{5^{m+1}}$$

Proposed by D.M. Bătineţu – Giurgiu, Daniel Sitaru-Romania

X.7. In ΔABC the following relationship holds:

$$(m_a^4+m_b^4+m_c^4)(w_a^2+w_b^2+w_c^2)\geq 27\sqrt{3}S^3;S=[ABC]$$

Proposed by D.M. Bătineţu – Giurgiu, Daniel Sitaru-Romania

X.8. If $x,y,z\in\left(0,\frac{\pi}{2}\right)$ then in ΔABC the following relationship holds:

$$\frac{\sin x+\tan x}{\sin y+\sin z}\cdot a^2+\frac{\sin y+\tan y}{\sin z+\sin x}\cdot b^2+\frac{\sin z+\tan z}{\sin x+\sin y}>4\sqrt{3}S$$

Proposed by D.M. Bătineţu – Giurgiu, Daniel Sitaru-Romania

X.9. If $x,y>0;\Delta MNP$ – circumcevian triangle of incentre in ΔABC; G – centroid then in ΔABC the following relationship holds:

$$\frac{GM}{h_bh_c(xh_b+yh_c)}+\frac{GN}{h_ch_a(xh_c+yh_a)}+\frac{GP}{h_ah_b(xh_a+yh_b)}\geq\frac{3}{(x+y)s^2}$$

Proposed by D.M. Bătineţu – Giurgiu, Dan Nănuți-Romania

X.10. In ΔABC the following relationship holds:

$$\left(\frac{(w_a+w_c)^a}{w_ah_a}\right)^{m+1}+\left(\frac{(w_c+w_a)b}{w_bh_b}\right)^{m+1}+\left(\frac{(w_a+w_b)c}{w_ch_c}\right)^{m+1}\geq 4(m+1)\sqrt{3}-3m;m\in\mathbb{N}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.11. If $x,y,z\in\left(0,\frac{\pi}{2}\right)$ then in ΔABC the following relationship holds:

$$\frac{(\sin x+\tan x)^2a^2}{\sin y\sin z}+\frac{(\sin y+\tan y)^2b^2}{\sin z\sin x}+\frac{(\sin z+\tan z)^2c^2}{\sin x\sin y}>16\sqrt{3}S$$

Proposed by D.M. Bătineţu – Giurgiu, Claudia Nănuți-Romania

X.12. In ΔABC the following relationship holds:

$$\frac{\sqrt{R}}{r}\cdot\sum_{cyc}h_a\sqrt{h_a-2r}\geq\frac{\sqrt{2}}{3}\left(\sum_{cyc}m_a+\sum_{cyc}w_a\right)$$

Proposed by Bogdan Fustei – Romania

X.13 In ΔABC the following relationship holds:

$$\frac{6\sqrt{3}r}{\frac{a^2}{bR+cr}+\frac{b^2}{cR+ar}+\frac{c^2}{aR+br}}\leq R+r$$

Proposed by Seyran Ibrahimov- Azerbaijan

X.14. Solve for real numbers:

$$\begin{cases}\tan^{-1}x+\tan^{-1}y=z-\tan x\\ \tan^{-1}y+\tan^{-1}z=x-\tan y\\ \tan^{-1}z+\tan^{-1}x=y-\tan z\end{cases}$$

Proposed by Seyran Ibrahimov- Azerbaijan

X.15. Solve for real numbers: $x\neq y\ x\wedge y\in\mathbb{R},\begin{cases}x-y=\sqrt{\frac{y^6-3}{2}}\\ x^y=y^x\end{cases}$

Proposed by Urfan Aliyev-Azerbaijan

X.16. Prove that in any $\Delta\ ABC$ the following inequality holds:

$$R-2r\geq\frac{r((r_a-r_b)^2+(r_b-r_c)^2+(r_c-r_a)^2)}{3p^2}$$

Proposed by Adil Abdullayev- Azerbaijan

X.17. Prove that in any $\Delta\ ABC$ the following inequality holds:

$$\sum m_a\sqrt{\frac{r_a}{h_a}}\geq\sqrt{\frac{(r_a+r_b+r_c+AI+BI+CI)(h_a+h_b+h_c)\sum\frac{h_bh_c}{h_a}}{m_a+m_b+m_c+h_a+h_b-3r}}$$

Proposed by Bogdan Fustei-Romania

X.18. Let $a\in\mathbb{R}$. Solve in $\mathbb{R}$:

$$\sqrt{1+ax}+\sqrt{1+2ax}+\sqrt{1+3ax}+\sqrt{1+4ax}+\sqrt{1+5ax}=5$$

Proposed by Nguyen Van Nho- Vietnam

X.19. Let $a,b,c\in[0;+\infty)$. Prove: $(1+a^2)(1+b^2)(1+c^2)\leq e^{a+b+c}$

Proposed by Nguyen Van Nho- Vietnam

X.20. Let $a,b,c\in[1;2]\wedge a^2+b^2+c^2=6$. Prove: $a^4+b^4+c^4\leq 18$

Proposed by Nguyen Van Nho- Vietnam

X.21. Let $n\in\mathbb{N}\wedge n\geq 2$. Solve in $\mathbb{R}$:

$$\frac{\sum_{k=1}^{n}\log_2(2-kx)}{\sum_{k=1}^{n}\sqrt{1+kx}}=1$$

Proposed by Nguyen Van Nho- Vietnam

X.22. Let $a,b,c\in(0;+\infty)\wedge\sum_{cyc(a,b,c)}a^7=3$. Prove:

$$\sum_{cyc(a,b,c)} a \leq 3$$

Proposed by Nguyen Van Nho- Vietnam

X.23. Prove that in any ΔABC the following inequality holds:

$$\sum l_a\sqrt{h_a} \leq p\sum\sqrt{h_a-2r} \leq \sum m_a\sqrt{h_a}$$

Proposed by Bogdan Fustei-Romania

X.24. Prove that in any ΔABC the following inequality holds:

$$\frac{m_a m_b m_c}{S^2} \geq \sqrt{3\left(\frac{1}{ha^2}+\frac{1}{hb^2}+\frac{1}{hc^2}\right)}$$

Proposed by Bogdan Fustei-Romania

X.25. Prove that in any ΔABC the following inequality holds:

$$\frac{l_a+l_b+l_c}{2r} \geq \sum\sqrt{\frac{m_a}{r_a}}+\frac{1}{2}\sum\frac{l_a}{h_a}$$

Proposed by Bogdan Fustei-Romania

X.26. Prove that in any ΔABC the following inequality holds:

$$\sum\frac{m_a}{h_a} \geq \left(1+\frac{R}{r}\right)\sqrt{\frac{2r}{R}}$$

Proposed by Bogdan Fustei-Romania

X.27. In all ΔABC prove:

$$\frac{(4R+r)^3-12RS^2}{S^3} > 3\sqrt{3}-\frac{3\pi}{2}$$

Proposed by Rovsen Pirguliyev-Azerbaijan

X.28. Prove that in any ΔABC the following inequality holds:

$$\sum\sqrt{\frac{p^2-r_b r_c}{h_b h_c}} \geq 3\sqrt{2}\frac{r_a+r_b+r_c+AI+BI+CI}{m_a+w_b+m_c+h_a+h_b+h_c-3r}$$

Proposed by Bogdan Fustei-Romania

X.29. Let $ABCD$ be any tetrahedron , r_A, r_B, r_C, r_D the exradii of the spheres and r = inradii of the sphere. Prove that:

a) $\frac{1}{\sqrt{r_a}}+\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}}+\frac{1}{\sqrt{r_D}} \leq \frac{4}{\sqrt{2r}}$

b) $\sqrt{r_a}+\sqrt{r_b}+\sqrt{r_c}+\sqrt{r_D} \geq \frac{4}{\sqrt{2r}}$

Proposed by Marian Ursărescu – Romania

X.30. Prove that in any ΔABC the following inequality holds:

$$\frac{a^3}{a\sin B+b\sin C}+\frac{b^3}{b\sin C+c\sin A}+\frac{c^3}{c\sin A+a\sin B} \geq 4S$$

Proposed by Marian Ursărescu – Romania

X.31. Prove that in any ΔABC the following inequality holds:

$$\frac{r_a + r_b + r_c + AI + BI + CI}{m_a + m_b + m_c + h_a + h_b + h_c - 3r} \leq \frac{m_a + m_b + m_c}{2\sum h_a - \sum \frac{h_b h_c}{h_a}}$$

Proposed by Bogdan Fustei-Romania

X.32. Prove that in any ΔABC the following inequality holds:

$$\sum \frac{m_a}{r_a} \geq \left(2\frac{R}{r} - 1\right)\sqrt{\frac{2r}{R}}$$

Proposed by Bogdan Fustei-Romania

X.33. Prove that in any ΔABC the following inequality holds:

$$\left(\sum m_a + \sum h_a - 3r\right)\sqrt{r_a r_b r_c} \geq \left(\sum r_a + \sum AI\right)\sqrt{h_a h_b h_c}$$

Proposed by Bogdan Fustei-Romania

X.34. Let be $\Delta ABC, I_a, I_b, I_c$ excentres. Prove that:

$$\frac{9}{2}r \leq \frac{S_{I_aBC}}{a} + \frac{S_{I_bAC}}{b} + \frac{S_{I_cAB}}{c} \leq \frac{9}{4}R$$

Proposed by Marian Ursărescu – Romania

X.35. Prove that in any acute-angled triangle the following inequality holds:

$$\min\{(b+c)\cot A, (a+c)\cot B, (a+b)\cot C\} \leq 2R \leq \max\{(b+c)\cot A, (a+c)\cot B, (a+b)\cot C\}$$

Proposed by Marian Ursărescu – Romania

X.36. Prove that in any ΔABC the following inequality holds:

$$\min\left(\left(\frac{a}{b+c}\right)^2, \left(\frac{b}{a+c}\right)^2, \left(\frac{c}{a+b}\right)^2\right) \leq \frac{R-r}{R+2r} \leq \max\left(\left(\frac{a}{b+c}\right)^2, \left(\frac{b}{a+c}\right)^2, \left(\frac{c}{a+b}\right)^2\right)$$

Proposed by Marian Ursărescu – Romania

X.37. Let $ABCD$ be a tetrahedron, G the centroid and M any point in space. Prove that:

$$\sum MA^2 \geq \frac{2}{3}\sum (GA \cdot GB)$$

Proposed by Marian Ursărescu – Romania

X.38. Prove that in any ΔABC the following inequality holds:

$$\sqrt{\frac{R}{2r}} \geq \frac{\frac{h_b h_c}{h_a} + \frac{h_a h_c}{h_b} + \frac{h_a h_b}{h_c}}{h_a + h_b + h_c}$$

Proposed by Bogdan Fustei-Romania

X.39. Prove that in any ΔABC the following inequality holds:

$$\frac{al_a^2 + bl_b^2 + cl_c^2}{m_a + m_b + m_c} \leq \sqrt{2Rr_a r_b r_c}$$

Proposed by Bogdan Fustei-Romania

X.40. Prove that in any ΔABC the following inequality holds:

$$\sqrt{\frac{R}{2r}} \cdot \sum \sqrt{m_a r_a} \geq \sum m_a \sqrt{\frac{r_a}{h_a}}$$

Proposed by Bogdan Fustei-Romania

X.41. Let $A_1A_2A_3A_4$ be a tetrahedron and I_1, I_2, I_3, I_4 the excenters of the sphere of the tetrahedron. (centrele sferelor exinscrise tetraedrului). Prove that:

$$V_{I_1A_2A_3A_n} + VI_2A_1A_3A_4 + VI_3A_1A_2A_4 + VI_4A_1A_2A_3 \geq 2V_{ABCD}$$

Proposed by Marian Ursărescu – Romania

X.42. Prove that in any acute-angled triangle the following relationships holds:

a) $S \geq 18r^2 \min\left\{\frac{a}{b+c}\cot A, \frac{b}{a+c}\cot B, \frac{c}{a+c}\cot C\right\}$

b) $S \leq \frac{9}{2}R^2 \max\left\{\frac{a}{b+c}\cot A, \frac{b}{a+c}\cot B, \frac{c}{a+c}\cot C\right\}$

Proposed by Marian Ursărescu – Romania

X.43. Prove that in any acute-angled ΔABC the following relationship holds:

$$\sqrt{m_a h_b} + \sqrt{m_b h_c} + \sqrt{m_c h_a} \leq 3(R + r)$$

Proposed by Marian Ursărescu – Romania

X.44. Let be ΔABC, I – incentre and R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta AIC, \Delta AIB$. Prove that:

$$2R^2 - 2Rr - r^2 \leq \left(\frac{R_a^2}{2R}\right)^2 + \left(\frac{R_b^2}{2R}\right)^2 + \left(\frac{R_c^2}{2R}\right)^2 \leq 4R^2 - 8Rr + 3r^2$$

