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A "PROBABILISTIC" METHOD FOR PROVING INEQUALITIES

By Daniel Sitaru and Claudia Nănuți

In this paper we solve a class of inequalities using an identity familiar from probability theory and classical mechanics. In the year 2000, Fuhua Wei and Shan – He Wu from the Departament of Mathematics and Computer Science, Longyan University, Longyan, Fujian 364012, P.R. China published the article: "Several proofs and generalizations of a fractional inequality with constraints." In this article, they give ten different proofs for the 2nd problem of the 36th IMO, held at Toronto (Canada) in 1995. In proof 5, the authors used a method based on a key random variable to prove that if a, b, c are positive real numbers with $abc = 1$ then:

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

Proof. We make the substitutions $x := bc, y := ca, z := ab$, and $s := x + y + z$. Then:

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} = \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} = \frac{x^2}{s-x} + \frac{y^2}{s-y} + \frac{z^2}{s-z}.$$

We consider the random variable ξ defined as follows:

$$\xi = \begin{cases} \frac{x}{s-x}: & \left(p = \frac{s-x}{2s}\right), \\ \frac{y}{s-y}: & \left(p = \frac{s-y}{2s}\right), \\ \frac{z}{s-z}: & \left(p = \frac{s-z}{2s}\right). \end{cases}$$

It follows that: $E(\xi) = \frac{x}{s-x} \cdot \frac{s-x}{2s} + \frac{y}{s-y} \cdot \frac{s-y}{2s} + \frac{z}{s-z} \cdot \frac{s-z}{2s} = \frac{x+y+z}{2s} = \frac{1}{2}$ and also

$$E(\xi^2) = \left(\frac{x}{s-x}\right)^2 \cdot \frac{s-x}{2s} + \left(\frac{y}{s-y}\right)^2 \cdot \frac{s-y}{2s} + \left(\frac{z}{s-z}\right)^2 \cdot \frac{s-z}{2s} = \frac{1}{2s}\left(\frac{x^2}{s-x} + \frac{y^2}{s-y} + \frac{z^2}{s-z}\right).$$

Now, the variance of ξ is given by $V(\xi) = E(\xi^2) - \left(E(\xi)\right)^2$. This is always non-negative, and positive unless ξ can take only value (in which case $x = y = z$ and $a = b = c$.) We thus have

$$\frac{1}{2s}\left(\frac{x^2}{s-x} + \frac{y^2}{s-y} + \frac{z^2}{s-z}\right) \geq \frac{1}{4}$$

and so $\frac{x^2}{s-x} + \frac{y^2}{s-y} + \frac{z^2}{s-z} \geq \frac{1}{2}s = \frac{1}{2}(x+y+z) \overset{AM-GM}{\geq} \frac{3}{2}\sqrt[3]{xyz} = \frac{3}{2}$. Hence

$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$ and this is strict unless $a = b = c$.

The method of proof used here is based on the positivity of variance:

$E(\xi^2) - \left(E(\xi)\right)^2 = V(\xi) = E\left(\left(\xi - E(\xi)\right)^2\right) \geq 0$, where $E(\xi^2) \geq \left(E(\xi)\right)^2$.

It can be applied to other problems as well. The technique is to construct a random variable such that its variance is the quantity, or difference, that we wish to show positive. (Readers familiar

with classical mechanics may prefer to consider this in terms of the parallel axix theorem for moments of intertia – a "mechanical" method of proof?)

Example 1. Prove that if $x, y, z > 0$ then:

$$\sqrt{\frac{x}{y}} + 2\sqrt{\frac{y}{z}} + 3\sqrt{\frac{z}{x}} \leq \sqrt{6\left(\frac{x}{y} + \frac{2y}{z} + \frac{3z}{x}\right)}$$

Solution. Define a random variable

$$\xi = \begin{cases} \sqrt{\frac{x}{y}}: & \left(p = \frac{1}{6}\right), \\ \sqrt{\frac{y}{z}}: & \left(p = \frac{2}{6}\right), \\ \sqrt{\frac{z}{x}}: & \left(p = \frac{3}{6}\right), \end{cases} \quad \text{then} \quad \xi^2 = \begin{cases} \frac{x}{y}: & \left(p = \frac{1}{6}\right), \\ \frac{y}{z}: & \left(p = \frac{2}{6}\right), \\ \frac{z}{x}: & \left(p = \frac{3}{6}\right). \end{cases}$$

It follows that: $E(\xi) = \frac{1}{6}\sqrt{\frac{x}{y}} + \frac{2}{6}\sqrt{\frac{y}{z}} + \frac{3}{6}\sqrt{\frac{z}{x}}$ and $E(\xi^2) = \frac{1}{6}\left(\frac{x}{y} + \frac{2y}{z} + \frac{3z}{x}\right)$. As $E(\xi^2) \geq \left(E(\xi)\right)^2$,

we have: $\frac{1}{6}\left(\frac{x}{y} + \frac{2y}{z} + \frac{3z}{x}\right) \geq \left[\frac{1}{6}\left(\sqrt{\frac{x}{y}} + 2\sqrt{\frac{y}{z}} + 3\sqrt{\frac{z}{x}}\right)\right]^2$, $\frac{x}{y} + \frac{2y}{z} + \frac{3z}{x} \geq \frac{1}{6}\left(\sqrt{\frac{x}{y}} + 2\sqrt{\frac{y}{z}} + 3\sqrt{\frac{z}{x}}\right)^2$,

$\sqrt{\frac{x}{y}} + 2\sqrt{\frac{y}{z}} + 3\sqrt{\frac{z}{x}} \leq \sqrt{6\left(\frac{x}{y} + \frac{2y}{z} + \frac{3z}{x}\right)}$ and, again, equality holds only for $x = y = z$.

Example 2. Prove that if $a, b, c > 0$ then:

$$\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}} \leq \sqrt{7\left(\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}\right)}$$

Solution. Define a random variable

$$\xi = \begin{cases} \sqrt{\frac{a}{b+c}}: & \left(p = \frac{1}{7}\right), \\ \sqrt{\frac{b}{c+a}}: & \left(p = \frac{2}{7}\right), \\ \sqrt{\frac{c}{a+b}}: & \left(p = \frac{4}{7}\right). \end{cases}$$

As before we get $E(\xi) = \frac{1}{7}\left(\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}}\right)$ and $E(\xi^2) = \frac{1}{7}\left(\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}\right)$,

and the inequality $\frac{1}{7}\left(\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}\right) \geq \frac{1}{49}\left(\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}}\right)^2$.

Therefore

$$\frac{1}{\sqrt{7}}\cdot\sqrt{\frac{a}{b+c}+\frac{2b}{c+a}+\frac{4c}{a+b}}\geq\frac{1}{7}\left(\sqrt{\frac{a}{b+c}}+2\sqrt{\frac{b}{c+a}}+4\sqrt{\frac{c}{a+b}}\right),$$

and

$$\sqrt{\frac{a}{b+c}}+2\sqrt{\frac{b}{c+a}}+4\sqrt{\frac{c}{a+b}}\leq\sqrt{7\left(\frac{a}{b+c}+\frac{2b}{c+a}+\frac{4c}{a+b}\right)},$$

with equality only for $a=b=c$.

Application 3. Prove that in any triangle ABC the following relationship holds for the medians m_a, m_b, m_c and altitudes h_a, h_b, h_c:

$$3\sqrt{\frac{m_a}{m_b}+\frac{2m_b}{m_c}+\frac{6m_c}{m_a}}\geq\sqrt{\frac{h_a}{m_b}}+2\sqrt{\frac{h_b}{m_c}}+6\sqrt{\frac{h_c}{m_a}}$$

Solution. Let be the probability distribution sequence of random variable ξ below:

Define a random variable

$$\xi=\begin{cases}\sqrt{\frac{m_a}{m_b}}: & \left(p=\frac{1}{9}\right)\\ \sqrt{\frac{m_b}{m_c}}: & \left(p=\frac{2}{9}\right)\\ \sqrt{\frac{m_c}{m_a}}: & \left(p=\frac{6}{9}\right).\end{cases}$$

It follows that: $E(\xi)=\frac{1}{9}\left(\sqrt{\frac{m_a}{m_b}}+2\sqrt{\frac{m_b}{m_c}}+6\sqrt{\frac{m_c}{m_a}}\right)$ and $E(\xi^2)=\frac{1}{9}\left(\frac{m_a}{m_b}+\frac{2m_b}{m_c}+\frac{6m_c}{m_a}\right)$,

and, $m_a\geq h_a, m_b\geq h_b$, and $m_c\geq h_c$, we have

$$\frac{1}{9}\left(\frac{m_a}{m_b}+\frac{2m_b}{m_c}+\frac{6m_c}{m_a}\right)\geq\frac{1}{81}\left(\sqrt{\frac{m_a}{m_b}}+2\sqrt{\frac{m_b}{m_c}}+6\sqrt{\frac{m_c}{m_a}}\right)^2$$

$$\geq\frac{1}{81}\left(\sqrt{\frac{h_a}{m_b}}+2\sqrt{\frac{h_b}{m_c}}+6\sqrt{\frac{h_c}{m_a}}\right)^2,$$

whence

$$9\left(\frac{m_a}{m_b}+\frac{2m_b}{m_c}+\frac{6m_c}{m_a}\right)\geq\left(\sqrt{\frac{h_a}{m_b}}+2\sqrt{\frac{h_b}{m_c}}+6\sqrt{\frac{h_c}{m_a}}\right)^2$$

and

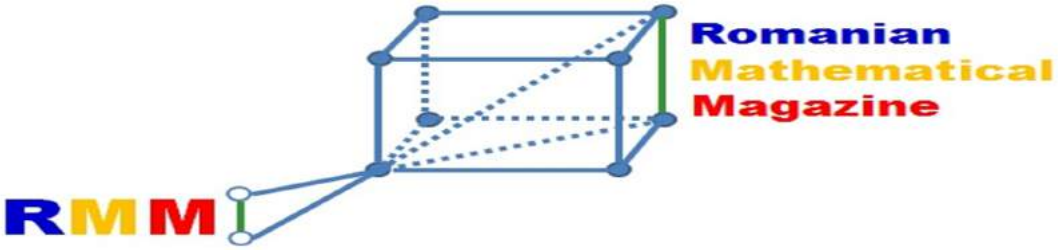
$$3\sqrt{\frac{m_a}{m_b}+\frac{2m_b}{m_c}+\frac{6m_c}{m_a}} \geq \sqrt{\frac{h_a}{m_b}}+2\sqrt{\frac{h_b}{m_c}}+6\sqrt{\frac{h_c}{m_a}},$$

which completes the solution. Of course, applying this process in reverse is an intriguing way to invent new inequalities!

References:

[1] Shan – He Wu, Mihàly Bencze, *Selected problems and theorems of analytic inequalities.* Studis Publishing House, Iași, Romania, 2012.

[2] Daniel Sitaru, *Math Phenomenon.* Paralela 45 Publishing House, Pitești, Romania, 2016.



ASUPRA CALCULULUI UNOR LIMITE DE FUNCȚII ȘI ASUPRA CALCULULUI UNOR LIMITE DE ȘIRURI

By D.M. Bătinețu Giurgiu, N Mușuroia, Daniel Sitaru-Romania

În lucrarea [1] a fost introdus conceptul de șir Lalescu iar în [2] au fost definite funcțiile Euler-Lalescu, arătându-se că: $\lim_{x\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}-\left(\Gamma(x+1)\right)^{\frac{1}{x}}\right)=\frac{1}{e}$ (1)

Unde, $\Gamma:\mathbb{R}_+^*\to\mathbb{R}_+^*=(0,\infty),\Gamma(x)=\int_0^\infty t^{x-1}\cdot e^{-t}\cdot dt$, este funcția lui Euler de speța a doua.În prezentul articol ne propunem să extindem unele dintre aceste rezultate iar pentru aceasta vom nota $\mathcal{F}(\mathbb{R}_+^*)=\{f|f:\mathbb{R}_+^*\to\mathbb{R}_+^*\}$ (2)

Definiția 1. Fie $(a,t)\in\mathbb{R}_+^*\times\mathbb{R}_+$. Spunem că funcția $f\in\mathcal{F}(\mathbb{R}_+^*)$ este o (a,t) – funcție Lalescu, dacă există $\lim_{x\to\infty}\frac{f(x+1)}{f(x)\cdot x^{t+1}}=a$ și există $\lim_{x\to\infty}\left(\frac{1}{x}\cdot\left(f(x)\right)^{\frac{1}{x}}\right)$

Definiția 2. Fie $(a,s)\in\mathbb{R}_+^*\times\mathbb{R}_+$. Spunem că funcția $f\in\mathcal{F}(\mathbb{R}_+^*)$ este o (a,s) – funcție Bătinețu-Giurgiu dacă există: $\lim_{x\to\infty}\frac{f(x+1)x^{s+1}}{f(x)}=a$ și există $\lim_{x\to\infty}\left(x^{s+1}\left(f(x)\right)^{\frac{1}{x}}\right)$

Definiția 3. Fie $a\in\mathbb{R}_+^*$. Spunem că funcția $f\in\mathcal{F}(\mathbb{R}_+^*)$ este o funcție Ianculescu, dacă există:

$\lim_{x\to\infty}\frac{f(x+1)}{f(x)}=a$ și există $\lim_{x\to\infty}\left(f(x)\right)^{\frac{1}{x}}$

În legătură cu conceptele definite mai sus vom demonstra unele propoziții și unele teoreme, care vor scoate în evidență proprietăți interesante ale acestor concepte.

Propoziția 1. Dacă $f \in \mathcal{F}(\mathbb{R}_+^*)$ este o (a,t) – funcție Lalescu, atunci:

$$\lim_{x\to\infty}\left(\frac{1}{x^{t+1}}\cdot f\left((x)\right)^{\frac{1}{x}}\right) = a\cdot e^{-(t+1)} \quad (3)$$

Demonstrație. Deoarece există $\lim_{x\to\infty}\left(\frac{1}{x^{t+1}}\left(f(x)\right)^{\frac{1}{x}}\right)$ atunci:

$$\lim_{x\to\infty}\left(\frac{1}{x^{t+1}}\left(f(x)\right)^{\frac{1}{x}}\right) = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}^*}}\left(\frac{1}{n^{t+1}}\cdot\left(f(n)\right)^{\frac{1}{n}}\right) = \lim_{n\to\infty}\frac{\sqrt[n]{f(n)}}{n^{t+1}} =$$

$$= \lim_{n\to\infty}\sqrt[n]{\frac{f(n)}{n^{n(t+1)}}} \overset{CDA}{=} \lim_{n\to\infty}\frac{f(n+1)}{(n+1)^{(n+1)(t+1)}}\cdot\frac{n^{n(t+1)}}{f(n)} = \lim_{n\to\infty}\frac{f(n+1)}{f(n)\cdot n^{t+1}}\cdot\left(\frac{n}{n+1}\right)^{(n+1)(t+1)}$$

$$= a\cdot\frac{1}{e^{t+1}} = a\cdot e^{-(t+1)}$$

Propoziția 2. Dacă $f \in \mathcal{F}(\mathbb{R}_+^*)$ este o (a,s) – funcție Bătinețu – Giurgiu, atunci:

$$\lim_{x\to\infty}\left(x^{s+1}\left(f(x)\right)^{\frac{1}{x}}\right) = a\cdot e^{s+1} \quad (4)$$

Demonstrație. Deoarece există $\lim_{x\to\infty}\left(x^{s+1}\left(f(x)\right)^{\frac{1}{x}}\right)$ atunci

$$\lim_{x\to a}\left(x^{s+1}\left(f(x)\right)^{\frac{1}{x}}\right) = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}^*}}\left(n^{s+1}\cdot\left(f(n)\right)^{\frac{1}{n}}\right) = \lim_{n\to\infty}\left(n^{s+1}\cdot\sqrt[n]{f(n)}\right)$$

$$= \lim_{n\to\infty}\sqrt[n]{f(n)\cdot n^{n(s+1)}} \overset{CDA}{=} \lim_{n\to\infty}\frac{f(n+1)(n+1)^{(n+1)(s+1)}}{f(n)\cdot n^{n(s+1)}}$$

$$= \lim_{n\to\infty}\frac{f(n+1)\cdot n^{s+1}}{f(n)}\cdot\left(\frac{n+1}{n}\right)^{(n+1)(s+1)} = a\cdot e^{s+1}$$

Propoziția 3. Dacă $f \in \mathcal{F}(\mathbb{R}_+^*)$ este o funcție lanculescu atunci:

$$\lim_{x\to\infty}\left(f(x)\right)^{\frac{1}{x}} = a \quad (5)$$

Demonstrație. Deoarece există $\lim_{x\to\infty}\left(f(x)\right)^{\frac{1}{x}}$ atunci:

$$\lim_{x\to\infty} f\left(f(x)\right)^{\frac{1}{x}} = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}^*}}\left(f(n)\right)^{\frac{1}{n}} = \lim_{n\to\infty}\sqrt[n]{f(n)} \overset{CDA}{=} \lim_{n\to\infty}\frac{f(n+1)}{f(n)} = a$$

Teorema 1. Dacă $f \in \mathcal{F}(\mathbb{R}_+^*)$ este o (a,t) – funcție Lalescu unde $(a,t) \in \mathbb{R}_+^* \times \mathbb{R}_+$, atunci:

$$\lim_{x\to\infty}\frac{1}{x^t}\left(\left(f(x+1)\right)^{\frac{1}{x+1}} - \left(f(x)\right)^{\frac{1}{x}}\right) = a(t+1)e^{-(t+1)} \quad (6)$$

Demonstrație. Fie $u: \mathbb{R}_+^* \to \mathbb{R}_+^*, u(x) = f\big((x+1)\big)^{\frac{1}{x+1}} \cdot \big(f(x)\big)^{-\frac{1}{x}}$ atunci:

$$\lim_{x\to\infty} u(x) = \lim_{x\to\infty} \frac{\big(f(x+1)\big)^{\frac{1}{x+1}}}{\big(f(x)\big)^{\frac{1}{x}}} = \lim_{x\to\infty} \frac{\big(f(x+1)\big)}{(x+1)^{t+1}} \cdot \frac{x^t}{\big(f(x)\big)^{\frac{1}{x}}} \left(\frac{x+1}{x}\right)^{t+1} =$$

$= a \cdot \frac{1}{a} \cdot 1 = 1$ și deci $\lim_{x\to\infty} \frac{u(x)-1}{\ln u(x)} = 1$ precum și

$$\lim_{x\to\infty} \big(u(x)\big)^x = \lim_{x\to\infty} \frac{f(x+1)}{f(x)} \cdot \frac{1}{\big(f(x+1)\big)^{\frac{1}{x+1}}} =$$

$$= \lim_{x\to\infty} \frac{f(x+1)}{f(x)\cdot x^{t+1}} \cdot \frac{(x+1)^{t+1}}{\big(f(x+1)\big)^{\frac{1}{x+1}}} \cdot \left(\frac{x}{x+1}\right)^{t+1} = a \cdot \frac{1}{a\cdot e^{-(t+1)}} \cdot 1 = e^{t+1}$$

Prin urmare, $B(x) = \frac{1}{x^t} \cdot \left(\big(f(x+1)\big)^{\frac{1}{x+1}} - \big(f(x)\big)^{\frac{1}{x}}\right) = \frac{(f(x))^{\frac{1}{x}}}{x^t}(u(x)-1) =$

$= \frac{(f(x))^{\frac{1}{x}}}{x^t} \cdot \frac{u(x)-1}{\ln u(x)} \ln u(x) = \frac{(f(x))^{\frac{1}{x}}}{x^{t+1}} \cdot \frac{u(x)-1}{\ln u(x)} \cdot \ln\big(u(x)\big)^x$ și atunci:

$$\lim_{x\to\infty} B(x) = \frac{a}{e^{t+1}} \cdot 1 \cdot \ln e^{t+1} = \frac{a(t+1)}{e^{t+1}} = a(t+1)\cdot e^{-(t+1)}$$

Teorema 2. Dacă $f \in \mathcal{F}(\mathbb{R}_+^*)$ este o (a,s) – funcție Bătinețu – Giurgiu unde $(a,s) \in \mathbb{R}_+^* \times R_+$, atunci:

$$\lim_{x\to\infty} \left((x+1)^{s+2} \cdot \big(f(x+1)\big)^{\frac{1}{x+1}} - x^{s+2}\big(f(x)\big)^{\frac{1}{x}}\right) = a \cdot e^{s+1} \quad (7)$$

Demonstrație. Fie $v: \mathbb{R}_+^* \to \mathbb{R}_+^*, v(x) = \left(\frac{x+1}{x}\right)^{s+2} \cdot \frac{(f(x+1))^{\frac{1}{x+1}}}{(f(x))^{\frac{1}{x}}}$ atunci:

$$\lim_{x\to\infty} v(x) = \lim_{x\to\infty} \left((x+1)^{s+1} \cdot \big(f(x+1)\big)^{\frac{1}{x+1}} \cdot \frac{1}{x^{s+1}\big(f(x)\big)^{\frac{1}{x}}} \cdot \frac{x+1}{x}\right) =$$

$= a \cdot e^{s+1} \cdot \frac{1}{a\cdot e^{s+1}} \cdot 1 = 1$ și deci $\lim_{x\to\infty} \frac{v(x)-1}{\ln v(x)} = 1$ iar

$$\lim_{x\to\infty} \big(v(x)\big)^x = \lim_{x\to\infty} \left(\left(\frac{x+1}{x}\right)^{(s+2)x} \cdot \frac{f(x+1)}{f(x)} \cdot \frac{1}{\big(f(x+1)\big)^{\frac{1}{x+1}}}\right) =$$

$$= e^{s+2} \cdot \lim_{x\to\infty} \frac{x^{s+1} f(x+1)}{f(x)} \cdot \frac{1}{(x+1)^{s+1} \cdot \big(f(x+1)\big)^{\frac{1}{x+1}}} \cdot \left(\frac{x+1}{x}\right)^{s+1} =$$

$$= e^{s+2} \cdot a \cdot \frac{1}{a\cdot e^{s+1}} \cdot 1 = e$$

Prin umare,

$$G(x) = (x+1)^{s+2}\cdot\left(f(x+1)^{\frac{1}{x+1}} - x^{s+2}\left(f(x)\right)^{\frac{1}{x}}\right) = x^{s+2}\left(f(x)\right)^{\frac{1}{x}}\cdot(v(x)-1) =$$

$$= x^{s+2}\left(f(x)\right)^{\frac{1}{x}}\cdot\frac{v(x)-1}{\ln v(x)}\cdot\ln v(x) = x^{s+1}\left(f(x)\right)^{\frac{1}{x}}\cdot\frac{v(x)-1}{\ln v(x)}\cdot\ln\left(v(x)\right)^{x} \text{ și atunci:}$$

$$\lim_{x\to\infty} G(x) = a\cdot e^{s+1}\cdot 1\cdot\ln e = a\cdot e^{s+1}.$$

Teorema 3. Dacă $f\in\mathcal{F}(\mathbb{R}_+^*)$ este o a – funcție lanculescu, unde $a\in\mathbb{R}_+^*$, atunci:

$$\lim_{x\to\infty}\left((x+1)\left(f(x+1)\right)^{\frac{1}{x+1}} - x\cdot\left(f(x)\right)^{\frac{1}{x}}\right) = a \quad (8)$$

Demonstrație. Fie $w\in\mathcal{F}(\mathbb{R}_+^*), w(x) = \frac{x+1}{x}\cdot\frac{\left(f(x+1)\right)^{\frac{1}{x+1}}}{\left(f(x)\right)^{\frac{1}{x}}}$, atunci, $\lim_{x\to\infty} w(x) = 1\cdot\frac{a}{a} = 1$ și deci $\lim_{x\to\infty}\frac{w(x)-1}{\ln w(x)} = 1$ iar $\lim_{x\to\infty}\left(w(x)\right)^{x} = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{x}\cdot\frac{f(x+1)}{f(x)}\cdot\frac{1}{\left(f(x+1)\right)^{\frac{1}{x+1}}} = e\cdot a\cdot\frac{1}{a} = e$. Prin urmare, $D(x) = (x+1)\cdot\left(f(x+1)\right)^{\frac{1}{x+1}} - x\cdot\left(f(x)\right)^{\frac{1}{x}} = x\left(f(x)\right)^{\frac{1}{x}}\cdot(w(x)-1) = x\left(f(x)\right)^{\frac{1}{x}}\cdot\frac{w(x)-1}{\ln w(x)}\cdot\ln w(x) = \left(f(x)\right)^{\frac{1}{x}}\cdot\frac{w(x)-1}{\ln w(x)}\cdot\ln\left(w(x)\right)^{x}$

și atunci $\lim_{x\to\infty} D(x) = a\cdot 1\ln e = a$

Aplicații

A.1. Dacă în definiția 1 considerăm $f\in\mathcal{F}(\mathbb{R}_+^*), f(x) = \Gamma(x+1)$ și $t = 0$ obținem

$$a = \lim_{x\to\infty}\frac{f(x+1)}{xf(x)} = \lim_{x\to\infty}\frac{\Gamma(x+2)}{x\cdot\Gamma(x+1)} = \lim_{x\to\infty}\frac{x\cdot\Gamma(x+1)}{x\cdot\Gamma(x+1)} = 1$$

și atunci avem $\lim_{x\to\infty}\frac{\left(f(x)\right)^{\frac{1}{x}}}{x} = \lim_{x\to\infty}\frac{\left(\Gamma(x+1)\right)^{\frac{1}{x}}}{x} = \frac{1}{e}$ prin urmare am obținut rezultatul (1).

In particular avem $\lim_{\substack{x\to\infty\\ n\in\mathbb{N}^*}}\left(\left(\Gamma(n+2)\right)^{\frac{1}{n+1}} - \left(\Gamma(n+1)\right)^{\frac{1}{n}}\right) = \lim_{x\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) = e^{-1}$ adică am obținut limita șirului lui Traian Lalescu.

A.2. Deci în definiția 2 considerăm $f\in\mathcal{F}(\mathbb{R}_+^*), f(x) = \frac{1}{\Gamma(x+1)} = \left(\Gamma(x+1)\right)^{-1}$ obținem,

pentru $s = 0$ că $a = \lim_{x\to\infty}\frac{f(x+1)x}{f(x)} = \lim_{x\to\infty}\frac{x\cdot\Gamma(x+1)}{\Gamma(x+2)} = 1$ și deci relația (7) devine

$$\lim_{x\to\infty}\left((x+1)^2\frac{1}{\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}} - x^2\cdot\frac{1}{\left(\Gamma(x+1)\right)^{\frac{1}{x}}}\right) = e \quad (9)$$

În particular relația (9) ne conduce la faptul că:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\left(\Gamma(n+2)\right)^{\frac{1}{n+1}}}-\frac{n^2}{\left(\Gamma(n+1)\right)^{\frac{1}{n}}}\right)=\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\right)=e$$

adică am abținut limita șirului Bătinețut – Giurgiu.

A.3. Dacă în definiția 3 considerăm $f(x)=x$ atunci: $a=\lim_{x\to\infty}\frac{x+1}{x}=1$ și atunci relația (8) devine: $\lim_{x\to\infty}\left((x+1)(x+1)^{\frac{1}{x+1}}-x\cdot x^{\frac{1}{x}}\right)=\lim_{x\to\infty}\left((x+1)^{1+\frac{1}{x+1}}-x^{1+\frac{1}{x}}\right)=1$

in particular rezultă: $\lim_{n\to\infty}\left((n+1)\sqrt[n+1]{n+1}-n\cdot\sqrt[n]{n}\right)=1$ adică am obținut limita șirului lui Romeo T. Ianculescu (Gazeta Matematică, anul 1914).

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UN NUMAR REMARCABIL

$$\sqrt[p]{n+\sqrt[p]{n+\sqrt[p]{n+\sqrt[p]{\ldots+\sqrt[p]{n}}}}}$$

By Chirfot Carmen – Victoriţa-Romania

Fie numărul $N=\underbrace{\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{\ldots+\sqrt{n}}}}}}_{k\ radicali}, n\in\mathbb{N}, k\in\mathbb{N}^*$. Pentru $n=1\cdot 2=2\Rightarrow$

$$\Rightarrow 1<\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\ldots+\sqrt{2}}}}}}_{k\ radicali}<2, k\in\mathbb{N}^*.\ \text{Pentru}\ n=2\cdot 3=6\Rightarrow$$

$\Rightarrow 2 < \underbrace{\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{...+\sqrt{6}}}}}}_{k\ radicali} < 3$ .Dacă $n = 3 \cdot 4 = 12 \Rightarrow$

$\Rightarrow 3 < \underbrace{\sqrt{12+\sqrt{12+\sqrt{12+\sqrt{...+\sqrt{12}}}}}}_{k\ radicali} < 4$. Analog, dacă $n = p \cdot (p+1), p \in \mathbb{N}^*$, se demonstrează că

$p < \underbrace{\sqrt{p(p+1)+\sqrt{p(p+1)+\sqrt{p(p+1)+\sqrt{...+p(p+1)}}}}}_{k\ radicali} < p+1$. Relația rămâne valabilă și pentru $p \in \mathbb{R}_+^*$. Partea dreaptă a acestei inegalități se poate demonstra prin inducție matematică după k. Partea stângă rezultă din

$$p < \sqrt{p(p+1)} \leq \underbrace{\sqrt{p(p+1)+\sqrt{p(p+1)+\sqrt{p(p+1)+\sqrt{...+p(p+1)}}}}}_{k\ radicali}$$

Să observăm, de exemplu, că $2 < \underbrace{\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{...+\sqrt{6}}}}}}_{k\ radicali} < 3$ și

$3 < \underbrace{\sqrt{12+\sqrt{12+\sqrt{12+\sqrt{...+\sqrt{12}}}}}}_{k\ radicali} < 4$. Dacă analizăm numerele

$\underbrace{\sqrt{10+\sqrt{10+\sqrt{10+\sqrt{...+\sqrt{10}}}}}}_{k\ radicali}$ și $\underbrace{\sqrt{11+\sqrt{11+\sqrt{11+\sqrt{...+\sqrt{11}}}}}}_{k\ radicali}$, observăm că sunt mai mari decât 3 (evident mai mici decât 4).

Dar și $\underbrace{\sqrt{9+\sqrt{9+\sqrt{9+\sqrt{...+\sqrt{9}}}}}}_{k\ radicali} > \sqrt{9} = 3$, $\underbrace{\sqrt{8+\sqrt{8+\sqrt{8+\sqrt{...+\sqrt{8}}}}}}_{k\ radicali} >$

$> \underbrace{\sqrt{8+\sqrt{8+\sqrt{8+\sqrt{...+\sqrt{8+2}}}}}}_{k-1\ radicali} > \cdots > 3$ și $\underbrace{\sqrt{7+\sqrt{7+\sqrt{7+\sqrt{...+\sqrt{7}}}}}}_{k\ radicali} > 3$ pentru

$k > 1$. Deci, primul număr natural n pentru care $\underbrace{\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{...+\sqrt{n}}}}}}_{k\ radicali} > 3$ este

$n = 7 = 2 \cdot 3 + 1$. Dacă analizăm numărul $\underbrace{\sqrt{13+\sqrt{13+\sqrt{13+\sqrt{...+\sqrt{13}}}}}}_{k\ radicali}$, observăm că

$4 < \underbrace{\sqrt{13+\sqrt{13+\sqrt{13+\sqrt{...+\sqrt{13}}}}}}_{k\ radicali}$, pentru $k > 1$. Deci, primul număr n pentru care

$\underbrace{\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{...+\sqrt{n}}}}}}_{k\ radicali} > 4$ și $k > 1$, este $n = 13 = 3 \cdot 4 + 1$.