Proposed by Marian Ursărescu – Romania

X.45. Prove that in any ΔABC the following inequality holds:

$$\sum \frac{\sin A + \sin B}{\sin^2 A + \sin^2 B} \leq \frac{2(R + r)^2}{[ABC]}, [ABC] = Area\ \Delta ABC$$

Proposed by Marian Ursărescu – Romania

X.46. ΔABC (sides a, b, c,),$\Delta A'B'C'$ (sides a', b', c') . Prove that:

$$\frac{R}{r'} \geq \frac{2}{3}\left(\frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'}\right)$$

Proposed by Bogdan Fustei-Romania

X.47. Does there exists non-constant functions $f: (1; +\infty) \to \mathbb{R}$ such that:

$$f(x) = f(x^2) + f(x^3) + \cdots + f(x^{2018})\ ?$$

Proposed by Rovsen Pirguliyev-Azerbaijan

X.48. In ΔABC the following relationship holds:

$$\sum (h_a + r_a)\sqrt{\frac{h_a}{r_a}} \geq \sqrt{\frac{r}{2R}}\left(\frac{4}{3}\sum m_a + \frac{8}{3}\sum l_a\right)$$

Proposed by Bogdan Fustei-Romania

X.49. In ΔABC the following relationship holds:

$$\frac{l_b l_c}{r_b r_c} + \frac{l_a l_c}{r_a r_c} + \frac{l_a l_b}{r_a r_b} \leq \frac{r_a}{l_a} + \frac{r_b}{l_b} + \frac{r_c}{l_c}$$

Proposed by Bogdan Fustei-Romania

X.50. In ΔABC the following relationship holds:

$$\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \geq \frac{R}{2r} \cdot \sum \frac{h_b + h_c}{r_a} \sin\frac{A}{2}$$

Proposed by Bogdan Fustei-Romania

X.51. In ΔABC the following relationship holds:

$$\frac{h_a h_b h_c}{l_a l_b l_c} \geq \frac{(h_a - 2r)(h_b - 2r)(h_c - 2r)}{r^3}$$

Proposed by Bogdan Fustei-Romania

X.52. In ΔABC the following relationship holds:

$$\frac{m_a}{l_a} + \frac{m_b}{l_b} + \frac{m_c}{l_c} \geq \sqrt{\frac{r}{2R}}\left(\frac{h_b + h_c}{h_a} + \frac{h_a + h_c}{h_b} + \frac{h_a + h_b}{h_c}\right)$$

Proposed by Bogdan Fustei-Romania

X.53. In ΔABC the following relationship holds:

$$\sqrt{2R}\left(4 + \frac{r}{R}\right) \geq \sum \frac{h_a}{\sqrt{h_a - 2r}}$$

Proposed by Bogdan Fustei-Romania

X.54. In ΔABC the following relationship holds:

$$\frac{1}{2}\left(\sqrt{\frac{r_a}{m_a}} + \sqrt{\frac{r_b}{m_b}} + \sqrt{\frac{r_c}{m_c}}\right) \geq \sum \sin\frac{A}{2}$$

Proposed by Bogdan Fustei-Romania

X.55. In ΔABC the following relationship holds:

$$\frac{m_a}{l_a} + \frac{m_b}{l_b} + \frac{m_c}{l_c} \geq \frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}} + \frac{h_b}{l_b}\sqrt{\frac{h_b}{r_b}} + \frac{h_c}{l_c}\sqrt{\frac{h_c}{r_c}}$$

Proposed by Bogdan Fustei-Romania

X.56. In ΔABC the following relationship holds:

$$\sqrt{\frac{R}{2r}}\left(2 + \frac{R}{r}\right) > \frac{m_a}{l_a}\sqrt{\frac{r_a}{h_a}} + \frac{m_b}{l_b}\sqrt{\frac{r_b}{h_b}} + \frac{m_c}{l_c}\sqrt{\frac{r_c}{h_c}}$$

Proposed by Bogdan Fustei-Romania

X.57. In ΔABC the following relationship holds: $\frac{R_b R_c}{a} + \frac{R_a R_c}{b} + \frac{R_a R_b}{c} \geq R\sqrt{3}$, R_a, R_b, R_c – circumradii (razele cercurilor circumscrise), $\Delta BIC, \Delta AIC, \Delta AIB$.

Proposed by Bogdan Fustei-Romania

X.58. Prove that in any ABC triangle the following inequality holds:

$$R\sum\sqrt{\frac{h_a}{s-a}} \geq \frac{2}{3\sqrt[4]{27}}\left(\sum m_a + 2\sum l_a\right)$$

Proposed by Bogdan Fustei – Romania

X.59. Fie $n \geq 3$ și $z_1, z_2, \ldots, z_n \in \mathbb{C}$ astfel încât

$|S| = |S-z_1| + |S-z_2| + |S-z_3| + \ldots + |S-z_n|$, unde $S = z_1 + z_2 + \ldots + z_n$.

Proposed by Butaru Zizi, Bețiu Anicuța-Romania

X.60. Prove that in any ABC triangle the following inequality holds:

$$\frac{m_a^2}{a} + \frac{m_b^2}{b} + \frac{m_c^2}{c} \geq 6S\left(\frac{r}{R}\right)^2$$

Proposed by Seyran Ibrahimov –Azerbaijan

X.61. Let a, b, c, d be strictly positive real numbers such that $a + b + c + d = 1$. Prove that:

$$\frac{ab}{1+c+d} + \frac{ac}{1+b+d} + \frac{ad}{1+b+c} + \frac{bc}{1+a+d} + \frac{bd}{1+a+c} + \frac{cd}{1+a+b} \leq \frac{1}{4}$$

Proposed by Vasile Mircea Popa-Romania

X.62. Prove that the following is a perfect square number.

$$\binom{n}{3}\binom{n}{4}\binom{n}{5}\cdots\binom{n}{99}\binom{n}{100}$$

Find the smallest value of n such that above number is perfect square number.

Proposed by Naren Bhandari-Nepal

X.63. In ΔABC the following relationship holds:

$$3p \leq \sum r_a^2\left(\frac{1}{b}+\frac{1}{c}\right) \leq p\left(\frac{4R}{r}-5\right).$$

Proposed by Marin Chirciu-Romania

X.64. In ΔABC the following relationship holds:

$$3p\left(\frac{2r}{R}\right)^2 \leq \sum h_a^2\left(\frac{1}{b}+\frac{1}{c}\right) \leq 3p\,.$$

Proposed by Marin Chirciu-Romania

X.65. In ΔABC the following relationship holds:

$$\frac{18r}{R} \leq \sum h_a^2\left(\frac{1}{b}+\frac{1}{c}\right)^2 \leq 9\,.$$

Proposed by Marin Chirciu-Romania

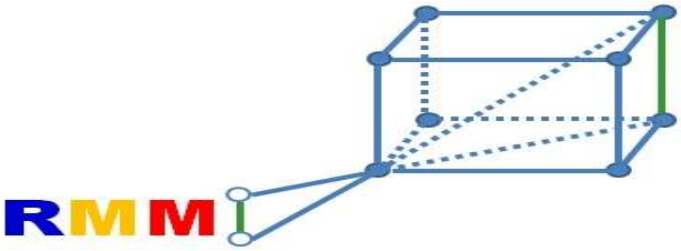
X.66. In ΔABC the following relationship holds:

$$\sum r_a^2\left(\frac{1}{b}+\frac{1}{c}\right)^2 \geq \frac{9R}{2r}.$$

Proposed by Marin Chirciu-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

11-CLASS-STANDARD



Romanian Mathematical Magazine

XI.1. Let be $f:\left(0,\frac{\pi}{2}\right)\to(0,\infty); f(t)=\tan t+\sin t$ and $x,y,z\in\left(0,\frac{\pi}{2}\right)$. Prove that in ΔABC the following relationship holds:

$$\frac{f(x)\cdot f(y)}{\sin^2 z}\cdot ab+\frac{f(y)\cdot f(z)}{\sin^2 x}\cdot bc+\frac{f(z)\cdot f(x)}{\sin^2 y}\cdot ca\geq 16\sqrt{3}S$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți-Romania

XI.2. If $a,b,c>0; b<c; n\in\mathbb{N}; n\geq 2; x_k,y_k\in(0,\infty); b\leq\frac{x_k}{y_k}\leq a+c; k\in\overline{1,n}; t_k>0;$ $k\in\overline{1,n}$ then:

$$\sum_{k=1}^{n} t_k x_k^2+b(a+c)\sum_{k=1}^{n} t_k y_k^2\leq(a+b+c)\sum_{k=1}^{n} t_k x_k y_k\,; n\in\mathbb{N}^*$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

XI.3. If $x,y,z>0; f:(0,\infty)\to(0,\infty); f(t)=\frac{t^8+2t^2}{\sqrt{t^6+1}}$ then:

$$\frac{f(x)}{yz}a^2+\frac{f(y)}{zx}b^2+\frac{f(z)}{xy}c^2>8\sqrt{3}S$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

XI.4 Let be $f:(0,+\infty)\to\mathbb{R}, f(x)=x^p+\ln x\,, p\in\mathbb{N}^*$ and the sequence $a_n=f^{-1}\left(1+\frac{1}{n}\right), n\in\mathbb{N}^*$. Find: $\lim_{n\to\infty} a_n^n$.

Proposed by Marian Ursărescu – Romania

XI.5. Let be $A,B\in M_n(\mathbb{R})$ such that $\sqrt{5-\sqrt{5}}(A^2+B^2)=\sqrt{3+\sqrt{5}}(AB-BA)$ and $AB-BA$ is invertible. Prove that: $n\vdots 5$. ***Proposed by Marian Ursărescu – Romania***

XI.6. Let be $x_0>0$ and $x_{n+1}=\sqrt[p]{1+px_n}-1, p\in\mathbb{N}, p\geq 2$. Find: $\lim_{n\to\infty} nx_n$

Proposed by Marian Ursărescu – Romania

XI.7. Find:

$$\lim_{n\to\infty}\frac{1}{n^4}\sum_{k=1}^{n}k(k-1)\sqrt[3]{2n^3-k(k-1)(2k-1)}$$

Proposed by Marian Ursărescu – Romania

XI.8. Find:

$$\lim_{n\to\infty}n^{2p+1}\left(\frac{\arcsin\frac{1}{n^p}}{n^p}-\frac{\arcsin\frac{1}{(n+1)^p}}{(n+1)^p}\right),p\in\mathbb{N}^*$$

Proposed by Marian Ursărescu – Romania

XI.9. Let be $A\in M_{3,2}(\mathbb{C})$ and $B\in M_{2,3}(\mathbb{C})$ such that:

$$AB=\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$$

Find: $(BA)+(BA)^{-1}$.

Proposed by Marian Ursărescu – Romania

XI.10.

$$\Omega_n(a)=\sum_{k=0}^{n}(k^2-a^2+1)(a+k)!,a,n\in\mathbb{N}$$

Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\Omega_n(a)-(a+1)!}$$

Proposed by Daniel Sitaru -Romania

XI.11. Let be $x_n=1+\frac{1}{3^4}+\frac{1}{5^4}+\cdots+\frac{1}{(2n-1)^4}$. Find:

$$\lim_{n\to\infty}n^3\left(x_n^{\frac{\pi^4}{90}}-\left(\frac{\pi^4}{90}\right)^{x_4}\right)$$

Proposed by Marian Ursărescu – Romania

XI.12. Let be $A,B\in M_2(\mathbb{R})$ such that: $tr(A^2B^2)=tr\,A^2\cdot tr\,B^2$. Prove that:

$$\det(A^2+\alpha^2B^2)+\det\left(A^2+\frac{1}{\alpha^2}B^2\right)\geq(\det A+\det B)^2,\forall\alpha\in\mathbb{R}^*$$

Proposed by Marian Ursărescu – Romania

XI.13. Let be $A\in M_4(\mathbb{R})$ invertible, such that $\det(A^2+I_4)=0$. Prove that: $tr\,A^{-1}=\frac{tr\,A}{\det A}$

Proposed by Marian Ursărescu – Romania

XI.14. Find $x,y,z\geq 0$ such that:

$$e^x=\pi^y+z^2+2z$$
$$e^y=\pi^z+x^2+2x$$

$$e^z = \pi^x + y^2 + 2y$$

Proposed by Seyran Ibrahimov- Azerbaijan

XI.15. If $0 \le x, y, z < 1$ solve the system:

$$\begin{cases} \arcsin x + \arcsin y = \arctan z \\ \arcsin y + \arcsin z = \arctan x \\ \arcsin x + \arcsin z = \arctan z \end{cases}$$

Proposed by Seyran Ibrahimov- Azerbaijan

XI.16. Let $f, g : R_+^* \to R_+^*$ be functions such that:

$\lim\limits_{x\to\infty}\big(f(x+1)-f(x)\big)=a\in R_+^*$, $\lim\limits_{x\to\infty}\dfrac{g(x+1)}{xg(x)}=b\in R_+^*$ and there exists $\lim\limits_{x\to\infty}\dfrac{f(x)}{x}$, and $\lim\limits_{x\to\infty}\dfrac{\big(g(x)\big)^{\frac{1}{x}}}{x}$. For $t\in R$ calculate the limit $\lim\limits_{x\to\infty}\big(f(x)\big)^{\cos^2 t}\left(\big(g(x)\big)^{\frac{\sin^2 t}{x+1}}-\big(g(x)\big)^{\frac{\sin^2 t}{x}}\right)$.