Prin inducție după k demonstrăm că $m < \underbrace{\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{...+\sqrt{n}}}}}}_{k\ radicali} < m + 1$ pentru

$n = \overline{(m-1) \cdot m + 1, m \cdot (m+1)}$ cu $m \in \mathbb{N}^*$ și $k > 1$. Pentru $k = 2$, avem

$m < \sqrt{n+\sqrt{n}} < m + 1, n = \overline{(m-1) \cdot m + 1, m \cdot (m+1)}$, și demonstrăm că propoziția este adevărată. Din $(m-1)m + 1 < n < m(m+1) \Rightarrow$

$\sqrt{(m-1)m+1} < \sqrt{n} < \sqrt{m(m+1)}$. Dar

$m - 1 < \sqrt{(m-1)m+1} < \sqrt{n} < \sqrt{m(m+1)} < m + 1 \Rightarrow m - 1 < \sqrt{n} < m + 1 \Rightarrow$

$\Rightarrow m + n - 1 < n + \sqrt{n} < m + n + 1.$

Cum $m + (m-1)m + 1 - 1 < m + n - 1 < n + \sqrt{n} < m + m(m+1) + 1 \Rightarrow$

$\Rightarrow m^2 < m + n - 1 < n + \sqrt{n} < m^2 + 2m + 1 \Rightarrow m < \sqrt{n+\sqrt{n}} < m + 1$. Pentru partea stângă a inegalității de demonstrat, avem $m < \sqrt{n+\sqrt{n}} \leq \underbrace{\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{...+\sqrt{n}}}}}}_{k\ radicali} \Rightarrow$

$m < \underbrace{\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{...+\sqrt{n}}}}}}_{k\ radicali}, k \geq 3$. Pentru partea dreaptă a inegalității, presupunem

$$\underbrace{\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{...+\sqrt{n}}}}}}_{k\ radicali} < m+1 \text{ cu}$$

$n = \overline{(m-1)\cdot m+1, m\cdot m\cdot(m+1)}, m \in \mathbb{N}^*, k > 1$ și demonstrăm că și

$$\underbrace{\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{...+\sqrt{n}}}}}}_{k+1\ radicali} < m+1 \text{ este adevărată. Din}$$

$$\underbrace{\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{...+\sqrt{n}}}}}}_{k\ radicali} < m+1 \Rightarrow n+\underbrace{\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{...+\sqrt{n}}}}}}_{k\ radicali} < m+1+n, \text{ deci}$$

$$n+\underbrace{\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{...+\sqrt{n}}}}}}_{k\ radicali} < m+1+n < m+1+m(m+1) \Rightarrow$$

$$\Rightarrow \underbrace{\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{...+\sqrt{n+\sqrt{n}}}}}}}_{k+1\ radicali} < m+1. \text{ Conform principiului inducției matematice, rezultă}$$

$$\text{că } m < \underbrace{\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{...+\sqrt{n}}}}}}_{k\ radicali} < m+1 \text{ cu } n = \overline{(m-1)\cdot m+1, m\cdot(m+1)} =$$

$= \overline{m^2-m+1, (m+1)^2-(m+1)}, m \in \mathbb{N}^*$ pentru orice număr natural $k > 1$.

Observații:

1) Din $x = p\cdot(p+1)$ cu $p \in \mathbb{R}_+^*$, deci $x \in \mathbb{R}_+^*$, rezultă că $p^2+p-x = 0 \Rightarrow$

$$\Rightarrow p = \frac{-1+\sqrt{1+4x}}{2}, \text{ deci } \frac{-1+\sqrt{1+4x}}{2} < \underbrace{\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{...+\sqrt{x}}}}}}_{k\ radicali} < \frac{1+\sqrt{1+4x}}{2}, x \in \mathbb{R}_+^*.$$

2) Dacă $y = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{...}}}}, x \geq 0$, atunci $y^2 = x + \sqrt{x + \sqrt{x + \sqrt{...}}} \Rightarrow$

$\Rightarrow y^2 - y - x = 0$. Deci, $y = \frac{1+\sqrt{1+4x}}{2} \Rightarrow \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{...}}}} = \frac{1+\sqrt{1+4x}}{2}, x \geq 0.$

3) Pentru că $\frac{1+\sqrt{1+4x}}{2} \in \mathbb{N}, x \in \mathbb{N} \Rightarrow \sqrt{1+4x} \in \mathbb{N}$ şi să fie impar. Deci,

$x = m(m+1)$ şi $\sqrt{1+4x} = 2m+1 \in \mathbb{N}$.

4) Tot prin inducţie matematică după k se poate demonstra că

$$m < \underbrace{\sqrt[3]{n + \sqrt[3]{n + \sqrt[3]{n + \sqrt[3]{... + \sqrt[3]{n}}}}}}_{k\ radicali} < m+1, m, n \in \mathbb{N}^* \text{ şi } k \in \mathbb{N}^* - \{1\} \text{ are loc pentru}$$

$n = \overline{(m-1)m(m+1)+1, m(m+1)(m+2)} = \overline{m^3 - m + 1, (m+1)^3 - (m+1)}.$

5) Pentru radicalul de ordin 4, prin inducţie matematică după k se poate demonstra că

$$m < \underbrace{\sqrt[4]{n + \sqrt[4]{n + \sqrt[4]{n + \sqrt[4]{... + \sqrt[4]{n}}}}}}_{k\ radicali} < m+1, m, n \in \mathbb{N}^* \text{ şi } k \in \mathbb{N}^* - \{1\} \text{ are loc pentru}$$

$n = \overline{m^4 - m + 1, (m+1)^4 - (m+1)}.$

Toate aceste observaţii ne duc la concluzia că

$$m < \underbrace{\sqrt[p]{n + \sqrt[p]{n + \sqrt[p]{n + \sqrt[p]{... + \sqrt[p]{n}}}}}}_{k\ radicali} < m+1, m, n \in \mathbb{N}^* \text{ şi } k \in \mathbb{N}^* - \{1\} \text{ are loc pentru}$$

$n = \overline{m^p - m + 1, (m+1)^p - (m+1)}$, demonstraţie care se poate face tot prin inducţie matematică.

Observaţie: Dacă $p = n \Rightarrow m < \underbrace{\sqrt[n]{n + \sqrt[n]{n + \sqrt[n]{n + \sqrt[n]{... + \sqrt[n]{n}}}}}}_{k\ radicali} < m+1, n \in \mathbb{N}^* - \{1\}$, pentru $n = \overline{m^n - m + 1, (m+1)^n - (m+1)}$. Verificăm dacă

$n = \overline{m^n - m + 1, (m+1)^n - (m+1)}$ este o mulţime corect definită pentru valorile lui n. Din $m^n - m + 1 \leq n \Rightarrow m(m^{n-1} - 1) + 1 \leq n$. Dacă $m \geq 2 \Rightarrow 2(2^{n-1} - 1) + 1 \leq n \Rightarrow$

$\Rightarrow 2^n \leq n + 1 \Rightarrow n \in \{0,1\}$, fals. Deci, $m \in \{0,1\}$. Dar $m = 0$ nu verifică

$m < \underbrace{\sqrt[n]{n+\sqrt[n]{n+\sqrt[n]{n+\sqrt[n]{\ldots+\sqrt[n]{n}}}}}}_{k\ radicali} < m+1, n \in \mathbb{N}^* - \{1\}$, atunci $m = 1 \Rightarrow n = \overline{1,2^n-2}$. Dar $n \neq 1 \Rightarrow n = \overline{2,2^n-2}$. Dacă puterea $n = 2 \Rightarrow n = \overline{2,2}$, dacă $n = 3 \Rightarrow n = \overline{2,6}$, dacă $n = 4 \Rightarrow n = \overline{2,14}$. În mod analog $n \in \{2,3,4,5,\ldots\}$ şi

$$1 < \underbrace{\sqrt[n]{n+\sqrt[n]{n+\sqrt[n]{n+\sqrt[n]{\ldots+\sqrt[n]{n}}}}}}_{k\ radicali} < 2, k \in \mathbb{N}^*. \text{ Deci,}$$

$$1 < \sqrt[n]{n+\sqrt[n]{n+\sqrt[n]{n+\sqrt[n]{\ldots}}}} < 2, n \in \mathbb{N}^* - \{1\}.$$

În continuare, voi propune un set de probleme ca aplicaţii la noţiunile prezentate.

Probleme propuse:

1) Să se găsească valorile naturale ale lui n pentru care

$$2018 < \underbrace{\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{\ldots+\sqrt{n}}}}}}_{2020\ radicali} < 2019.$$

2) Să se găsească partea întreagă a fiecăruia dintre numerele

$$\underbrace{\sqrt{42+\sqrt{42+\sqrt{42+\sqrt{\ldots+\sqrt{42}}}}}}_{k\ radicali}, \underbrace{\sqrt{99+\sqrt{99+\sqrt{99+\sqrt{\ldots+\sqrt{99}}}}}}_{k\ radicali},$$

$$\underbrace{\sqrt{101+\sqrt{101+\sqrt{101+\sqrt{\ldots+\sqrt{101}}}}}}_{2020\ radicali}, k \in \mathbb{N}^*.$$

3) Să se găsească partea întreagă a numărului

$$\underbrace{\sqrt{3+2\sqrt{2}+\sqrt{3+2\sqrt{2}+\sqrt{3+2\sqrt{2}+\sqrt{\ldots+\sqrt{3+2\sqrt{2}}}}}}}_{k\ radicali}, k \in \mathbb{N}^*$$

4) Să se determine partea întreagă a numărului

$$\underbrace{\sqrt{100+\sqrt{100+\sqrt{100+\sqrt{\ldots+\underbrace{\sqrt{2100+\sqrt{2100}+\sqrt{+\cdots+\sqrt{2100}}}}_{100\ radicali}}}}}}_{2200\ radicali},\ k\in\mathbb{N}^*.$$

5) Să se determine câți radicali $\underbrace{\sqrt[3]{n+\sqrt[3]{n+\sqrt[3]{n+\sqrt[3]{\ldots+\sqrt[3]{n}}}}}}_{k\ radicali}$ cu $k\in\mathbb{N}^*$ și $n\in\mathbb{N}^*$ au partea întreagă 5.

6) Să se calculeze

$$\left[\underbrace{\sqrt[4]{625+\sqrt[4]{625+\sqrt[4]{625+\sqrt[4]{\ldots+\sqrt[4]{625}}}}}}_{625\ radicali}-\underbrace{\sqrt{30+\sqrt{30+\sqrt{30+\sqrt{\ldots+\sqrt{30}}}}}}_{30\ radicali}\right],$$ unde $[x]$ reprezintă partea întreagă a numărului real x.

7) Să se calculeze

$$\left[\underbrace{\sqrt{100+\sqrt{100+\sqrt{100+\sqrt{\ldots+\sqrt{100}}}}}}_{k+1\ radicali}-\underbrace{\sqrt{100+\sqrt{100+\sqrt{100+\sqrt{\ldots+\sqrt{100}}}}}}_{k-1\ radicali}\right],$$

$k\in N^*-\{1\}$, unde $[x]$ reprezintă partea întreagă a numărului real x.

8) Să se determine n natural pentru care $\left[\underbrace{\sqrt[3]{n+\sqrt[3]{n+\sqrt[3]{n+\sqrt[3]{\ldots+\sqrt[3]{n}}}}}}_{2020\ radicali}\right]=10$ și

$$\left[\underbrace{\sqrt[3]{n+1+\sqrt[3]{n+1+\sqrt[3]{n+1+\sqrt[3]{\ldots+\sqrt[3]{n+1}}}}}}_{2020\ radicali}\right]=11.$$

9) Să se determine n natural pentru care $\sqrt[3]{n+\sqrt[3]{n+\sqrt[3]{n+\sqrt[3]{\ldots}}}}=10$.

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APLICAȚII LA TEOREMA TRANSVERSALEI

By Marian Ursărescu-Romania

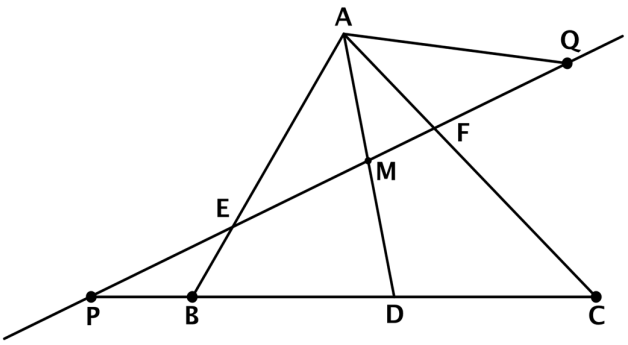
În cadrul acestei note matematice vom rezolva câteva probleme de geometrie plană cu ajutorul teoremei transversalei. Pentru început vom enunța și demonstra teorema transversalei și consecințele acestei teoreme.

Teorema transversalei

Fie un triunghi ABC și punctele $D \in BC, E \in AB, F \in AC$ și $M \in AD$. Atunci punctul

$$M \in EF \Leftrightarrow CD \cdot \frac{EB}{EA} + BD \cdot \frac{FC}{FA} = BC \cdot \frac{MD}{MA}.$$

Demonstrație



"⇒"

Dacă $EF \parallel BC$ demonstrația este banală. Să presupunem $EF \nparallel BC$. Ducem o paralelă prin A la BC și notăm cu Q intersecția dintre EF cu această paralelă și cu P intersecția dintre EF și BC.

$$\Delta PBE \sim \Delta QAE \Rightarrow \frac{EB}{EA} = \frac{PB}{AQ} \mid \cdot CD \Rightarrow CD \cdot \frac{EB}{EA} = CD \cdot \frac{PB}{AQ} \quad (1)$$

$$\Delta PCF \sim \Delta QFA \Rightarrow \frac{FC}{FA} = \frac{PC}{AQ} \mid \cdot BD \Rightarrow BD \cdot \frac{FC}{FA} = BD \cdot \frac{PC}{AQ} \quad (2)$$

Din (1) + (2) $\Rightarrow CD \cdot \frac{EB}{EA} + BD \cdot \frac{FC}{FA} = CD \cdot \frac{PB}{AQ} + BD \cdot \frac{PC}{AQ}$ (3), deci rămâne de arătat că

$$\frac{CD \cdot PB + BD \cdot PC}{AQ} = BC \cdot \frac{MD}{MA}$$

Dar $\Delta PDM \sim \Delta QAM \Rightarrow \frac{MD}{MA} = \frac{PD}{AQ} \mid \cdot BC \Rightarrow BC \cdot \frac{MD}{MA} = BC \cdot \frac{PD}{AQ} \Rightarrow$ că trebuie să arătăm că

$$\frac{CD \cdot PB + BD \cdot PC}{AQ} = \frac{BC \cdot PD}{AQ} \Leftrightarrow CD \cdot PB + BD \cdot PC = BC \cdot PD \Leftrightarrow$$

$$\Leftrightarrow CD \cdot PB + BD \cdot PC = (BD + DC) \cdot PD \Leftrightarrow BD(PC - PD) = DC(PD - PB) \Leftrightarrow$$

$$\Leftrightarrow BD \cdot DC = CD \cdot BD, \text{ relație adevărată.}$$

"⇐" Reciproca rezultă imediat folosind metoda reducerii la absurd și folosind relația directă pe care am demonstrat-o.

Observație. Teorema poate fi demonstrată și vectorial faptul că vectorii $\overrightarrow{EM}$ și $\overrightarrow{MF}$ sunt coliniari. În continuare vom enunța două consecințe interesante ale teoremei transversalei.

Consecința 1. Fie ABC un triunghi, G centrul de greutate al triunghiului ABC, $E \in AB$ și $F \in AC$. Atunci $G \in EF \Leftrightarrow \frac{EB}{EA} + \frac{FC}{FA} = 1$.

Demonstrație. Aplicăm teorema transversalei pentru $M = G$. Atunci avem:

$$CD = BD = \frac{1}{2}BC \text{ și } \frac{MD}{MA} = \frac{GD}{GA} = \frac{1}{2} \Rightarrow \frac{1}{2}BC \cdot \frac{EB}{EA} + \frac{1}{2}BC \cdot \frac{FC}{FA} = BC \cdot \frac{1}{2} \Rightarrow \frac{EB}{EA} + \frac{FC}{FA} = 1.$$

Consecința 2. Fie ABC un triunghi, I punctul de intersecție al bisectoarelor interioare ale triunghiului, $E \in AB$ și $F \in AC$. Atunci: $I \in EF \Leftrightarrow AC \cdot \frac{EB}{EA} + AB \cdot \frac{FC}{FA} = BC$.

Demonstrație:

Aplicăm teorema transversalei pentru $M = I \Rightarrow CD \cdot \frac{EB}{EA} + BD \cdot \frac{FC}{FA} = BC \cdot \frac{ID}{IA} | : BC \Rightarrow$

$$\Rightarrow \frac{CD}{BC} \cdot \frac{EB}{EA} + \frac{BD}{BC} \cdot \frac{FC}{FA} = \frac{ID}{IA} \quad (1)$$

Din teorema bisectoarei $\Rightarrow \frac{BD}{DC} = \frac{AB}{AC} \Rightarrow \frac{BD}{BC} = \frac{AB}{AC+AB}$ (2) și $\frac{CD}{BC} = \frac{AC}{AC+AB}$ (3)

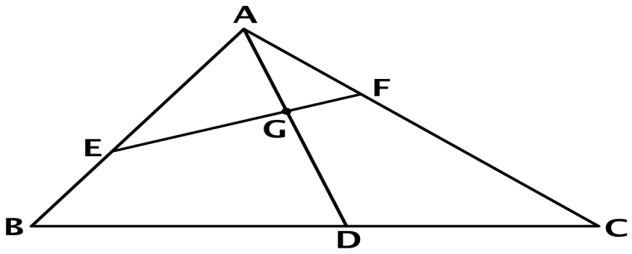
$$\frac{ID}{IA} = \frac{BD}{AB} = \frac{BC \cdot AB}{(AC+AB)AB} = \frac{BC}{AC+AB} \quad (4)$$

Din (1), (2), (3) și (4) ⇒ concluzia consecinței.

În continuare vom prezenta câteva aplicații la teorema transversalei și consecințele sale.

Problema 1. (Gazeta matematică): Fie ABC un triunghi și G centrul de greutate al triunghiului. O dreaptă care trece prin G intersectează laturile AB și AC în punctele E, respectiv F. Să se arate că: $\frac{BE}{EA} \cdot \frac{CF}{FA} \leq \frac{1}{4}$.

Demonstrație:



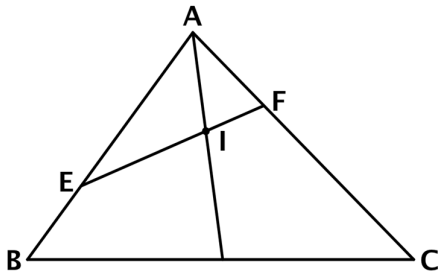
$G \in EF$, aplicăm consecința 1 $\Rightarrow \frac{BE}{EA} + \frac{CF}{FA} = 1$ (1). Dar din inegalitatea mediilor $\Rightarrow$

$\frac{BE}{EA} + \frac{CF}{FA} \geq 2\sqrt{\frac{BE}{EA} \cdot \frac{CF}{FA}}$ (2). Din (1) + (2) $\Rightarrow 2\sqrt{\frac{BE}{EA} \cdot \frac{CF}{FA}} \leq 1 \Rightarrow \frac{BE}{EA} \cdot \frac{CF}{FA} \leq \frac{1}{4}$.

Problema 2. (concurs de matematică)

Fie ABC un triunghi dreptunghic ($A = 90°$) și I punctul de intersecție al bisectoarelor anterioare. O dreaptă care trece prin I intersectează AB și AC în E respectiv F. Să se arate că: $\left(\frac{EB}{EA}\right)^2 + \left(\frac{FC}{FA}\right)^2 \geq 1$.

Demonstrație:



Aplicăm consecința 2 $\Rightarrow AC \cdot \frac{EB}{FA} + AB \cdot \frac{CF}{FA} = BC$ (1)

Din inegalitatea lui Cauchy $\Rightarrow (AC^2 + AB^2)\left(\left(\frac{EB}{EA}\right)^2 + \left(\frac{FC}{FA}\right)^2\right) \geq \left(AC \cdot \frac{EB}{EA} + AB \cdot \frac{FC}{FA}\right)^2$ (2)

$A = 90° \Rightarrow AC^2 + AB^2 = BC^2$ (3)

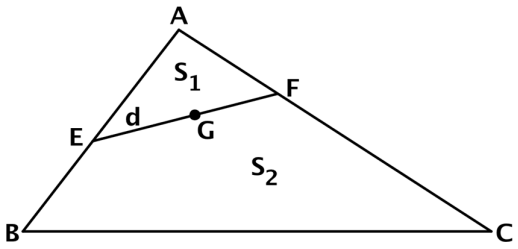
Din (1) + (2) + (3) $\Rightarrow BC^2\left(\left(\frac{EB}{EA}\right)^2 + \left(\frac{FC}{FA}\right)^2\right) \geq BC^2 \Rightarrow \left(\frac{EB}{FA}\right)^2 + \left(\frac{FC}{FA}\right)^2 \geq 1$.

Problema 3. (Gazeta matematică)

Fie un triunghi ABC și centrul său de greutate. O dreaptă d care trece prin centrul de greutate împarte triunghiul în două suprafețe de arii S_1, respectiv S_2.

a) Să se determine dreptele d pentru care $S_1 = S_2$.b) Să se determine dreptele d pentru care $\frac{S_1}{S_2} = \frac{4}{5}$. c) Să se arate că nu există nici o dreaptă d pentru care $\frac{S_1}{S_2} = \frac{3}{4}$.

Demonstrație:



a) evident că medianele triunghiului au proprietatea că $S_1 = S_2$ vom arăta că sunt singurele drepte. Notăm cu E, respectiv F intersecțiile dreptei d cu laturile AB și AC.

Conform consecinței 1 avem: $\frac{EB}{EA} + \frac{FC}{FA} = 1 \Rightarrow \frac{AB-EA}{EA} + \frac{AC-FA}{FA} = 1 \Rightarrow \frac{AB}{EA} - 1 + \frac{AC}{FA} - 1 = 1 \Rightarrow$

$$\Rightarrow \frac{AB}{EA} + \frac{AC}{FA} = 3 \ (1)$$

$$S_1 = S_2 \Rightarrow \frac{S_{AEF}}{S_{EFBC}} = 1 \Rightarrow \frac{S_{AEF}}{S_{ABC}} = \frac{1}{2} \Rightarrow \frac{AE \cdot AF \cdot \sin A}{AB \cdot AC \cdot \sin A} = \frac{1}{2} \Rightarrow \frac{AE}{AB} \cdot \frac{AF}{AC} = \frac{1}{2} \Rightarrow \frac{AB}{AE} \cdot \frac{AC}{FA} = 2 \ (2)$$

Notăm $\frac{AB}{AE} = x, \frac{AC}{AF} = y \Rightarrow \begin{cases} x + y = 3 \\ xy = 2 \end{cases}$ sistem care are soluțiile $\begin{cases} x = 1 \\ y = 2 \end{cases}$ sau $\begin{cases} x = 2 \\ y = 1 \end{cases}$, adică dreapta d este mediană.

b) $\frac{S_1}{S_2} = \frac{4}{5} \Rightarrow \frac{S_{AEF}}{S_{ABC}} = \frac{4}{9}$ cu notațiile de la punctul a $\Rightarrow \begin{cases} xy = \frac{4}{9} \\ x + y = 3 \end{cases}$ sistem care are soluția

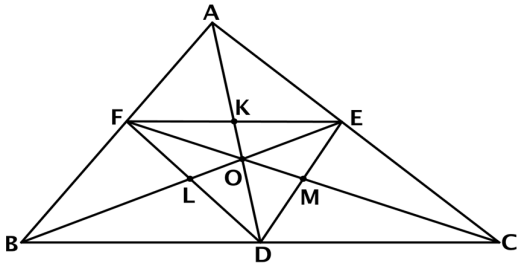
$x = y = \frac{3}{2} \Rightarrow$ dreapta d trebuie să fie paralelă cu una din laturile triunghiului.

c) $\frac{S_1}{S_2} = \frac{3}{4} \Rightarrow \frac{S_{AEF}}{S_{ABC}} = \frac{3}{7} \Rightarrow \begin{cases} xy = \frac{7}{3} \\ x + y = 3 \end{cases}$ x, y sunt rădăcinile ecuație $3t^2 - 9t + 7 = 0$ care are discriminantul negativ $\Rightarrow$ sistemul nu are soluții reale $\Rightarrow$ că nu există nici o dreaptă d.

Problema 4. (Olimpiadă Iugoslavia)

Fie ABC un triunghi și $O \in Int(\Delta ABC)$. Notăm cu D, E și F intersecțiile dintre AO, BO și CO cu BC, AC, respectiv AB. Fie K, L și M intersecțiile dintre AO, BO și CO cu EF, FD, respectiv DE. Să se arate că: $\frac{AK}{KD} \cdot \frac{BL}{LE} \cdot \frac{CM}{MF} \geq 1$.

Demonstrație.



Fie $\frac{BD}{DC} = x, \frac{EC}{AE} = y, \frac{AF}{FB} = z. BC = a, AC = b, AB = c$. Din teorema lui Ceva $\Rightarrow x \cdot y \cdot z = 1$.

$\frac{BD}{DC} = x \Rightarrow \frac{BD}{BC} = \frac{x}{x+1} \Rightarrow BD = \frac{ax}{x+1}$ și $CD = \frac{a}{x+1}$. Din teorema transversalei

$$\Rightarrow BC \cdot \frac{KD}{AK} = CD \cdot \frac{FB}{FA} + BD \cdot \frac{EC}{CD} \Rightarrow BC \cdot \frac{KD}{AK} = \frac{BC}{x+1} \cdot \frac{1}{z} + \frac{xBC}{x+1} \cdot y \Rightarrow$$

$\Rightarrow \frac{KD}{AK} = \frac{1}{z(x+1)} + \frac{xy}{x+1} \Rightarrow \frac{KD}{AK} = \frac{1+xyz}{z(x+1)} = \frac{2}{z(x+1)} \Rightarrow \frac{AK}{KD} = \frac{z(x+1)}{2}$ analog $\frac{BL}{LE} = \frac{x(y+1)}{2}, \frac{CM}{MF} = \frac{y(z+1)}{2}$

$$\Rightarrow \frac{AK}{KD} \cdot \frac{BL}{LE} \cdot \frac{CM}{MF} = \frac{(x+1)(y+1)(z+1)}{8} \geq \frac{2\sqrt{x} \cdot 2\sqrt{y} \cdot 2\sqrt{z}}{8} = 1$$

(am aplicat inegalitatea mediilor).

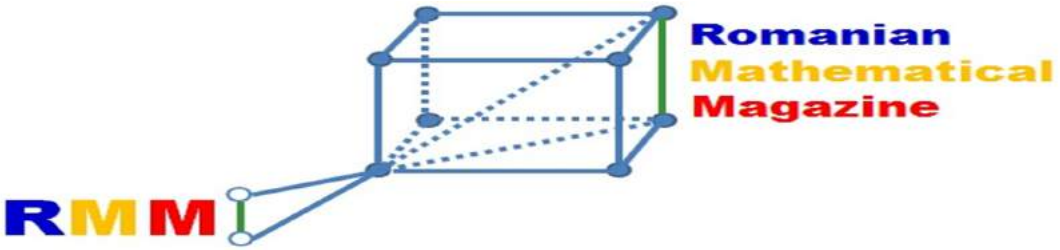
În încheiere propunem câteva probleme, pentru care în rezolvarea lor trebuie folosită teorema transversalei.

1. Fie ABC un triunghi ascuțitunghic, $E \in AB$ și $F \in AC$. Notăm cu H ortocentrul triunghiului ABC. Atunci $H \in EF \Leftrightarrow \tan B \cdot \frac{AB}{EA} + \tan C \cdot \frac{AC}{FA} = \tan A \tan B \tan C$.

2. Fie ABC un triunghi, $E \in AB$ și $F \in AC$. Notăm cu O centrul cercului circumscris. Atunci $O \in EF \Leftrightarrow \sin 2B \cdot \frac{EB}{EA} + \sin 2C \cdot \frac{FC}{FA} = \sin 2A$.

3. Fie un triunghi ABC și G centrul său de greutate. O dreaptă care trece prin G intersectează AB și AC din punctele E, respectiv F. Să se arate că: $\left(\frac{BE}{AE}\right)^4 + \left(\frac{CF}{AF}\right)^4 \geq \frac{1}{8}$ (concurs de matematică)

4. Fie ABC un triunghi și punctele $D \in BC, E \in AC, F \in AB$ astfel încât dreptele AD, BE, CF sunt concurente în M. Notăm: $\{P\} = AD \cap EF, \{Q\} = BE \cap FD, \{R\} = CF \cap DE$. Să se arate că: $\frac{AP}{PM} + \frac{BQ}{QM} + \frac{CR}{RM} \geq 9$. (Gazeta matematică).



NECESSARY AND SUFFICIENT CONDITIONS FOR THE EQUATION OF DEGREE III

By Marius Drăgan, Bucharest, Neculai Stanciu-Romania

We denote $T_1 = x + y + z, T_2 = xy + yz + zx, T_3 = xyz$, where x,y,z $\in R$. Also we denote $\Delta = (x-y)^2(y-z)^2(z-x)^2$. It is well-known that the equation

$$u^3 - T_1u^2 + T_2u - T_3 = 0, (1)$$

has three real positive roots if and only if $\Delta \geq 0, T_1 \geq 0, T_2 \geq 0, T_3 \geq 0$, (2).

We shall determine a necessary and sufficient condition such that the equation (1) has three real roots in the interval [a,b]

Theorem 1. The equation (1) has the roots x,y,z$\in (-\infty, b]$ if and only if:

$$\Delta \geq 0, T_1 - 3b \leq 0, 3b^2 - 2T_1b + T_2 \geq 0, b^2 - T_1b^2 + T_2b - T_3 \geq 0$$

Proof. We consioder the function $f: R \to R, f(u) = u^3 - T_1u^2 + T_2u - T_3$ and u=b-t.

From (2) we get that the equation f(a)=0 three roots in the interval $(-\infty, b]$ if and only if that f(b-t)=0 <=>

$$t^3 - (3b - T_1)t^2 + (3b^2 - 2T_1b + T_2)t - (b^2 - T_1b^2 + T_2b - T_3) = 0$$

has three real positive solutions; i.e. $\Delta \geq 0, T_1' = 3b - T_1 \geq 0, T_2' = 3b^2 - 2T_1b + T_2 \geq 0,$

$$T_3' = b^2 - T_1 b^2 + T_2 b - T_3 \geq 0$$

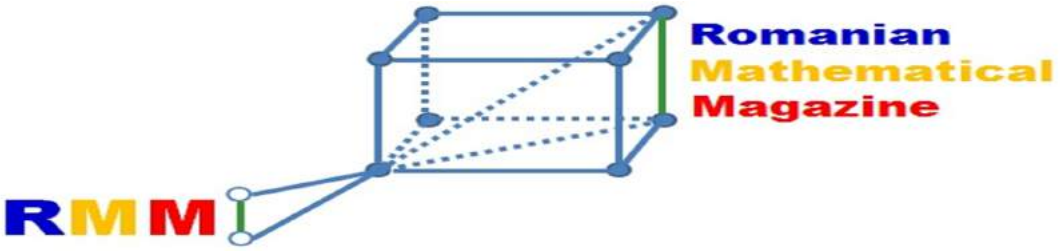
Theorem 2 . The equation (1) has the roots x,y,z$\in [a, +\infty)$ if and only if

$$\Delta \geq 0, -3a + T_1 \geq 0, 3a^2 - 2T_1 a + T_2 \geq 0, -a^3 + T_1 a^2 - T_2 a + T_3 \geq 0$$

Proof. The equation (1) has the roots x,y,z$\in [a, +\infty)$ iff the equation f(t+a)=0 has three real roots , i.e. f(t+a)=0 <=> $t^3 - (-3a + T_1)t^2 + (3a^2 - 2T_1 a + T_2)t - (-a^3 + T_1 a^2 - T_2 a + T_3) = 0$. So, $\Delta \geq 0, T_1'' = -3a + T_1 \geq 0, T_2'' = 3a^2 - 2T_1 a + T_2 \geq 0$, $-a^3 + T_1 a^2 - T_2 a + T_3 \geq 0$. By theorem 1 and theorem 2, yields the following result:

Theorem 3. The equation (1) has three real solutions in the interval [a,b] if and only if

$$\Delta \geq 0, T_1', T_2', T_3', T_1'', T_2'', T_3'' \geq 0.$$



SOME RESULTS ABOUT THE FIBONACCI NUMBERS AND THE GOLDEN RATIO

By D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Abstract: In this paper we present new results related to the Fibonacci numbers and the golden ratio. **Keywords:** Fibonacci sequence, Lucas sequence, Euler sequence, Lalescu sequence, Golden ratio. AMS classification: 39A10, 11B39.