Proposed by D.M. Bătineţu – Giurgiu, N. Stanciu – Romania

XI.17. Let $f, g : R_+^* \to R_+^*$ such that: $\lim\limits_{x\to\infty}\big(f(x+1)-f(x)\big)=a\in R_+^*$, $\lim\limits_{x\to\infty}\dfrac{g(x+1)}{xg(x)}=b\in R_+^*$ and there is $\lim\limits_{x\to\infty}\dfrac{f(x)}{x}$, $\lim\limits_{x\to\infty}\dfrac{\big(g(x)\big)^{\frac{1}{x}}}{x}$. For $t\in R$, calculate:

$$\lim_{x\to\infty}\big(f(x)\big)^{\sin^2 t}\left(\big(g(x)\big)^{\frac{\cos^2 t}{x+1}}-\big(g(x)\big)^{\frac{\cos^2 t}{x}}\right).$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

XI.18. Let be the sequence $x_0 > 0, x_{n+1} = \frac{1}{x_1^p} + \frac{1}{x_2^p} + \cdots + \frac{1}{x_n^p}, p \in \mathbb{N}^*$. Find:

$$\Omega = \lim_{n\to\infty}\frac{\sqrt{\sum_{1\le i<j\le n} x_i x_j}}{\sqrt[p+1]{n^{p+2}}}$$

Proposed by Marian Ursărescu – Romania

XI.19. Let be the sequence $x_0 > 0$ and $\sqrt[p]{x_n} = \frac{1+x_n-x_{n+1}}{x_{n+1}-x_n}, \forall n \in \mathbb{N}, p \in \mathbb{N}^*, p \ge 2$. Find:

$$\lim_{n\to\infty}\frac{x_n^{p+1}}{n^p}$$

Proposed by Marian Ursărescu – Romania

XI.20. Fie $g_n = \dfrac{(n+2)^{n+1}}{(n+1)^n}$, $\forall n \in N$ şi $x \in R$. Să se calculeze $\lim\limits_{n\to\infty} n^{\sin^2 x}\left(g_{n+1}^{\cos^2 x} - g_n^{\cos^2 x}\right)$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

XI.21. For $\{a_n\}_{n\geq 0}, a_n = \frac{(n+2)^{n+1}}{(n+1)^n}, x \in (-\infty, \infty)$, $\{b_n(x)\}_{n\geq 1}, b_n(x) = n^{\sin^2 x}\left(a_{n+1}^{\cos^2 x} - a_n^{\cos^2 x}\right)$, find $\lim_{n\to\infty} b_n(x)$.

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

XI.22. Find:

$$\lim_{n\to\infty} \frac{\sum_{1\leq i<j\leq n} \frac{1}{\sqrt{ij}}}{\sqrt[n]{1\cdot\sqrt{2!}\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}$$

Proposed by Marian Ursărescu – Romania

XI.23. Let be $A \in M_4(\mathbb{R})$ such that $\det(A^2 + I_4) = 0$. Prove that: $Tr\, A^* = 1 + \det A$

Proposed by Marian Ursărescu – Romania

XI.24. Find the continuous functions $f, g, h: \mathbb{R} \to \mathbb{R}$ with the property:

$$f\left(\frac{x+y}{2}\right) = \frac{g(x)+h(y)}{2}, \forall x, y \in \mathbb{R}$$

Proposed by Marian Ursărescu – Romania

XI.25. Dacă $x \in R$, atunci şirul $\left(L_n(x)\right)_{n\geq 2}, L_n(x) = n^{\sin^2 x}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{\cos^2 x} - \left(\sqrt[n]{n!}\right)^{\cos^2 x}\right)$ este convergent. Să se calculeze $\lim_{n\to\infty} L_n(x)$.

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

XI.26. Let be $A \in M_6(\mathbb{R})$, invertible such that: $\det(A^2 + I_6) = 0$. Prove that:

$$Tr\, A^* + \det A = 1 + \det A \cdot Tr(A^*)^{-1}$$

Proposed by Marian Ursărescu – Romania

XI.27. Fie $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}, x_n \in R_+^*, y_n \in R$ astfel încât există $\lim_{n\to\infty}(x_{n+1} - x_n) = x \in R_+^*$,

$\lim_{n\to\infty} y_n = y \in R$ şi $\lim_{n\to\infty}((y_n - y)n) = z \in R^*$. Să se calculeze : $\lim_{n\to\infty}(x_{n+1}y - x_n y_n)$.

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

XI.28. Dacă $(x_n)_{n\geq 1}, x_n \in R_+^*, \forall n \in N^*$ astfel încât există $\lim_{n\to\infty}(x_{n+1} - x_n) = x \in R_+^*$.

Să se calculeze :

$$\lim_{n\to\infty}\left(x_{n+1}\sqrt[n+1]{n+1} - x_n\sqrt[n]{n}\right).$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

XI.29 Prove that if $A, B \in M_{2n+1}(\mathbb{C})$ such that: $(AB - BA)^2 = I_{2n+1}$

Proposed by Marian Ursărescu – Romania

XI.30. Let be $A \in M_n(\mathbb{Z})$. If $\det A$ is odd, then: $\det(A - 2kI_n) \neq 0, \forall k \in \mathbb{Z}$.

Proposed by Marian Ursărescu – Romania

XI.31. Fie $x \in R$. Să se calculeze: $\lim\limits_{n\to\infty}\left(n^{\sin^2 x}\left(\left(\sqrt[n+1]{n+1}\right)^{(n+2)\cos^2 x} - \left(\sqrt[n]{n}\right)^{(n+1)\cos^2 x}\right)\right)$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

XI.32. Fie $x \in R$. Să se calculeze: $\lim\limits_{n\to\infty}\left(n^{\cos^2 x}\left(\left(\sqrt[n+1]{n+1}\right)^{(n+2)\sin^2 x} - \left(\sqrt[n]{n}\right)^{(n+1)\sin^2 x}\right)\right)$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

XI.33. Find:

$$\lim_{n\to\infty}\frac{1}{\sqrt[p]{n}}\left(\arctan n^p + \frac{1}{2}\arctan(n-1)^p + \cdots + \frac{1}{n}\arctan 1\right); p \in \mathbb{N}^*, p \geq 2$$

Proposed by Marian Ursărescu – Romania

XI.34. Let be $A \in M_2(\mathbb{Z})$. Find that:

$\det(A + A^T + A^*) + \det(-A + A^T + A^*) + \det(-A^T + A + A^*) + \det(-A^* + A + A^T) \vdots 12$

Proposed by Marian Ursărescu – Romania

XI.35. Find:

$$\Omega = \lim_{p\to\infty}\sqrt[p]{\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k^2 + kp - 1}{(p+k+1)!}\right)}$$

Proposed by Daniel Sitaru – Romania

XI.36.

$$\Omega(a) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{3^k}\sin^3(3^k \sin a)$$

If $a, b, c \in \left[0, \frac{\pi}{2}\right)$ then:

$$4\left(b\Omega(a) + c\Omega(b) + a\Omega(c)\right) \leq 3(a^2 + b^2 + c^2)$$

Proposed by Daniel Sitaru – Romania

XI.37.

$$\Omega_n = \sum_{k=1}^{n}\left(\int_{-\frac{1}{k}}^{\frac{1}{k}}\left((2x^8 + 3x^6 + 1)\cdot\cos^{-1}(kx)\right)dx\right)$$

Find:

$$\Omega = \lim_{n\to\infty}(\Omega_n - \pi \cdot H_n)$$

Proposed by Daniel Sitaru – Romania

XI.38. Find:

$$\Omega = \sum_{n=1}^{\infty}\left(\log\left(\frac{n^2+3n+2}{n^2+3n}\right)+\frac{1}{4}\log\left(\frac{(n-1)^2+3(n-1)+2}{(n-1)^2+3(n-1)}\right)+\cdots+\frac{1}{n^2}\log\frac{3}{2}\right)$$

Proposed by Daniel Sitaru – Romania

XI.39. Find:

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{k+n^{14}}}$$

Proposed by Daniel Sitaru – Romania

XI.40. Let be $x_n, y_n > 0, x_n \neq y_n$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = p, p > 0$. Find:

$$\Omega = \lim_{n\to\infty}\frac{x_n^{y_n}-y_n^{x_n}}{\left(\frac{1}{x_n}\right)^{\frac{1}{y_n}}-\left(\frac{1}{y_n}\right)^{\frac{1}{x_n}}}$$

Proposed by Marian Ursărescu – Romania

XI.41. Let be the sequences: $x_0 \in (0,1), x_{n+1} = x_n\sqrt[p]{1-x_n}, y_0 > 0, y_{n+1} = y_n + \frac{1}{y_n^{p-1}}$, $\forall n \in \mathbb{N}, p \in \mathbb{N}, p \geq 2$. Find:

$$\lim_{n\to\infty}\sqrt[p]{x_n}\cdot y_n$$

Proposed by Marian Ursărescu – Romania

XI.42.

$$\Omega(n) = \sum_{k=1}^{\infty}\frac{2k^2+2nk+k-1}{(2k+2n+2)!!}$$

Find:

$$\Omega = \lim_{n\to\infty}\left(n!\cdot\Omega(n)\right)$$

Proposed by Daniel Sitaru – Romania

XI.43. If $x > 0$ then:

$$\left(e^{x^2}+e^{(x+3)^2}\right)\left(\frac{1}{1+e^x}+\frac{1}{1+e^{x+3}}\right) > \left(e^{(x+1)^2}+e^{(x+2)^2}\right)\left(\frac{1}{1+e^{x+1}}+\frac{1}{1+e^{x+2}}\right)$$

Proposed by Daniel Sitaru – Romania

XI.44. Solve the equation:

$\sin\frac{\pi}{[x]}+\cos\frac{\pi}{[x]} = \tan\frac{\pi}{[2x]}$, where $[\cdot]$ denote the integer part.