1. Introduction

The *Fibonacci* numbers F_n and the *Lucas* numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1;\ L_{n+2} = L_{n+1} + L_n, L_0 = 2, L_1 = 1.$$

Also, $\alpha = \frac{1+\sqrt{5}}{2}$ (the golden ratio), $\beta = \frac{1-\sqrt{5}}{2}$, $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$, and $L_n = \alpha^n + \beta^n$. We have

$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{\sqrt{5}+1}{2} = \alpha$. Indeed,

$$F_{n+2} - F_{n+1} - F_n = 0, \forall n \in \mathbb{N}^* \Leftrightarrow \frac{F_{n+2}}{F_n} - \frac{F_{n+1}}{F_n} - 1 = 0 \Leftrightarrow \frac{F_{n+2}}{F_{n+1}} \cdot \frac{F_{n+1}}{F_n} - \frac{F_{n+1}}{F_n} - 1 = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{F_{n+2}}{F_{n+1}} \cdot \frac{F_{n+1}}{F_n} - \frac{F_{n+1}}{F_n} - 1\right) = 0 \Leftrightarrow x^2 - x - 1 = 0, \text{where } x = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}, \text{so } x = \frac{1\pm\sqrt{5}}{2}.$$

Since $\frac{F_{n+1}}{F_n} > 0$, yields $x = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{\sqrt{5}+1}{2} = \alpha$.

Other proof: $\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\lim_{n\to\infty}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha^n-\beta^n}=\frac{\alpha}{e}\lim_{n\to\infty}\frac{1-\left(\frac{\beta}{\alpha}\right)^{n+1}}{1-\left(\frac{\beta}{\alpha}\right)^n}=\frac{\alpha}{e}\cdot\frac{1-0}{1-0}=\frac{\alpha}{e}$.

Similarly $\lim_{n\to\infty}\frac{L_{n+1}}{L_n}=\frac{\sqrt{5}+1}{2}=\alpha$. For more on the *Fibonacci* and the *Lucas* numbers see e.g. [3].

The *Traian Lalescu's* sequence, [4], is: $(TL_n)_{n\geq 1}$ where $TL_1=\sqrt{2}-1$ and $TL_n=\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}$, $\forall n\in\mathbb{N}^*-\{1\}$. It is well-known that: $\lim_{n\to\infty}TL_n=\frac{1}{e}$. Let $e_n=\left(1+\frac{1}{n}\right)^n$, it is well-known that $\lim_{n\to\infty}e_n=e$ (*Euler's* number). In [1] and [2], was proved some *Euler-Fibonacci-Lalescu-Lucas* collaboration and the golden ratio:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!\,F_{n+1}}-\sqrt[n]{n!\,F_n}\right)=\frac{\alpha}{e};\ \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!\,L_{n+1}}-\sqrt[n]{n!\,L_n}\right)=\frac{\alpha}{e};$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!\,F_{n+1}}-\sqrt[n]{(2n-1)!!\,F_n}\right)=\frac{2\alpha}{e};$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!\,L_{n+1}}-\sqrt[n]{(2n-1)!!\,L_n}\right)=\frac{2\alpha}{e};$$

$$\lim_{n\to\infty}\left(e\cdot\sqrt[n+1]{(n+1)F_{n+1}}-e_n\cdot\sqrt[n]{n!\,F_n}\right)=\frac{3\alpha}{2};\ \lim_{n\to\infty}\left(e\cdot\sqrt[n+1]{(n+1)!\,L_{n+1}}-e_n\cdot\sqrt[n]{n!\,L_n}\right)=\frac{3\alpha}{2};$$

$$\lim_{n\to\infty}\left(e_{n+1}\cdot\sqrt[n+1]{(n+1)!\,F_{n+1}}-e_n\cdot\sqrt[n]{n!\,F_n}\right)=\alpha;$$

$$\lim_{n\to\infty}\left(e_{n+1}\cdot\sqrt[n+1]{(n+1)!\,L_{n+1}}-e_n\cdot\sqrt[n]{n!\,L_n}\right)=\alpha;$$

$$\lim_{n\to\infty}\left(e_{n+1}\cdot\sqrt[n+1]{(2n+1)!!\,F_{n+1}}-e_n\cdot\sqrt[n]{(2n-1)!!\,F_n}\right)=2\alpha;$$

$$\lim_{n\to\infty}\left(e_{n+1}\cdot\sqrt[n+1]{(2n+1)!!\,L_{n+1}}-e_n\cdot\sqrt[n]{(2n-1)!!\,L_n}\right)=2\alpha;$$

$$\lim_{n\to\infty}\left(e\cdot\sqrt[n+1]{(2n+1)!!\,F_{n+1}}-e_n\cdot\sqrt[n]{(2n-1)!!\,F_n}\right)=3\alpha;$$

$$\lim_{n\to\infty}\left(e\cdot\sqrt[n+1]{(2n+1)!!\,L_{n+1}}-e_n\cdot\sqrt[n]{(2n-1)!!\,L_n}\right)=3\alpha;$$

2. Main results

Theorem. Let $m>0$ and $\Gamma\colon(0,\infty)\to(0,\infty)$ be the gamma function.

$$\lim_{n\to\infty}\int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}}\Gamma\left(\frac{x}{n}\sqrt[n]{F_n^m}\right)dx=\frac{\Gamma(\alpha^m e^{-1})}{e}$$

Proof. We have $\lim_{n\to\infty}\sqrt[n]{F_n}=\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\alpha=\frac{\sqrt{5}+1}{2}$, and

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}.$$

By the mean value theorem for definite integrals we have that there exists

$\xi_n \in \left[\sqrt[n]{n!}, \sqrt[n+1]{(n+1)!}\right]$ such that

$$B_n = \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} \Gamma\left(\frac{x}{n}\sqrt[n]{F_n^m}\right)dx = \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right)\Gamma\left(\frac{\xi_n}{n}\sqrt[n]{F_n^m}\right)$$

$$\sqrt[n]{n!} \le \xi_n \le \sqrt[n+1]{(n+1)!} \Leftrightarrow \frac{\sqrt[n]{n!}}{n} \le \frac{\xi_n}{n} \le \frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n+1}{n} \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} \le \lim_{n\to\infty}\frac{\xi_n}{n} \le \lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n+1}{n} \Rightarrow \lim_{n\to\infty}\frac{\xi_n}{n} = \frac{1}{e}.$$

Let $u_n = \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}, \forall n \ge 2$. We have $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n+1}{n}\cdot\frac{n}{\sqrt[n]{n!}} = \frac{1}{e}\cdot 1\cdot e = 1$, so $\lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1$. Also we have

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}} = \lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}} = e.$$

Yields that $B_n = \sqrt[n]{n!}\,(u_n-1)\Gamma\left(\frac{\xi_n}{n}\sqrt[n]{F_n^m}\right) = \frac{\sqrt[n]{n!}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\Gamma\left(\frac{\xi_n}{n}\sqrt[n]{F_n^m}\right)\cdot\ln u_n^n$. Hence

$$\lim_{n\to\infty} B_n = \frac{1}{e}\cdot 1\cdot\lim_{n\to\infty}\Gamma\left(\frac{\xi_n}{n}\sqrt[n]{F_n^m}\right)\cdot\ln\left(\lim_{n\to\infty} u_n^n\right) = \frac{1}{e}\cdot\Gamma\left(\lim_{n\to\infty}\frac{\xi}{n}\cdot\left(\lim_{n\to\infty}\sqrt[n]{F_n}\right)^m\right) =$$

$$= \frac{1}{e}\Gamma\left(\frac{1}{e}\alpha^m\right) = \frac{\Gamma(\alpha^m e^{-1})}{e}.$$

Remark. We note that instead of $\Gamma\colon (0,\infty)\to(0,\infty)$ we can take any $f\colon (0,\infty)\to(0,\infty)$, continue function.

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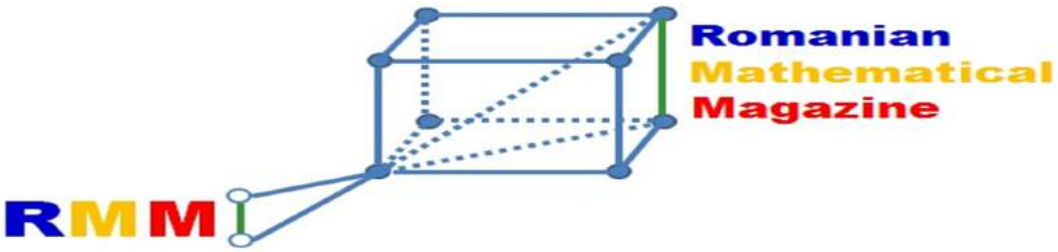
[2] D.M. Bătineţu-Giurgiu and N. Stanciu, *Problem B-1160*, The Fibonacci Quarterly, Vol. 52, No. 4, November, 2014, 368. http://www.fq.math.ca/Problems/ElemProbSolnNov14.pdf

[3] D.M. Bătineţu-Giurgiu, N. Stanciu, G. Tica, *Din tainele numerelor Fibonacci şi Lucas*, Editura Sitech, Craiova, 2013.

[4] Traian Lalescu, Problema 579, Gazeta Matematică, Vol. 6, nr. 6/1900, 148.



ABOUT SOME SPECIAL CLASS OF TRIANGLES

By Andrei Mihalcea Ştefan-Romania

Abstract: *In this paper are presented some applications for a special class of triangles with diameter of circumcircle 1*

$$2R = 1 \leftrightarrow \begin{cases} a = \sin A \\ b = \sin B \\ c = \sin C \end{cases}$$

$\sin(A + B) = \sin(\pi - C) = \sin C$ and $\cos A = \sqrt{1 - a^2}, \cos B = \sqrt{1 - b^2}$ (using the fact that ΔABC is acute). Hence: $c = a\sqrt{1 - b^2} + b\sqrt{1 - a^2}$ and analogous.

Let be $f: (0,1) \to \mathbb{R}, f(x) = \sqrt{1 - x^2}, f''(x) < 0 \to f - concave$

$$4m_a^2 = 2\big(af(c) + cf(a)\big)^2 + 2\big(af(b) + bf(a)\big)^2 - \big(c\sqrt{1 - b^2} + b\sqrt{1 - c^2}\big)^2 \quad (*)$$

$$af(c) + cf(a) \leq (a + c)f\left(\frac{2ac}{a+c}\right) \leftrightarrow af(c) + cf(a) \leq \sqrt{(a + c)^2 - 4a^2c^2} \quad (1)$$

By Jensen's inequality: $c\sqrt{(1 - b)(1 + b)} + b\sqrt{(1 - c)(1 + c)} \overset{AM-GM}{\leq} \frac{2b(1-c^2)}{2} + \frac{2c(1-b^2)}{2} = (b + c)(1 - bc)$ (2), By (1), (2), (*):

$$4\sum m_a^2 \leq 4\sum(a + b)^2 - 8\sum a^2(b^2 + c^2) - \sum(a + b)^2(1 - ab)^2 \quad (3).$$

$$b^2 + c^2 \geq 2bc \to -8\sum a^2(b^2 + c^2) \leq -16\sum a^2bc = -16abc\sum a \quad (4).$$

From $\begin{cases} 4\sum m_a^2 = 3\sum a^2 \\ 4\sum(a + b)^2 = 8\sum a^2 + 8\sum ab \end{cases}$ we have: $4\sum(a + b)^2 - 4\sum m_a^2 = 4(\sum a)^2 + \sum a^5 \quad (5)$

Using (4), (5) in (3) $\to \sum(a + b)^2 (1 - ab)^2 \leq \sum a^2 + 4(\sum a)(\sum a - 4abc)$.

Another proof for last inequality:

By (2) $\to a^2 \geq (b + c)^2(1 - bc)^2 \to \sum(a + b)^2(1 - ab)^2 \leq \sum a^2$.

Remains to prove: $4(\sum a)(\sum a - 4abc) \geq 0 \leftrightarrow 4abc \leq \sum a . \sum a \geq 3\sqrt[3]{abc}$ by AM-GM.

$$3\sqrt[3]{abc} \geq 4abc \leftrightarrow abc \leq \frac{3\sqrt{3}}{8} \leftrightarrow \sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}.$$

Let be $g:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}, g(x)=\ln\sin x, g''(x)=\frac{2}{\cos 2x-1}<0-g\ concave$

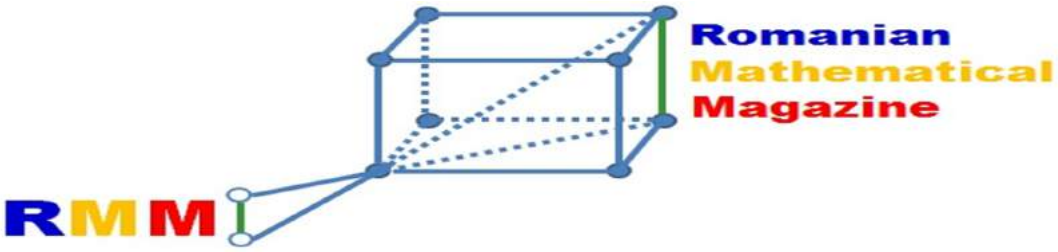
By Jensen's inequality: $\sum\ln\sin A\le 3\ln\sin\frac{\pi}{3}\leftrightarrow\sin A\sin B\sin C\le\frac{3\sqrt{3}}{8}$.

By (1) $\to c^2+4a^2b^2\le a^2+b^2+2ab\to 4\sum a^2b^2\le\sum a^2+2\sum ab,$

Hence: $\sqrt{a^2b^2+b^2c^2+c^2a^2}\le s$. By (2) $\to 2\sum m_a^2\ge\sum a^4+\sum a^2b^2$.

Proposed problems:

If $R=\frac{1}{2}\ then:\begin{cases}\sum(a+b)^2(1-ab)^2\le\sum a^2+4(\sum a)(\sum a-4abc)\\ \sqrt{a^2b^2+b^2c^2+c^2a^2}\le s\\ 2\sum m_a^2\ge\sum a^4+\sum a^2b^2\end{cases}$.



A GENERALIZATION OF *J.RADON*'S INEQUALITY

By D.M. Bătineţu – Giurgiu, Daniel Sitaru and Neculai Stanciu-Romania

Theorem . If $a,b,c,d,x_k,y_k\in R_+^*,\forall k=\overline{1,n}$ and $X_n=\sum_{k=1}^{n}x_k, Y_n=\sum_{k=1}^{n}y_k$,

$m,p,q,s\in R_+, r\in[1,\infty)$ such that $cY_n^s>d\max_{1\le k\le n}y_k^s,\forall k=\overline{1,n}$, then:

$$\sum_{k=1}^{n}\frac{\left(aX_n^{\ p}+bx_k^q\right)^{m+1}x_k^{r(m+1)}}{\left(cY_n^s-dy_k^s\right)^m y_k^m}\ge\frac{\left(an^qX_n^{\ p+r}+bX_n^{\ q+r}\right)^{m+1}}{\left(cn^s-d\right)^m Y_n^{m(s+1)}}\cdot\frac{1}{n^{(m+1)(q+r-1)-ms}}.$$

Proof. We denoted

$$\sum_{k=1}^{n}\frac{\left(aX_n^{\ p}+bx_k^q\right)^{m+1}x_k^{r(m+1)}}{\left(cY_n^s-dy_k^s\right)^m y_k^m}\ge\frac{\left(an^qX_n^{\ p+r}+bX_n^{\ q+r}\right)^{m+1}}{\left(cn^s-d\right)^m Y_n^{m(s+1)}}\cdot\frac{1}{n^{(m+1)(q+r-1)-ms}}\qquad(1)$$

$u_k=\left(aX_n^{\ p}+bx_k^q\right)x_k^r, v_k=\left(cY_n^s-dy_k^s\right)y_k,\forall k=\overline{1,n}, V_n=\sum_{k=1}^{n}v_k$ and the LHS of (1) becomes:

$$\sum_{k=1}^{n}\frac{u_k^{m+1}}{v_k^m}=\sum_{k=1}^{n}v_k\left(\frac{u_k}{v_k}\right)^{m+1}=V_n\sum_{k=1}^{n}\frac{v_k}{V_n}\left(\frac{u_k}{v_k}\right)^{m+1}$$

Since the function $f:R_+^*\to R_+^*,\ f(x)=x^{m+1}$ is convex on R_+^*, we use *Jensen*'s inequality and we obtain that:

$$\sum_{k=1}^{n}\frac{v_k}{V_n}f\left(\frac{u_k}{v_k}\right)\geq\sum_{k=1}^{n}f\left(\frac{v_k}{V_n}\cdot\frac{u_k}{v_k}\right)=\sum_{k=1}^{n}f\left(\frac{u_k}{V_n}\right)=\frac{\left(\sum_{k=1}^{n}u_k\right)^{m+1}}{V_n^{m+1}}\Leftrightarrow\sum_{k=1}^{n}\frac{v_k}{V_n}\left(\frac{u_k}{v_k}\right)^{m+1}\geq\frac{\left(\sum_{k=1}^{n}u_k\right)^{m+1}}{V_n^{m+1}}$$

Therefore,

$$\sum_{k=1}^{n}\frac{u_k^{m+1}}{v_k^m}\geq V_n\cdot\frac{\left(\sum_{k=1}^{n}u_k\right)^{m+1}}{V_n^{m+1}}=\frac{\left(\sum_{k=1}^{n}u_k\right)^{m+1}}{V_n^m}\Leftrightarrow$$

$$\Leftrightarrow\sum_{k=1}^{n}\frac{\left(aX_n^{\,p}+bx_k^q\right)^{m+1}x_k^{r(m+1)}}{\left(cY_n^s-dy_k^s\right)^m y_k^m}\geq\frac{\left(\sum_{k=1}^{n}\left(aX_n^{\,p}+bx_k^q\right)x_k^r\right)^{m+1}}{\left(\sum_{k=1}^{n}\left(cY_n^s-dy_k^s\right)y_k\right)^m}=$$

$$=\frac{\left(\sum_{k=1}^{n}\left(aX_n^{\,p}x_k^r+bx_k^{q+r}\right)\right)^{m+1}}{\left(\sum_{k=1}^{n}\left(cY_n^s y_k-dy_k^{s+1}\right)\right)^m}=\frac{\left(aX_n^{\,p}\sum_{k=1}^{n}x_k^r+b\sum_{k=1}^{n}x_k^{q+r}\right)^{m+1}}{\left(cY_n^s\sum_{k=1}^{n}y_k-d\sum_{k=1}^{n}y_k^{s+1}\right)^m}=\frac{\left(aX_n^{\,p}\sum_{k=1}^{n}x_k^r+b\sum_{k=1}^{n}x_k^{q+r}\right)^{m+1}}{\left(cY_n^{s+1}-d\sum_{k=1}^{n}y_k^{s+1}\right)^m}$$

Because the functions $g,h,k:R_+^*\to R_+^*,\ g(x)=x^r,\ h(x)=x^{q+r},\ k(y)=y^{s+1}$ are convex on R_+^*, also by *Jensen*'s inequality we have:

$$\sum_{k=1}^{n}x_k^r=\sum_{k=1}^{n}g(x_k)\geq ng\left(\frac{1}{n}\cdot\sum_{k=1}^{n}x_k\right)=n\cdot\frac{X_n^{\,r}}{n^r}=\frac{X_n^{\,r}}{n^{r-1}},$$

$$\sum_{k=1}^{n}x_k^{q+r}=\sum_{k=1}^{n}h(x_k)\geq nh\left(\frac{1}{n}\cdot\sum_{k=1}^{n}x_k\right)=n\cdot\frac{X_n^{\,q+r}}{n^{q+r}}=\frac{X_n^{\,q+r}}{n^{q+r-1}},$$

$$\sum_{i=1}^{n}y_i^{s+1}=\sum_{i=1}^{n}k(y_i)\geq nk\left(\frac{1}{n}\cdot\sum_{i=1}^{n}y_i\right)=n\cdot\frac{Y_n^{s+1}}{n^{s+1}}=\frac{Y_n^{s+1}}{n^s}.$$

Then, we deduce that:

$$\sum_{k=1}^{n}\frac{\left(aX_n^{\,p}+bx_k^q\right)^{m+1}x_k^{r(m+1)}}{\left(cY_n^s-dy_k^s\right)^m y_k^m}\geq\frac{\left(a\cdot\dfrac{X_n^{\,p+r}}{n^{r-1}}+b\cdot\dfrac{X_n^{\,q+r}}{n^{q+r-1}}\right)^{m+1}}{\left(cY_n^{s+1}-\dfrac{dY_n^{s+1}}{n^s}\right)^m}=$$

$$=\frac{\left(an^qX_n^{\,p+r}+bX_n^{\,q+r}\right)^{m+1}}{\left(cn^s-d\right)^mY_n^{m(s+1)}}\cdot\frac{n^{ms}}{n^{(m+1)(q+r-1)}}=\frac{\left(an^qX_n^{\,p+r}+bX_n^{\,q+r}\right)^{m+1}}{\left(cn^s-d\right)^mY_n^{m(s+1)}}\cdot\frac{1}{n^{(m+1)(q+r-1)-ms}},\text{ and we are done.}$$

Observation 1.1. *If* $p=q=s=0$, *then (1) becomes:*

$$\sum_{k=1}^{n}\frac{(a+b)^{m+1}x_k^{r(m+1)}}{(c-d)^my_k^m}\geq\frac{(a+b)^{m+1}X_n^{\,r(m+1)}}{(c-d)^mY_n^m}\cdot\frac{1}{n^{(m+1)(r-1)}}\Leftrightarrow\sum_{k=1}^{n}\frac{x_k^{r(m+1)}}{y_k^m}\geq\frac{X_n^{\,r(m+1)}}{Y_n^{\,m}n^{(m+1)(r-1)}}\qquad(1')$$

If we consider $r=1$, *then by* $(1')$ *we obtain:*

$$\sum_{k=1}^{n}\frac{x_k^{m+1}}{y_k^m}\geq\frac{X_n^{\,m+1}}{Y_n^m}\qquad\textbf{(R)}$$

i.e that is just the inequality of J. Radon, with equality if and only if there exists $t\in R_+^*$ *such that* $x_k=ty_k,\forall k=\overline{1,n}.$

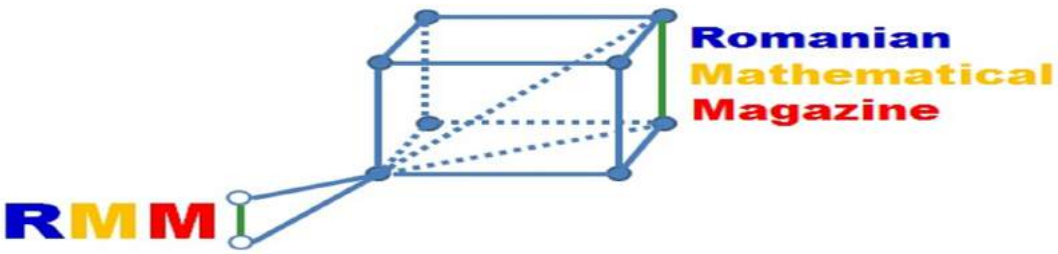
Observation 1.2. *If* $m=1$, *then* (1) *becomes*:

$$\sum_{k=1}^{n}\frac{\left(aX_n^{\,p}+bx_k^q\right)^2x_k^{2r}}{\left(cY_n^s-dy_k^s\right)y_k}\geq\frac{\left(an^qX_n^{\,p+r}+bX_n^{\,q+r}\right)^2}{\left(cn^s-d\right)Y_n^{s+1}}\cdot\frac{1}{n^{2(q+r-1)-s}}\qquad(1'')$$

If we take $p=q=s=0,r=1$, *then by* $(1'')$ *we obtain*:

$$\sum_{k=1}^{n}\frac{x_k^2}{y_k}\geq\frac{X_n^{\,2}}{Y_n}\qquad\textbf{(B)}$$

but that is just the inequality of H. Bergström.



FEW SOLUTIONS AND TWO REFINEMENTS FOR BOGDAN FUSTEI'S INEQUALITY

By Daniel Sitaru – Romania

Abstract: In this paper we present 5 solutions and a refinement for an inequality proposed by Bogdan Fustei in Romanian Mathematical Magazine, March 2018: In any triangle ABC the following relationship holds:

$$\sqrt{\frac{a}{s-a}}+\sqrt{\frac{b}{s-b}}+\sqrt{\frac{c}{s-c}}\geq 3\sqrt{2}$$

Solution 1 by Daniel Sitaru – Romania

$$a=y+z, b=z+x, c=x+y$$

$$3\sqrt{2}=\frac{3}{\sqrt{xyz}}\cdot\sqrt{2xyz}=\frac{3}{\sqrt{xyz}}\cdot\sqrt[6]{8x^3y^3z^3}=\frac{3}{\sqrt{xyz}}\cdot\sqrt[6]{x^2y^2z^2\cdot 8xyz}\leq$$

$$\overset{CESARO}{\leq}\frac{3}{\sqrt{xyz}}\cdot\sqrt[6]{x^2y^2z^2(y+z)(z+x)(x+y)}=$$

$$=\frac{1}{\sqrt{xyz}}\cdot 3\sqrt[3]{\sqrt{yz(y+z)}\cdot\sqrt{zx(z+x)}\cdot\sqrt{xy(x+y)}}\overset{GM-AM}{\leq}\frac{1}{\sqrt{xyz}}\cdot\sum\sqrt{yz(y+z)}=$$

$$=\sum\sqrt{\frac{y+z}{x}}=\sum\sqrt{\frac{a}{s-a}}=\sqrt{\frac{a}{s-a}}+\sqrt{\frac{b}{s-b}}+\sqrt{\frac{c}{s-c}}$$

Solution 2 by Mehmet Sahin – Ankara – Turkey.

Using the function $f(x)=\sqrt{\frac{x}{s-x}}$, where $s=\frac{a+b+c}{2}$, $f(x)$ is a convex function in $(0,s)$. Using Jensen's Inequality: $f\left(\frac{a+b+c}{3}\right)\leq\frac{1}{3}[f(a)+f(b)+f(c)]$

$$f(a)+f(b)+f(c)\geq 3f\left(\frac{a+b+c}{3}\right)\geq 3\sqrt{\frac{\frac{a+b+c}{3}}{\frac{a+b+c}{2}-\frac{a+b+c}{3}}}\geq 3\sqrt{2}\ \therefore$$

Solution 3 by Bogdan Fustei – Romania

ABC any triangle. Starting from $\cos\frac{B-C}{2}\geq\sqrt{\frac{2R}{R}}$ *(and analogs)*

$$\cos\frac{B-C}{2}=\frac{h_a}{w_a}\geq\sqrt{\frac{2r}{R}}\Rightarrow\frac{1}{w_a}\geq\frac{1}{h_a}\sqrt{\frac{2r}{R}}$$ *(and analogs)*

$$\Rightarrow \frac{1}{w_a}+\frac{1}{w_c}+\frac{1}{w_c} \geq \sqrt{\frac{2r}{R}}\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right); \text{ But } \frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{R}$$

$$\Rightarrow \frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c} \geq \frac{1}{r}\sqrt{\frac{2r}{R}} = \sqrt{\frac{2r}{Rr^2}} = \sqrt{\frac{2}{Rr}}$$

So, finally, we have the following inequality:

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c} \geq \sqrt{\frac{2}{Rr}}$$

$$\left.\begin{aligned}\cos\frac{B-C}{2} &= \frac{b+c}{2}\sin\frac{A}{2}\\ \sin\frac{A}{2} &= \sqrt{\frac{rr_a}{2Rh_a}}\end{aligned}\right\} \Rightarrow \cos\frac{B-C}{2} = \frac{b+c}{a}\sqrt{\frac{r}{2R}}\cdot\sqrt{\frac{r_a}{h_a}} \geq \sqrt{\frac{2r}{R}}$$

$$\frac{b+c}{a}\sqrt{\frac{r_a}{h_a}} \geq \sqrt{2}\cdot\sqrt{2} = 2$$

So, finally we have the inequality: $\sqrt{\frac{r_a}{h_a}} \geq 2\cdot\frac{a}{b+c}$ *(and the analogs).*

So, finally we have:

$$\sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{r_b}{h_b}}+\sqrt{\frac{r_c}{h_a}} \geq 2\left(\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\right)$$

$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \geq \frac{3}{2}$ *– Nesbit's inequality* $\Rightarrow \sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{r_b}{h_b}}+\sqrt{\frac{r_c}{h_c}} \geq 3$

We know the inequality:

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \geq \frac{w_a}{r_b+r_c}+\frac{w_b}{r_a+r_c}+\frac{w_c}{r_a+r_c} \Rightarrow$$

$$\Rightarrow \sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{r_b}{h_b}}+\sqrt{\frac{r_c}{h_c}} \geq 2\left(\frac{w_a}{r_b+r_c}+\frac{w_b}{r_a+r_c}+\frac{w_c}{r_a+r_c}\right)$$

$\frac{r_a}{h_a} = \frac{S}{s-a}\cdot\frac{1}{h_a} = \frac{h_a\cdot a}{2(s-a)}\cdot\frac{1}{h_a} = \frac{a}{2(s-a)} \Rightarrow \frac{r_a}{h_a} = \frac{a}{2(s-a)}$ *(and the analogs)*

$$\Rightarrow \sqrt{\frac{r_a}{h_a}} = \frac{1}{\sqrt{2}}\sqrt{\frac{a}{s-a}} \Rightarrow \sqrt{\frac{a}{s-a}}+\sqrt{\frac{b}{s-b}}+\sqrt{\frac{c}{s-c}} \geq 3\sqrt{2}$$

$$\Rightarrow \sqrt{\frac{a}{s-a}}+\sqrt{\frac{b}{s-b}}+\sqrt{\frac{c}{s-c}} \geq 2\sqrt{2}\left(\frac{w_a}{r_{+r_c}}+\frac{w_b}{r_a+r_c}+\frac{w_c}{r_a+r_b}\right)$$

Solution 4 by Neculai Stanciu, Titu Zvonaru – Romania.

After squaring, the inequality we have to prove can be written:

$$\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}+2\left(\sqrt{\frac{ab}{(s-a)(s-b)}}+\sqrt{\frac{bc}{(s-b)(s-c)}}+\sqrt{\frac{ca}{(s-c)(s-a)}}\right)\geq 18$$

$$\Leftrightarrow \frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}+2\left(\csc\frac{A}{2}+\csc\frac{B}{2}+\csc\frac{C}{2}\right)\geq 18, \text{ which follows from the known}$$

$$\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\geq 6 \text{ and } \csc\frac{A}{2}+\csc\frac{B}{2}+\csc\frac{C}{2}\geq 6$$

(see point 2.51 from "Geometric Inequalities" of O. Bottema, Groningen, 1969).

For completeness, the inequality $\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\geq 6$ *it follows from the identity*

$$\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}-6=\frac{(a-b)^2}{(s-a)(s-b)}+\frac{(s-c)^2}{(s-b)(s-c)}+\frac{(c-a)^2}{(s-c)(s-a)}$$

Solution 5 by Neculai Stanciu, Titu Zvonaru – Romania

We will use the notation $\sum$ *for* $\sum_{cyc}$ *. We have:*

$$\sum\sqrt{\frac{a}{s-a}}-3\sqrt{2}=\sum\left(\sqrt{\frac{a}{s-a}}-\sqrt{2}\right)=\sum\frac{\frac{a}{s-a}-2}{\sqrt{\frac{a}{s-a}}+\sqrt{2}}=$$

$$=\sum\frac{2a-b-c}{\sqrt{s(s-a)}+(s-a)\sqrt{2}}=\sum\frac{a-b}{\sqrt{a(s-a)}+(s-a)\sqrt{2}}+\sum\frac{a-c}{\sqrt{a(s-a)}+(s-a)\sqrt{2}}$$

$$=\sum\frac{a-b}{\sqrt{a(s-a)}+(s-a)\sqrt{2}}+\sum\frac{s-a}{\sqrt{b(s-b)}+(s-b)\sqrt{2}}=$$

$$=\sum\frac{(a-b)\left[\sqrt{b(s-b)}-\sqrt{a(s-a)}+(s-b)\sqrt{2}-(s-a)\sqrt{2}\right]}{\left(\sqrt{a(s-a)}+(s-a)\sqrt{2}\right)\left(\sqrt{b(s-b)}+(s-b)\sqrt{2}\right)}$$

Because $\sqrt{s(s-b)}-\sqrt{a(s-a)}=\frac{s(s-b)-a(s-a)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}=\frac{a^2-b^2-s(a-b)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}=$

$$=\frac{(a-b)(a+b-s)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}=\frac{(a-b)(s-c)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}} \text{ and}$$

$(s-b)\sqrt{2}-(s-a)\sqrt{2}=(a-b)\sqrt{2}$*, it follows that:*

$$\sum\sqrt{\frac{a}{s-a}}-3\sqrt{2}=\sum\frac{(a-b)^2\left(\frac{s-c}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}+\sqrt{2}\right)}{\left(\sqrt{a(s-a)}+(s-a)\sqrt{2}\right)\left(\sqrt{b(s-b)}+(s-b)\sqrt{2}\right)}\geq 0 \text{ and}$$

the desired inequality is proved. We have equality if and only if $a=b=c$*.*

Remark. Thus more elaborate, the second solution allows obtaining a refinement of the inequality from enunciation. Because:

$$\left(\sqrt{a(s-a)}+(s-a)\sqrt{2}\right)\left(\sqrt{b(s-b)}+(s-b)\sqrt{2}\right)=$$

$$=\sqrt{(s-a)(s-b)}\left(\sqrt{ab}+\sqrt{2a(s-b)}+\sqrt{2b(s-a)}+\sqrt{2(s-a)\cdot 2(s-b)}\right)\leq$$

$$\leq\sqrt{(s-a)(s-b)}\cdot\frac{a+b+2a+s-b+2b+s-a+2s-2a+2s-2b}{2}=$$

$$=3s\sqrt{(s-a)(s-b)},\ \textit{we deduce the inequality:}$$

$$\sum\sqrt{\frac{a}{s-a}}-3\sqrt{2}\geq\frac{\sqrt{2}}{3s}\sum\frac{(s-b)^2}{\sqrt{(s-a)(s-b)}}$$

Refinement by Neculai Stanciu, Titu Zvonaru – Romania

Prove that in any triangle the following inequality holds:

$$\sum\sqrt{\frac{a}{s-a}}-3\sqrt{2}\geq\frac{\sqrt{2}}{3s}\sum\frac{(a-b)^2}{\sqrt{(s-a)(s-b)}}$$

Proof.

$$\sum\sqrt{\frac{a}{s-a}}-3\sqrt{2}=\sum\left(\sqrt{\frac{a}{s-a}}-\sqrt{2}\right)=\sum\frac{\frac{a}{s-a}-2}{\sqrt{\frac{a}{s-a}}+\sqrt{2}}=$$

$$=\sum\frac{2a-b-c}{\sqrt{a(s-a)}+(s-a)\sqrt{2}}=\sum\frac{a-b}{\sqrt{a(s-a)}+\sqrt{(s-a)\sqrt{2}}}+\sum\frac{a-c}{\sqrt{a(s-a)}+(s-a)\sqrt{2}}=$$

$$=\sum\frac{a-b}{\sqrt{a(s-a)}+(s-a)\sqrt{2}}+\sum\frac{b-a}{\sqrt{b(s-b)}+(s-b)\sqrt{2}}=$$

$$=\sum\frac{(a-b)\left[\sqrt{b(s-b)}-\sqrt{a(s-a)}+(s-b)\sqrt{2}-(s-a)\sqrt{2}\right]}{\left(\sqrt{a(s-a)}+(s-a)\sqrt{2}\right)\left(\sqrt{b(s-b)}+(s-b)\sqrt{2}\right)}$$

Because $\sqrt{b(s-b)}-\sqrt{a(s-a)}=\frac{b(s-b)-a(s-a)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}=\frac{a^2-b^2-s(s-b)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}=$

$$=\frac{(a-b)(a+b-p)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}=\frac{(a-b)(s-c)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}\ \textit{and}$$

$(s-b)\sqrt{2}-(s-a)\sqrt{2}=(a-b)\sqrt{2}$, *it follows that:*

$$\sum\sqrt{\frac{a}{s-a}}-3\sqrt{2}=\sum\frac{(a-b)^2\left(\frac{s-c}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}+\sqrt{2}\right)}{\left(\sqrt{a(s-a)}+(s-a)\sqrt{2}\right)\left(\sqrt{b(s-b)}+(s-b)\sqrt{2}\right)}\geq 0$$

Because:

$$\left(\sqrt{a(s-a)}+(s-a)\sqrt{2}\right)\left(\sqrt{b(s-b)}+(s-b)\sqrt{2}\right)=$$

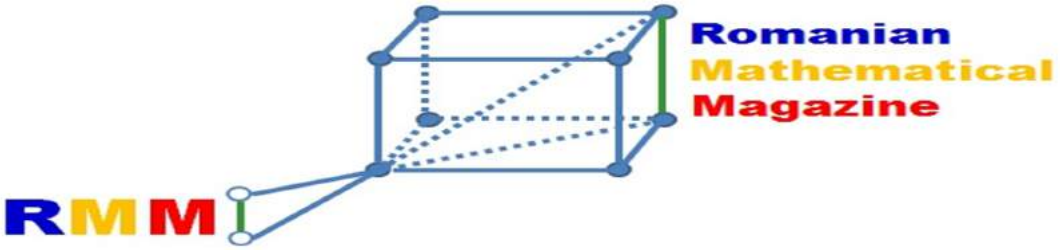
$$= \sqrt{(s-a)(s-b)}\left(\sqrt{ab} + \sqrt{2a(s-b)} + \sqrt{2b(s-a)} + \sqrt{2(s-a)\cdot 2(s-b)}\right) \leq$$

$$\leq \sqrt{(s-a)(s-b)} \cdot \frac{a+b+2a+s-b+2b+s-a+2s-2a+2s-2b}{2} =$$

$= 3s\sqrt{(s-a)(s-b)}$, *we deduce the inequality:* $\sum \sqrt{\frac{a}{s-a}} - 3\sqrt{2} \geq \frac{\sqrt{2}}{3s}\sum \frac{(a-b)^2}{\sqrt{(s-a)(s-b)}}$, *q.e.d*

We have equality if and only if $a = b = c$.

Remark.: *The inequality proposed above refines the inequality:* $\sqrt{\frac{a}{s-a}} + \sqrt{\frac{b}{s-b}} + \sqrt{\frac{c}{s-c}} \geq 3\sqrt{2}$.



GENERALIZATION OF THE LIMITS OF THE SEQUENCES OF BĂTINEŢU, GHERMĂNESCU, IANCULESCU, LALESCU AND OTHER COLLABORATIONS

By D.M. Bătineţu – Giurgiu, Neculai Stanciu-Romania

I. If $B_n(t) = n^{1-t}\left(\frac{(n+1)^{2t}}{\left(\sqrt[n+1]{(n+1)!}\right)^t} - \frac{n^{2t}}{\left(\sqrt[n]{n!}\right)^t}\right)$, with $t > 0$, then $\lim\limits_{n\to\infty} B_n(t) = te^t$.

Proof. $$B_n(t) = n^{1-t}\cdot\frac{n^{2t}}{\left(\sqrt[n]{n!}\right)^t}\cdot(u_n - 1) = \left(\frac{n}{\sqrt[n]{n!}}\right)^t \cdot \frac{u_n - 1}{\ln u_n}\cdot \ln u_n^n, \forall n \geq 2, \quad (1)$$

where we denoting $u_n = \left(\frac{n+1}{n}\right)^{2t}\cdot\left(\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}\right)^t$, $\forall n \geq 2$.

We have $\lim\limits_{n\to\infty} u_n = 1$ and then we obtain $\lim\limits_{n\to\infty}\frac{u_n - 1}{\ln u_n} = 1$.We also have:

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(e_n^{2t}\cdot\left(\frac{n!}{(n+1)!}\cdot\sqrt[n+1]{(n+1)!}\right)^t\right) = e^{2t}\cdot\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\right)^t = e^{2t}\cdot e^{-t} = e^t.$$

By (1) and above we obtain that: $\lim\limits_{n\to\infty} B_n(t) = e^t\cdot 1\cdot \ln e^t = te^t$

Observation. For $t=1$ we obtain that:

$$\lim_{n\to\infty} B_n(1)=\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\right)=e,$$

i.e. the limit of well-known ***D.M Bătineţu-Giurgiu'*** **s sequence.**

II. Let $\forall t\in R^*$ and the sequence $\left(I_n(t)\right)_{n\geq 2}$, $I_n(t)=n^{1-t}\left(\left(n+1\right)^t\left(\sqrt[n+1]{n+1}\right)^t-n^t\left(\sqrt[n]{n}\right)^t\right)$, then

$$\lim_{n\to\infty} I_n(t)=t$$

Proof. $$I_n(t)=n\cdot\sqrt[n]{n^t}\cdot\left(u_n-1\right)=\left(\sqrt[n]{n}\right)^t\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n,\forall n\geq 2 \qquad (1)$$

where $u_n=\left(\frac{n+1}{n}\right)^t\cdot\left(\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}\right)^t,\forall n\geq 2$. So, $\lim_{n\to\infty}u_n=1$, $\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$. Also we have

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}e_n^t\cdot\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}\right)^{nt}=e^t\cdot\lim_{n\to\infty}\left(\frac{n+1}{n}\cdot\frac{1}{\sqrt[n+1]{n+1}}\right)^t=e^t\cdot 1=e^t.$$

Hence: $\lim_{n\to\infty}I_n(t)=1\cdot 1\cdot\ln\left(\lim_{n\to\infty}u_n^n\right)=\ln e^t=t$.

Observation. For $t=1$ we get that

$\lim_{n\to\infty}I_n=\lim_{n\to\infty}I_n(1)=1$, i.e. the limit of **the sequence of** ***Romeo T. Ianculescu***.

III. Let $g_n=\frac{(n+2)^{n+1}}{(n+1)^n},\forall n\in N$, $x\in R$. $\lim_{n\to\infty}n^{\sin^2 x}\left(g_{n+1}^{\cos^2 x}-g_n^{\cos^2 x}\right)=\cos^2 x\cdot e^{\cos^2 x}$

Proof. $$G_n(x)=n^{1-\cos^2 x}\left(g_{n+1}^{\cos^2 x}-g_n^{\cos^2 x}\right)=n\cdot\left(\frac{g_n}{n}\right)^{\cos^2 x}\left(u_n-1\right)=$$

$$=\left(\frac{g_n}{n}\right)^{\cos^2 x}\frac{u_n-1}{\ln u_n}\ln u_n^n=\frac{(n+2)^{(n+1)\cos^2 x}}{n^{\cos^2 x}(n+1)^{n\cos^2 x}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n=$$

$$=\left(\frac{n+2}{n+1}\right)^{(n+1)\cos^2 x}\left(\frac{n+1}{n}\right)^{\cos^2 x}\frac{u_n-1}{\ln u_n}\ln u_n^n,\forall n\in N^* \qquad (1)$$

where $u_n=\left(\frac{g_{n+1}}{g_n}\right)^{\cos^2 x}=\left(\frac{(n+3)^{n+2}}{(n+2)^{n+1}}\cdot\frac{(n+1)^n}{(n+2)^{n+1}}\right)^{\cos^2 x},\forall n\in N$.

Therefore, $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\left(\left(\frac{n+3}{n+2}\right)^{n+2}\left(\frac{n+1}{n+2}\right)^{n}\right)^{\cos^2 x} = \left(e\cdot\frac{1}{e}\right)^{\cos^2 x} = 1$, şi deci $\lim\limits_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$.

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{(n+3)^{n+2}}{(n+2)^{n+1}}\cdot\frac{(n+1)^n}{(n+2)^{n+1}}\right)^{n\cos^2 x} = \lim_{n\to\infty}\left(\left(\frac{(n+1)(n+3)}{(n+2)^2}\right)^{n+1}\cdot\frac{n+3}{n+1}\right)^{n\cos^2 x} =$$

$$= \lim_{n\to\infty}\left(\frac{n^2+4n+3}{n^2+4n+4}\right)^{(n^2+n)\cos^2 x}\cdot\lim_{n\to\infty}\left(\frac{n+3}{n+1}\right)^{n\cos^2 x} = e^{-\cos^2 x}\cdot e^{2\cos^2 x} = e^{\cos^2 x}.$$

Hence we obtain that:

$$\lim_{n\to\infty} G_n(x) = e^{\cos^2 x}\cdot 1\cdot 1\cdot\ln\left(\lim_{n\to\infty} u_n^n\right) = e^{\cos^2 x}\cdot\ln e^{\cos^2 x} = \cos^2 x\cdot e^{\cos^2 x}.$$

Observation. For $x=0$, yields that $\lim\limits_{n\to\infty} G_n(0) = \lim\limits_{n\to\infty} G_n = e$, i.e. the limit of **the sequence of *Mihail Ghermănescu*.**

IV. Euler-Mascheroni-Lalescu collaboration

If $a,b\in R, a+b=1$, then $\lim\limits_{n\to\infty}\left((n+1)^a\sqrt[n+1]{((n+1)!c_n)^b} - n^a\sqrt[n]{(n!e_n)^b}\right) = \frac{a+b\ln c}{e^b}$, where

$$e_n = \left(1+\frac{1}{n}\right)^n \text{ and } c_n = -\ln n + \sum_{k=1}^{n}\frac{1}{k}.$$

Proof: $B_n = (n+1)^a\sqrt[n+1]{((n+1)!c_n)^b} - n^a\sqrt[n]{(n!e_n)^b} = n^a\sqrt[n]{(n!e_n)^b}(u_n-1) =$

$$= n^a\sqrt[n]{(n!e_n)^b}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n = n\cdot\frac{\sqrt[n]{(n!e_n)^b}}{n^b}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n = \left(\frac{\sqrt[n]{n!e_n}}{n}\right)^b\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n, \forall n\geq 2,$$

$$\text{unde } u_n = \left(\frac{n+1}{n}\right)^a\left(\frac{\sqrt[n+1]{(n+1)!c_n}}{\sqrt[n]{n!e_n}}\right)^b, \forall n\geq 2.$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!c_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!c_n}{n^n}} = \lim_{n\to\infty}\left(\frac{(n+1)!c_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!c_n}\right) = \lim_{n\to\infty}\frac{c_{n+1}}{c_n}\cdot\frac{1}{e_n} = \frac{1}{e};$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!e_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!e_n}{n^n}} = \lim_{n\to\infty}\left(\frac{(n+1)!e_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!e_n}\right) = \lim_{n\to\infty}\frac{e_{n+1}}{e_n}\cdot\frac{1}{e_n} = \frac{1}{e}.$$

So $\lim\limits_{n\to\infty} u_n = 1$ and then $\lim\limits_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$.

Also we have that $\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} e_n^a \left(\frac{(n+1)!c_n}{n!e_n} \cdot \frac{1}{\sqrt[n+1]{(n+1)!c_n}} \right)^b = e^a \cdot \lim_{n\to\infty} \left(\frac{c_n}{e_n} \cdot \frac{n+1}{\sqrt[n+1]{(n+1)!c_n}} \right) =$

$$= e^a \left(\frac{c}{e} \cdot e \right)^b = e^a c^b .$$

Hence: $\lim_{n\to\infty} B_n = \left(\frac{1}{e} \right)^b \cdot 1 \cdot \ln\left(\lim_{n\to\infty} u_n^n \right) = \frac{1}{e^b} \cdot \ln\left(e^a c^b \right) = \frac{a + b\ln c}{e^b}$.

V. Euler-Mascheroni-Bătineţu collaboration

Let $e_n = \left(1 + \frac{1}{n} \right)^n$ and $\gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$, for any positive integer n, then

$$\lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!!\gamma_n}} - \frac{n^2}{\sqrt[n]{(2n-1)!!e_n}} \right) = \frac{e}{2}(2 - \ln\gamma).$$

Proof. (1) $x_n = \frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!!c_n}} - \frac{n^2}{\sqrt[n]{(2n-1)!!e_n}} = \frac{n^2}{\sqrt[n]{(2n-1)!!e_n}}(u_n - 1) =$

$$= \frac{n}{\sqrt[n]{(2n-1)!!e_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \forall n \geq 2,$$ where we denoted

$$u_n = \left(\frac{n+1}{n} \right)^2 \frac{\sqrt[n]{(2n-1)!!e_n}}{\sqrt[n+1]{(2n+1)!!c_n}}, \forall n \geq 2.$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!e_n}}{n} = \lim_{n\to\infty} \frac{(2n+1)!!e_{n+1}}{(2n-1)!!e_n} \left(\frac{n}{n+1} \right)^n \left(\frac{1}{n+1} \right) = \lim_{n\to\infty} \left((2n+1) \cdot \frac{1}{n+1} \cdot \frac{1}{e_n} \right) = \frac{2}{e},$$

and analogous, $\lim_{n\to\infty} \frac{\sqrt[n+1]{(2n+1)!!\gamma_n}}{n+1} = \frac{2}{e}$. Yields, $\lim_{n\to\infty} u_n = 1$, and $\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$.

We have (2) $\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \left(e_n^2 \cdot \frac{(2n-1)!!e_n}{(2n+1)!!\gamma_n} \cdot \sqrt[n+1]{(2n+1)!!\gamma_n} \right) = \frac{e^3}{\gamma} \lim_{n\to\infty} \frac{\sqrt[n+1]{(2n+1)!!\gamma_n}}{2n+1} =$

$$= \frac{e^3}{\gamma} \lim_{n\to\infty} \frac{\sqrt[n+1]{(2n+1)!!\gamma_n}}{n+1} \cdot \frac{n+1}{2n+1} = \frac{e^3}{\gamma} \cdot \frac{2}{e} \cdot \frac{1}{2} = \frac{e^2}{\gamma}$$

Hence, taking to limt in (1) with $n \to \infty$ and considering (2) we obtain that

$$\lim_{n\to\infty} x_n = \frac{e}{2}(2 - \ln\gamma).$$

SOME NEW APPLICATIONS OF BLUNDON'S THEOREM IN AN ACUTE TRIANGLE

By Marius Drăgan , Neculai Stanciu-Romania

Abstract. In this paper we present two new applications of [1].
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MSC 2010: 51M16.

The article from [1] presented a method to find ''the best'' constant k for which the inequality (1) $F(s,R,r)\le k$, or the reverse is true in every acute triangle where $F(s,R,r)$ is a monotone function and homogenous of zero degree in s. In [2] is proved the following:

Theorem (Blundon in an acute triangle). *In every acute triangle is true the inequality* $s_1\le s\le s_2$ *if* $\frac{R}{r}\in\left[2,\sqrt{2}+1\right]$ *and* $s_3\le s\le s_2$ *if* $\frac{R}{r}\in\left[\sqrt{2}+1,\infty\right)$ *where* s_1,s_2 *represents the semiperimeter of two isosceles triangles* $A_1B_1C_1$ *and* $A_2B_2C_2$ *with the sides* $a_1=2\sqrt{(R+r-d)(R-r+d)}$, $b_1=c_1=\sqrt{2R(R+r-d)}$; $a_2=2\sqrt{(R+r+d)(R-r-d)}$, $b_2=c_2=2\sqrt{2R(R+r+d)}$; *and* s_3 *the semiperimeter of right triangle* $A_3B_3C_3$ *with the sides* $a_3=2R$, $b_3=R+r+\sqrt{R^2-2Rr-r^2}$, $c_3=R+r-\sqrt{R^2-2Rr-r^2}$, *where* $d=\sqrt{R^2-2Rr}$.

If F is an decreasing function, from (1) and the above theorem in s it follows that

(2) $$F(s,R,r)\le F(s_1,R,r)\le k_1 \text{ if } \frac{R}{r}\in\left[2,\sqrt{2}+1\right] \text{ and}$$

$$F(s,R,r)\le F(s_3,R,r)\le k_2 \text{ if } \frac{R}{r}\in\left[\sqrt{2}+1,\infty\right).$$

We denote $t=\sqrt{(R+r-d)(R-r+d)}$, $x=\sqrt{\frac{\sqrt{2R}-\sqrt{R-r+d}}{\sqrt{2R}+\sqrt{R-r+d}}}$.

From theorem we have $a_1=2t$, $b_1=c_1=t\cdot\frac{1+x^2}{1-x^2}$, $s_1=\frac{2t}{1-x^2}$, $r_1=r=tx$,

$$R_1=R=t\cdot\frac{(1+x^2)^2}{4x(1-x^2)}.$$

Since $2\le\frac{R}{r}\le\sqrt{2}+1$ we obtain $2\le\frac{(1+x^2)^2}{4x^2(1-x^2)}\le\sqrt{2}+1$ or after we perform some calculation

$$x\in\left[\sqrt{2}-1,\sqrt{\frac{2\sqrt{2}}{7}+\frac{1}{7}}\right].$$

We have $F(s,R,r)=F\left(\frac{2}{1-x^2},\frac{(1+x^2)^2}{4x(1-x^2)},x\right)$. So if $\frac{R}{r}\in\left[2,\sqrt{2}+1\right]$, then from (2) we obtain that $k_1=\sup f_1(x)$, where $f_1:\left[\sqrt{2}-1,\sqrt{\frac{2\sqrt{2}}{7}+\frac{1}{7}}\right]\to R$, $f_1(x)=F\left(\frac{2}{1-x^2},\frac{(1+x^2)^2}{4x(1-x^2)},x\right)$.

If $\frac{R}{r} \geq \sqrt{2}+1$ we denote $w = 2R$. Let x such that $b_3 = w \cdot \frac{1-x^2}{1+x^2} < a_3$, it follows $x \in (0,1)$ and since $a_3^2 = b_3^2 + c_3^2$ we get that $c_3 = w \cdot \frac{2x}{1+x^2}$, $s_3 = w \cdot \frac{1+x}{1+x^2}$, $r_3 = r = \frac{wx(1-x)}{1+x^2}$, $R_3 = R = \frac{w}{2}$, so we get $F(s_3, R, r) = F\left(w \cdot \frac{1+x}{1+x^2}, \frac{w}{2}, \frac{wx(1-x)}{1+x^2}\right) = F\left(\frac{1+x}{1+x^2}, \frac{1}{2}, \frac{x(1-x)}{1+x^2}\right)$.

Because $\frac{R}{r} \geq \sqrt{2}+1$, yields that $\frac{1+x^2}{2(x-x^2)} \geq \sqrt{2}+1$ or $(x-\sqrt{2}-1)^2 \geq 0$, it remain $x \in (0,1)$

so

$$k_2 = \sup f_2(x), \text{ where } f_2 : (0,1) \to R,\ f_2(x) = F\left(\frac{1+x}{1+x^2}, \frac{1}{2}, \frac{x(1-x)}{1+x^2}\right).$$

Therefore, in the case when F is a decreasing function in s we have that

(3) $$\overline{k_1} = \begin{cases} \sup\limits_{\sqrt{2}-1 \leq x \leq \sqrt{\frac{2\sqrt{2}}{7}+\frac{1}{7}}} f_1(x), \text{if } \frac{R}{r} \in \left[2, \sqrt{2}+1\right] \\ \sup\limits_{0<x<1} f_2(x), \text{if } \frac{R}{r} \in \left[\sqrt{2}+1, \infty\right) \end{cases}$$, are f_1, f_2 are the functions defined above.

If F is a increasing function in s from (1) and theorem we have that

$$F(s, R, r) \leq F(s_2, R, r) \leq k \text{ for any } \frac{R}{r} \geq 2.$$

We denote $u = \sqrt{(R+r+d)(R-r-d)}$, $x = \sqrt{\frac{\sqrt{2R}-\sqrt{R-r-d}}{\sqrt{2R}+\sqrt{R-r-d}}}$, $x \in (0,1)$, so we obtain

$$a_2 = 2u,\ b_2 = c_2 = u \cdot \frac{1+x^2}{1-x^2},\ s_2 = \frac{2u}{1-x^2}\ r_2 = r = ux, R_2 = R = \frac{(1+x^2)^2 u}{4x(1-x^2)}.$$

Hence, $F\left(\frac{2u}{1-x^2}, \frac{(1+x^2)^2 u}{4x(1-x^2)}, xu\right) = F\left(\frac{2}{1-x^2}, \frac{(1+x^2)^2}{4x(1-x^2)}, x\right)$, for any $x \in (0,1)$.

Therefore, $\overline{k_2} = \sup\limits_{x \in (0,1)} f(x)$, where $f : (0,1) \to R,\ f(x) = F\left(\frac{2}{1-x^2}, \frac{(1+x^2)^2}{4x(1-x^2)}, x\right)$.

In the same way like above we obtain the best constant $\overline{k_3}$ for which is true the inequality $F(s, R, r) \geq \overline{k_3}$ in every acute triangle in the case of decreasing function F in s, i.e.

(5) $$\overline{k_3} = \inf\limits_{x \in (0,1)} f(x), \text{ where } f : (0,1) \to R,\ f(x) = F\left(\frac{2}{1-x^2}, \frac{(1+x^2)^2}{4x(1-x^2)}, x\right).$$

In the case of increasing function the best constant for which the inequality $F(s, R, r) \geq \overline{k_4}$ is true in every acute triangle is (6) $$\overline{k_4} = \begin{cases} \inf\limits_{\sqrt{2}-1 \leq x \leq \sqrt{\frac{2\sqrt{2}}{7}+\frac{1}{7}}} f_1(x), \text{if } \frac{R}{r} \in \left[2, \sqrt{2}+1\right] \\ \inf\limits_{0<x<1} f_2(x), \text{if } \frac{R}{r} \in \left[\sqrt{2}+1, \infty\right) \end{cases}$$, where

$f_1:\left[\sqrt{2}-1,\sqrt{\frac{2\sqrt{2}}{7}+\frac{1}{7}}\right]\to R$, $f_1(x)=F\left(\frac{2}{1-x^2},\frac{(1+x^2)^2}{4x(1-x^2)},x\right)$ and $f_2:(0,1)\to R$,

$$f_2(x)=F\left(\frac{1+x}{1+x^2},\frac{1}{2},\frac{x(1-x)}{1+x^2}\right).$$

Applications

I. The Hadwiger-Finsler inequality in an acute triangle. *Find the best constant k for which the inequality (7) $a^2+b^2+c^2\geq 4\sqrt{3}S+k[(a-b)^2+(b-c)^2+(c-a)^2]$ is true in every acute triangle ABC with area S.*

Solution. The inequality from the statement can be written as

$F(s,R,r)=\frac{s^2-2\sqrt{3}rs-r^2-4Rr}{s^2-3r^2-12Rr}\geq k$, where F is a decreasing function in s.

From (5) we have $\overline{k_3}=\inf\limits_{x\in(0,1)}f(x)$, where $f:(0,1)\to R$,

$f(x)=F\left(\frac{2}{1-x^2},\frac{(1+x^2)^2}{4x(1-x^2)},x\right)=\left(\frac{x+\sqrt{3}}{x\sqrt{3}+1}\right)^2$, and it results that $\overline{k_3}=\inf\limits_{x\in(0,1)}f(x)=1$(see [3]).

II.*Find the best constant such that* $(x^2+y^2+z^2)(x^2+y^2+z^2-xy-yz-zx)\geq$

$\geq k(x+y+z)(x^2y+yx^2+y^2z+yz^2+z^2x+zx^2-6xyz)$, $\forall x,y,z>0$, *and*

$xy+yz+zx\geq\max\{x^2,y^2,z^2\}$.

Solution. From [3] we have that $F(s,R,r)=\left(\frac{(4R+r)^2}{s^2}-2\right)\frac{[(4R+r)^2-3s^2]}{(4R+r)(4R-8r)}\geq k$, with F a decreasing function in s. The condition $xy+yz+zx\geq\max\{x^2,y^2,z^2\}$ is equivalent with $A,B,C\in\left(0,\frac{\pi}{2}\right]$, so we are in the case (5) and it results that $\overline{k_3}=\inf\limits_{x\in(0,1)}f(x)=\sqrt{6}-2$ (see [3]).

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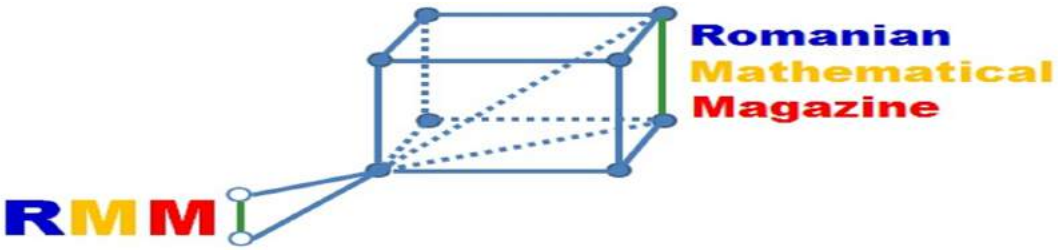
PROPOSED PROBLEMS

5-CLASS-STANDARD



V.1. Aflați $x, y \in \mathbb{N}^*$ pentru care $x^y + y^x = 2017$.

Proposed by Petre Stângescu – Romania

V.2. Într-o magazie sunt depozitate 747 televizoare de același tip, în două tipuri de containere. În primul tip de container intră exact 32 televizoare, iar în al doilea intră exact 33 televizoare. Știind că toate containerele sunt pline aflați numărul lor.

Proposed by Petre Stângescu – Romania

V.3. Aflați $x, y, z \in \mathbb{N}$ dacă $x^2 + 2017^y = 2016^z$.

Proposed by Petre Stângescu – Romania

V.4. Să se rezolve în $\mathbb{N}$ ecuația: $x^2 = 3^y + 135$.

Proposed by Petre Stângescu – Romania

V.5. Să se determine numerele naturale a, b, c, d și e, știind că $15^{2019} + 225^{1009} = a^2 + b^2 + c^2 + d^2 + e^2$. ***Proposed by Draga Tatucu Mariana-Romania***

V.6. Un creion costă cât o riglă și o gumă împreună. Dacă voi cumpăra 3 rigle, 5 gume și 2 creioane voi plăti 17 lei. Iar dacă voi cumpăra 8 rigle și 10 gume, voi plăti 26 lei. Aflați cât costă o riglă, o gumă și un creion. ***Proposed by Draga Tatucu Mariana-Romania***

V.7. Find the pairs of prime numbers $(\overline{at};\ \overline{ac})$ having the property $10|\overline{ab}^2 + \overline{ac}^2$ in the following cases: 1) $\overline{ab}^2 + \overline{ac}^2$ – minium, 2) $\overline{ab}^2 + \overline{ac}^2$ – maximum. $a \neq b \neq c$

Proposed by Ștefan Marica – Romania

V.8. Find $\overline{abcd}$ from the relationship: $\overline{abcd} = a^{11} - a^5 + a$

Proposed by Ștefan Marica – Romania

V.9. Find $\overline{abcd}$ knowing that the following relationships hold:

1) $a + b + c + d$ – perfect square, 2) $c\overline{d}$ – perfect square, 3) $c + d$ – perfect square
4) $b = a + 1$ ***Proposed by Ștefan Marica – Romania***

V.10. Find $\overline{ab}_1 < \overline{ab}_2 < \overline{ab}_3$ consecutive natural numbers such that:

$$\overline{ab}_1 + \overline{ab}_2 + \overline{ab}_3 = (a-1)^3$$

Proposed by Ștefan Marica – Romania

V.11. Find $\overline{ab}$ knowing that the following relationships holds: 1) $a! + b! = \overline{bb}^2$, 2) $a! \cdot b! = 720$ ***Proposed by Ștefan Marica – Romania***

V.12. 1. Find $\overline{abcd}$ such that: $\overline{ab} + \overline{cd} = p^2q^2 + 1$; $\overline{ab} - \overline{cd} = q; p, q$ - prime numbers.