Proposed by Rovsen Pirguliyev- Azerbaijan

XI.45. For all $x > 0$, prove:

$$\frac{\sin^2 x - x}{\ln\left(\frac{\sin^2 x}{x}\right)^{\sqrt{x}}}+\frac{\cos^2 x - x}{\ln\left(\frac{\cos^2 x}{x}\right)^{\sqrt{x}}} > |\sin x|+|\cos x|$$

Proposed by Rovsen Pirguliyev- Azerbaijan

XI.46. Prove without computer: $\frac{2+e^2}{2+\pi^2} > \sqrt[3]{\frac{e^{2e}}{e^{2\pi}}}$

Proposed by Rovsen Pirguliyev- Azerbaijan

XI.47. Let be $A \in M_5(\mathbb{R})$, invertible such that $\det(A^2 + I_5) = 0$. Prove that:

$$tr\ A + \det A = \det A\left(tr(A^{-1})\right)^*$$

Proposed by Marian Ursărescu – Romania

XI.48 Let be $A, B \in M_2(\mathbb{R})$ such that $\det A = \det B$. Prove that:

$$\det(xAB + yBA) \geq xy \det(AB + BA), \forall x, y \in \mathbb{R}$$

Proposed by Marian Ursărescu – Romania

XI.49. Find:

$$\lim_{n\to\infty} \frac{\left(1 + 2^p\sqrt{2} + 3^p\sqrt[3]{3} + \cdots + n^p\sqrt[n]{n}\right)^{q+1}}{(1^q + 3^q + \cdots + (2n-1)^q)^{p+1}},$$

$p, q \in \mathbb{N}^*$. ***Proposed by Marian Ursărescu – Romania***

XI.50. Let be $x_0, y_0 > 0, x_{n+1} = x_n - x_n^{p+1}, y_{n+1} = y_n + \frac{1}{y_n^{p-1}}, p \in \mathbb{N}^*, p \geq 2$. Find:

$$\lim_{n\to\infty}(x_n y_n)$$

Proposed by Marian Ursărescu – Romania

XI.51. $\Omega_n(x) = \int_1^x \left(\frac{t^n-1}{t-1}\right) dt, n \in \mathbb{N}, n \geq 1$. Find:

$$\Omega = \lim_{x\to 1} \frac{tan^{-1}(nx - n) - \Omega_n(x)}{(x-1)^2}$$

Proposed by Ahmad Albaw-Jordan

XI.52. Find:

$$\Omega = \lim_{n\to\infty} n\left(1 - \frac{log\left(1 + \frac{\sqrt[n]{10}}{n}\right)^{n+1}}{log\left(1 + \frac{\sqrt[n+1]{10}}{n+1}\right)^n}\right)$$

Proposed by Daniel Sitaru – Romania

XI.53. $x_1 = 1, x_2 = 3, x_n = x_{n-2} + 2x_{n-1}, n \geq 3$. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{(-1)^{n+1}(x_n^2 - 2x_n x_{n-1} - x_{n-1}^2)}{n}\right)$$

Proposed by Daniel Sitaru – Romania

XI.54. Let be the sequences: $x_1 > 0, x_{n+1} = \ln(1 + x_n), \forall \in \mathbb{N}$ and $y_1 > 0$,

$y_{n+1} = e^{\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}}$. Find: $\lim_{n\to\infty}(x_n y_n)$. ***Proposed by Marian Ursărescu – Romania***

XI.55.

$$\alpha(x) = \frac{4}{3}\sum_{n=1}^{\infty}\left(\left(-\frac{1}{3}\right)^n \cdot \cos^3(3^{n-1}x)\right), x \in \left(0, \frac{\pi}{2}\right), \beta(x) = \alpha\left(\frac{\pi}{2} - x\right)$$

Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\lim_{\substack{x\to 0\\ x>0}}\left(\frac{\beta(x)\beta(3x)\cdot \ldots \cdot \beta((2n-1)x)}{\beta(2x)\beta(4x)\cdot \ldots \cdot \beta(2nx)}\right)}$$

Proposed by Daniel Sitaru – Romania

XI.56. $a,b,c > 0, 2e(a+b+c) = 3e+2, \Omega(a) = \lim_{n\to\infty} n\left(\left(\frac{n+1}{n}\right)^{n+a} - e\right)$. Prove that:

$$\Omega(a)\cdot\Omega(b)\cdot\Omega(c) \leq \frac{1}{27}$$

Proposed by Daniel Sitaru – Romania

XI.57. Find:

$$\Omega = \sum_{n=0}^{\infty}\left(\frac{1}{3^{n+1}}\sum_{k=1}^{n}\left(3^k\cdot tan^{-1}\left(\frac{3}{k^2-k-1}\right)\right)\right)$$

Proposed by Daniel Sitaru – Romania

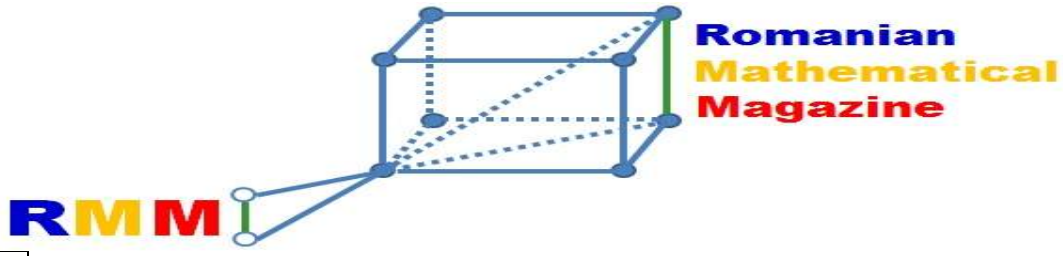
XI.58 $\Omega(x) = -\frac{1}{2} + 4\sum_{n=1}^{\infty}\frac{n+x}{(n+1)(n+2)(n+3)}, x\in\mathbb{R}$

If $a\in(0,1), b>1$ then: $\left(\Omega(a)\right)^{\Omega(b)} + \left(\Omega(b)\right)^{\Omega(a)} < 1+\Omega(a)\cdot\Omega(b)$

Proposed by Daniel Sitaru – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

12-CLASS-STANDARD



XII.1. Find:

$$\int \ln\left[\frac{\sin(\arctan x)}{\sqrt{1+x^2}}\right]dx$$

Proposed by Abdul Mukhtar-Nigeria

XII.2. Find:

$$\int \frac{\cos x + \sin x}{\sqrt{\sin(2x) - \sin^2(2x)}}dx$$

Proposed by Abdul Mukhtar-Nigeria

XII.3. Find:

$$\int \frac{2x^4 + 5x^3 + 6x^2 + 6x + 12}{(x^2 + 2x + 2)\sqrt{x^2 + 2x + 2}}\,dx$$

Proposed by Abdul Mukhtar-Nigeria

XII.4. Find:

$$\int_{-1}^{2} \frac{dx}{(4 + 2x + x^2)^{\frac{5}{2}}} + \int_{0}^{\frac{1}{2}} \frac{\ln(1-x)}{2x^2 - 2x + 1}\,dx$$

Proposed by Abdul Mukhtar-Nigeria

XII.5. Let $n \in \mathbb{N}^*$ and $a_k \in \mathbb{R}, \forall k = \overline{1;n}$. Find $\Omega = \int \ln\left(\prod_{k=1}^{n}(x - a_k)\right) dx$.
$\left(x > \max\{a_k | \forall k \in \overline{1;n}\}\right)$

Proposed by Nguyen Van Nho-Vietnam

XII.6. Let be $f: \mathbb{R} \to \mathbb{R}$ continuous such that $\left(f(x)\right)^5 + \left(f(x)\right)^3 + pf(x) = -x, \forall x \in \mathbb{R}, p >$ 0. Find: $\int_0^{p+2} f(x)\,dx$. ***Proposed by Marian Ursărescu – Romania***

XII.7. Find:

$$\int \frac{x^4 + 2x^3 + 3x^2 + 4x + 5}{x^7 - 7x^6 + 5x^5 + 5x^4 + 9x^2 + 52x - 60}\,dx$$

Proposed by Seyran Ibrahimov- Azerbaijan

XII.8. Find:

$$\gamma = \int \frac{x + 3x^2 + 5x^3 + 7x^5}{7x^8 + 14x^7 - 2x^6 - x^5 + 2x^4 - 11x^3 - 6x^2}\,dx, x \in \mathbb{R}$$

Proposed by Seyran Ibrahimov- Azerbaijan

XII.9. Find:

$$\Omega = \int \frac{x^2 + x^3 + x^4 + x^5}{x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{18} + x^{19}}\,dx, x > 0$$

Proposed by Naren Bhandari-Nepal

XII.10. Find:

$$\alpha = \int \frac{2x^3 + 3x^2 + 9x + 4}{x^4 + 2x^3 + 9x^2 + 8x + 16}\,dx$$

Proposed by Seyran Ibrahimov- Azerbaijan

XII.11. Find:

$$\int \frac{x^3 + x^2 + x + 1}{x^5 + 10x^4 + 28x^3 + 28x^2 + 27x + 18}\,dx, x > 0$$

Proposed by Seyran Ibrahimov- Azerbaijan

XII.12. Find:

$$\Omega = \lim_{n\to\infty} \int_{\frac{1}{n}}^{n} \frac{1}{x}\left(\frac{\cos 3x}{1+3x} - \frac{\cos 2x}{1+2x}\right) dx$$

Proposed by Daniel Sitaru– Romania

XII.13. Find:

$$\Omega = \lim_{x\to 0} \frac{\int_0^x \frac{\sqrt[3]{1+2x}\cdot\sqrt[5]{1+3x}-1}{x} dx}{\int_0^x \frac{\sqrt[3]{1+3x}\cdot\sqrt[5]{1+2x}-1}{x} dx}$$

Proposed by Daniel Sitaru– Romania

XII.14. Find:

$$\Omega = \lim_{a\to\infty} \int_0^a \frac{2x+3}{x(x+1)(x+2)(x+3)+a} dx$$

Proposed by Daniel Sitaru– Romania

XII.15. Find:

$$\Omega = \lim_{n\to\infty} (n+1)\left(\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx\right)^n$$

Proposed by Nho Nguyen Van-Vietnam

XII.16. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n(n+1)} \sum_{k=1}^{n} k \sin\left(\frac{k^2+k}{n^2+n}\right)$$

Proposed by Daniel Sitaru– Romania

XII.17. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n(n+1)(2n+1)} \sum_{k=1}^{n} k^2 \tan^{-1}\left(\frac{k(k+1)(2k+1)}{n(n+1)(2n+1)}\right)$$

Proposed by Daniel Sitaru– Romania

XII.18. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n^3} \sum_{1\le i<j<k\le n} \sin\left(2+\frac{3i}{n}\right)\sin\left(2+\frac{3j}{n}\right)\sin\left(2+\frac{3k}{n}\right)$$

Proposed by Daniel Sitaru – Romania

XII.19.

$$\omega(n) = \int_{-1}^{1} \frac{(1+2x+x^2)^n(1-2x+x^2)^n}{(1-x^2)(1+2x^2+x^4)^n} dx, n \in \mathbb{N}, n \ge 2$$

Find:

$$\Omega = \lim_{n\to\infty} \frac{\omega(n+2)-\omega(n+1)}{\omega(n+1)-\omega(n)}$$

Proposed by Daniel Sitaru – Romania

XII.20. Find: $\Omega = \lim_{n\to\infty} \frac{\int_0^1 \left(\frac{2x}{1+x^2}\right)^{n+1} dx}{\int_0^1 \left(\frac{2x}{1+x^2}\right)^{n} dx}$

Proposed by Daniel Sitaru– Romania

XII.21. Let f: [0,1] → (−1,1) be a continuous function so that $\int_0^1 f(x)\,dx \notin \{-1,1\}$.Prove that: $\frac{e^{\int_0^1 f(x)dx}}{1+e^{\int_0^1 f(x)dx}} \leq \int_0^1 \frac{e^{f(x)}}{1+e^{f(x)}}\,dx$.

Proposed by Soumitra Mandal - India

XII.22. If $0 < a < b < \frac{\pi}{2}$ then:

$$7\int_a^b x^3 \tan x \left(\sin\frac{x}{2} + \tan\frac{x}{2}\right)\sqrt{\sin x \tan x}\,dx > b^7 - a^7$$

Proposed by Daniel Sitaru -Romania

XII.23. Let f: $\mathbb{R}^+ \to \mathbb{R}^+$ be integrable and satisfying

$$f(xt + (1-t)y) \leq tf(x) + (1-t)f(y) \text{ where } x, y \in \mathbb{R}^+ \text{ and } t \in (0,1)$$

Show that:

$$\frac{1}{\ln a}\int_1^a f(x^4)\,dx + \frac{1}{\ln b}\int_1^b f(x^4)\,dx \geq 2\int_0^1 (ab)^{\frac{x}{2}} f(a^{4x} - a^{3x}b^x + a^{2x}b^{2x} - a^x b^{3x} + b^{4x})dx$$

where $a, b > 0$.