Proposed by Ștefan Marica – Romania

V.13. Find $\overline{au}, \overline{bu}, \overline{cu}, \overline{du}$ – prime numbers such that $\overline{au} + \overline{bu} + \overline{cu} + \overline{du}$ is a perfect square.

Proposed by Ștefan Marica – Romania

V.14. If p is a prime number, solve for natural numbers: $2x^3 + x^2 - x = p$

Proposed by Gheorghe Calafeteanu – Romania

V.15. Solve for natural numbers: $x^4 + y^4 + z^4 = 2019^{2019}$

Proposed by Gheorghe Calafeteanu – Romania

V.16. Un număr natural îl numim de tip *dependent de* 19 dacă suma cifrelor sale este divizibilă cu 19. a) Demonstrați că există o infinitate de numere de tip *dependent de* 19, care sunt pătrate perfecte. b) Demonstrați că există o infinitate de numere de tip *dependent de* 19, care sunt cuburi perfecte. c) Se pot construi o infinitate de perechi de numere naturale consecutive $(n, n+1)$ astfel ca n și $n+1$ să fie simultan numere de tip *dependent de* 19?

Proposed by Dan Nedeianu-Romania

V.17. Să se determine ultimele trei cifre ale numărului 7^{2021}.

Proposed by Dan Nedeianu-Romania

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6-CLASS-STANDARD

VI.1. Determinați numerele naturale a și b cu proprietatea: $1 \cdot 2 \cdot 3 \cdot \ldots \cdot a + 12 = b^2$

Proposed by Marin Chirciu – Romania

VI.2. Arătați că numărul $n = 71^a + 17^{4b+2}$, este multiplu de 5, oricare ar fi numerele naturale a și b. ***Proposed by Marin Chirciu – Romania***

VI.3. Dacă împărțim un număr natural n la numărul natural a, unde $a \geq 2$, obținem restul $a-2$. Determinați restul împărțirii numărului n la numărul natural b, unde $b \geq 2$, știind că b divide a. ***Proposed by Marin Chirciu – Romania***

VI.4. Arătați că numărul $A = 9 + 99 + 999 + \cdots + \underbrace{99\ldots9}_{"n"\ cifre\ 9}$ se scrie folosind numai cifra 1.

Proposed by Marin Chirciu – Romania

VI.5. Să se arate că suma primelor 2005 numere naturale n, pentru care fracția $\frac{5n+2}{4n+3}$ este reductibilă, este un număr natural divizibil cu 2005.

Proposed by Marin Chirciu – Romania

VI.6. Dacă un număr natural x este o sumă de trei pătrate perfecte distincte, de forma $1 + a^2 + b^2$, unde $a, b \in \mathbb{N}$, să se arate că x^n se scrie ca o sumă de trei pătrate perfecte distincte, oricare ar fi numărul natural nenul n. ***Proposed by Marin Chirciu – Romania***

VI.7. Fie $a, b, n \in \mathbb{N}$ astfel încât b divide a^2 și $b \geq 10$. Determinați restul împărțirii numărului $a^{2n} - 10$ la b. ***Proposed by Marin Chirciu – Romania***

VI.8. Fie $A \subset N$ o mulțime care satisface condițiile:i) $1 \in A$;ii) dacă $x \in A$, atunci $5x \in A$; iii) dacă $7x - 1 \in A$. ***Proposed by Marin Chirciu – Romania***

VI.9. Fie $a, b, c \in \mathbb{N}^*$ astfel încât $a = b + 1 = c + 2$. Fie $n \in \mathbb{N}$, care împărțit la a dă restul b și împărțit la b dă restul c. Determinați restul împărțirii lui n la ab.

Proposed by Marin Chirciu – Romania

VI.10. Fie $k \in \mathbb{N}; k \geq 2$. Dacă $a, b, c \in \mathbb{N}$ astfel încât $2k^a = k^b + k^c$, atunci $a = b = c$.

Proposed by Petre Stângescu – Romania

VI.11. Fie $a, b \in \mathbb{N}$ astfel încât numerele $3a + 1$ și $5b + 2$ sunt direct proporționale cu 2 și 3. Să se demonstreze că ab nu este pătrat perfect.

Proposed by Petre Stângescu – Romania

VI.12. Find $\overline{abc}$ knowing that the following relationship holds:

$$a^7 \cdot b \cdot c + ab^2c + b^2 = 2019; \text{ then compare } \overline{abc} \text{ with } a^8$$

Proposed by Ştefan Marica – Romania

VI.13. Let be the sets: $A = \left\{4; 5; 8\frac{1}{2}; 9; 10\right\}$, $B = \left\{0{,}4; 5\frac{1}{2}; 9\right\}$. Prove that the arithmetic mean of the elements $A \cup B$ is equal with the geometric mean of the elements from the set $A \cap B$. ***Proposed by Ştefan Marica – Romania***

VI.14. Find $a; b; c$, natural numbers from the relationship: $\frac{5}{13} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Proposed by Ştefan Marica – Romania

VI.15. Prove without computer: $1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 13 > 2^{28}$

Proposed by Ştefan Marica – Romania

VI.16. Find $\overline{abc}$ such that $\overline{abbc} = \overline{b^2c^2}$. ***Proposed by Ştefan Marica – Romania***

VI.17. Find $\overline{abcd}$ such that $\overline{abbc} = \overline{dc^2}$. ***Proposed by Ştefan Marica – Romania***

VI.18. Find $\overline{ab}$; $\overline{ac}$ such that: $\overline{ab} = \overline{ac} + 1$; $(\overline{ac})^2 - \left(\overline{ab}\right)^2 = b^3$

Proposed by Ştefan Marica – Romania

VI.19. If $\Omega = \overline{abbc} - \overline{cbbba}; a > c$ then Ω can't be a perfect square.

Proposed by Daniel Sitaru – Romania

VI.20. Prove that: $\Omega = \overline{abcd2} + \left(\overline{abcd2}\right)^2 + \left(\overline{abcd2}\right)^2 + \cdots + \left(\overline{abcd2}\right)^{2020}$ is divisible with 10. ***Proposed by Daniel Sitaru – Romania***

VI.21. Find the number of solutions for the following equation in natural numbers: $\overline{aaaab} = 9999 + \overline{baaaa}$. ***Proposed by Daniel Sitaru – Romania***

VI.22. If $a, b, c \in \mathbb{R}$ then: $2\sum a^4 \pm 2\sum a^3 + 2\sum a^2 \pm 2\sum a + 3 > 0$

Proposed by Daniel Sitaru – Romania

VI.23. Find $\overline{abcd}$ such that: $\overline{abca} = \left(\overline{da}\right)^2$. ***Proposed by Ştefan Marica – Romania***

VI.24. Find $\overline{au}, \overline{bu}, \overline{cu}$ – prime numbers such that $\overline{au} + \overline{bu} + \overline{cu}$ is a perfect square.

Proposed by Ştefan Marica – Romania

VI.25. In ΔABC: $m(\hat{A}) = \frac{1}{3}m(\hat{B}) = \frac{1}{5}m(\hat{C})$. Find $m(\hat{A});\ m(\hat{B});\ m(\hat{C})$.

Proposed by Gheorghe Calafeteanu – Romania

VI.26. Fie $k \in \mathbb{N}^*$, k este fixat. Determinați mulțimea:

$$A = \{n \in \mathbb{N} | n^2 - n + 4k - 1 = 1 + 2 + 3 + \cdots + 4k\}.$$

Proposed by Marin Chirciu-Romania

VI.27. Se consideră triunghiul ABC, cu $m(\sphericalangle A) = 45°, AD \perp BC, D \in (BC), AD > BC$. Pe segmentul $[AD]$ se ia punctul M astfel încât $[AM] \equiv [BC]$. Dacă $BM \cap AC = \{O\}$, să se calculeze $m(\sphericalangle BOC)$. ***Proposed by Dan Nedeianu-Romania***

VI.28. Determinați restul împărțirii numărului p^{2018} la 30, unde $p \in \mathbb{N}, p$ număr prim.

Proposed by Dan Nedeianu-Romania

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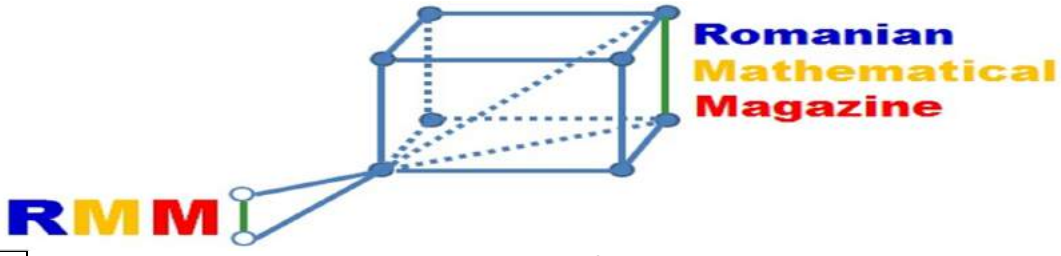
7-CLASS-STANDARD



VII.1. Aflați $a, n \in \mathbb{N}$ și p număr prim pentru care: $a^3 = p^n + 1$.

Proposed by Petre Stângescu – Romania

VII.2. Să se afle numerele raționale a și b pentru care are loc

$$\sqrt{5(a+2)^2} + |b+3| \cdot \sqrt{7} + \left|5\sqrt{7} - 7\sqrt{5}\right| - 8\sqrt{5} = 0$$

Proposed by Draga Tatucu Mariana-Romania

VII.3. Prove that: $(a+b+c)^2 - (c-a-b)^2 = 4ac + 4bc$.

Proposed by Ştefan Marica – Romania

VII.4. Write the following expression : $c(x;y;z) = 4(x+y)(x+z) + 4(y+z)(z+x)$ as a difference of two squares. ***Proposed by Ştefan Marica – Romania***

VII.5. Find two triplets of natural numbers $(x;y;z)$ such that:

$$\sqrt{16x^2y^2 + 9x^2 + 16z^2 + 6x - 8xy + 2} \in \mathbb{N}$$

Proposed by Ştefan Marica – Romania

VII.6. Find the natural number a such that:

$$\sqrt{2a+1}+\sqrt{4a+3}+\sqrt{6a+5}+\sqrt{8a+7}\leq 100$$

Proposed by Ștefan Marica – Romania

VII.7. If in ΔABC: $a^2\leq 26a+b-c-176, b^2\leq 24b+c-a-136, c^2\leq 10c+a-b-26$ then the triangle is a rightangled one. ***Proposed by Gheorghe Calafeteanu – Romania***

VII.8. Solve for natural numbers: $x^2+2x+13=y^2+4y$.

Proposed by Ștefan Marica – Romania

VII.9. If $ABCD$ is a square; $AB=a$ and $M\in(BC); N\in(CD); P\in(DA); Q\in(AO); R\in(OB)$; $\{O\}=AC\cap BD; AQ=\frac{1}{4}AC; BR=\frac{1}{4}BD$ then find $[MNPQR]$ in terms of a and then find a such that the perimeter of $MNPQR$ is a natural number.

Proposed by Ștefan Marica – Romania

VII.10. Solve for natural numbers: $x+\sqrt{xy}=18$. ***Proposed by Ștefan Marica – Romania***

VII.11 If ABC parallelogram; $MN\parallel AB; M\in(AD); N\in(BC); AC\cap BM=\{E\}, BD\cap AN=\{F\}$ then: $AE\cdot AF=CE\cdot FN, AE\cdot DF=CE\cdot BF$

Proposed by Ștefan Marica – Romania

VII.12. In any convexe quadrilater $ABCD, AB=a, BC=b, CD=c, DA=d$ the following relationship holds: $2[ABCD]\leq ab+cd; 2[ABCD]\leq ad+bc$.

Proposed by Ștefan Marica – Romania

VII.13. Find $n\in\mathbb{N}$ such that $n^2+(n+1)^2+(n+2)^2$ is divisible with 7.

Proposed by Ștefan Marica – Romania

VII.14. If $a,b,c\in\mathbb{R}^*$ then: $\left(\frac{a}{b}\right)^{16}+\left(\frac{b}{c}\right)^{16}+\left(\frac{c}{a}\right)^{16}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$

Proposed by Gheorghe Calafeteanu – Romania

VII.15. Pe prelungirile laturilor AB și AC ale triunghiului echilateral ABC se construiesc punctele M, respectiv N astfel ca $[BM]\cap[CN]=\{A\}$ și $\{BM\}\equiv[AN]$. Să se arate că triunghiul CMN este isoscel. ***Proposed by Dan Nedeianu-Romania***

VII.16. Se consideră triunghiul obtuzunghic $ABC, m(\hat{A})>90°$, în care mediana $[AM]$ este perpendiculară pe latura $[AB], M\in[BC]$. Dacă se construiește $MO\perp AC, O\in[AC]$ și $AC=8MO$, să se calculeze măsura unghiului $\widehat{BAC}$. ***Proposed by Dan Nedeianu-Romania***

VII.17. Fie ABCD un pătrat de latura l și M, N, P trei puncte oarecare pe laturile (AD), (DC) și (AB). Să se arate că $A_{MNP}\leq\frac{l^2}{2}$.

Proposed by Mirea Mihaela Mioara,Ciulcu Claudiu-Romania

VII.18. Să se arate că nu există numere întregi x, y, z astfel încât: $x^2 + xy + yz = y^2 + yz + zx = z^2 + zx + xy = 2018^{2018} + 2019^{2019}$.

Proposed by Tutescu Lucian, Mirea Mihaela Mioara-Romania

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8-CLASS-STANDARD

VIII.1. Aflați $x, y \in \mathbb{Z}$ pentru care: $\sqrt{x+2} + \sqrt{x+18} = \frac{3y^2+13y+2}{2y+1}$

Proposed by Petre Stângescu – Romania

VIII.2. If $x, y, z > 0$ then the following relationship holds:

$$\frac{x^7}{x+2y} + \frac{y^7}{y+2z} + \frac{z^7}{z+2x} \geq x^2y^2z^2$$

Proposed by Marian Ursărescu – Romania

VIII.3. Find the type of ABC triangle, knowing that between the lengths of the sides the following relationship holds: $a^4 - b^4 - c^4 + 2bc(b^2 + c^2 - bc) = 0$

Proposed by Ştefan Marica – Romania

VIII.4. Find the prime numbers $a; b; c; d$ which satisfy the relationship:

$$a^2(d^2 - c^2) + b^2(b^2 + c^2) - b^2(a^2 - d^2) - 1008 = 0$$

Proposed by Ştefan Marica – Romania

VIII.5. Solve in real numbers set, the following equation:

$$x^3 - a(\sqrt{6}+1)x^2 + a^2(\sqrt{6}+1)x = a^3$$

Proposed by Ştefan Marica – Romania

VIII.6. 1. Find $\overline{ab}$ such that: $(a^2b + b)^2 = (\overline{ab})^2$. ***Proposed by Ştefan Marica – Romania***

VIII.7. In rightangled paralleliped $ABCDA'B'C'D'; M \in (AC); N \in (CD'); P \in (AD')$ such that: $MA \cdot NC \cdot PD' = MC \cdot ND' \cdot PA$. Prove that ΔMNP is rightangled and:

$$AC \cdot ND' \cdot PD' = MD' \cdot (NC \cdot PD' + PA \cdot ND')$$

Proposed by Ştefan Marica – Romania

VIII.8. Find $x > 0$ such that: $(2018^5 + x^5)(2018 + x) \leq (2018^3 + x^3)^2$

Proposed by Carmen Victoriţa – Chirfot – Romania

VIII.9. Se consideră piramida $VABC$, cu $VA \perp VB \perp VC \perp VA$. Se duce $VO \perp (ABC), O \in (ABC)$ și se construiesc punctele A', B', C' care sunt respectiv proiecțiile punctului O pe planele (VBC), (VAC), respectiv (VAB). Să se arate că $VA \cdot VB \cdot VC \geq 27 \cdot OA' \cdot OB' \cdot OC'$.

Proposed by Dan Nedeianu-Romania

VIII.10. Determinați $x, y \in \mathbb{Z}$, știind că $xy(x+y) \geq 0$, iar $(x^2+y^2)(x+y) = x^2y^2(x+y-2)$. ***Proposed by Dan Nedeianu-Romania***

VIII.11. Există numere întregi n cu proprietatea că numărul $3n^2 + 3n + 25$ este cub perfect?

Proposed by Dană Camelia, Mirea Mihaela Mioara-Romania

VIII.12. Să se arate că nu există numere întregi x, y, z, t astfel încât: $x^6 + y^6 + z^6 = t^6 + 4$.

Proposed by Tutescu Lucian, Mirea Mihaela Mioara-Romania

VIII.13. Solve for real positive numbers:

$$\begin{cases} 27\sqrt{\left(x^2+\frac{1}{y^2}\right)\left(y^2+\frac{1}{z^2}\right)\left(z^2+\frac{1}{x^2}\right)} = 8(x+y+z)^3 \\ x+y+z = \frac{1}{xyz} \end{cases}$$

Proposed by Daniel Sitaru – Romania

VIII.14. If $a, b \geq 0, a+b+c+d = 0$ then:

$$4\sum a^3 \geq 3(a+b)(ac+ad+bc+bd+4cd)$$

Proposed by Daniel Sitaru – Romania

VIII.15. If $a, b, c, d, e, f > 0$ then:

$$\frac{(a^3+b^3)^4}{(c^6+d^6)^5} \cdot \frac{(c^5+d^5)^6}{(e^8+f^8)^7} \cdot \frac{(e^7+f^7)^8}{(a^4+b^4)^3} > 1$$

Proposed by Daniel Sitaru – Romania

VIII.16. Prove that if $x, y, z > 0$ then:

$$\sqrt{\frac{x}{y}} + 2\sqrt{\frac{y}{z}} + 3\sqrt{\frac{z}{x}} \leq \sqrt{6\left(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\right)}$$

Proposed by Daniel Sitaru – Romania

VIII.17. Prove that if $a, b, c > 0$ then:

$$\sqrt{\frac{a}{b+c}}+2\sqrt{\frac{b}{c+a}}+4\sqrt{\frac{c}{a+b}}\leq\sqrt{7\left(\frac{a}{b+c}+\frac{2b}{c+a}+\frac{4c}{a+b}\right)}$$

Proposed by Daniel Sitaru – Romania

VIII.18. If a,b,c be positive real number such that $a\leq b\leq c$ then

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3\geq(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Pham Quoc Sang-Ho Chi Minh-Vietnam

VIII.19. If $a,b,c,d>0, abcd=1$ then:

$$a\left(\frac{b}{b+a}+\frac{d}{d+a}\right)+c\left(\frac{b}{b+c}+\frac{d}{d+c}\right)\leq\frac{1}{2}(ab+bc+cd+da)$$

Proposed by Daniel Sitaru – Romania

VIII.20. Let a,b,c be positive real numbers such that:

$$\begin{cases} ab>6 \\ \frac{a}{8}+3b+\frac{2c}{3}=\frac{abc}{9}+\frac{67}{4a} \end{cases}$$

Find the minimum value of the expression: $P=3a+2b+c$

Proposed by Do Quoc Chinh-Vietnam

VIII.21. If $0<a\leq b$ and $c,d,e\geq 0$ then:

$$a^3\leq\frac{\left(a+c\sqrt{ab}+b\right)\left(a+d\sqrt{ab}+b\right)\left(a+e\sqrt{ab}+b\right)}{(c+2)(d+2)(e+2)}\leq b^3$$

Proposed by Daniel Sitaru – Romania

VIII.22. If $0\leq x,y,z\leq a$ then:

$$\sqrt{x^2-xz+z^2}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2}\leq a\left(1+\sqrt{2}+\sqrt{3}\right)$$

Proposed by Daniel Sitaru – Romania

VIII.23. If $x,y,z,t\geq 1$ then:

$$\frac{xy+2yz+2zt+2xz+ty+tx+9}{2x+2y+3z+2t}\geq 2$$

Proposed by Daniel Sitaru – Romania

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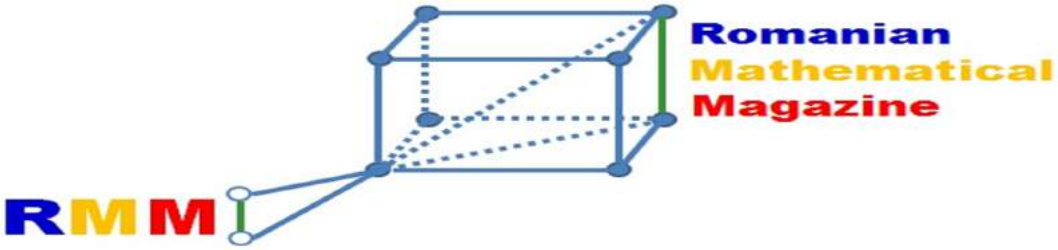
9-CLASS-STANDARD



IX.1. Fie $\Delta ABC, BF$ și CE simedianele din B respectiv C. Dacă punctele E, F și I (centrul cercului înscris) sunt coliniare atunci $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$.

Proposed by Marian Ursărescu – Romania

IX.2. Fie ΔABC și Γ punctul lui Lemoine. Să se arate că:

$$\frac{1}{S_{\Gamma BC}} + \frac{1}{S_{\Gamma AC}} + \frac{1}{S_{\Gamma AB}} \leq \frac{\sqrt{3}}{2} \cdot \frac{R}{r^3}$$

Proposed by Marian Ursărescu – Romania

IX.3. Fie ΔABC și I-centrul cercului înscris. Să se arate că:

$$S_{BIC}^2 + S_{AIC}^2 + S_{AIB}^2 \leq r^2(2R^2 + r^2)$$

Proposed by Marian Ursărescu – Romania

IX.4. Fie $a, b, c > 0$ astfel încât $abc = 1$. Să se arate că:

$$\frac{1}{a^4 + b^4 + c} + \frac{1}{a^4 + c^4 + b} + \frac{1}{b^4 + c^4 + a} \leq 1$$

Proposed by Marian Ursărescu – Romania

IX.5. Fie un ΔABC ascuțitunghic și neisoscel. Fie AA_1, AA_2 și AA_3 înălțimea, simediana și mediana din A. Analog pentru BB_1, BB_2, BB_3 și CC_1, CC_2, CC_3. Să se arate că:

$$\frac{A_1A_2}{A_2A_3} + \frac{B_1B_2}{B_2B_3} + \frac{C_1C_2}{C_2C_3} > 3$$

Proposed by Marian Ursărescu – Romania

IX.6. Let $ABCD$ be a cyclic quadrilateral with perimeter 2. Denote $AB = a, BC = b, CD = c, DA = d$. Prove that:

$$4 \leq \sum tan\frac{A}{2} \leq \frac{2(a + c)(b + d)}{\sqrt{(1 - a)(1 - b)(1 - c)}}$$

Proposed by Andrei Ștefan Mihalcea-Romania

IX.7. Let DEF be pedal triangle of Gergone point – Ge.Prove that:

$$A(DEF) \geq \frac{3s^3r^3}{[R(r+4R)]^2}$$

Proposed by Abdilkadir Altintas-Turkey

IX.8. Let DEF be pedal triangle of Nagel point – Na.Prove that:

$$A(DEF) \geq s \cdot R^{-2} \cdot r^3$$

Proposed by Abdilkadir Altintas-Turkey

IX.9. If $m \geq 0$ then in ΔABC the following relationship holds:

$$a^{m+1}r_a b^{m+1}r_b + c^{m+1}r_c \geq \frac{2^{m+1} \cdot s^{m+1}(2R-r)^{m+1}}{(4R+r)^m}$$

Proposed by D.M.Bătineţu-Giurgiu-Romania,Martin Lukarevski-Macedonia

IX.10. If $a, b \geq 0; a+b, x, y, z > 0$ then:

$$\frac{(x^2+yz+zx)^2}{ax+by} + \frac{(y^2+zx+xy)^2}{ay+bz} + \frac{(z^2+xy+yz)^2}{az+bx} \geq \frac{(x+y+z)^3}{a+b}$$

Proposed by D.M.Bătineţu-Giurgiu-Romania,Martin Lukarevski-Macedonia

IX.11. If $a, b, c, m, n > 0$ then:

$$\frac{(m+n)a+c^2}{mb+nc} + \frac{(m+n)b+a^2}{mc+na} + \frac{(m+n)c+b^2}{ma+nb} \geq 3 + \frac{a+b+c}{m+n}$$

Proposed by D.M.Bătineţu-Giurgiu-Romania,Martin Lukarevski-Macedonia

IX.12. Let $x, y, z > 0$ be positive real numbers and F be the area of the triangle ABC. Then:

$$\frac{x+y}{z}a^2 + \frac{y+z}{x}b^2 + \frac{z+x}{y}c^2 \geq 8\sqrt{3}F$$

Proposed by D.M.Bătineţu-Giurgiu-Romania,Martin Lukarevski-Macedonia

IX.13. If $a, b, c, m, n > 0$ then:

$$\frac{a}{mb+nc} + \frac{b}{mc+na} + \frac{c}{ma+nb} + \frac{(a+b+c)^2}{(m+n)(ab+bc+ca)} \geq \frac{6}{m+n}$$

Proposed by D.M.Bătineţu-Giurgiu-Romania,Martin Lukarevski-Macedonia

IX.14. If $x, y > 0$ then in ΔABC the following relationship holds:

$$\sum \frac{a^4}{(xr_a^2+yr_b^2)\cos^2 A} \geq \frac{64s^4}{9(x+y)((4R+r)^2-2s^2)}$$

Proposed by D.M.Bătineţu-Giurgiu-Romania,Martin Lukarevski-Macedonia

IX.15. If $a, b, c, d > 0$ then:

$$(ab+bc+cd+da)(a^4+b^4+c^4+d^4) \geq abcd(a+b+c+d)^2$$

Proposed by D.M.Bătineţu-Giurgiu-Romania,Martin Lukarevski-Macedonia

IX.16. If $m \geq 0; x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{x^{m+1}\cdot a^{2m+2}}{(y+z)^{m+1}}+\frac{y^{m+1}\cdot b^{2m+2}}{(z+x)^{m+1}}+\frac{z^{m+1}\cdot c^{2m+2}}{(x+y)^{m+1}}\geq 2^{m+1}\left(\sqrt{3}\right)^{1-m}S^{m+1}$$

Proposed by D.M. Bătineţu – Giurgiu; Claudia Năuţi – Romania

IX.17. If $x,y>0$ then in ΔABC the following relationship holds:

$$\frac{1}{xr_a^2+yr_br_c}+\frac{1}{xr_b^2+yr_cr_a}+\frac{1}{xr_c^2+yr_ar_b}\geq\frac{9}{(4R+r)^2x+(y-2x)s^2}$$

Proposed by D.M. Bătineţu – Giurgiu; Claudia Năuţi – Romania

IX.18. If $x\in\mathbb{R}$ then in ΔABC the following relationship holds:

$$\frac{a^3}{a\cos^2 x+b\sin^2 x}+\frac{b^3}{b\cos^2 x+c\sin^2 x}+\frac{c^3}{c\cos^2 x+a\sin^2 x}\geq 4\sqrt{3}S$$

Proposed by D.M. Bătineţu – Giurgiu; Claudia Năuţi – Romania

IX.19. If $x,y>0$ then in ΔABC the following relationship holds:

$$\sum\frac{a^4}{\left(x\cos^2\frac{B}{2}+y\cos^2\frac{C}{2}\right)\cos^2 A}\geq\frac{256s^4}{81(x+y)}$$

Proposed by D.M. Bătineţu – Giurgiu; Claudia Năuţi – Romania

IX.20. If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{ax}{(y+z)h_a}+\frac{by}{(z+x)h_b}+\frac{cz}{(x+y)h_c}\geq\sqrt{3}$$

Proposed by D.M. Bătineţu – Giurgiu; Claudia Năuţi – Romania

IX.21. If $m,n\in\mathbb{N};a,b,c>0,u\geq 0;u$ – fixed then:

$$\sum(m+a^{m+1})\left(n+\frac{1}{(b+c+u)^{m+1}}\right)\geq\frac{3(m+1)(n+1)(a+b+c)}{2(a+b+c)+3u}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu – Romania

IX.22. If $a,b,c>0,a+b+c=1$ then:

$$2\sqrt{abc}\sum\frac{a}{1+a^2}\leq\frac{1+a^2+b^2+c^2}{3+a^2+b^2+c^2}$$

Proposed by Andrei Ştefan Mihalcea-Romania

IX.23. Să se determine funcțiile $f:\mathbb{R}\to\mathbb{R}$, cu proprietatea:

$$(x-y)^2\left(f^2(x)-f^2(y)\right)=(x+y)f^3(x-y),\forall x,y\in\mathbb{R}$$

Proposed by Marian Ursărescu – Romania

IX.24. Fie $x,y,t>0$ astfel încât $xyz=1$. Să se arate că: $x^5y+y^5z+z^5x\geq x^3+y^3+z^3$

Proposed by Marian Ursărescu – Romania

IX.25. Să se determine funcția $f:\mathbb{R}\to\mathbb{R}$ care verifică condițiile

$$x+y\leq 2^xf(x)+2^yf(y)\leq 2^{x+y}f(x+y),\forall x,y\in\mathbb{R}$$

Proposed by Marian Ursărescu – Romania

IX.26. Fie un triunghi ABC și A',B',C' punctele de tangență ale cercului înscris cu laturile triunghiului. Să se arate că: $aAA'^2+bBB'^2+cCC'^2\geq 54\sqrt{3}\cdot r^3$

Proposed by Marian Ursărescu – Romania

IX.27. Fie $ABCD$ un paralelogram și punctele M, N, P și Q pe laturile AB, BC, CD, respectiv AD. Să se arate că cel puțin unul din triunghiurile AMQ, BMN, CPN și DPQ are aria mai mică sau egală cu $\frac{1}{8}$ decât aria paralelogramului. ***Proposed by Marian Ursărescu – Romania***

IX.28. Fie ABC un triunghi și I_a, I_b, I_c centrele cevienelor exînscrise triunghiului ABC. Să se arate că: $\sigma_{I_aBC} + \sigma_{I_bAC} + \sigma_{I_cAB} \geq 3\sigma_{ABC}$. ***Proposed by Marian Ursărescu – Romania***

IX.29. Să se arate că în orice triunghi ABC are loc inegalitatea:

$$\sqrt[3]{(h_a + r_a)(h_b + r_b)(h_c + r_c)} \geq 6r$$

Proposed by Marian Ursărescu – Romania

IX.30. Să se arate că în orice Δ ascuțitunghic ABC are loc inegalitatea:

$A\Omega + B\Omega + C\Omega \leq 3R$, unde Ω este centrul cercului lui Euler.