Proposed by Soumitra Mandal-India

XII.24. If $0 < a < b < \frac{\pi}{2}$ then:

$$\int_a^b \frac{\cos x}{\sin x + 4x\cos x}\,dx < \frac{1}{5}\left(\log\frac{b}{a}\right)^{\frac{4}{5}} \cdot \sqrt[5]{\log\left(\frac{\sin b}{\sin a}\right)}$$

Proposed by Daniel Sitaru -Romania

XII.25 Find: $\Omega = \lim_{n\to\infty} \int_0^{2018} \left|\left[nx + \frac{1}{2}\right] - nx\right| dx$, $[*]$ - great integer function

Proposed by Daniel Sitaru -Romania

XII.26. Find:

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_\varepsilon^{\frac{\pi}{2}} \left(\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\right)dx$$

Proposed by Abdul Mukhtar-Nigeria

XII.27. If $0 < a < 1, f: \left[a, \frac{1}{a}\right] \to \mathbb{R}, f$ convexe and increasing function then:

$$\frac{1-a^2}{a} f\left(\frac{1+a^2}{2a}\right) \leq \int_a^{\frac{1}{a}} \left(\frac{1+x^2}{2x^2}\right) f(x)dx \leq \frac{1-a^2}{2a}\left(f(a) + f\left(\frac{1}{a}\right)\right)$$

Proposed by Abdallah El Farissi-Bechar-Algerie

XII.28. If $a, b, c \in \left(0, \frac{\pi}{4}\right)$ then:

$$0 \leq \int_0^a \left(\int_0^b \left(\int_0^c \left(\sum (\tan x + 2\tan y \tan z) + 4\prod \tan x\right) dx\right) dy\right) dz \leq abc$$

Proposed by Daniel Sitaru – Romania

XII.29. If $a, b, c > 0$ then:

$$e^{\sum a \int_b^{\frac{cx^4+1}{x^6+1}} dx} \leq \left(\frac{c}{b}\right)^a \cdot \left(\frac{a}{c}\right)^b \cdot \left(\frac{b}{a}\right)^c$$

Proposed by Daniel Sitaru-Romania

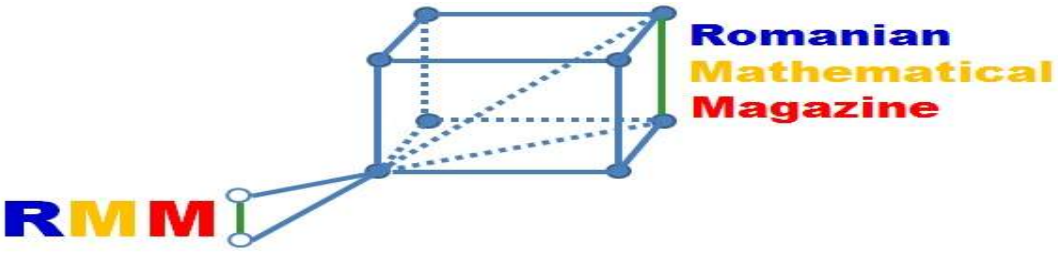
XII.30. If $a, b, c \geq 0$, $\Omega(a) = \int_0^a \sqrt{\frac{x(x^2+x+1)}{(x+1)(x^4+x^2+1)}} dx$ then:

$$\left(\Omega(a) + \Omega(b) + \Omega(c)\right)^3 \geq \prod \log\left(\sqrt{a^3+1} + \sqrt{a^3}\right)$$

Proposed by Daniel Sitaru – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS



U.1. An intriguing mock exponential integral with an improper closed-form.

$$\int_0^{\infty} e^{-x} \frac{(e^{2x}-1)}{(e^{4x}+1)} \cdot \frac{\tanh^2(x)}{x} dx = -\frac{4c}{\pi} - 3\left(\log\left(\sqrt{2}\right) - \log(7)\right) +$$

$$+5(\log(3)-\log(5))+\log\left(\frac{\Gamma\left(\frac{13}{8}\right)^2\Gamma\left(\frac{9}{4}\right)^3}{\Gamma\left(\frac{11}{8}\right)^2\Gamma\left(\frac{11}{4}\right)^3}\right)$$

C – Catalan Constant.

Proposed by Srinivasa Raghava-AIRMC-India

U.2. Application of the Fourier series:

$$\sum_{n=0}^{\infty}\frac{\sin\big((2n+1)\beta\big)}{k^2+(2n+1)^2}(-1)^n=\frac{\pi}{4k}\cdot\frac{\sinh(\beta k)}{\cosh\left(\frac{\pi k}{2}\right)}$$

provided $|\beta|\le\frac{\pi}{2}\ \&\ k>0, m>0$

$$\sum_{n=-\infty}^{\infty}\frac{\sin\left((2n+1)\frac{\pi}{2}\right)}{k^2+m^2(2n+1)^2}(-1)^n=\frac{\pi}{2}\cdot\frac{\tanh\left(\frac{\pi k}{2m}\right)}{km}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3. Find:

$$\Omega=\int_0^{\infty}\frac{\pi x-2\log x}{\left(\frac{\pi^2}{4}+\log^2 x\right)(1+x^2)^2}dx$$

Proposed by Khalef Ruhemi-Jarash-Jordan

U.4. Find:

$$\int_0^{\infty}\frac{\ln^2\left(x+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}dx$$

Proposed by Abdul Mukhtar-Nigeria

U.5. Evaluate the following integral:

$$\int_0^{\frac{\pi}{3}}\sin^{-1}\left(\sqrt{\sin(x)}\right)\sin(2x)\,dx$$

and prove that

$$\int_0^{\frac{\pi}{3}}\sin^{-1}\left(\sqrt{\sin(x)}\right)\sin(2x)\,dx\approx\frac{13}{20}$$

Proposed by K. Srinivasa Raghava – AIRMC – India

U.6. If we define the function $Z(n)$ for any $n>1$

$$Z(n)=\int_0^1\log^2(x)\left(\tanh^{-1}(nx)-\tanh^{-1}(x)\right)\frac{dx}{\sqrt{x}}$$

Then prove that:

$$\lim_{n\to\infty}Z(n)=14\zeta(3)-\frac{1}{2}\pi(\pi(\pi-4)+8)+8\log(2)-16G-8\pi i$$

Proposed by K. Srinivasa Raghava – AIRMC – India

U.7. Find:

$$\Omega=\int_0^{\infty}\frac{x\cdot\tan^{-1}x}{x^4+x^2+1}dx$$

Proposed by Vasile Mircea Popa – Romania

U.8. Prove that:

$$2 + \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{\sin(mx)}{\sinh(2x)}(x\coth(x)-1)dx\right)^2 dm = \frac{2\pi}{3}+\frac{5\pi^3}{36}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.9. If

$$\int_0^{\infty}(\coth(x)+\cos(x))\,e^{-mx}xdx = \phi(m) + \int_0^{\infty}(\tanh(x)+\sin(x))\,e^{-mx}xdx$$

then we have

$$\int_0^1\left(\phi\left(\sqrt{m}\right)+\phi\left(-\sqrt{m}\right)\right)\frac{dm}{\sqrt{m}} = 2(\pi-1)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.10. For sufficiently large positive integer T (some times we can say T approaches to infinity), we have:

$$\int_2^T\left(\sum_{n=1}^{\infty}\frac{1}{n^2(an^2-1)}\right)da \geq \log(T)+2\gamma$$

here γ = .57721566 … Euler's constant

Proposed by Srinivasa Raghava-AIRMC-India

U.11. Show that:

$$\frac{1}{\sqrt{2}}+\int_0^{\infty}\left(\sin\left(\frac{\pi x}{2}\right)+\tanh\left(\frac{\pi x}{2}\right)\right)e^{-2\pi x}\frac{dx}{\sqrt{x}} =$$

$$= 2\left(1+(\sqrt{2}-1)\zeta\left(\frac{1}{2}\right)\right)+\sqrt{\frac{\sqrt{17}-4}{17}}$$

Proposed by K. Srinivasa Raghava – AIRMC – India

U.12. Evaluate the integral:

$$\int_0^1\left(\ln\left(\sqrt{1+x^2}-\ln\left(\sqrt{1-x^2}\right)\right)\right)\ln(x)\,dx$$

and then prove this sharp inequality

$$\int_0^1\left(\ln\left(\sqrt{1+x^2}-\ln\left(\sqrt{1-x^2}\right)\right)\right)\ln(x)\,dx \leq 4$$

Proposed by Srinivasa Raghava-AIRMC-India

U.13. Prove that:

$$\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\frac{(\tan^2(x)+\tan^2(y))\log(\tan(x))\log(\tan(y))\,Ti_2(\tan(x)\tan(y))}{\tan(x)\tan(y)}dx\,dy$$

$$= \frac{1}{368640}\psi^{(5)}\left(\frac{1}{4}\right)-\frac{21}{32}\zeta(6)-\frac{2}{3}G^3,$$

Where G is the Catalan's constant, $\psi^{(n)}$ denotes the Polygamma function, ζ designates the Riemann zeta function and $Ti_2(x)$ represents the inverse tangent integral.

Proposed by Cornel Ioan Vălean – Romania

U.14. Evaluate the sum in closed form, for $m \neq 0$

$$\sum_{m,n\in Z} \frac{1}{\sinh\left(\frac{m\pi}{2}\right)^2 \left(m + i(2n+1)\right)^2}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.15. For any $n \geq 1$, we have

$$\int_0^{\frac{\pi}{2}} \left(2n\left(n\left(1 - \sqrt[n]{\cos(x)}\right) + \log(\cos(x))\right) + \log(\cos^2(x))\right) dx =$$

$$= \pi(n^2 - (n+1)\log(2)) - \frac{\sqrt{\pi} n^2 \Gamma\left(\frac{n+1}{2n}\right)}{\Gamma\left(1 + \frac{1}{2n}\right)}$$

$$\int_0^{\frac{\pi}{2}} \left(2n\left(n\left(1 - \sqrt[n]{\cos(x)}\right) + \log(\cos(x))\right) + \log^2(\cos(x))\right) dx =$$

$$= \pi n^2 - \frac{\sqrt{\pi} n^2 \Gamma\left(\frac{n+1}{2n}\right)}{\Gamma\left(1 + \frac{1}{2n}\right)} + \frac{1}{2}\pi\log(2)\left(\log(2) - 2n\right) + \frac{\pi^3}{24}$$

Inspired from Kartick Chandra Betal

Proposed by K. Srinivasa Raghava – AIRMC – India

U.16. Prove that:

$$\sum_{n=-\infty}^{\infty} \left(\frac{1}{5n+4} + \frac{1}{5n+1}\right)\left(\frac{1}{5n+3} + \frac{1}{5n+2}\right)^2 \frac{(-1)^n}{4n+1} \geq \gamma$$

where γ is Euler's constant.

Proposed by K. Srinivasa Raghava – AIRMC – India

U.17. Calculate the integral:

$$\int_0^{\infty} \frac{arctanx}{x^4 - x^2 + 1} dx$$

It is required to express the integral value with the usual mathematical constants, without using values of special functions. ***Proposed by Vasile Mircea Popa-Romania***

U.18. Find:

$$\Omega = \lim_{n\to\infty} \int_{\pi}^{2\pi} \frac{|\sin nx|}{x^2} dx$$

Proposed by Vasile Mircea Popa-Romania

U.19. Prove that:

$$\int_0^{\pi} \left(\frac{\sin(\alpha x)}{\sin(x)}\right)^{2\mathfrak{C}} (\cos^{2\omega}(x) - \cos^{2\varepsilon}(x))dx = \int_0^{\pi} \left(\frac{\sin(\beta(x))}{\sin(x)}\right)^{2\mathfrak{C}} (\cos^{2\omega}(x) - \cos^{2\varepsilon}(x))dx$$

where $\{\mathfrak{C}, \alpha, \omega, \varepsilon, \beta\}$ are integers, $\mathfrak{C} < \{\omega, \varepsilon\}a, \alpha \geq \left\{\frac{\omega}{\mathfrak{C}}, \frac{\varepsilon}{\mathfrak{C}}\right\}$

Proposed by Ahmed Albaw-India

U.20. Calculate the integral:

$$\int_0^{\sqrt{\frac{\pi}{2}}} \left(\frac{\ln\left(\frac{x\sqrt{2}+\sqrt{\pi}}{\sqrt{2}}\right)^2}{2x^2+\pi} \right) dx$$

Proposed by Ekpo Samuel-Nigeria

U.21. Calculate the integral:

$$\int_0^{\infty} \log(x^x)\, e^{-x^3} \sin^3(x^3)\, dx$$

Proposed by Zaharia Burghelea-Romania

U.22. Find:

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon<0}} \left(\int_{-1}^{\varepsilon} \sqrt{\frac{1+e^x}{1-e^x}}\, dx \right)$$

Proposed by Ekpo Samuel-Nigeria

U.23.

$$x! + n \le p \le x! + x$$

Is there any prime number p such that given inequality above can be true, where $1 \le n \le x$ and $x > 2$?