Proposed by Marian Ursărescu – Romania

IX.31. Prove in any acute – angled ΔABC, the following relationship holds:

$$\frac{r}{R} \leq \max\left(\frac{\sin\frac{A}{2}}{\cos\left(\frac{B-C}{2}\right)}, \frac{\sin\frac{B}{2}}{\cos\left(\frac{A-C}{2}\right)}, \frac{\sin\frac{C}{2}}{\cos\left(\frac{A-B}{2}\right)}\right)$$

Proposed by Marian Ursărescu – Romania

IX.32. Să se arate că în orice Δ are loc inegalitatea:

$$\frac{a}{b^4 + c^4 + bca^2} + \frac{b}{a^4 + c^4 + acb^2} + \frac{c}{a^4 + b^4 + abc^2} \leq \frac{1}{24\sqrt{3}r^3}$$

Proposed by Marian Ursărescu – Romania

IX.33. Fie $a, b, c > 0$ astfel încât: $a + b + c = 3$. Să se arate că:

$$5(a^4 + b^4 + c^4) \geq a^5 + b^5 + c^5 + 12$$

Proposed by Marian Ursărescu – Romania

IX.34. a.Prove that: $1 + 3 + 5 + \cdots + (2n-1) = n^2, \forall n \in \mathbb{N}^*$. b.Using a. Prove that:

$1^2 + 2^2 + \cdots + n^2 = n + 3(n-1) + 5(n-2) + 7(n-3) + \cdots + (2n-1)\cdot 1, \forall n \in \mathbb{N}^*$

c.Using b. find: $1^2 + 2^2 + \cdots + n^2, n \in \mathbb{N}^*$. d.Find the smallest number $n \in \mathbb{N}$ such that: $S_n > 2020$. ***Proposed by Carmen Victoriţa – Chirfot – Romania***

IX.35. Let x, y, z be positive real numbers such that: $x + y + z = 3$. Find the minimum of the expression:

$$P = \frac{x^3}{y\left(\sqrt{2\left(y^4 + z^4\right)} + yz\right)^2} + \frac{y^3}{z\left(\sqrt{2\left(z^4 + x^4\right)} + zx\right)^2} + \frac{z^3}{x\left(\sqrt{2\left(x^4 + y^4\right)} + xy\right)^2} + \frac{\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z}}{27}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

IX.36. Let a, b, c be positive real numbers. Prove that:

$$\sqrt[8]{\frac{a^8+b^8}{2}}+\sqrt[8]{\frac{b^8+c^8}{2}}+\sqrt[8]{\frac{c^8+a^8}{2}}\le(a+b+c)^{19}\left(\frac{1}{9a}+\frac{1}{9b}+\frac{1}{9c}\right)^9$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

IX.37. If $m,n\in\mathbb{N},u,v\in[0,\infty);u+v=2$ then in ΔABC the following relationship holds:

$$\sum\left(m+a^{2(m+1)}\right)(n+1)\ge 4\sqrt{3}(m+1)(n+1)S$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu – Romania

IX.38. If $m,n\in\mathbb{N},x,y,z,u,v\in(0,\infty)$ then:

$$\sum(m+(ux)^{m+1})\left(n+\left(\frac{v}{y+z}\right)^{n+1}\right)\ge\frac{3uv(m+1)(n+1)}{2}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu – Romania

IX.39. If $m\ge 0;x,y,A,B,C,D,E,F>0;A+B+C=(x+y)^{2n},D+E+F=(x+y)^{3n}$ then: $\frac{A^{m+1}}{D^m}+\frac{B^{m+1}}{F^m}+\frac{C^{m+1}}{E^m}\ge(x+y)^{n(2-m)}$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu – Romania

IX.40. If $m,n\in\mathbb{N}$ then in ΔABC the following relationship holds:

$$\sum(m+a^{m+1})\,(n+b^{m+1})\ge 4\sqrt{3}(m+1)(n+1)S$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu – Romania

IX.41. If $a,b,c>0,a+b+c=1$ then:

$$\sum ab^2\ge\left(\sum ab\right)^2-10\sum ab+12abc$$

Proposed by Andrei Ştefan Mihalcea-Romania

IX.42. If $x,y>0$ then in ΔABC the following relationship holds:

$$\sum\frac{a^2}{(r_a+r_br_c)(bx+cy)^2}\ge\frac{9}{(x+y)^2(4R+r+s^2)}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu – Romania

IX.43. If $x,y>0$ then in ΔABC the following relationship holds:

$$\sum\frac{a^3}{(bx+cy)^3\cos^4\frac{A}{2}}\ge\frac{108R^2}{(4R+r)^2(x+y)^3}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu – Romania

IX.44. If $a,b>0$ then:

$$\frac{(1+ab)(ab-a-b-1)}{(a^2+1)(b^2+1)}+\frac{4(a^2+b^2+a+b)}{(2+a+b)(a^2+b^2+a+b)-2(a+b)^2}\le 1$$

Proposed by Andrei Ştefan Mihalcea-Romania

IX.45. In ΔABC the following relationship holds:

$$\sqrt{\frac{r_a}{h_a}} + \sqrt{\frac{r_b}{h_b}} + \sqrt{\frac{r_c}{h_c}} \geq 3$$

Proposed by Bogdan Fustei-Romania

IX.46. In ΔABC the following relationship holds:

$$\sqrt{\frac{r_a}{h_a}} + \sqrt{\frac{r_b}{h_b}} + \sqrt{\frac{r_c}{h_c}} \geq 2\left(\frac{l_a}{r_b + r_c} + \frac{l_b}{r_a + r_c} + \frac{l_c}{r_a + r_c}\right)$$

Proposed by Bogdan Fustei-Romania

IX.47. In ΔABC the following relationship holds:

$$\sqrt{\frac{a}{p-a}} + \sqrt{\frac{b}{p-b}} + \sqrt{\frac{c}{p-c}} \geq 3\sqrt{2}$$

Proposed by Bogdan Fustei-Romania

IX.48. In ΔABC the following relationship holds:

$$\sqrt{\frac{a}{p-a}} + \sqrt{\frac{b}{p-b}} + \sqrt{\frac{c}{p-c}} \geq 2\sqrt{2}\left(\frac{l_a}{r_b + r_c} + \frac{l_b}{r_a + r_c} + \frac{l_c}{r_a + r_c}\right)$$

Proposed by Bogdan Fustei-Romania

IX.49. In ΔABC the following relationship holds:

$$\sum \sqrt{\frac{h_a}{h_a}} \geq \sum \frac{r_a - r}{r_a + r} + \sum \frac{l_b}{r_b + r_c}$$

Proposed by Bogdan Fustei-Romania

IX.50. In ΔABC the following relationship holds:

$$\left(5 + \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c}\right) \sum \frac{l_b l_c}{m_a h_a} \leq \frac{8(2R - r)}{r}$$

Proposed by Bogdan Fustei-Romania

IX.51. In ΔABC the following relationship holds: $abc \geq \sqrt{m_a m_b m_c l_a l_b l_c}$

Proposed by Bogdan Fustei-Romania

IX.52. In ΔABC the following relationship holds: $\left(5 + \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c}\right) \sum \frac{l_a l_b}{l_c} > \frac{4p^2}{R}$

Proposed by Bogdan Fustei-Romania

IX.53. In ΔABC the following relationship holds: $\sqrt{h_a + \sqrt[3]{l_a + \sqrt[4]{m_a}}} \geq \sqrt[32]{h_a l_a m_a}$

Proposed by Bogdan Fustei-Romania

IX.54. In ΔABC the following relationship holds:

$$\frac{a^4 + b^4 + c^4}{2r} \geq \left(5 + \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c}\right) l_a l_b l_c$$

Proposed by Bogdan Fustei-Romania

IX.55. 1. Solve for natural numbers: $8a^{2x} - 7ya^x - y^2 - 450 = 0$.

Proposed by Ştefan Marica – Romania

IX.56. Solve for integers: $(x^2 - 5\alpha x + 5\alpha^2)(x^2 - 5\alpha x + 6\alpha^2) = 9$.

Proposed by Ştefan Marica – Romania

IX.57. If in ΔABC; $2c^2 = a^2 + b^2$ then: $\cos A + \cos B \leq 1$.

Proposed by Ştefan Marica – Romania

IX.58. Dacă x, y, z sunt termenii consecutivi ai unei progresii geometrice cu termeni pozitivi, să se arate că: $(x + y + z)^n + y^n \geq (x + y)^n + (y + z)^n, \forall n \in \mathbb{N}^*$

Proposed by Dan Nedeianu-Romania

IX.59. Se consideră dreptunghiul $ABCD$ cu toate vârfurile situate în interiorul unui cerc de centru O. Prelungirile laturilor dreptunghiurilor intersectează cercul în punctele $A_1, A_2, \ldots, A_8$. Să se arate că pentru orice punct M din planul dreptunghiului are loc relația

$$\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} + \overrightarrow{MD} + 2\overrightarrow{MO} = \sum_{i=1}^{6} MA_i$$

Proposed by Dan Nedeianu-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

10-CLASS-STANDARD

X.1. Prove that if $m > 0$ in any triangle ABC with usual notations holds:

$$\frac{1}{w_a^{2m}} + \frac{1}{w_b^{2m}} + \frac{1}{w_c^{2m}} \geq \frac{3^{m+1}}{s^{2m}},$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.2. În orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{r_a^3}{r_b + r_c} + \frac{r_b^3}{r_c + r_a} + \frac{r_c^3}{r_b + r_c} \geq \frac{((4R + r)^2 - 2p^2)^2}{2p^2}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.3. Dacă $x, y \in \mathbb{R}_+^*, x > y$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{1}{x(b^2+c^2)-ya^2}+\frac{1}{x(c^2+a^2)-yb^2}+\frac{1}{x(a^2+b^2)-yc^2}\geq\frac{9}{2(2x-y)(p^2-r^2-4Rr)}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.4. Dacă $x, y \in \mathbb{R}_+^*, m \in \mathbb{R}_+$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{a^{m+2}}{(xb+yc)^m}+\frac{b^{m+2}}{(xc+ya)^m}+\frac{c^{m+2}}{(xa+yb)^m}\geq\frac{p^2+r^2+4Rr}{(x+y)^m}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.5. Dacă $x, y \in \mathbb{R}_+^*$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{a^3}{xb+yc}+\frac{b^3}{xc+ya}+\frac{c^3}{xa+yb}\geq\frac{p^2+r^2+4Rr}{x+y}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.6. Dacă $x, y \in \mathbb{R}_+^*, m \in \mathbb{R}_+$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{a^{m+2}}{(xa+yb)^m}+\frac{b^{m+2}}{(xb+yc)^m}+\frac{c^{m+2}}{(xc+ya)^m}\geq\frac{2(p^2-r^2-4Rr)}{(x+y)^m}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.7. Dacă $x, y \in \mathbb{R}_+^*$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{a^3}{xa+yb}+\frac{b^3}{xb+yc}+\frac{c^3}{xc+ya}\geq\frac{2(p^2-r^2-4Rr)}{x+y}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.8. Dacă $x, y \in \mathbb{R}_+^*, m \in \mathbb{R}_+$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{r_a^{m+2}}{(xr_a+yr_b)^m}+\frac{r_b^{m+2}}{(xr_b+yr_c)^m}+\frac{r_c^{m+2}}{(xr_c+yr_a)^m}\geq\frac{(4R+r)^2-2p^2}{(x+y)^m}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.9. Dacă $x, y \in \mathbb{R}_+^*$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{r_a^3}{xr_a+yr_b}+\frac{r_b^3}{xr_b+yr_c}+\frac{r_c^3}{xr_c+yr_a}\geq\frac{(4R+r)^2-2p^2}{x+y}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.10. Dacă $x, y \in \mathbb{R}_+^*, m \in \mathbb{R}_+$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{1}{(xa+yb)^m}+\frac{1}{(xb+yc)^m}+\frac{1}{(xc+ya)^m}\geq\frac{3^{m+1}}{\big(2p(x+y)\big)^m}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.11. Dacă $m, n \in \mathbb{R}_+^*$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{1}{(ma+nb)^2}+\frac{1}{(mb+nc)^2}+\frac{1}{(mc+na)^2}\geq\frac{9}{2(m+n)^2(p^2-r^2-4Rr)}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.12. Dacă $m,n\in\mathbb{R}_+^*$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{1}{(mr_a+nr_b)^2}+\frac{1}{(mr_b+nr_c)^2}+\frac{1}{(mr_c+nr_a)^2}\geq\frac{27}{(m+n)^2(4R+r)^2}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.13 Dacă $m,n\in\mathbb{R}_+^*$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{a}{mb+nc}+\frac{b}{mc+na}+\frac{c}{ma+nb}\geq\frac{4p^2}{(m+n)(p^2+r^2+4Rr)}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.14. Dacă $m,n\in\mathbb{R}_+^*$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{r_a}{mr_b+nr_c}+\frac{r_b}{mr_c+nr_a}+\frac{r_c}{mr_a+nr_b}\geq\frac{(4R+r)^2}{(m+n)p^2}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.15. Dacă $m,n\in\mathbb{R}_+^*$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{ab^2}{ma+nb}+\frac{bc^2}{mb+nc}+\frac{ca^2}{mc+na}\geq\frac{(p^2+r^2+4Rr)^2}{(2m+n)p^2+(n-2m)r^2+4(n-2m)Rr}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.16. Dacă $m,n\in\mathbb{R}_+^*$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{r_ar_b^2}{mr_a+nr_b}+\frac{r_br_c^2}{mr_b+nr_c}+\frac{r_cr_a^2}{mr_c+nr_a}\geq\frac{p^4}{(4R+r)^2m+(n-2m)p^2}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.17. Prove that: If $m\geq 0$, then in any triangle ABC holds:

$$\cot^{m+1}\frac{A}{2}+\cot^{m+1}\frac{B}{2}+\cot^{m+1}\frac{C}{2}\geq 3^{\frac{m+3}{2}}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.18. Dacă $m,n\in\mathbb{R}_+^*$, atunci în orice triunghi ABC (cu notațiile obișnuite) are loc inegalitatea:

$$\frac{1}{(ma+nb)^2}+\frac{1}{(mb+nc)^2}+\frac{1}{(mc+na)^2}\geq\frac{27}{4(m+n)^2p^2}$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.19. In ΔABC the following relationship holds:

$$4-\frac{2r}{R}\leq\sum\frac{h_bh_c}{l_a^2}\leq 1+\frac{R}{r}$$

Proposed by Marin Chirciu – Romania

X.20. In ΔABC the following relationship holds:

$$\frac{27r}{4}\leq\sum l_a\cos\frac{B}{2}\cos\frac{C}{2}\leq\frac{p^2}{2R}$$

Proposed by Marin Chirciu – Romania

X.21. In ΔABC the following relationship holds:

$$\frac{9Sr}{R}\leq\sum l_bl_c\cos\frac{A}{2}\leq\frac{9R}{2}$$

Proposed by Marin Chirciu – Romania

X.22. In ΔABC the following relationship holds:

$$\frac{81r^2}{4}\leq\sum l_bl_c\cos\frac{B}{2}\cos\frac{C}{2}\leq\frac{81R^2}{16}$$

Proposed by Marin Chirciu – Romania

X.23. In ΔABC the following relationship holds:

$$\frac{9r}{4}\leq\sum l_a\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{9R}{8}$$

Proposed by Marin Chirciu – Romania

X.24. In ΔABC the following relationship holds:

$$\frac{27r^2}{2}\leq\sum l_bl_c\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{27R^2}{16}$$

Proposed by Marin Chirciu – Romania

X.25. In ΔABC the following relationship holds:

$$\frac{9r\sqrt{3}}{2}\leq\sum l_a\cos\frac{A}{2}\leq\frac{3p}{2}$$

Proposed by Marin Chirciu – Romania

X.26. Let ΔABC, with $a,b,c\leq 2R\sqrt{\frac{2}{3}}$. Prove that:

$$\frac{R(\sum a)(\sum a^2)}{3abc}\geq\sqrt{36R^2-\left(\sum a\right)^2}.$$

Proposed by Mihalcea Andrei Ştefan - Romania

X.27. In ΔABC the following relationship holds:

$$9r\leq\sum\frac{l_bl_c}{l_a}\leq\frac{9R}{2}$$

Proposed by Marin Chirciu – Romania

X.28. In ΔABC the following relationship holds:

$$4\leq\sum\frac{bc}{l_a^2}\leq 2+\frac{R^2}{2r^2}$$

Proposed by Marin Chirciu – Romania

X.29. If , $y, z > 0$, $x + y + z = 3$ then:

$$\frac{1}{\sqrt{x + y^2 + z^2}} + \frac{1}{\sqrt{x^2 + z^2 + y}} + \frac{1}{\sqrt{x^2 + y^2 + z}} \leq \sqrt{3}$$

Proposed by Marian Ursărescu – Romania

X.30. If $z_1, z_2, z_3 \in$, $(z_1 + z_2)(z_2 + z_3)(z_3 + z_1) \neq 0$, $|z_1| = |z_2| = |z_3| = 1$

$\frac{z_1^2}{(z_1 + z_2)(z_1 + z_3)} + \frac{z_2^2}{(z_2 + z_1)(z_2 + z_3)} + \frac{z_3^2}{(z_3 + z_1)(z_3 + z_2)} = 3$, $A(z_1), B(z_2), C(z_3)$ then ΔABC is an equilateral one.

Proposed by Marian Ursărescu – Romania

X.31. In $\Delta\ ABC$ the following relationship holds:

$$\frac{2r}{R} + \frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab} \geq 4$$

Proposed by Marian Ursărescu – Romania

X.32. Fie $a, b, c \in R_+^*$ și $x, y, z \in (1, \infty)$. Să se arate că:

$$\log_{y^b z^c} x^a + \log_{z^b x^c} y^a + \log_{x^b y^c} z^a \geq \frac{3a}{b+c}.$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu – Romania

X.33. Aflați $m, n \in \mathbb{N}^*$ pentru care este adevărată egalitatea: $\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} = \frac{5m-5}{m+3}$

Proposed by Petre Stângescu – Romania

X.34. Fie ABC un triunghi, I centrul cercului înscris și D, E, F punctele de intersecție ale cevienelor AI, BI și CI cu cercul circumscris triunghiului ABC. Să se arate că:

$$\frac{DI}{AI} + \frac{EI}{BI} + \frac{FI}{CI} \geq 3$$

Proposed by Marian Ursărescu – Romania

X.35. Fie $z_1, z_2, z_3 \in \mathbb{C}^*$ astfel încât $|z_1| = |z_2| = |z_3| = 1$. Dacă

$$\sum \frac{1}{2 + |z_1 + z_2|} = 1$$

atunci z_1, z_2, z_3 sunt afixele vârfurilor un triunghi echilateral.

Proposed by Marian Ursărescu – Romania

X.36. Să se arate că în orice triunghi ABC are loc inegalitatea:

$$\sum \frac{\cos\left(\frac{A-B}{2}\right) \sin\frac{C}{2}}{r_c^2} \geq \frac{r}{3R^3}$$

Proposed by Marian Ursărescu – Romania

X.37. Să se rezolve ecuația: $2^x + 2^{\frac{10}{x}} + 2^{x+\frac{10}{x}} = 164$.

Proposed by Marian Ursărescu – Romania

X.38. Să se arate că în orice triunghi ascuțitunghic are loc inegalitatea:

$$\frac{r_a}{\sin A}+\frac{r_b}{\sin B}+\frac{r_c}{\sin C}\geq 6\sqrt{3}r$$

Proposed by Marian Ursărescu – Romania

X.39. Let ΔABC. Prove that:

a) $\sqrt{3a^2-2b^2+6c^2}+\sqrt{3a^2+6b^2-2c^2}+\sqrt{-a^2+2b^2+2c^2}\leq 4b+4c$;

b) $\sqrt{10a^2+15b^2-6c^2}+\sqrt{4a^2-3b^2+12c^2}+\sqrt{-2a^2+3b^2+6c^2}\leq 6b+6c$

Proposed by Mihalcea Andrei Ştefan-Romania

X.40. Let $a,b,c,d>0$, with $a+b+c+d=1$. Prove that:

a) $\sum abc+\sum a\left(\sqrt[3]{bc}+\sqrt[3]{cd}+\sqrt[3]{db}\right)^3\leq\frac{5+\sum a^2}{3}$

b) $\prod\left(\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}\right)\leq\sqrt[3]{\frac{\prod(1-abc)}{abcd}}$.

Proposed by Mihalcea Andrei Ştefan-Romania

X.41. Fie $z_1,z_2,z_3\in\mathbb{C}^*$ astfel încât $(z_1+z_2)(z_2+z_3)(z_3+z_1)\neq 0$ cu

$|z_1|=|z_2|=|z_3|=1$, $\frac{z_1^2}{(z_1+z_2)(z_1+z_3)}+\frac{z_2^2}{(z_2+z_1)(z_2+z_3)}+\frac{z_3^2}{(z_3+z_1)(z_3+z_2)}=3$

Să se arate că z_1,z_2,z_3 sunt afixele vârfurilor unui triunghi echilateral.

Proposed by Marian Ursărescu – Romania

X.42. Să se rezolve: $4^x+25^{\frac{1}{x}}+4^x\cdot 25^{\frac{1}{x}}=101$.

Proposed by Marian Ursărescu – Romania

X.43. Să se arate că în orice triunghi ABC are loc inegalitatea:

$$\frac{h_ah_bh_c}{i_ai_bi_c}\geq\left(\frac{2r}{R}\right)^4$$

Proposed by Marian Ursărescu – Romania

X.44. Să se arate că în orice triunghi ABC are loc inegalitatea:

$$\sum\frac{a^2\sin A\sin C}{a^2+b^2}\geq\frac{2r}{R}+\frac{1}{2}\left(\frac{r}{R}\right)^2$$

Proposed by Marian Ursărescu – Romania

X.45. Fie ΔABC și BF,CE simedianele din B,C. Să se arate că punctele E,O,F sunt coliniare $\Leftrightarrow \cot B+\cot C=\cot A$. ***Proposed by Marian Ursărescu – Romania***

X.46. Să se arate că în orice ABC au loc inegalitățile:

a) $\frac{\sin\frac{A}{2}}{\cos\left(\frac{B-C}{2}\right)}+\frac{\sin\frac{B}{2}}{\cos\left(\frac{A-C}{2}\right)}+\frac{\sin\frac{C}{2}}{\cos\left(\frac{A-B}{2}\right)}\geq\frac{3}{2}$

b) $(\sin A+\sin B+\sin C)\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)\geq 6\left(\frac{\sin\frac{A}{2}}{\cos\left(\frac{B-C}{2}\right)}+\frac{\sin\frac{B}{2}}{\cos\left(\frac{A-C}{2}\right)}+\frac{\sin\frac{C}{2}}{\cos\left(\frac{A-B}{2}\right)}\right)$

Proposed by Marian Ursărescu – Romania

X.47. Fie $a,b,c>1$. Să se arate că:

$$\frac{1}{\log_a c+2\log_a b}+\frac{1}{\log_b a+2\log_b c}+\frac{1}{\log_c b+2\log_c a}\geq 1$$

Proposed by Marian Ursărescu – Romania

X.48. In ΔABC the following relationship holds:

$$6 + \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{h_c} \geq 6\sqrt{3}\frac{l_a + l_b + l_c}{a + b + c}$$

Proposed by Bogdan Fustei-Romania

X.49. In ΔABC the following relationship holds:

$$\frac{a^4 + b^4 + c^4}{2} \geq \left(\frac{4\sqrt{3}S - r^2}{5R} + 5r\right) l_a l_b l_c$$

Proposed by Bogdan Fustei-Romania

X.50. In ΔABC the following relationship holds:

$$p \geq \sqrt{l_a l_b l_c \sqrt{\frac{2}{Rr}}} \Leftrightarrow p^2 \geq l_a l_b l_c \sqrt{\frac{2}{2r}}$$

Proposed by Bogdan Fustei-Romania

X.51. In ΔABC the following relationship holds:

$$\prod (a^2 + 4r_b r_c) \geq 8r_a r_b r_c \sqrt{\frac{m_a m_b m_c}{l_a l_b l_c}}$$

Proposed by Bogdan Fustei-Romania

X.52. In ΔABC the following relationship holds:

$$\frac{R}{r} + \frac{r}{R} \geq \frac{5}{2}\sqrt[5]{\frac{m_a m_b m_c}{h_a h_b h_c}}$$

Proposed by Bogdan Fustei-Romania

X.53. In ΔABC the following relationship holds:

$$\frac{R}{r} + \frac{r}{R} \geq \frac{5}{2}\sqrt[5]{\left(\frac{r_a + r_b + r_c}{m_a + m_b + m_c}\right)^6}$$

Proposed by Bogdan Fustei-Romania

X.54. In ΔABC the following relationship holds:

$$\sum \sqrt{\frac{m_a^2 - h_a^2}{l_a^2 - h_a^2}} \geq 4$$

Proposed by Bogdan Fustei-Romania

X.55. In ΔABC the following relationship holds:

$$\frac{8(2R - r)}{R} \geq \sum \left(\frac{h_b + h_c}{m_a}\right)^2$$

Proposed by Bogdan Fustei-Romania

X.56. In ΔABC the following relationship holds:

$$\sum m_a \sqrt{\frac{r_a}{h_a}} \geq \frac{(h_a + h_b + h_c)(r_a + r_b + r_c)}{m_a + m_b + m_c}$$

Proposed by Bogdan Fustei-Romania

X.57. In ΔABC the following relationship holds:

$$\sum m_a \sqrt{\frac{a}{p-a}} \geq \sqrt{2} \cdot \frac{(l_a + l_b + l_c)^2}{m_a + m_b + m_c}$$

Proposed by Bogdan Fustei-Romania

X.58. 1. Solve for integers: $(7 + 4\sqrt{3})^x - 15(2 + \sqrt{3})^x - (2 - \sqrt{3})^x + 15 = 0$.

Proposed by Ştefan Marica – Romania

X.59. Let be: $\Omega = \sin^2 3x + \cos^2 2x + 4\cos^2 x + 2\sin 3x\,; x \in [0,2\pi]$. Find: $\min \Omega\,;\max \Omega$.

Proposed by Ştefan Marica – Romania

X.60. Determinați funcțiile $f: \left(0,\frac{\pi}{2}\right) \to \mathbb{R}$ astfel încât $f(x)g(x) = \frac{1}{2\cos x}, \forall x \in \left(0,\frac{\pi}{2}\right)$ și

$$\cos^2 x \cdot \left(f^2(x) + g^2(x)\right) = f^2(x) - \cos x\,, \forall x \in \left(0,\frac{\pi}{2}\right)$$

Proposed by Dan Nedeianu-Romania

X.61. Determinați că dacă $a, b, c \in (0,\infty)$ atunci are loc inegalitatea:

$$b^a[(a+1)^{b-c} - 1] + c^a[(a+1)^{c-b} - 1] \geq 0$$

Proposed by Dan Nedeianu-Romania

X.62. For $z_1, z_2 \in \mathbb{C}$, satisfy: $|z_1 + z_2| = |z_1| + |z_2|$. Prove:

$$|z_1 - z_2| = max\{|z_1|;|z_2|\} - min\{|z_1|;|z_2|\}$$

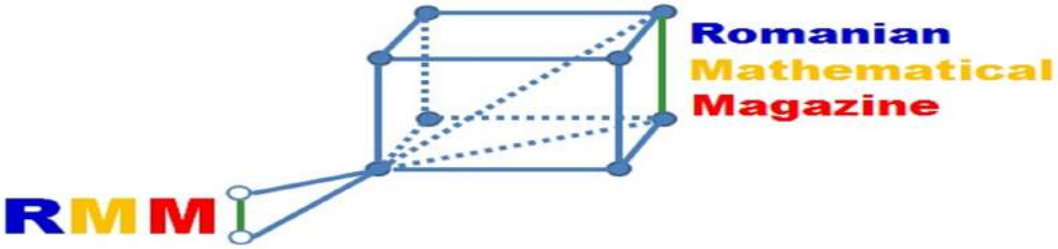
Proposed by Nguyen Van Nho-Nghe An-Vietnam

X.63. If $z \in \mathbb{C}\,, \alpha \geq 2$ then: $|Rez|^\alpha + |Imz|^\alpha \geq 2^{1-\frac{\alpha}{2}} \cdot |z|^\alpha$.

Proposed by Nguyen Van Nho-Nghe An-Vietnam

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

11-CLASS-STANDARD



XI.1. Să se calculeze $\lim\limits_{n\to\infty} \dfrac{(n+1)\ln(n+1)}{(n+1)^2 \ln(n+1) - n^2 \ln n}$.

Proposed by Chirfot Carmen – Victoriţa-Romania

XI.2. Fie $A, B \in M_3(\mathbb{C})$. Să se arate că:

$$\det(A + B) = \det A + \det B + Tr\,(AB^* + BA^*)$$

Proposed by Marian Ursărescu – Romania

XI.3. Să se calculeze

$$\lim_{n\to\infty} \sum_{i,j=1}^{n} \tan\left(\frac{ij}{n^4}\right)$$

Proposed by Marian Ursărescu – Romania

XI.4 Să se calculeze

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sum_{i=1}^{k}(8i^3 - 3i^2 + i)} = \frac{\pi^2}{6} + 8\ln 2 - 7$$

Proposed by Marian Ursărescu – Romania

XI.5. Fie $A, B \in M_2(\mathbb{R})$ inversabile astfel încât: $Tr(AB^{-1}) = \det(AB^{-1}) = 1$. Să se calculeze

$$\det(I_2 + A^{-1}B)$$

Proposed by Marian Ursărescu – Romania

XI.6. Fie $A \in M_n(\mathbb{R})$ o matrice simetrică și inversabilă. Să se arate că:

$$\det(A^2 + A^{-2} + 6(A + A^{-1}) + 11I_n) > 25^4$$

Proposed by Marian Ursărescu – Romania

XI.7. Să se determine funcțiile continue $f: (0, +\infty) \to \mathbb{R}$ cu proprietatea:

$$f(e^x) + f(e^{px}) = x, \forall x > 0, p \in \mathbb{R}, p > 1 \text{ fixat}$$

Proposed by Marian Ursărescu – Romania

XI.8. Fie $A \in M_2(\mathbb{R})$. Să se arate că: $\det(A^2 + 2A + 2) \geq (Tr\,A + 2)^2$

Proposed by Marian Ursărescu – Romania

XI.9. Fie șirurile $x_0 > 0, x_{n+1} = x_n - x_n^{p+1}, \forall n \in \mathbb{N}$ și $y_0 > 0, y_{n+1} = \frac{y_n}{\sqrt[q]{1+y_n^q}}, \forall n \in \mathbb{N}$, iar $p, q \in \mathbb{N}, p, q \geq 2$. Să se calculeze: $\lim_{n\to\infty} \frac{\sqrt[q]{x_n}}{\sqrt[p]{y_n}}$.

Proposed by Marian Ursărescu – Romania

XI.10. $\Omega_n \in [1, \infty), n \geq 1, \lim_{n\to\infty} \Omega_n = \Omega \in \mathbb{R}$. Find:

$$\lim_{n\to\infty} e^{\sqrt[n]{(1+\ln\Omega_1)(1+\ln\Omega_2)\cdot\ldots\cdot(1+\ln\Omega_n)}-1}$$

Proposed by Daniel Sitaru -Romania

XI.11. Fie $a_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$. Să se calculeze:

$$\lim_{n\to\infty} \sqrt{n} \prod_{k=1}^{n} \left(1 - \frac{1}{\sqrt{k+1}a_{k+1}}\right)$$

Proposed by Marian Ursărescu – Romania

XI.12. Prove that: $e^e(1 - e^{\tan e}) > e^\pi - \pi^\pi$

Proposed by Rovsen Pirkuliyev-Azerbaijan

XI.13. Solve for natural numbers:

$$\begin{vmatrix} 1 & 1 & 1 \\ A_n^3 & A_n^2 & A_n^1 \\ C_n^3 & C_n^2 & C_n^1 \end{vmatrix} = 8n^2$$

Proposed by Ștefan Marica – Romania

XI.14. Fie $\alpha \in \mathbb{R}$ și $i \in \mathbb{C}, i^2 = -1$. Pentru matricele $A, B \in M_n(\mathbb{R})$, se consideră matricea $C = A \cdot {}^tA + B \cdot {}^tB + (A \cdot {}^tB + B \cdot {}^tA)\cos\alpha + i(B \cdot {}^tA - A \cdot {}^tB)\sin\alpha$, unde ${}^tA$ este transpusa matricei A. Să se arate că determinantul matricei C este un număr real pozitiv.

Proposed by Dan Nedeianu-Romania

XI.15. Să se calculeze: $\lim_{n\to\infty} n^\alpha \left(\frac{1}{\sqrt{n^2+1}} + \frac{2}{\sqrt{n^2+2}} + \frac{3}{\sqrt{n^2+3}} + \cdots + \frac{2n+1}{(n+1)}\right)$, unde $\alpha \in \mathbb{R}$.