Proposed by Naren Bhandari-Nepal

U.24. Prove that for $x_n \in \mathbb{Z}^+$

$$\sqrt{x_0^{\sqrt{x_1^{\sqrt{x_2^{\sqrt{\cdots\sqrt{x_n}}}}}}}} = \sqrt{x_1}^{\sqrt{x_2}^{\sqrt{x_3}^{\sqrt{\cdots\sqrt{x_n}}}}}$$

Setting $x_0 = x_1 = x_2 = \cdots = x_n = 2$ for $n = 5$ also prove that:

$$\sqrt{2^{\sqrt{2^{\sqrt{2^{\sqrt{2^{\sqrt{2}}}}}}}}} = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}}} < 2$$

Proposed by Naren Bhandari-Nepal

U.25.

$$\cos\left(\frac{\pi}{128}\right) = \left(\frac{\sqrt{\sqrt{\sqrt{\sqrt{1+\sqrt{2}}+\sqrt{2\sqrt{2}}}+\sqrt{2\sqrt{2\sqrt{2}}}}+\sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}}+\cdots+\sqrt{2\sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}}}}{2\sqrt{2\sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}}} \right)^{2^{-1}}$$

Proposed by Naren Bhandari-Nepal

U.26.

$$\sum_{n=1}^{\infty}\frac{\left(nH_n^{(1)}-1\right)\left(\left(H_n^{(1)}\right)^3+3H_n^{(1)}H_n^{(2)}+H_n^{(3)}\right)}{n^3}+\sum_{n=0}^{\infty}\frac{Si\left(2\pi(2n+1)\right)}{(2n+1)^3}=$$

$$=60A-\frac{B}{4}\pi^3, A, B\in\mathbb{R}, Si(x)-\text{sine integral function}$$

Find:

$$\Omega=A+B+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cos\left(\frac{n\pi}{4}\right)$$

Proposed by Arafat Rahman Akib-Bangladesh

U.27.

$$\int_0^{\infty}\frac{e^{-st-\frac{x^2}{4t}}}{\sqrt{t}}dt+\frac{1}{\sqrt{s}}\int_0^{\infty}\frac{e^{-t-\frac{x^2}{4t}}}{\sqrt{t}}dt=\sqrt{\frac{\pi}{s}}\cdot A(x)$$

Find: $\Omega=\sqrt{\mathcal{L}\left(A(x)\right)\cdot\log\sqrt[4]{2s}}$, $\mathcal{L}$ - Lapalce's transform

Proposed by Arafat Rahman Akib-Bangladesh

U.28. Let

$$S_a(m)=1+(a-1)\sum_{k=0}^{m}\binom{m+1}{k+1}\frac{(-1)^k}{ak+1}$$

then for any $a\geq 1, m\geq 1$, we have

$$\frac{\partial}{\partial m}S_a(m)=S_a(m)\left(H_{m+1}-H_{m+\frac{1}{a}}\right)$$

Note: $S_1(m)=H_{m+1}$

H_m – Harmonic Number

Proposed by Srinivasa Raghava-AIRMC-India

U.29. An Unique Integral

$$\int_0^1\int_0^1\int_0^1\frac{xy+xz+yz}{(\log(x)+\log(y)+\log(z))^3}dz\,dy\,dx=3$$

Proposed by Srinivasa Raghava-AIRMC-India

U.30. If,

$$\Phi(y)=\int_0^{\infty}e^{-yx}\sum_{n=0}^{\infty}\frac{n\cosh(\pi n)\,(ix)^{2n+1}}{(2n)!}dx$$

then

$$\int_0^{\infty}\phi(y)\,y^2dy=2\int_0^{\infty}\phi(y)\ln(y)\,dy$$

where $i=\sqrt{-1}$

Proposed by Srinivasa Raghava-AIRMC-India

U.31. $F(m,\alpha,\beta)=\int_{-\infty}^{+\infty}x^me^{-\alpha x^2}\sin\beta x^2\,dx;\alpha,\beta>0, m=-2,-1,0,1\ldots$

Find:

$$\Omega = \lim_{\alpha\to 0}\left(\lim_{m\to 0}\left(\lim_{\beta\to 0}\left(\frac{F(m,\alpha,\beta)}{\frac{\beta}{\alpha\sqrt{4\alpha\pi}}}\right)\right)\right)$$

Proposed by Feti Sinani-Kosovo

U.32. Find:

$$\lim_{x\to\infty}\frac{\left(\left(\frac{\zeta\left(1+\frac{1}{\Gamma(1+x)}\right)}{\Gamma(1+x)}\right)\Gamma\left(1+\frac{1}{\Gamma(1+x)}\right)\right)^{\frac{\Gamma(1+x)}{x}}-1}{\left(\frac{\zeta\left(1+\frac{1}{x}\right)}{x}\right)\Gamma\left(1+\frac{1}{x}\right)-1}$$

where $\zeta(x)$ is Riemann Zeta function and $\Gamma(x)$ is Gamma function.

Proposed by Feti Sinani-Kosovo

U.33. Calculate the integrals:

$$\int_0^{\frac{\pi}{2}} \ln(1+\tan^2 x+\tan^4 x)\,dx,\ \int_0^1 \frac{1}{\ln(1+\tan^2 x+\tan^4 x)}\,dx,\ \int_0^{\frac{\pi}{2}} \tan^{-1}\frac{\left(\sqrt{1+x^2}-1\right)}{x}\,dx$$

Proposed by Abdul Mukhtar-Nigeria

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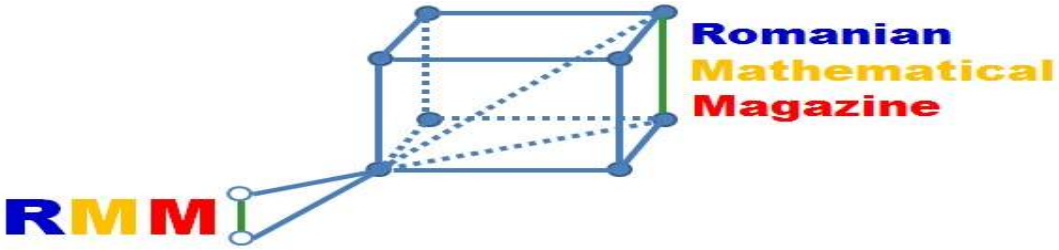
ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-sUMMER 2018



PROBLEMS FOR JUNIORS

JP.136. Let x, y, z be positive real numbers such that: $xyz = 1$. Find the maximum of the expression:

$$Q = \frac{1}{\sqrt[3]{2x^5+y^4-x^2+4}} + \frac{1}{\sqrt[3]{2y^5+z^4-y^2+4}} + \frac{1}{\sqrt[3]{2z^5+x^4-z^2+4}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.137. Let $x, y \geq 1$. Prove that:

$$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\geq 2+\frac{4(x-y)^2}{(2x+xy+1)(2y+xy+1)}$$

Proposed by Andrei Ștefan Mihalcea – Romania

JP.138. Let $a,b,c>0$, with $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$. Prove that:

$$(4a-3)(4b-3)(4c-3)\geq 243\sqrt[3]{abc}$$

Proposed by Andrei Ștefan Mihalcea – Romania

JP.139. Let x,y,z be positive real numbers such that: $x^2+y^2+z^3=3$. Find the minimum of the expression:

$$P=\frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}}+3yz}+\frac{y}{\sqrt[4]{\frac{z^8+x^8}{2}}+3zx}+\frac{z}{\sqrt[4]{\frac{x^8+y^8}{2}}+3xy}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.140. Let $a,b,c>0$. Prove that:

$$\sum\frac{\sqrt{a+b}}{a}\leq\left(\sum\frac{1}{a}\right)\sqrt{\sum a-\frac{\sum ab}{\sum a}}$$

Proposed by Andrei Ștefan Mihalcea – Romania

JP.141. Let $a,b,c>0$. Prove that:

$$\left(\sum\sqrt{\frac{b+c}{a}}\right)^2\leq\frac{2(\sum ab)^3}{3a^2b^2c^2}$$

Proposed by Andrei Ștefan Mihalcea – Romania

JP.142. Let $a,b,c\geq 1$. Prove that:

$$\sum\sqrt{\frac{a-1}{bc}}\leq\left(\sum\frac{1}{ab}\right)\sqrt{abc-\frac{\sum a}{3}}$$

Proposed by Andrei Ștefan Mihalcea – Romania

JP.143. In any ABC triangle the following relationship holds:

$$\frac{w_a^2}{h_b\cdot h_c}+\frac{w_b^2}{h_c\cdot h_a}+\frac{w_c^2}{h_a\cdot h_b}\leq\left(\frac{R}{r}\right)^2-1$$

all notations are usual sense.

Proposed by Mehmet Şahin – Ankara – Turkey

JP.144. In any ABC triangle the following relationship holds:

$$\frac{a}{w_a}+\frac{b}{w_b}+\frac{c}{w_c}\geq\frac{4s}{3R}$$

Proposed by Mehmet Şahin – Ankara – Turkey

JP.145. In any ABC triangle the following relationship holds:

$$\frac{m_a}{r+r_a}+\frac{m_b}{r+r_b}+\frac{m_c}{r+r_c}\leq\frac{s}{2r}$$

Proposed by Mehmet Şahin – Ankara – Turkey

JP.146. Let x, y, z be positive real numbers such that: $xyz = 1$. Find the maximum of the expression:

$$P=\frac{1}{\sqrt[3]{2(x^5-x^3+4)}}+\frac{1}{\sqrt[3]{2(y^5-y^3+4)}}+\frac{1}{\sqrt[3]{2(z^5-z^3+4)}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.147. Let a, b, c be positive real numbers such that: $a^2+b^2+c^2=3abc$. Find the minimum of the expression:

$$P=\frac{a^2}{\sqrt[3]{4(b^3+c^3)}}+\frac{b^2}{\sqrt[3]{4(c^3+a^3)}}+\frac{c^2}{\sqrt[3]{(a^3+b^3)}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.148. Let a, b, c be positive real numbers such that: $ab+bc+ca=12$. Prove that:

$$\frac{a^3+b^3}{2b^2-bc+2c^2}+\frac{b^3+c^3}{2c^2-ca+2a^2}+\frac{c^3+a^3}{2a^2-ab+2b^2}\geq 4$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.149. Find all functions: $f:(0,+\infty)\to\mathbb{R}$ which verify the relationship:

$$\ln(xy)\leq xf(x)+yf(y)\leq xyf(xy),\forall x,y>0$$

Proposed by Marian Ursărescu – Romania

JP.150. Let be $z_1, z_2, z_3\in\mathbb{C}^*$ different in pairs such that $|z_1|=|z_2|=|z_3|$. If $(z_1+$ *z2z2+z3z3+z1+z1z2z3=0*, then *z1,z2, z3* are the affixes of an equilateral triangle.

Proposed by Marian Ursărescu – Romania

PROBLEMS FOR SENIORS

SP.136. Let x, y, z be positive real numbers such that: $x^4+y^4+z^4=xy+yz+zx$. Find the maximum of the expression:

$$P = \sqrt[3]{\frac{x^6 + y^6}{2}} + \sqrt[3]{\frac{y^6 + z^6}{2}} + \sqrt[3]{\frac{z^6 + x^6}{2}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.137. Let $a, b, c > 0$ such that: $a + b + c = 3$. Prove that:

$$\frac{a}{\sqrt[3]{4(b^6 + c^6) + 7bc}} + \frac{b}{\sqrt[3]{4(c^6 + a^6) + 7ca}} + \frac{c}{\sqrt[3]{4(a^6 + b^6) + 7ab}} + \frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{12} \geq \frac{7}{12}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.138. Let a, b, c be positive real numbers such that: $a + b + c = 3$. Prove that:

$$\frac{a^2}{\sqrt{5(b^4 + 4)}} + \frac{b^2}{\sqrt{5(c^4 + 4)}} + \frac{c^2}{\sqrt{5(a^4 + 4)}} \geq \frac{3}{5}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.139. In ABC triangle the lengths of sides BC, CA, AB are a, b, c. Let h_a, h_b, h_c be the distances from A, B, C to BC, CA, AB; l_a, l_b, l_c are the lengths of the bisectors A, B, C. Prove that:

$$\frac{l_a l_b}{l_c} + \frac{l_b l_c}{l_a} + \frac{l_c l_a}{l_b} \geq \frac{h_a h_b}{h_c} + \frac{h_b h_c}{h_a} + \frac{h_c h_a}{h_b}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.140. Let a, b, c be positive real numbers. Prove that:

$$\frac{b + c}{a} + \frac{c + a}{b} + \frac{a + b}{c} \geq \frac{4(a^2 + b^2 + c^2)}{ab + bc + ca} + \frac{2(ab + bc + ca)}{a^2 + b^2 + c^2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.141. Let $a, b, c > 0$ such that: $a + b + c = 3$. Prove that:

$$\frac{a^4}{b^4(2ab - \sqrt{c} + 2)} + \frac{b^4}{c^4(2bc - \sqrt{a} + 2)} + \frac{c^4}{a^4(2ca - \sqrt{b} + 2)} \geq \frac{a^2 + b^2 + c^2}{3}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.142. Let a, b, c be positive real numbers such that: $abc = 1$. Prove that:

$$\frac{a^2 b^2}{a^4 - 2a + b^2 + 2} + \frac{b^2 c^2}{b^4 - 2b + c^2 + 2} + \frac{c^2 a^2}{c^4 - 2c + a^2 + 2} \leq \frac{a^2 + b^2 + c^2 + 3}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.143. Let x, y, z be non-negative real numbers. Prove that:

$$x\sqrt{3x^2 + yz} + y\sqrt{3y^2 + zx} + z\sqrt{3z^2 + xy} \geq x^2 + y^2 + z^2 + xy + yz + zx$$

Proposed by Do Quoc Chinh – Ho Chi Minh – Vietnam

SP.144. Let A, B, C be the corners in a triangle ABC. Prove that:

$$\left(\frac{\sin\frac{A}{2}}{\tan\frac{B}{2}}\right)^2+\left(\frac{\sin\frac{B}{2}}{\tan\frac{C}{2}}\right)^2+\left(\frac{\sin\frac{C}{2}}{\tan\frac{A}{2}}\right)^2\geq\frac{9}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.145. If $1<a\leq b$ then:

$$\int_a^b\int_a^b\int_a^b\frac{dxdydz}{1+\sqrt[3]{xyz}}\leq\log\left(\sqrt[3]{\frac{b+1}{a+1}}\right)^{(b-a)^2}$$

Proposed by Daniel Sitaru – Romania

SP.146. Let be $A, B\in M_3(\mathbb{R})$ such that:

$$AB=\begin{pmatrix}2&1&1\\0&-1&1\\0&0&-1\end{pmatrix}$$

Find: $\det((BA)^2-3I_3)$.