Proposed by Dan Nedeianu-Romania

XI.16. If $\in \mathbb{N}^*$, $x_k \in \mathbb{R}_+^*, k = \overline{1,n}$, then:

$$\sum_{k=1}^{n} (\arctan x_k)^2 + \sum_{k=1}^{n} \left(\arctan \frac{1}{x_k}\right)^2 \geq \frac{n\pi^2}{8}$$

Proposed by D.M. Bătinețu – Giurgiu, N. Stanciu – Romania

XI.17. If $\in N^*, m \in \mathbb{R}^*$, $x_k \in \mathbb{R}_+^*, k = \overline{1,n}$ then:

$$\sum_{k=1}^{n} \left((\arctan x_k)^{m+1} + \left(\arctan \frac{1}{x_k}\right)^{m+1}\right) \geq \frac{n \cdot \pi^{m+1}}{2^{2m+1}}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

XI.18. If $A \in M_2(\mathbb{R})$ then: $det(A^2 + 2A + 2I_2) \geq (2 + Tr\ A)^2$

Proposed by Marian Ursarescu-Romania

XI.19. $A, B \in M_2(\mathbb{R}), det\ A \neq 0, det\ B \neq 0, Tr(AB^{-1}) = det(AB^{-1}) = 1$

Find: $\Omega = det(I_2 + A^{-1}B)$

Proposed by Marian Ursarescu-Romania

XI.20. If $A, B \in M_5(\mathbb{R}), A^3 + 7I_5 = A^2, B^3 + 9I_5 = B^2$ then: $det(AB) > 0$

Proposed by Daniel Sitaru – Romania

XI.21. If $A, B \in M_5(\mathbb{R}), A^3 - 2I_5 = A^2, B^3 - 3I_5 = B^2$ then: $det(AB) > 0$

Proposed by Daniel Sitaru – Romania

XI.22. Find $A, B \in M_2(\mathbb{R})$ such that: $det\ A < 0, det(A - B) > 0,$

$$det(A + B) < 0, det(2A + B) > 0$$

Proposed by Marian Ursarescu-Romania

XI.23. If $A \in M_4(\mathbb{C}), det\ A \neq 0, Tr\ A = 0$ then: $Tr\ (A^3) = 3(det\ A)(Tr\ A^{-1})$

Proposed by Marian Ursarescu-Romania

XI.24. Find:

$$\Omega = \lim_{x\to 0}\left(\lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\tan^2\frac{x}{2^k}\right)\right)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

XI.25. Find:

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\frac{1}{n}\sum_{k=2}^{n}\frac{1+k^2}{(k-1)k(k+1)!}}$$

Proposed by Daniel Sitaru – Romania

XI.26. If $A, B \in M_2(\mathbb{C}), det(A+B) = 1$ then: $det(A\cdot det\, B + B\cdot det\, A) = det(AB)$

Proposed by Marian Ursarescu-Romania

XI.27. If $A \in M_2(\mathbb{R}), B \in M_3(\mathbb{R}), C \in M_4(\mathbb{R})$,

$A^2 - A = I_2, B^2 - B = I_3, C^2 - C = I_4$ then: $|det\, A + det\, B + det\, C| < 28$

Proposed by Daniel Sitaru – Romania

XI.28. If $A, B, C, D \in M_n(\mathbb{C}), n \in \mathbb{N}, n \geq 2, det(ABCD) \neq 0$ then:

$$rank(AB\cdot det(CD) + CD\cdot det(AB)) = rank\left(\frac{1}{det\, C\cdot det\, D}B^{-1}A^{-1} + \frac{1}{det\, A\cdot det\, B}D^{-1}C^{-1}\right)$$

Proposed by Daniel Sitaru – Romania

XI.29 If $A, B \in M_2(\mathbb{C}), det(A+B) = 1$ then: $det(A\cdot det\, B + B\cdot det\, A) = det(AB)$

Proposed by Marian Ursărescu – Romania

XI.30. $A \in M_n(\mathbb{R}), det\, A \neq 0, \alpha \in (-1,1), A^2 + A^{-2} = \alpha(A + A^{-1})$' Find: $|det\, A|$

Proposed by Marian Ursărescu – Romania

XI.31. Find:

$$\Omega = \lim_{n\to\infty}\frac{\sqrt{2\sqrt{3\sqrt{4\sqrt{5\ldots\sqrt{n}}}}}}{n}$$

Proposed by Daniel Sitaru – Romania

XI.32. Solve for real numbers:

$$\begin{vmatrix} 1 & 3+\sin x & 2+3\sin x & 2\sin x \\ 1 & 2+\sin x+\cos x & 2\sin x+\sin x\cos x & \sin 2x \\ 1 & 1+\sin x+\cos x & \sin x+\cos x+\sin x\cos x & \sin x\cos x \\ 1 & 3+\cos x & 2+3\cos x & 2\cos x \end{vmatrix} = 0$$

Proposed by Daniel Sitaru -Romania

XI.33. Fie $p \in \mathbb{N}, p \geq 3$. Să se rezolve în $M_3(\mathbb{R})$ ecuația:

$$x + x^3 + \cdots + x^{2p-1} = \begin{pmatrix} p & p & 0 \\ 0 & p & p \\ 0 & 0 & p \end{pmatrix}$$

Proposed by Marian Ursărescu – Romania

XI.34. Fie $A, B \in M_2(\mathbb{C})$ astfel încât $Tr((AB)^2) = Tr(A^2B^2)$. Să se calculeze $Tr[(AB-BA)^n], n \in \mathbb{N}^*, n \geq 2$. ***Proposed by Marian Ursărescu – Romania***

XI.35. Find:

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \arctan\left(\frac{9}{9+(3k+5)(3k+8)}\right)$$

Proposed by Daniel Sitaru – Romania

XI.36. Find:

$$\Omega = \lim_{n\to\infty} \frac{\left(\sqrt[n]{n!}+n\right)^n}{(2n)!}$$

Proposed by Daniel Sitaru – Romania

XI.37. $x_0 > 0 \wedge x_{n+1} = a^{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}, \forall n \in \mathbb{N}, a > 1$.Find:

$$\Omega = \lim_{n\to\infty} \frac{\sum_{1\leq i<j\leq k\leq n} x_i x_j x_k}{n^6}$$

Proposed by Marian Ursărescu – Romania

XI.38. Find:

$$\Omega = \lim_{n\to\infty} n\sqrt[n]{\frac{\left((2n)!!\right)^2}{(2n)!}}$$

Proposed by Daniel Sitaru – Romania

XI.39. Find:

$$\Omega = \lim_{n\to\infty} \frac{\sqrt{1! + \sqrt{2! + \sqrt{3! + \sqrt{4! \ldots \sqrt{n!}}}}}}{n}$$

Proposed by Daniel Sitaru – Romania

XI.40. Fie $A \in M_4(\mathbb{C})$ cu proprietatea: $\det A \neq 0$ și $Tr\, A = 0$. Să se arate că:

$$Tr\, A^3 = 3 \det A \cdot Tr\, A^{-1}$$

Proposed by Marian Ursărescu – Romania

XI.41. $x_0 > 0 \wedge x_{n+1} = \frac{\sqrt[3]{x_n^3 + 3x_n^2} + 2x_n}{3}, \forall n \in \mathbb{N}$.Find: $\Omega = \lim_{n\to\infty} \frac{\sum_{1\leq i<j\leq n} x_i x_j}{n^4}$.

Proposed by Marian Ursărescu – Romania

XI.42. Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x) + f(3x) + f(9x) = 91x^2 + 26x + 3, \forall x \in \mathbb{R}$$

Proposed by Marian Ursărescu – Romania

XI.43. Find:

$$\Omega = \lim_{n\to\infty} \frac{13 \cdot 25 \cdot 37 \cdot \ldots \cdot (12n - 11)}{7 \cdot 19 \cdot 31 \cdot \ldots \cdot (12n - 5)}$$

Proposed by Daniel Sitaru – Romania

XI.44. $x_0 = 1 \wedge x_{u+1} = 3x_u + \sqrt{8x_u^2 + 1}, \forall n \in \mathbb{N}$.Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{x_n}$$

Proposed by Marian Ursărescu – Romania

XI.45. Să se determine matricele $A \in M_n(\mathbb{C})$ care verifică relațiile:

$$\begin{cases} A^T = \bar{A} \\ A^{2p+1} + A^{2p-1} = 2I_n, p \in \mathbb{N}^* \end{cases}$$

Proposed by Marian Ursărescu – Romania

XI.46. If $A \in M_n(\mathbb{C})$, $\det A \neq 0, aA + bA^n = cI_n, a, b, c \in \mathbb{C}^*$, $|a| = |b| = |c|$ then:

$$\left(\frac{\sqrt{5} - 1}{2}\right)^n \leq |\det A| \leq \left(\frac{\sqrt{5} + 1}{2}\right)^n$$

Proposed by Marian Ursărescu – Romania

XI.47. Fie $A, B \in M_2(\mathbb{C})$ astfel încât $\det(A + B) = 1$. Să se arate că:

$$\det(\det BA + \det AB) = \det(AB)$$

Proposed by Marian Ursărescu – Romania

XI.48 Fie $A, B \in M_2(\mathbb{R})$ astfel încât $\det(A-B)\det(A+B) \geq 0$. Să se arate că cel puțin unul din numerele: $\det(A^2-B^2)$ sau $\det(AB-BA)$ este mai mare sau egal cu 0.

Proposed by Marian Ursărescu – Romania

XI.49. Să se calculeze

$$\lim_{n\to\infty} \prod_{1\leq i<j<k\leq n} \left(1+\frac{ijk}{n^6}\right)$$

Proposed by Marian Ursărescu – Romania

XI.50. Fie $A \in M_3(\mathbb{R})$. Să se arate că: $\det(I_3 + A^2) = 0 \Leftrightarrow tr\, A = \det A$ și $tr\, A^* = 1$.

Proposed by Marian Ursărescu – Romania

XI.51. Fie $A \in M_4(\mathbb{R})$ astfel încât: $\det(A^2+3I_4) = \det(A^2+2A+2I_4) = 0$

Să se calculeze $\det A$. ***Proposed by Marian Ursărescu – Romania***

XI.52. Fie $A \in M_4(\mathbb{R})$ o matrice simetrică și inversabilă. Să se arate că:

$$\det(A^2 + A^{-2} + 2A + 2A^{-1} + 3I_n) \geq 9^n$$

Proposed by Marian Ursărescu – Romania

XI.53. Să se calculeze

$$\lim_{n\to\infty}\left(\frac{1}{2}+\sum_{k=1}^{n}\frac{k^p}{k^{4p}+k^{2p}+1}\right)^{n^{3p-1}}, p \in \mathbb{N}^*$$

Proposed by Marian Ursărescu – Romania

XI.54. Fie $A, B \in M_3(\mathbb{R})$ cu $tr^2(AB) = tr(AB)^2$.Dacă $(AB)^3 = O_3$ atunci $(BA)^3 = O_3$

Proposed by Marian Ursărescu – Romania

XI.55. Să se arate că $\forall a, b \in \mathbb{R}$ și $\forall A, B \in M_4(\mathbb{R})$ cu proprietatea: $AB = BA$, atunci are loc inegalitatea: $\det(I_n + 2(a^2+b^2)(A^2+B^2) + 2(a+b)(A+B) + 8abAB) \geq 0$

Proposed by Marian Ursărescu – Romania

XI.56. Să se determine funcțiile continue $f: (0,+\infty) \to \mathbb{R}$ cu proprietatea:

$$f(x) = f\left(\sqrt{2x^2-2x+1}\right), \forall x > 0$$

Proposed by Marian Ursărescu – Romania

XI.57. Să se arate că nu există $A, B \in M_2(\mathbb{R})$ care să verifice relațiile:

$\det A < 0, \det(A-B) > 0, \det(A+B) < 0$ și $\det(2A+B) > 0$

Proposed by Marian Ursărescu – Romania

XI.58. a) Să se arate că $\forall\, n \in \mathbb{N}, n \geq 2$ ecuația $x + \arctan(x-1) = \sqrt[n]{2018}$, are o singură rădăcină reală notată cu x_n.

b) Să se calculeze $\lim_{n\to\infty} x_n$.

c) Să se calculeze $\lim_{n\to\infty} n(x_n - 1)$.

Proposed by Marian Ursărescu – Romania

XI.59. Fie $A \in M_n(\mathbb{R})$ o matrice simetrică și inversabilă. Să se arate că:

$$\det[4(A^2 + A^{-2}) + 25(A + A^{-1}) + 42] > 1$$

Proposed by Marian Ursărescu – Romania

XI.60. Find:

$$L = lim_{n\to\infty} \sqrt[n]{p! + \frac{(p+1)!}{1!} + \frac{(p+2)!}{2!} + \cdots + \frac{(p+n)!}{n!}}, p \in \mathbb{N}^*, p - \text{fixed}$$

Proposed by Marian Ursărescu – Romania

XI.61. $(x_n)_{n\geq 0}, x_0 > 0, x_{n+1} = x_n + \frac{1}{x_n^p}, p \in \mathbb{N}^*, p$ – fixed. Find:

$$\Omega = \lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n^{\sqrt[p+1]{n}}}$$

Proposed by Marian Ursărescu – Romania

XI.62. Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{2} + \sum_{k=1}^{n} \frac{k^p}{k^{4p} + k^{2p} + 1}\right)^{n^{3p-1}}, p \in \mathbb{N}^*$$

Proposed by Marian Ursărescu – Romania

XI.63. Find:

$$\Omega = \lim_{n\to\infty} (n+1)! \left(e - \sum_{k=0}^{n} \frac{1}{k!}\right)$$

Proposed by D.M.Batinetu-Giurgiu, Neculai Stanciu-Romania

XI.64. Compute:

$$\lim_{n\to\infty} \sqrt[4]{n!} \prod_{k=1}^{n} \left((k+1)^{\frac{3}{4}} - k^{\frac{3}{4}}\right)$$

Proposed by Mihály Bencze-Romania

XI.65. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n(n+1)} \sum_{k=1}^{n} k \sin\left(\frac{k^2+k}{n^2+n}\right)$$

Proposed by Daniel Sitaru – Romania

XI.66. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n(n+1)(2n+1)} \sum_{k=1}^{n} k^2 \tan^{-1}\left(\frac{k(k+1)(2k+1)}{n(n+1)(2n+1)}\right)$$

Proposed by Daniel Sitaru – Romania

XI.67. If $a, b, c, d > 0$, are different in pairs, $\Delta = \begin{vmatrix} 1 & a & a^3 & a^4 \\ 1 & b & b^3 & b^4 \\ 1 & c & c^3 & c^4 \\ 1 & d & d^3 & d^4 \end{vmatrix}$ then:

$$\frac{8\Delta}{(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)} < 3(a+b+c+d)^2$$

Proposed by Daniel Sitaru – Romania

XI.68. Find:

$$\Omega = \lim_{n\to\infty} \frac{\sin n}{n}\left(\left(1+\frac{\sqrt[n]{n}}{n}\right)^{\frac{1}{n}} + \left(1-\frac{\sqrt[n]{n}}{n}\right)^{\frac{1}{n}}\right)$$

Proposed by Daniel Sitaru – Romania

XI.69. Find:

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{k+n^7}}$$

Proposed by Daniel Sitaru – Romania

XI.70. Find:

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(k+n)^5}{7 + \tan^{-1}(k+n) + (k+n)^6}$$

Proposed by Daniel Sitaru – Romania

XI.71. Find:

$$\Omega = \lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{k^4}{n^5}\right)^{1+\frac{k^4}{n^5}}$$

Proposed by Daniel Sitaru – Romania

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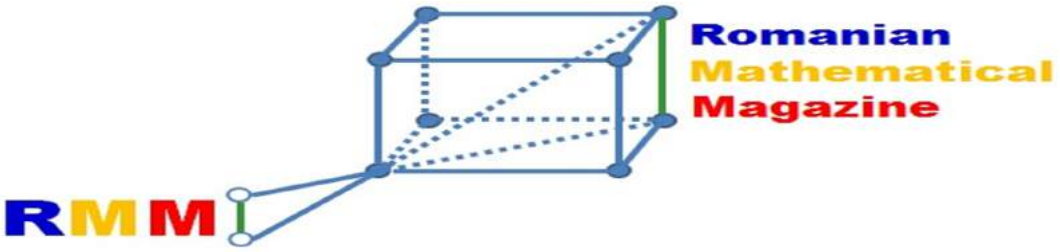
12-CLASS-STANDARD



XII.1. Fie $G = (-1,1)$ și a, b numere reale fixate cu $0 \leq a < b$.

a) Arătați că legea de compoziție $(x, y) \to x \circ y = \frac{a(x+y)+b(1+xy)}{b(1+xy)+a(x+y)}, \forall x, y \in G$ este internă pe G și că $(G,\circ)$ este un grup abelian.

b) Demonstrați că funcția $f: (G,\circ) \to (R, +), f(x) = \ln\left(\frac{b-a}{b+a} \cdot \frac{1-x}{1+x}\right)$ este un izomorfism de grupuri.

Proposed by Marin Chirciu – Romania

XII.2. Fie $a, b \in \mathbb{R}$ astfel încât $a^2 + b^2 \neq 0$ și

$G = \left\{A(x) | A(x) = \begin{pmatrix} 1 & 0 & 0 \\ ax & 1 & 0 \\ a^2x^2 + bx & 2ax & 1 \end{pmatrix}, x \in \mathbb{R}\right\}$ o submulțime a lui $M_3(\mathbb{R})$.

Demonstrați că $(G,\cdot)$ este un grup abelian izomorf cu grupul $(R, +)$.

Proposed by Marin Chirciu – Romania

XII.3. Fie $m, n \in \mathbb{N}$. Determinați primitivele funcției

$$f(x) = \frac{1}{x(1-x^m)^n}, x \in (0,1)$$

Proposed by Marin Chirciu – Romania

XII.4. Fie $a, b, c, d \in \mathbb{R}$ astfel încât $a + b = t \neq 0, ab = cd$ și

$M = \left\{A(x) | A(x) = \begin{pmatrix} ax + 1 & 0 & cx \\ 0 & 0 & 0 \\ dx & 0 & bx + 1 \end{pmatrix}, x \in R - \left\{-\frac{1}{t}\right\}\right\}$ o submulțime a lui $M_3(R)$.

Demonstrați că $(M,\cdot)$ este un grup abelian izomorf cu grupul $(G,\circ)$, unde $G = R - \left\{-\frac{1}{t}\right\}$ și $x \circ y = x + y + txy$.

Proposed by Marin Chirciu – Romania

XII.5. Fie $P \in \mathbb{R}[x]$, un polinom de gradul $n \geq 2$, cu coeficienți pozitivi. Să se arate că:

$$P\left(1 + \frac{1}{n}\right) \geq P(1) + \frac{P'(1)}{n}$$

Proposed by Marian Ursărescu – Romania

XII.6. Fie șirul de polinoame definit astfel: $P_0(x) = 1, P_1(x) = x$ și $P_n(x) = 2xP_{n-1}(x) - P_{n-2}(x), \forall n \in \mathbb{N}, n \geq 2$. Să se afle restul împărțirii lui P_n la $x^3 - x^2 - x + 1, n \geq 3$.

Proposed by Marian Ursărescu – Romania

XII.7. Să se calculeze:

$$\lim_{n\to\infty} \int_0^1 \frac{\arctan nx \cdot \arctan x}{x + 1} dx$$

Proposed by Marian Ursărescu – Romania

XII.8. Fie $b > a > 0$ și $f: [a, b] \to [0, +\infty)$ o funcție derivabilă cu derivata pozitivă. Să se arate că:

$$\int_a^b x^2 f(x)dx \geq ab\int_a^b f(x)\,dx$$

Proposed by Marian Ursărescu – Romania

XII.9. Solve for integers: $\begin{cases}(x+y)\perp(x-y)=12\\(x+y)\Delta(x-y)=11\end{cases}$

where: $x\perp=2(x+y)+xy+1; x\Delta y=2xy+x+y+1$.

Proposed by Ștefan Marica – Romania

XII.10. Pe mulțimea $G=(0,\infty)$ se introduce legea de compoziție:

$$x\circ y=\ln[(e^x-1)(e^y-1)+1],\forall x,y\in G$$

a) Să se arate că legea "$\circ$" este corect definită pe G.b) Demontrați că perechea $(G,\circ)$ este grup abelian. c) Să se arate că $(G,\circ)\simeq(R,+)$. ***Proposed by Dan Nedeianu-Romania***

XII.11. Să se calculeze:

$$\int\frac{2x^4\ln x+1}{x^3e^{x^2}}dx, x\in(0,\infty)$$

Proposed by Dan Nedeianu-Romania

XII.12. Find:

$$\Omega=\int_a^{\frac{\pi}{2}}\left(\frac{\sin 7x}{\sin x}\right)^2 dx, 0<a\leq\frac{\pi}{2}$$

Proposed by Daniel Sitaru– Romania

XII.13. Find:

$$\Omega=\int\frac{x(\tan x+2\tan 2x+4\tan 4x)}{\cot x-8\cot 8x}dx$$

Proposed by Daniel Sitaru– Romania

XII.14. Find:

$$\Omega=\int\tan^2 x\,(\tan x+2\tan 2x+4\tan 4x+8\cot 8x)dx, x\in\left(0,\frac{\pi}{2}\right)$$

Proposed by Daniel Sitaru– Romania

XII.15. Find:

$$\Omega=\int\frac{\tanh^2 x+\tanh^2 x\,(1+\tanh^2 x)^2}{(1+\tanh^2 x)^2}dx, x\in\mathbb{R}$$

Proposed by Daniel Sitaru– Romania

XII.16. Find:

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{\tan^4 x\,(\tan^2 x-2)}{(1-\tan^2 x)^2}dx$$

Proposed by Daniel Sitaru– Romania

XII.17. Find:

$$\Omega = \int \frac{tan^2x \cdot tan^4x \cdot tan^6x \cdot tan^8x \cdot \ldots \cdot tan^{2n}x}{sin^2x\sqrt{1-tan^{n^2+n+1}x} - \sqrt{1-tan^{n^2+n+1}x} - cos^2x\sqrt{1-tan^{n^2+n+1}x}}dx$$

Proposed by Vural Ozap-Turkey

XII.18. If $0 < a < b < 1$ then:

$$\frac{1}{2} + \frac{2}{a+b} < \frac{2(ln\, b - ln\, a)}{b-a} + \frac{1}{b-a}\int_a^b \frac{dx}{ln(1-x)}$$

Proposed by Daniel Sitaru – Romania

XII.19. If $0 < a < b < 1$ then:

$$\frac{2(\ln b - \ln a)}{b-a} + \frac{1}{b-a}\int_a^b \frac{dx}{\ln(1-x)} < 1 + \frac{1}{\sqrt{ab}}$$

Proposed by Daniel Sitaru – Romania

XII.20. Find:

$$\Omega = \lim_{x\to 0} \frac{\int_0^x \frac{\sqrt[3]{1+2x}\cdot\sqrt[5]{1+3x}-1}{x}dx}{\int_0^x \frac{\sqrt[3]{1+3x}\cdot\sqrt[5]{1+2x}-1}{x}dx}$$

Proposed by Daniel Sitaru– Romania

XII.21. Find:

$$\Gamma = \lim_{n\to\infty}\big(\Omega(n) - \Omega(n+1)\big), \Omega(n) = \int_1^e \frac{dx}{x(1+x^3)^n}, n \in \mathbb{N}^*$$

Proposed by Daniel Sitaru -Romania

XII.22. If $0 < a \leq b$ then:

$$2\int_a^b \frac{1}{1+e^{2x^2}}dx \geq \frac{tan^{-1}\left(e^{b^2}\right) - tan^{-1}\left(e^{a^2}\right)}{be^{b^2}}$$

Proposed by Daniel Sitaru -Romania

XII.23. If $a, b, c \in \left(0, \frac{1}{2}\right)$ then:

$$b\int_a^{2a} \frac{15x+2}{36x^3+1}dx + c\int_a^{2b} \frac{15x+2}{36x^3+1}dx + a\int_a^{2c} \frac{15x+2}{36x^3+1}dx \leq (a+b+c)\, ln\, 2$$

Proposed by Daniel Sitaru -Romania

XII.24. If $0 \leq a < b$ then:

$$(1 + ab - a^2)e^{a^2} < \frac{1}{b-a}\int_a^b e^{x^2}dx < (1 - ab + b^2)\,e^{b^2}$$

Proposed by Daniel Sitaru -Romania

XII.25. If $0 < a < \frac{\pi}{2}$ then:

$$a\pi + \pi\int_a^{\frac{\pi}{2}} \frac{\sin x}{x}dx > (\pi - 2)(1 + a - \sin a)$$

Proposed by Daniel Sitaru -Romania

XII.26. $a, b, c \in \mathbb{N}^*, \Omega(a) = lim_{n\to\infty}\, n\int_0^1 \frac{x^n}{(x+\pi)^a}dx$.Prove that:

$$(1 + \pi)^b\Omega(a) + (1 + \pi)^c\Omega(b) + (1 + \pi)^a\Omega(c) \geq 3$$

Proposed by Daniel Sitaru -Romania

XII.27. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\int_a^b \frac{\cos x \cdot \sin^2(\sin x)}{\sin^2 x}dx \geq tan^{-1}\left(\frac{\sin b - \sin a}{1 + \sin a \cdot \sin b}\right)$$

Proposed by Daniel Sitaru -Romania

XII.28. For $0 < a < b$. Prove:

$$\int_a^b \sin\left(\sqrt[3]{x}\right)dx \leq (b - a)\sqrt[3]{b}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

XII.29.

$$F(a) = \int_0^a \frac{\cos^7 x}{(\cos 6x + 6\cos 4x + 15\cos 2x + 10)}dx, a \in \left[0, \frac{\pi}{2}\right]$$

Prove that:

$$F(a)F(b)F(c)[F(a) + F(b) + F(c)] \leq 2^{-20}(a^4 + b^4 + c^4)$$

Proposed by Daniel Sitaru-Romania

XII.30. Prove that:

$$\frac{\pi^{\frac{5}{2}}}{12\sqrt{2}} \leq \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,(x)^{\frac{3}{2}}dx \leq \frac{\pi^2}{8}$$

Proposed by Sagar Kumar-Kolkata-India

XII.31. Prove:

$$\int_0^{\frac{\pi}{2}} \left(\frac{9^{\sin x}}{4^{\sin x} + 5^{\sin x}} + \frac{16^{\sin x}}{3^{\sin x} + 5^{\sin x}} + \frac{25^{\sin x}}{3^{\sin x} + 4^{\sin x}} \right)^3 dx \geq \frac{1593}{8 \ln 60}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

XII.32. For $n \in \mathbb{N}^* \wedge n \geq 2$. Prove:

$$\int_0^1 \left(\sum_{k=1}^{n} e^{x^k} \right) dx > n + \frac{n}{((n+1)!)^{\frac{1}{n}}}.$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

XII.33.

$$\Omega(n) = \int_{-1}^{1} x \log(1 + n^{3x})\, dx, n \in \mathbb{N}^*$$

Prove that:

$$9\left(1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}\right)^2 > 4e^{2\Omega(n)}\left(1 + e^{\Omega(n)}\right)$$

Proposed by Daniel Sitaru -Romania

XII.34. Let be $\Omega: \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$,

$$\Omega(x) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_{\varepsilon}^{x} \frac{\sin t}{t} dt$$

Prove that:

$$(a + b + c)\Omega\left(\frac{a^2 + b^2 + c^2}{a + b + c}\right) \geq a\Omega(a) + b\Omega(b) + c\Omega(c)$$

Proposed by Daniel Sitaru -Romania

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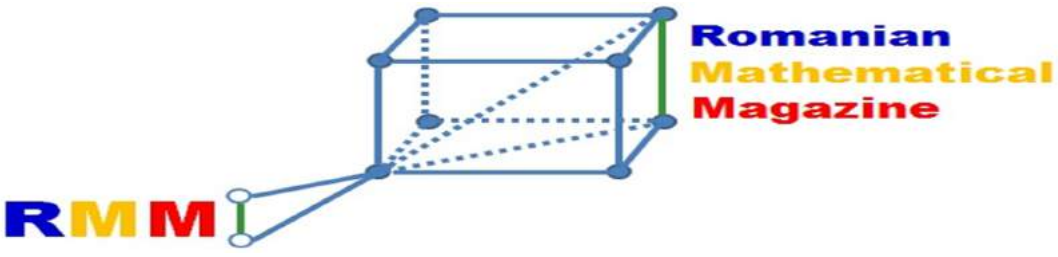
UNDERGRADUATE PROBLEMS



U.1. If $\psi(m,n) = \int_0^{\infty} (e^{-x} - e^{-x \cdot m}) \cdot \sin(m \cdot n \cdot x)^2 \frac{dx}{x}$ then prove that

$$\lim_{m\to i}\left(\frac{1}{m}\sum_{n=1}^{\infty}\left(\frac{\partial^2\psi(m,n)}{\partial m\partial n}\right)\right)=\frac{1}{2}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2. Prove that:

$$\int_{-\infty}^{\infty}\frac{e^{\frac{i\pi x}{3}}\sin(3\pi x)}{x^2+x+e^{\frac{i\pi}{3}}}dx=\frac{\pi\left(\sqrt{3}-i\right)}{2\sqrt{1+2i\sqrt{3}}}\cdot e^{-\frac{4\pi\sqrt{1+2i\sqrt{3}}}{3}\left(e^{-\frac{\pi\sqrt{1+2i\sqrt{3}}}{3}}+1\right)}$$

where $i=\sqrt{-1}$ is imaginary number.