Proposed by Marian Ursărescu – Romania

SP.147. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ having the property:

$$f(x)+2f(2x)+f(4x)=25x^2+9x+4,\forall x\in\mathbb{R}.$$

Proposed by Marian Ursărescu – Romania

SP.148. Let be $x_0>0$ and $x_{n+1}=\arctan\frac{x_n}{1+x_n},\forall n\in\mathbb{N}$. Find: $\lim_{n\to\infty}n\cdot x_n$.

Proposed by Marian Ursărescu – Romania

SP.149. Let be the sequence $(x_n)_{n\in\mathbb{N}}: x_0>1$ and $x_{n+1}=1+\ln\left(\frac{2x_n}{1+x_n}\right),\forall n\in\mathbb{N}$. Find $\lim_{n\to\infty}n\ln x_n$.

Proposed by Marian Ursărescu – Romania

SP.150. Let be $f\in\mathbb{Z}, f=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, such that $a_1,a_2,\ldots,a_n\in\{\pm1,\pm2,\ldots,\pm n\}$. If a_0 is a prime number, $a_0>n^2$ then f is irreducible over $\mathbb{Z}$.

Proposed by Marian Ursărescu – Romania

UNDERGRADUATE PROBLEMS

UP.136. Prove that:

$$\sum_{k=0}^{n} T_{4k}(x) = \frac{1}{4}\left[\frac{2 + U_{4n+2}(x)}{x\sqrt{1-x^2}}\right]$$

where, $T_n(x)$ and $U_n(x)$ denotes the Chebyshev Polynomials of first and second kind.

Proposed by Shivam Sharma – New Delhi – India

UP.137. Let $f, g: \mathbb{R}_+^* \to \mathbb{R}_+^*$ be functions such that:

$\lim_{n\to\infty}\big(f(x+1) - f(x)\big) = a \in \mathbb{R}_+^*$, $\lim_{x\to\infty} \frac{g(x+1)}{xg(x)} = b \in \mathbb{R}_+^*$ and exists $\lim_{x\to\infty} \frac{f(x)}{x}$ and

$\lim_{x\to\infty} \frac{(g(x))^{\frac{1}{x}}}{x}$. For $t \in \mathbb{R}$ calculate the limit:

$$\lim_{x\to\infty}\big(f(x)\big)^{\cos^2 t}\left(\big(g(x)\big)^{\frac{\sin^2 t}{x+1}} - \big(g(x)\big)^{\frac{\sin^2 t}{x}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.138. Let $f, g: \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that:

$\lim_{x\to\infty}\big(f(x+1) - f(x)\big) = a \in \mathbb{R}_+^*$, $\lim_{x\to\infty} \frac{g(x+1)}{xg(x)} = b \in \mathbb{R}_+^*$ and exists

$\lim_{x\to\infty} \frac{f(x)}{x}, \lim_{x\to\infty} \frac{(g(x))^{\frac{1}{x}}}{x}$.

For $t \in \mathbb{R}$, calculate:

$$\lim_{x\to\infty}\big(f(x)\big)^{\sin^2 t}\left(\big(g(x)\big)^{\frac{\cos^2 t}{x+1}} - \big(g(x)\big)^{\frac{\cos^2 t}{x}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.139. Calculate:

$$\lim_{x\to\infty}\left(x^{\cosh^2 t}\left(\big(\Gamma(x+1)\big)^{-\frac{\sinh^2 t}{x}}\right) - \left(\big(\Gamma(x+2)\big)^{-\frac{\sinh^2 t}{x+1}}\right)\right)$$

where $t \in \mathbb{R}$ and Γ is the Gamma function (Euler integral of the second kind).

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.140. Calculate:

$$\lim_{x\to\infty}\left(x^{\sin^2 t}\left(\big(\Gamma(x+2)\big)^{\frac{\cos^2 t}{x+1}} - \big((\Gamma(x+1)\big)^{\frac{\cos^2 t}{x}}\right)\right)$$

where $t \in \mathbb{R}$ and Γ is the Gamma function (Euler integral of the second kind).

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.141. For $\{a_n\}_{n\geq 0}, a_n = \frac{(n+2)^{n+1}}{(n+1)^n}, x \in (-\infty, \infty), \{b_n(x)\}_{n\geq 1},$

$b_n(x) = n^{\sin^2 x}\left(a_{n+1}^{\cos^2 x} - a_n^{\cos^2 x}\right)$**, find** $\lim_{n\to\infty} b_n(x)$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.142. Let $(x_n)_{n\geq 1}$ **be a sequence which satisfy:**

$$-\ln(mn + x_n) + \sum_{k=1}^{mn} \frac{1}{k} = \gamma$$

where m **is positive integer and** γ **is Euler-Mascheroni's constant. Compute:** $\lim_{n\to\infty} x_n$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.143. Let $a, b \in \mathbb{R}_+, \gamma_n(a,b) = -\ln(n+a) + \sum_{k=1}^{n} \frac{1}{k+b}$ **with** $\lim_{n\to\infty} \gamma_n(a,b) = \gamma(a,b) \in \mathbb{R}$**. Calculate:**

$$\lim_{n\to\infty}\left(\ln\frac{e}{n+a} + \sum_{k=1}^{n}\frac{1}{k+b} - \gamma(a,b)\right)^n.$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.144. Let $a \in (0,\infty)$ **and** $e(x) = \left(1+\frac{1}{x}\right)^x$**, for any** $x \in (0,\infty)$**. Calculate:**

$$\lim_{n\to\infty}\left(\lim_{n\to\infty}\left(\left(\frac{x}{n}\right)^2\left(\sum_{k=1}^{n} e(x+ka) - ne(x)\right)\right)\right).$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.145. Let be $(x_n)_{n\geq 1}, x_n \in \mathbb{R}_+^*, \forall n \in \mathbb{N}^*$**, such that exists** $\lim_{n\to\infty}(x_{n+1} - x_n) = x \in \mathbb{R}_+^*$**. Find:**

$$\lim_{n\to\infty}\left(\frac{(n+1)_{x_{n+1}}}{\sqrt[n+1]{(2n+1)!!}} - \frac{nx_n}{\sqrt[n]{(2n-1)!!}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.146. Let $f:(0,\infty)\to(0,\infty)$ **be a function with** $\lim_{n\to\infty}\frac{f(x)}{x} = a \in (0,\infty)$ **and** $\in \mathbb{R}$**. Find:**

$$\lim_{n\to\infty}\left((n+1)^{\sin^2 t}\cdot\sqrt[n+1]{\left(f(1)f(2)\dots f(n)f(n+1)\right)^{\cos^2 t}} - n^{\sin^2 t}\cdot\sqrt[n]{\left(f(1)f(2)\dots f(n)\right)^{\cos^2 t}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.147. In an ABC **triangle let be** a, b, c **the lengths of** BC, CA, AB**, and** r_a, r_b, r_c **exradii. Prove that:**

$$\frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{r_b^2}{\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{r_c^2}{\tan\frac{A}{2}\tan\frac{B}{2}}\geq\frac{9(a^2+b^2+c^2)}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

UP.148. Let a,b,c be positive real numbers such that: $a+b+c=3$. Prove that:

$$2(a^2+b^2+c^2)+3\geq 3\sqrt{3abc(a^3b+b^3c+c^3a)}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

UP.149. Prove that:

$$\sum_{k=-l}^{l}\left[(-1)^k\binom{2l}{l+k}\binom{2m}{m+k}\binom{2n}{n+k}\right]=\frac{(l+m+n)!\,(2l)!\,(2m)!\,(2n)!}{(l+m)!\,(l+n)!\,(m+n)!\,(l)!\,(m)!\,(n)!}$$

Proposed by Shivam Sharma – New Delhi – India

UP.150. If $a,b,c>0$ then:

$$\int_0^1\int_0^1\int_0^1\frac{dxdydz}{1+(x^ay^bz^c)^{\frac{1}{a+b+c}}}\geq(a+b+c)\log 2$$

Proposed by Daniel Sitaru – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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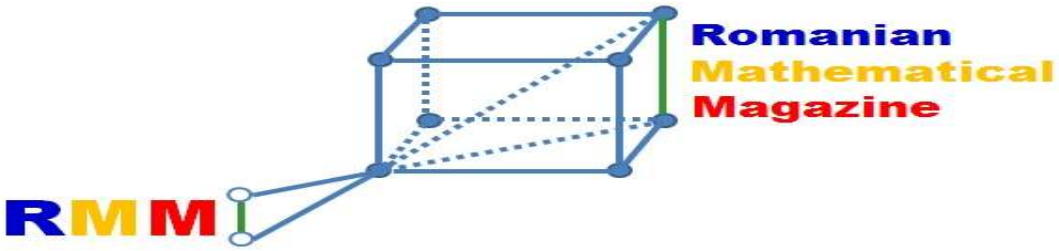

PROBLEMS FOR JUNIORS

JP.151. Let a,b be positive real numbers. Find the maximum of k such that inequality is true:

$$\frac{1}{a^4}+\frac{1}{b^4}+\frac{k}{a^4+b^4}\geq\frac{8k+32}{(a+b)^4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.152. Let ABC; h_a, h_b, h_c denote the lengths of the highways from the peaks A, B, C; l_a, l_b, l_c are the lengths of the symmetric divergence lines from the peaks A, B, C; r_a, r_b, r_c are the radius of the circle next to the corners A, B, C. Prove that:

$$\frac{h_a r_a}{l_a^2} + \frac{h_b r_b}{l_b^2} + \frac{h_c r_c}{l_c^2} \geq 3$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.153. In ABC the lengths BC, CA, AB are a, b, c. Let l_a, l_b, l_c are the length of the bisectors from the vertices A, B, C in triangle ABC. Prove that:

$$\frac{l_a l_b}{\sin\frac{C}{2}} + \frac{l_b l_c}{\sin\frac{A}{2}} + \frac{l_c l_a}{\sin\frac{B}{2}} \leq \frac{3}{2}\sqrt{3abc(a+b+c)}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.154. In triangle ABC with sides $BC = a, CA = b, AB = c$. r_a, r_b, r_c are exradii, h_a, h_b, h_c are distances from A, B, C to BC, CA, AB. Prove that:

$$\frac{r_a r_b}{\sin\frac{A}{2}\sin\frac{B}{2}} + \frac{r_b r_c}{\sin\frac{B}{2}\sin\frac{C}{2}} + \frac{r_c r_a}{\sin\frac{C}{2}\sin\frac{A}{2}} \geq \frac{h_a h_b}{\sin^2\frac{C}{2}} + \frac{h_b h_c}{\sin^2\frac{A}{2}} + \frac{h_c h_a}{\sin^2\frac{B}{2}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.155. Let a, b, c be positive real numbers such that: $a + b + c = 3$. Prove that:

$$\frac{a^2}{b^4 c\sqrt[3]{4(b^6+1)}} + \frac{b^2}{c^4 a\sqrt[3]{(c^6+1)}} + \frac{c^2}{a^4 b\sqrt[3]{4(a^6+1)}} \geq \frac{3}{2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.156. Let ABC be a triangle having the area S. Let be $A' \in BC$ such that the incircles of $\Delta AA'B, \Delta AA'C$ have the same radius. Analogous, we obtain the points $B' \in AC, C' \in AB$. Prove that:

$$S = \frac{AA' \cdot BB' \cdot CC'}{s}$$

Proposed by Marian Ursărescu – Romania

JP.157. Prove that in ΔABC, the following inequality holds:

$$\sum_{cyc}\left(a \cdot \cot\frac{A}{2} \cdot \sin^2 B \cdot \sin^2 C\right) \geq \frac{81r^4}{2R^2(2R-r)}$$

Proposed by Marian Ursărescu – Romania

JP.158. Let $a, b, c > 0$. Prove that:

$$\sum_{cyc}\frac{1}{a}+\sum_{cyc}\frac{a}{b^2+c^2}\geq 3\sum_{cyc}\frac{1}{b+c}$$

Proposed by Andrei Ștefan Mihalcea – Romania

JP.159. Prove that in any ΔABC the following inequality holds:

$$a^2h_bh_c+b^2h_ah_c+c^2h_ah_b\leq 4(R+r)^4$$

Proposed by Marian Ursărescu – Romania

JP.160. Prove that for all non-negative real numbers x, y, z the following inequality holds:

$$\frac{y+z}{(1+x)^2}+\frac{z+x}{(1+y)^2}+\frac{x+y}{(1+z)^2}\geq\frac{x}{1+yz}+\frac{y}{1+zx}+\frac{z}{1+xy}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

JP.161. Let a, b, c, d be positive real numbers such that $abcd\geq 1$. Prove that:

$$\frac{a^2+b^2+c^2+1}{a^3+b^3+c^3+1}+\frac{b^2+c^2+d^2+1}{b^3+c^3+d^3+1}+\frac{c^2+d^2+a^2+1}{c^3+d^3+a^3+1}+$$
$$+\frac{d^2+a^2+b^2+1}{d^3+a^3+b^3+1}\leq 4$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

JP.162. Let x, y, z be non-negative real numbers such that $x+y+z=1$. Prove that:

(a) $\frac{3}{\sqrt{10}}\leq\frac{x}{\sqrt{1+yz}}+\frac{y}{\sqrt{1+zx}}+\frac{z}{\sqrt{1+xy}}\leq 1.$

(b) $\sqrt{\frac{3}{5}}\leq\frac{x}{\sqrt{1+y+z}}+\frac{y}{\sqrt{1+z+x}}+\frac{z}{\sqrt{1+x+y}}\leq 1$. **When do the equalities occur?**

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

JP.163. Let ABC be an acute triangle with standard notations. Prove that:

$$\frac{m_a}{a}+\frac{m_b}{b}+\frac{m_c}{c}\leq\frac{a+b+c}{4r}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

JP.164. If $a, b\geq 0$ then: $\left(a+b+\sqrt{a^2+b^2}\right)^2\geq 6\sqrt{3}ab$

Proposed by Daniel Sitaru – Romania

JP.165. If $a, b, c\geq 0$ then:

$$4(a+b+c)\leq\left(3\sqrt{3}-2\right)\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEMS FOR SENIORS

SP.151. Let a, b, c be positive real numbers such that: $a + b + c = 3$. Prove that:

$$\frac{a}{\sqrt{2(b^4 + c^4)} + 7bc} + \frac{b}{\sqrt{2(c^4 + a^4)} + 7ca} + \frac{c}{\sqrt{2(a^4 + b^4)} + 7ab} \geq \frac{1}{3}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.152. Let a, b, c be positive real numbers. Find the minimum of value:

$$P = \frac{1}{\sqrt{2(a^4 + b^4)}} + \frac{1}{\sqrt{2(b^4 + c^4)}} + \frac{1}{\sqrt{2(c^4 + a^4)}} + \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.153. Solve the system of equations:

$$\begin{cases} 2\left(\frac{x^3}{y^2} + \frac{y^3}{x^2}\right) = \sqrt[4]{8(x^4 + y^4)} + 2\sqrt{xy} \\ 16x^5 - 20x^3 + 5\sqrt{xy} = \sqrt{\frac{y+1}{2}} \end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.154. Let x, y, z be positive real numbers such that: $x + y + z = 3$. Find the minimum of value:

$$P = \frac{\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} + 1}{4\left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right)} + \frac{8}{3}(x^2 + y^2 + z^2)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.155. Let x, y, z be positive real numbers such that: $xyz = 1$. Find the minimum of value:

$$P = \frac{x^3}{(2y^2 - yz + 2z^2)^2} + \frac{y^3}{(2z^2 - zx + 2x^2)^2} + \frac{z^3}{(2x^2 - xy + 2y^2)^2} + \frac{xy + yz + zx}{3}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.156. Find all the polynomials $p \in \mathbb{R}[x]$ having the property $P(x) = P\left(x + \sqrt{x^2 + 1}\right)$, $\forall x \in \mathbb{R}$.

Proposed by Marian Ursărescu – Romania

SP.157. Let $f: [0, +\infty) \to [0, +\infty)$ a derivable function and $a > 1$. If $f'(x)(f(x) + x^2 + 2x + a) = 1, \forall x \geq 0$ then: $\lim_{x\to\infty} f(x)$ exists and it is finite.

Proposed by Marian Ursărescu – Romania

SP.158. Prove that for any real numbers $a_1, a_2, \ldots, a_n$, the inequality holds:

$$\sum_{i=1}^{n} a_i(a_i + a_{i+1})\left(a_i^2 + a_{i+1}^2\right)\ldots\left(a_i^{2^k} + a_{i+1}^{2^k}\right) \geq 0$$

where k is a positive integer and $a_{n+1} = a_1$. When does the equality occur?

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

SP.159. Prove that in any triangle ABC with standard notations, the inequality holds:

$$(a+b+c)(a^2+b^2+c^2) \geq 2(b+c)h_a^2 + 2(c+a)h_b^2 + 2(a+b)h_c^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

SP.160. Let a, b, c be positive real numbers such that $a+b+c \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Prove that:

$$3(a^3b + b^3c + c^3a) \geq (a+b+c)^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

SP.161. Prove that the following inequalities hold for all real numbers $a, b, c \in [0,1]$

(a) $(1-a+a^2)(1-b+b^2)(1-c+c^2) \leq (1-abc+a^2b^2c^2)$.

(b) $(1-a+a^2)^2(1-b+b^2)^2(1-c+c^2)^2 \leq$

$\leq (1-ab+a^2b^2)(1-bc+b^2c^2)(1-ca+c^2a^2)$

When does the equality occur?

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

SP.162. If $m \in [0,\infty), x, y, z, t \in (0,\infty)$, then in any triangle ABC, with usual notations holds:

$$\sum_{cyclic} \frac{\left(xa^2 + ym_b^2\right)^{m+1}}{\left(zw_b^2 + th_c^2\right)^m} \geq \frac{(4x+3y)^{m+1}}{3^m(x+t)^m}\sqrt{3}S$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

SP.163. If $m \in [0,\infty), x, y, z, t \in (0,\infty)$, then in any triangle ABC, with usual notations holds:

$$\sum_{cyclic} \frac{(xa^2 + yb^2)^{m+1}}{\left(zw_b^2 + tw_c^2\right)^m} \geq \frac{4^{m+1}(x+y)^{m+1}}{3^{m-\frac{1}{2}}(z+t)^m}S$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

SP.164. If $a, b, c > 0$ then:

$$(a+b)\sqrt{a^2+b^2-ab} + (b+c)\sqrt{b^2+c^2-bc} + (c+a)\sqrt{c^2+a^2-ca} \geq$$
$$\geq 2(ab+bc+ca)$$

Proposed by Daniel Sitaru – Romania

SP.165. If $a, b, c \geq 0$ then:

$$(a+b)\sqrt{a^2+b^2} + (b+c)\sqrt{b^2+c^2} + (c+a)\sqrt{c^2+a^2} \geq$$

$$\geq \left(2\sqrt{3}-1\right)(ab+bc+ca)$$

Proposed by Daniel Sitaru – Romania

UNDERGRADUATE PROBLEMS

UP.151. Given real numbers $a_1, a_2, \ldots, a_n \in [0,1]$. Find the maximum and minimum possible value of

$$\Omega = a_1 + a_2 + \cdots + a_n + (1-a_1)(1-a_2)\ldots(1-a_n)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

UP.152. Let $a_1, a_2, \ldots, a_n (n \geq 3)$ be positive real numbers such that:

$$\sum_{1\leq i<j\leq n} a_i a_j \geq \frac{n(n-1)}{2}$$

Prove that:

$$\sum_{i=1}^{n} \frac{1}{a_i^2+n-1} \leq 1$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

UP.153. Find:

$$\Omega = \int_0^1 \int_0^\infty \left(\frac{x^{100}}{100^x}\right)(y\ln y)^{100}\,dxdy$$

Proposed by Ekpo Samuel – Nigeria

UP.154. If $a_n \in (0,\infty); n \in \mathbb{N}; n \geq 1; \lim_{n\to\infty} \frac{a_{n+1}}{n^2 a_n} = a > 0$ then find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{a_n}} \cdot \sum_{k=1}^{n}\left[\left(\sqrt[2k]{k!} + \sqrt[2k+1]{(k+1)!}\right)^2\right]\right)$$

$[*]$ **- great integer function.**

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

UP.155. If $m \in \mathbb{N}; x, y, z > 0$ then in ΔABC the following relationship holds:

$$3m + \left(\frac{a^2x^2}{yz}\right)^{m+1} + \left(\frac{b^2y^2}{zx}\right)^{m+1} + \left(\frac{c^2z^2}{xy}\right)^{m+1} \geq 4(m+1)\sqrt{3}S$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.156. If $a, b, c > 0; b < c; b(a+c) \in \mathbb{N}; b(a+c) > 1; n \in \mathbb{N}; n \geq 2; x_k \in [b,c]; t_k > 0; k \in \overline{1,n}$ then:

$$\left(\sum_{k=1}^{n} t_k x_k\right)\left(\sum_{k=1}^{n} \frac{t_k}{x_k}\right)^{b(a+c)} \leq \left(\frac{a+b+c}{1+b(a+c)}\right)^{1+ab+bc} \cdot \left(\sum_{k=1}^{n} t_k\right)^{1+ab+bc}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.157. If $m \geq 0; x, y > 0$ then in ΔABC the following relationship holds:

$$\frac{h_a}{h_b^m h_c^m (xh_b + yh_c)^m} + \frac{h_b}{h_c^m h_a^m (xh_a + yh_a)^m} + \frac{h_c}{h_a^m h_b^m (xh_a + yh_b)^m} \geq$$

$$\geq \frac{9}{(x+y)^{m \cdot s^{2m} \cdot r^{m-1}}}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.158. Find:

$$\Omega = \int_{9}^{1} \left(x^2 \log(x) \cot\left(\frac{\pi x}{2}\right)\right) dx$$

Proposed by Arafat Rahman Akib – Dhaka – Bangladesh

UP.159. Find:

$$\Omega = \int_{-1}^{+1} (x^2 + 2x + 1)^{\frac{3}{2}} (1-x)^{\frac{1}{2}} dx$$

Proposed by Ekpo Samuel – Nigeria

UP.160. Let $x, y, z \in (0; +\infty)$ and $a \geq 1$. Prove:

$$\frac{1}{([x] + \{y\} + 1)^a} + \frac{1}{([y] + \{z\} + 1)^a} + \frac{1}{([z] + \{x\} + 1)^a} \geq \frac{3^{a+1}}{(x+y+z+3)^a}$$

$[x]$ is the integer part of the real number x;

$\{x\}$ is the fraction of real numbers x.

Proposed by Nguyen Van Nho – Nghe An – Vietnam

UP.161. Find:

$$\Omega = \int_{0}^{\infty} \int_{0}^{\infty} \frac{dxdy}{(x^4 + 2x^2 + 1)(y^2 + 25)^4}$$

Proposed by Ekpo Samuel – Nigeria

UP.162. If $ABCD$ tetrahedron; $AB = a_1; AC = a_2; AD = a_3; BC = a_4; BD = a_5; CD = a_6$ then:

$$\sum_{1\le i<j\le 6}\left(a_i+a_j\right)^2 \ge 4\sqrt{3}S[ABCD]$$

$S[ABCD]$ – total area of tetrahedron $ABCD$.

Proposed by Daniel Sitaru – Romania

UP.163. Let a, b, c be positive real numbers such that:

$(a^2+b^2)(b^2+c^2)(c^2+a^2)=8$. Find the minimum value of:

$$P=\frac{a}{b(b+2c+1)(a+3c)^2}+\frac{b}{c(c+2a+1)(b+3a)^2}+\frac{c}{a(a+2b+1)(c+3b)^2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

UP.164. Solve the equation: $\sqrt{2(x^4+1)}+2\sqrt{3x-2x^4}=7-3x$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

UP.165. Solve the system of equations:

$$\begin{cases}\dfrac{1}{\sqrt{a^3}}+\dfrac{1}{\sqrt{b^3}}+\dfrac{1}{\sqrt{c^3}}=3\\ \dfrac{a^2}{b}+\dfrac{b^2}{c}+\dfrac{c^2}{a}=\dfrac{(a^3+b^3+c^3)^2}{3}\end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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