Proposed by Srinivasa Raghava-AIRMC-India

U.3. For $-1<m<-\frac{1}{2}$, prove that:

$$\int_0^{\infty}x^m\cdot\sqrt{\frac{1+\sqrt{1+x^2}}{1+x^2}}dx=i\cdot\tan\left(\frac{m\pi}{2}\right)\int_0^{\infty}x^m\cdot\sqrt{\frac{1-\sqrt{1+x^2}}{1+x^2}}dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.4. If we define the following,

$$\chi(q)=1-\left(\frac{1\cdot3}{2\cdot4}\right)\cdot3\cdot q+\left(\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\right)\cdot5\cdot q^2-$$
$$-\left(\frac{1\cdot3\cdot5\cdot7\cdot9\cdot11}{2\cdot4\cdot6\cdot8\cdot10\cdot12}\right)\cdot7\cdot q^3+\left(\frac{1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13\cdot15}{2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot14\cdot16}\right)\cdot9\cdot q^4-\cdots+\infty$$

then we have

$$\int_0^{\infty}\frac{\chi(q)}{(1+q)}dq=\frac{5}{6}+\frac{\ln\left(3-2\sqrt{2}\right)}{4\sqrt{2}}+\sqrt{2}\ln\left(1+\sqrt{2}\right)$$

$$\int_0^{\infty}\frac{\chi(1+q)}{(1+q)}dq=\ln\left(\frac{1+\sqrt{2\cdot\left(\sqrt{2}-1\right)}}{1-\sqrt{2\cdot\left(\sqrt{2}-1\right)}}\right)-\sqrt{1+\sqrt{2}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.5. Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\int_0^{\infty}\frac{dx}{\left(x^2+\frac{1}{4}\right)^{n+1}}}$$

Proposed by Daniel Sitaru – Romania

U.6. Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\int_0^1x^{2n}\sqrt{(1-x^2)^{2n+1}}dx}$$

Proposed by Daniel Sitaru – Romania

U.7. If we define the following function: $f(x)=\sum_{m=0}^{\infty}\frac{(-1)^m x^{6m+2}}{(6m+2)!}\left(1+\frac{x^2}{6m+3}-\frac{x^4}{6m+4}\right)$

then evaluate the integral in a closed-form: $\int_{-\infty}^{\infty} \frac{f(x^2)}{x^2} e^{-x^2} dx$

Proposed by Srinivasa Raghava-AIRMC-India

U.8. Let for any complex number, $t > 1$, $\Phi(t) = \int_{-\infty}^{\infty} \operatorname{arccot}(x^2 t) \operatorname{arccot}\left(\frac{x^2}{t}\right) dx$

then prove that: $\int_0^{\infty} \frac{\Phi(t)}{t} dt = \pi^2\left(8 - 4\sqrt{2}\right)$

Proposed by Srinivasa Raghava-AIRMC-India

U.9. On the integral of Prof. Khalef Ruhemi: Let $\eta(p) = \int_0^1 x^p \arccos(x)\, dx$ then for any complex number $p, Re(p) > 0$, we have: $\frac{\partial}{\partial p}\eta(p) = \frac{\eta(p)}{2}\left(H_{\frac{p}{2}} - H_{\frac{p-1}{2}} - \frac{4}{1+p}\right)$

where H_p – Harmonic number

Proposed by Srinivasa Raghava-AIRMC-India

U.10. If $\phi(m) = \sum_{k=0}^{\infty}\left(e^{-\frac{\pi(k^2-k)}{k}} - e^{-\frac{\pi(k^2+k)}{k}}\right)^3 (-m)^k \cosh(\pi k)$ then we have

$$\int_0^{\infty} \frac{\phi(m)}{m+1} dm = \frac{\pi}{e^{\pi} + e^{3\pi}}(e^{2\pi} - 1)^2(1 + 3e^{2\pi})$$

Proposed by Srinivasa Raghava-AIRMC-India

U.11. If $f(k) = \int_0^{\infty} \left(\frac{e^x + e^{-x}}{2}\right)^{-k} dx$ we can observe easily that

$$f(1) = \frac{\pi}{2}, f(2) = 1, f(3) = \frac{\pi}{4}, f(4) = \frac{2}{3}, f(5) = \frac{3\pi}{16}, f(6) = \frac{8}{15}, f(7) = \frac{5\pi}{32}, f(8) = \frac{16}{35}$$

then prove this sharp inequality: $\int_0^1 f(k) + f(-k) dk \geq \frac{13+\sqrt{2}}{10}$

Proposed by Srinivasa Raghava-AIRMC-India

U.12. Generalization of the limit problem by Prof. Dan Sitaru

If we define, $\Omega_k(n) = \int_0^{\frac{\pi}{4}} \ln(1 + \tan(x)) \cdot (4 \cdot x^2 - \pi x + 4n^2)^k dx$

then the following is true for $n, k \in \mathbb{N}$: $\lim_{n\to\infty} \frac{\Omega_k(n)}{(1+2+3+4+\cdots n)^k} = 2^{3k-3}\pi \ln(2)$

Proposed by Srinivasa Raghava-AIRMC-India

U.13. A note on the problem by Alvaro Salas

For any positive interger n we have: $\int_0^{\infty} \frac{\log^{2n}(x)}{(1-x)^2} \frac{dx}{\sqrt{x}} = \left|\pi^{2n} n\, 4^n \left(E_{2\,n-1}(1) + \frac{1}{2}E_{2\,n-1}\left(\frac{1}{2}\right)\right)\right|$

Where $E_k(q)$ is the k^{th} order Euler Polynomial

Also we can express the above integral as a generating funciton of $x \tan(x)$

$$\sum_{k=0}^{\infty}\left(\frac{(4k-1)!!}{(4k)!!}\right)\left(\frac{(2k-1)!!}{(2k)!!}\right)^2 = \frac{3\left(\sqrt{2}-1\right)^{\frac{3}{2}}\Gamma\left(\frac{1}{8}\right)^6}{128\pi^2\Gamma\left(\frac{1}{3}\right)^3}\sum_{k=0}^{\infty}\frac{2^{-k}}{2k+2}\left(\frac{(4k-1)!!}{(4k)!!}\right)\left(\frac{(2k-1)!!}{(2k)!!}\right)$$

Where $(m)!!$ is double factorial.

Proposed by Srinivasa Raghava-AIRMC-India

U.14. For any complex number m & $Re(m) > -\frac{1}{2}$, $H_m = H_{m-\frac{1}{2}} + \frac{d}{dm}\left(\int_0^1 \frac{x^{2m}-x}{\ln(x)} \frac{dx}{1+x}\right)$

Where H_m – Harmonic Number

Proposed by Srinivasa Raghava-AIRMC-India

U.15. Calculate the integral: $\int_0^1 \frac{\sqrt{x}\ln x}{x^2-x+1}dx$ .It is required to express the integral value with the usual mathematical constants, without using values of special functions.

Proposed by Vasile Mircea Popa-Romania

U.16. Calculate the integral: $\int_0^{\frac{\pi}{2}} \sin x \ln(tgx+ctgx)dx$.

Proposed by Vasile Mircea Popa-Romania

U.17. Calculate the integral: $\int_0^1 \frac{\ln x}{x^2-x+1}dx$.

Proposed by Vasile Mircea Popa-Romania

U.18. Prove:

$$\sqrt[3]{2\left(\cos\frac{\pi}{13}+\cos\frac{5\pi}{13}\right)}-\sqrt[3]{2\left(\cos\frac{2\pi}{13}-\cos\frac{3\pi}{13}\right)}-\sqrt[3]{2\left(\cos\frac{4\pi}{13}+\cos\frac{6\pi}{13}\right)}=\sqrt[3]{7-3\sqrt[3]{13}}$$

Proposed by Vasile Mircea Popa-Romania

U.19. Evaluate the following integral in closed form:

$$\int_0^{\infty}(e^{-4\pi x}+e^{-2\pi x})\tanh(\pi x)\frac{dx}{x\cosh\left(\frac{\pi x}{4}\right)}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.20. Let, $F(a,b)=\int_0^{\infty}\frac{dx}{\sqrt{\left(x^2+\frac{a}{b}\right)\left(x^2+\frac{b}{a}\right)}}$ then prove that, $\lim_{a\to b}\frac{\partial^2}{\partial a\partial b}F\left(\frac{a}{b},\frac{b}{a}\right)(ab)=\frac{\pi}{4}$

Proposed by Srinivasa Raghava-AIRMC-India

U.21.

$$\frac{1}{2}\left(\pi+2i\ln\left(\sqrt{\frac{n^2-mn+\sqrt{(m-n)^2\sqrt{mn}}}{(m-n)^2}}-\sqrt[i]{\frac{m^2-mn-\sqrt{(m-n)^2\sqrt{mn}}}{(m-n)^2}}\right)\right)=x$$

then prove that

$$(m\sin(x)^2+n\cos(x)^2)^2=\sqrt{mn}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.22. Let, $F(a,b)=\int_0^{\infty}\frac{dx}{\sqrt{\left(x^2+\frac{a}{b}\right)\left(x^2+\frac{b}{a}\right)}}$ then prove that, $\lim_{a\to b}\frac{\partial^2}{\partial a\partial b}F\left(\frac{a}{b},\frac{b}{a}\right)(ab)=\frac{\pi}{4}$

Proposed by Srinivasa Raghava-AIRMC-India

U.23.

$$16\int_0^1 \frac{x\ln\left(\ln\left(\frac{1}{x}\right)\right)}{(1+x+x^2+x^3)^2}dx = \left(\ln\left(\frac{\Gamma\left(\frac{1}{8}\right)^8}{16\pi^6}\right)+\pi\ln\left(\frac{4\pi^3}{\Gamma\left(\frac{1}{4}\right)^4}\right)-\ln(2)^2\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.24. A very good approximation to an interesting sequence. If

$$\ln(1+\ln(2+\ln(3+\ln(4+\ln(5+\cdots))))) = \alpha \text{ then:}$$

$$\int_0^\infty \frac{x^{-\alpha}}{e^{\alpha x}+\frac{4}{5}}dx \approx \frac{99}{25}\cdot\frac{\alpha}{\gamma}e^{-\gamma}$$

where $\gamma = 0.57721566\ldots$ Euler's Constant.

Proposed by Srinivasa Raghava-AIRMC-India

U.25. If

$$\int_{-\infty}^\infty \frac{\sin(\alpha\pi x)^2}{(1+\cosh(\pi x))^2}dx = (1-Q(\alpha))\int_{-\infty}^\infty \frac{\cos(\alpha\pi x)^2}{(1+\cosh(\pi x))^2}dx$$

then evaluate the limit

$$\lim_{\alpha\to\infty}\left(\frac{1}{Q(\alpha)}\frac{d}{d\alpha}Q(\alpha)\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.26. Evaluate to a possible closed – form

$$\int_0^{\frac{2\pi i}{5}} \ln\left(1+e^{\frac{2\pi i}{5}}\sin(\lambda)\right)\sin(\lambda)\,d\lambda$$

Proposed by Srinivasa Raghava-AIRMC-India

U.27. Let

$$\theta(m,n) = \int_0^\infty \frac{\ln(\cosh(x))^m}{\sinh(x)^n}dx$$

then show that

$$\frac{2}{3}\theta(1,1) = \theta(2,1)^2+\theta(2,2)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.28. If

$$\frac{\pi^2}{12}+\frac{\ln(2)^2}{2}+\int_0^1 \ln\left(\frac{(e^{2q}-1)\left(e^{-\frac{2}{q}}-1\right)}{2q}\right)\frac{dq}{q} = 2+\frac{\gamma^2}{2}+T$$

then find the value of T, where $\gamma = 0.57721566\ldots$ Euler's constant.

Proposed by Srinivasa Raghava-AIRMC-India

U.29. Let, for $Re(m) > 0$

$$\omega(m) = \int_{-\infty}^\infty \frac{(\sin(\pi x)+\sin(mx))(\cos(\pi x)+\cos(mx))}{\sinh(mx)}dx$$

then show that

$$\omega\left(\frac{-i\pi^2}{2\pi n + i\pi}\right) = 0, n \in \mathbb{Z}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.30. A note on the special Logarithmic integral

Let's define the function $\chi(m)$ for any complex number $m, Re(m) > 0$

$$\chi(m) = \int_0^1 x \ln(x) \ln(x + m)\, dx$$

then we have the following,

$$\int_0^1 \chi(m) dm = \frac{\pi^2 - 18}{24}; \int_0^1 \chi(m) \ln(m)\, dm = \frac{1}{2}$$

$$\int_0^1 \chi\left(\frac{1}{m}\right) dm = \frac{3\zeta(3)}{4} - 1; \int_0^1 \chi\left(\frac{1}{m}\right) \ln(m)\, dm = 1 - \frac{7\pi^4}{720}$$

$$\int_0^1 \chi\left(\frac{1}{m}\right) \chi(m) dm = \frac{3}{4} + 6\ln(2) - 4\ln(2)^2 + \frac{(\pi^2 - 18)\pi^2}{144} - 2\zeta(3)$$

$$\int_0^\infty \frac{\chi(m)}{1 + m} dm = 2(1 - \zeta(3)); \int_0^\infty \frac{\chi(m)}{1 + m^2} dm = \frac{8\pi(G - 1) - 3\zeta(3)}{16}$$

$$\int_0^\infty \frac{\chi(m)}{(1 + m^2)^2} dm = \frac{\pi(5\pi - 24)}{96}; \int_0^\infty \frac{\chi(m) \ln(m)}{1 + m^2} dm = \frac{17\pi^4}{3840}$$

$$\int_0^\infty \frac{\chi(m) \ln(m)}{(1 + m^2)^2} dm = \frac{2\pi(8 - 8G + \pi \ln(2)) + 9\zeta(3)}{64}$$

where G is Catalan's constant.

Proposed by Srinivasa Raghava-AIRMC-India

U.31. If

$$\beta(m) = \int_{-\pi}^{\pi} \left(\frac{\sin\left(\frac{mx}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right)^6 \cos(x)\, dx$$

then prove that,

$$\sum_{m=0}^{\infty} \frac{\beta(m)}{m!} = \frac{541e\pi}{10} \approx 462$$

Proposed by Srinivasa Raghava-AIRMC-India

U.32. Let, $m > 1$ and

$$\phi(m) = \int_{-\infty}^{\infty} \operatorname{arccot}(my) \operatorname{arccot}\left(\frac{y}{m}\right) dy$$

then

$$\int_0^{\infty} \phi(m)\frac{dm}{1+m^2} = 2\int_0^{\infty} \frac{\phi(m)}{(1+m^2)^2}dm$$

Proposed by Srinivasa Raghava-AIRMC-India

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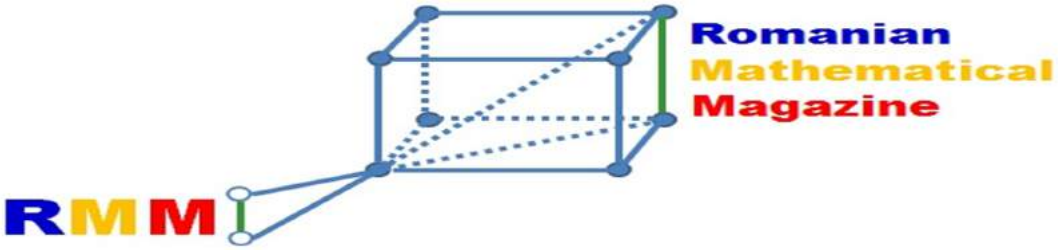

PROBLEMS FOR JUNIORS

JP.106. Let $a,b,c > 0$. Prove that:

$$\frac{8(a+b+c)^3}{(a+b)(b+c)(c+a)} + 5(ab+bc+ca) \geq 12 + 10(a+b+c)$$

Proposed by Nguyen Ngoc Tu – Ha Giang – Vietnam

JP.107. Prove that in any triangle ABC with incentre I the following relationship holds:

$$AI\cdot\frac{a^2}{w_a} + BI\cdot\frac{b^2}{w_b} + CI\cdot\frac{c^2}{w_c} \leq 3\sqrt{2}\frac{R^2}{r}\sqrt{R(R-r)},$$

Proposed by George Apostolopoulos – Messolonghi – Greece

JP.108. If $a,b > 0$ then:

$$4\sqrt{ab}\cdot\frac{\sin x}{x} + b\left(\frac{\tan x}{x}\right)^2 + a > 6\sqrt{ab}, \forall x \in \left(0,\frac{\pi}{2}\right)$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

JP.109. If $a,b > 0$, then:

$$(a+b)\cdot\frac{\sin x}{x} + \frac{2ab}{a+b}\cdot\frac{\tan x}{x} > \frac{6ab}{a+b}, \forall\, x \in \left(0,\frac{\pi}{2}\right)$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

JP.110. If $x,y,z \in (0,1)$ and ABC is a triangle, then prove that:

$$\frac{\sin^2\frac{A}{2}}{x(1-x^3)}+\frac{\sin^2\frac{B}{2}}{y(1-y^3)}+\frac{\sin^2\frac{C}{2}}{z(1-z^3)}\geq\frac{2\sqrt[3]{4}}{3R}(2R-r)$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

JP.111. Prove that:

(i) If $a,b,c,d,x,y,z,t\in R_+^*$, then:

$$4(a^2+b^2+c^2+d^2+x^2+y^2+z^2+t^2)+8\sqrt{(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+t^2)}\geq(a+b+c+d+x+y+z+t)^2;$$

(ii) If $a,b,c,m,n,p,x,y,z\in R_+^*$, then:

$$5(a^3+b^3+c^3+m^3+n^3+p^3+x^3+y^3+z^3)+3\sqrt[3]{(a^3+b^3+c^3)(m^3+n^3+p^3)(x^3+y^3+z^3)}\geq$$

$$\geq 2(a+b+c)(m+n+p)(x+y+z)$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

JP.112. Prove that if $a,b\in R_+,x,y,u,v\in R_+^*$ then:

$$\left(a\cdot\frac{x}{y}+b\cdot\frac{u}{v}\right)^2+\left(a\cdot\frac{y}{x}+b\cdot\frac{v}{u}\right)^2\geq 2(a+b)^2$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

JP.113. Let x,y,z be real numbers such that:

$$x^2y+y^2z+z^2x+4(xy^2+yz^2+zx^2)+13xyz=5$$

Find the minimum value of the expression:

$$P=(x^2+4xy+5y^2)(y^2+4yz+5z^2)(z^2+4zx+5x^2)+6x^2y^2z^2$$

Proposed by Do Quoc Chinh – Vietnam

JP.114. Let a,b,c be positive real numbers. Prove that:

(a) $\ln^3\left(\frac{ab}{c}\right)+\ln^3\left(\frac{bc}{a}\right)+\ln^3\left(\frac{ca}{b}\right)+24\ln a\cdot\ln b\cdot\ln c=\ln^3(abc)$

(b) $\ln(ab)\cdot\ln(bc)\cdot\ln(ca)=\frac{\ln^3(abc)-\ln^3 a-\ln^3 b-\ln^3 c}{3}$

Proposed by George Apostolopoulos – Messolonghi – Greece

JP.115. If a,b,c are positive real number such that $a^2+b^2+c^2=3$ then:

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)\geq 2+\frac{18}{a+b+c}$$

Proposed by Pham Quoc Sang – Ho Chi Minh – Vietnam

JP.116. If a,b,c are positive real number such that $abc=1$ then:

$$\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}\leq\frac{3}{2}$$

Proposed by Pham Quoc Sang – Ho Chi Minh – Vietnam

JP.117. If a,b,c are positive real number then:

$$\frac{ab}{c^2+ca}+\frac{bc}{a^2+ab}+\frac{ca}{b^2+bc}\geq\frac{3}{2}+\frac{3}{2(a+b+c)^2}\cdot\max\{(a-b)^2,(b-c)^2,(c-a)^2\}$$

Proposed by Pham Quoc Sang – Ho Chi Minh – Vietnam

JP.118. Let a,b,c be the three sides of a triangle. Prove that:

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\geq\frac{3(a^3+b^3+c^3)}{a+b+c}$$

Proposed by Nguyen Ngoc Tu – Ha Giang – Vietnam

JP.119. Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{a+b}{c^2(c^3+a+b)}+\frac{b+c}{a^2(a^3+b+c)}+\frac{c+a}{b^2(b^3+c+a)}\geq\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{a^2+b^2+c^2}}{\sqrt[3]{abc}}$$

Proposed by Do Quoc Chinh - Ho Chi Minh – Vietnam

JP.120. Let a,b,c be positive real numbers and $k\in[1;3]$. Prove that:

$$\frac{1}{a^2+ab+ca+kbc}+\frac{1}{b^2+bc+ab+kca}+\frac{1}{c^2+ca+bc+kab}\leq\frac{9}{(k+3)(ab+bc+ca)}$$

Proposed by Do Quoc Chinh – Ho Chi Minh – Vietnam

PROBLEMS FOR SENIORS

SP.106. In $\Delta\,ABC$ the following relationship holds:

$$(a\cot 20^\circ+b\cot 40^\circ+c\cot 80^\circ)^3>9\sqrt{3}\left(\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c}\right)$$

Proposed by Daniel Sitaru – Romania

SP.107. Prove that:

$$\left(\int_0^1\arctan^2 x\,dx\right)\left(\int_0^1\frac{dx}{\arctan^2\left(\frac{1}{x^2-x+1}\right)}\right)>\frac{1}{4}$$

Proposed by Daniel Sitaru – Romania

SP.108. If $a,b,c>0, a+b+c=abc$ then:

$$\frac{4(a+b)(a+c)}{(b+c)^2}+\frac{4(b+c)(b+a)}{(c+a)^2}+\frac{4(c+a)(c+b)}{(a+b)^2}\leq 3+a^2+b^2+c^2$$

Proposed by Daniel Sitaru – Romania

SP.109. If $a,b,c\geq 0; \Omega(a)=\int_0^a \sin\left(\frac{x}{x^2+1}\right)dx$ then:

$$e^{\pi\left(b\Omega(a)+c\Omega(b)+a\Omega(c)\right)}\geq (a^2+1)^b(b^2+1)^c(c^2+1)^a$$

Proposed by Daniel Sitaru – Romania

SP.110. Let $m,x,y,z>0$ be positive real numbers and F be the area of the triangle ABC. Prove that:

$$\frac{a^{2m+2}x^{m+1}}{(y+z)^{m+1}}+\frac{b^{2m+2}y^{m+1}}{(z+x)^{m+1}}+\frac{c^{2m+2}z^{m+1}}{(x+y)^{m+1}}\geq\frac{2^{m+1}}{\left(\sqrt{3}\right)^{m-1}}F^{m+1}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania, Martin Lukarevski-Macedonia

SP.111. Let $x,y,z>0$ be positive real numbers and F the area of the triangle ABC. Prove that:

$$\frac{(y+z)^2a^4}{x^2}+\frac{(z+x)^2b^4}{y^2}+\frac{(z+x)^2c^4}{z^2}\geq 64F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania, Martin Lukarevski-Macedonia

SP.112. Let $x,y,z>0$ be positive real numbers and F be the area of the triangle ABC with circumradius R. Prove that:

$$\frac{x}{y+z}\sin^2\frac{A}{2}+\frac{y}{z+x}\sin^2\frac{B}{2}+\frac{z}{x+y}\sin^2\frac{C}{2}\geq\frac{2\sqrt{3}F}{R^2}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania, Martin Lukarevski-Macedonia

SP.113. Let $x,y,z>0$ be positive real numbers and F be the area of the triangle ABC. Prove that:

$$\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\geq 8\sqrt{3}F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania, Martin Lukarevski-Macedonia

SP.114. Let $t,u,v,x,y,z>0$ be positive real numbers, $t\geq\max\{u.v\}$ and

$S=x+y+z$. Prove that:

$$\frac{tS-uy-vz}{uy+vz}a^2+\frac{tS-uz-vx}{uz+vx}b^2+\frac{tS-ux-vy}{ux+vy}c^2\geq\frac{4(3t-u-v)\sqrt{3}S}{u+v}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania, Martin Lukarevski-Macedonia

SP.115. Let a, b, c be the lengths of the sides of a triangle with inradius r and circumradius R. Let r_a, r_b, r_c be the exradii of triangle. Prove that

$$1728r^5 \leq \frac{a^6}{r_a} + \frac{b^6}{r_b} + \frac{c^6}{r_c} \leq 108R^4(R - r)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

SP.116. A triangle with side lengths a, b, c has perimeter equal to 3. Prove that:

$$a^3 + b^3 + c^3 + a^4 + b^4 + c^4 \geq 2(a^2b^2 + b^2c^2 + c^2a^2)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

SP.117. Let ABC be a triangle with inradius r and circumradius R. Prove that:

a. $\frac{8\sqrt{3}}{3} \leq \frac{1}{\cos^3\frac{A}{2}} + \frac{1}{\cos^3\frac{B}{2}} + \frac{1}{\cos^3\frac{C}{2}} \leq \frac{2\sqrt{3}}{3}\left(\frac{R}{r}\right)^2$

b. $9\left(\frac{r}{R}\right)^2 \leq \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} \leq \frac{9}{4}$

Proposed by George Apostolopoulos – Messolonghi – Greece

SP.118. Let $a, b, c > 0$ sucht that: $a^2 + b^2 + c^2 = 3$. Find the minimum of the expression:

$$P = \frac{a^3}{\sqrt[4]{\frac{b^8+c^8}{2}} + 5bc} + \frac{b^3}{\sqrt[4]{\frac{c^8+a^8}{2}} + 5ca} + \frac{c^3}{\sqrt[4]{\frac{a^8+b^8}{2}} + 5ab} + \frac{(a+b)(b+c)(c+a)}{16}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.119. Let $a, b, c > 0$ such that: $a + b + c = 3$. Find the minimum of the expression:

$$P = \frac{a}{\sqrt[3]{4(b^6 + c^6)} + 7bc} + \frac{b}{\sqrt[3]{4(c^6 + a^6)} + 7ca} + \frac{c}{\sqrt[3]{4(a^6 + b^6)} + 7ab} + \frac{(a+b)(b+c)(c+a)}{24}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.120. In ΔABC the following relationship holds: $\sqrt[3]{a^2B} + \sqrt[3]{b^2C} + \sqrt[3]{c^2A} \leq \sqrt[3]{4\pi s^2}$

Proposed by Daniel Sitaru – Romania

UNDERGRADUATE PROBLEMS

UP.106. In ABC triangle the following relationship holds:

$$\cos\frac{A}{4} + \cos\frac{B}{4} + \cos\frac{C}{4} \leq \frac{\sqrt{3}-1}{4\sqrt{6}}t + \frac{11\sqrt{3}+37}{8\sqrt{6}}, t = \frac{r}{R}.$$

Proposed by Vadim Mitrofanov – Kiev - Ukraine

UP.107. Let ABC be a triangle with inradius r and circumradius R. Prove that

$$2\sqrt{3}\left(\frac{r}{R}\right)^2 \leq \frac{\sum_{cyc}\sin^4 A}{\sum_{cyc}\sin^3 A} \leq \frac{\sqrt{3}}{4}\left(\frac{R}{r}\right)^2\left(1-\frac{r}{R}\right)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

UP.108. Let ABC be a triangle with circumradius R and inradius r. Prove that

$$\frac{3}{16} \leq \cos^4 A + \cos^4 B + \cos^4 C \leq 6\left(\frac{r}{R}\right)^2 - \frac{123}{8}\cdot\frac{r}{R} + \frac{51}{8}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

UP.109. Let a, b, c and d be positive real numbers. Prove or disprove that:

$$\frac{(a+b+c+d)^3}{abc+bcd+cda+dab} \geq 16$$

Proposed by George Apostolopoulos - Messolonghi - Greece

UP.110. Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC. Prove that:

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c} \leq \frac{R}{2r^2}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

UP.111. For an acute triangle ABC and a positive integer n, prove that:

$$\left(\sum(\sin A \sin B \sin C)^{\frac{1}{n}}\right)^n \leq \frac{3^{n+1}}{8}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

UP.112. Solve in positive real numbers:

$$\begin{cases} \frac{x^2}{y}+\frac{y^2}{x} = \sqrt[8]{128(x^8+y^8)} \\ 4x^3-3y = \sqrt{\frac{1+\sqrt{1-xy}}{2}} \end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

UP.113. Let x, y, z be positive real numbers. Prove that

$$\sqrt{\frac{3x^2+yz}{y^2+z^2}}+\sqrt{\frac{3y^2+zx}{z^2+x^2}}+\sqrt{\frac{3z^2+xy}{x^2+y^2}} \geq \sqrt{\frac{103}{6}+\frac{20x^2y^2z^2}{3(x^2+y^2)(y^2+z^2)(z^2+x^2)}}$$

Proposed by Do Quoc Chinh – Vietnam

UP.114. Let a, b, c be the sides and R and r the circumradius and the inradius of a triangle ABC respectively. Prove that:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq\frac{9}{4r(4R+r)}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania, Martin Lukarevski-Macedonia

UP.115. Evaluate

$$\int_0^{\infty}\frac{\ln(x)\sin(x)}{(x)^{\frac{1}{2}}}dx$$

Proposed by Arafat Rahman Akib – Dahka – Bangladesh

UP.116. $\sum_{n=1}^{\infty}\left((-1)^{n-1}\frac{H_n^4+6H_n^2H_n^{(2)}+3\left(H_n^{(2)}\right)^2+8H_nH_n^{(3)}+6H_n^{(4)}}{n}\right)=-24Li_5\left(\frac{1}{2}\right)$

Proposed by Ali Shather - An Nasiriyah - Iraq, Shivam Sharma - New Delhi-India

UP.117. Let $a,b,c\in\left[\frac{1}{2};3\right)$ be positive real numbers such that: $a+b+c=3$.
Prove that:

$$\frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}}{30}+\frac{11}{40}\geq\frac{3(ab+bc+ca-2)}{2\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

UP.118. Show that:

$$\int_0^{\infty}x^p\sqrt{\frac{1+\sqrt{1+x^2}}{1+x^2}}dx=2^{p+\frac{1}{2}}B\left(\frac{p+1}{2},-p-\frac{1}{2}\right);\ -1<p<-\frac{1}{2}$$

Proposed by Shivam Sharma - New Delhi - India

UP.119. Prove that:

$$\sum_{k=1}^{\infty}\left(\frac{H_k^{(2)}}{k^9}\right)=9\zeta(2)\zeta(9)+2\zeta(3)\zeta(8)+6\zeta(4)\zeta(7)+4\zeta(5)\zeta(6)-27\zeta(11)$$

Proposed by Ali Shather - An Nasiriyah - Iraq, Shivam Sharma - New Delhi-India

UP.120. Prove that in any acute – angled triangle ABC the following relationship holds:

$$\sqrt{2}\sum(\sin A+\cos A)>\sum\sin C\,(1+\cos 2(A-B))$$

Proposed by Daniel Sitaru – Romania

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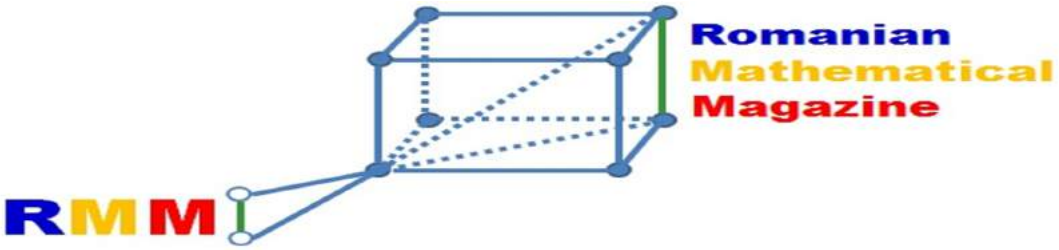

PROBLEMS FOR JUNIORS

JP.121. Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\sum \frac{\sin^3 A + \sin^3 B}{\sin^5 A + \sin^5 B} \leq \left(\frac{R}{r}\right)^2$$

Proposed by George Apostolopoulos – Messolonghi - Greece

JP.122. Prove that in ΔABC the following relationship holds:

$$\min\left(\frac{a}{s-a}, \frac{b}{s-b}, \frac{c}{s-c}\right) \leq 2\left(\frac{R}{r} - 1\right) \leq \max\left(\frac{a}{s-a}, \frac{b}{s-b}, \frac{c}{s-c}\right)$$

Proposed by Marian Ursărescu – Romania

JP.123. Solve for real numbers: $\log_a(b^x + a - b) = \log_b(a^x + b - a)$, $b > a > 1$

Proposed by Marian Ursărescu – Romania

JP.124. Let a, b, c be the lengths of the sides of a triangle ABC with inradius r. Prove that:

$$\frac{(2r)^{a+b+c}}{a^a \cdot b^b \cdot c^c} \leq \left(\tan\frac{A}{2}\right)^b \cdot \left(\tan\frac{B}{2}\right)^c \cdot \left(\tan\frac{C}{2}\right)^a \leq \left(\frac{\sqrt{3}}{36}\right)^{a+b+c} \cdot \frac{a^{2b} \cdot b^{2c} \cdot c^{2a}}{r^{2(a+b+c)}}$$

Proposed by George Apostolopoulos – Messolonghi - Greece

JP.125. Prove that in ΔABC the following relationship holds:

$$\frac{1}{3}(r_a^2 + r_b^2 + r_c^2) \geq 4\sqrt{3}S + (a-b)^2 + (b-c)^2 + (c-a)^2$$

Proposed by Marian Ursărescu – Romania

JP.126. Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\cot A + \cot B + \cot C \geq \sqrt{\frac{3R}{2r}}$$

Proposed by George Apostolopoulos – Messolonghi - Greece

JP.127. In any ΔABC non-equilateral the following relationship holds:

$$\tan\left(\frac{1}{2}m(\sphericalangle OIH)\right) < 3\sqrt{\frac{R+2r}{R-2r}}$$

Proposed by Marian Ursărescu – Romania

JP.128. In ΔABC the following relationship holds:

$$\frac{1}{m_a+m_b}+\frac{1}{m_b+m_c}+\frac{1}{m_c+m_a}\leq\frac{1}{2r}$$

Proposed by Marian Ursărescu – Romania

JP.129. Let a, b and c be the lengths of the sides of a triangle ABC with inradius r and circumradius R. Prove that:

(a) $\frac{a^3}{b^2+c^2}+\frac{b^3}{c^2+a^2}+\frac{c^3}{a^2+b^2}\leq\frac{3\sqrt{3}}{2}\cdot\frac{R(R-r)}{r}$

(b) $\frac{a}{\sqrt{b^2+c^2}}+\frac{b}{\sqrt{c^2+a^2}}+\frac{c}{\sqrt{a^2+b^2}}\leq\sqrt{\frac{6R-3r}{2r}}$

Proposed by George Apostolopoulos – Messolonghi - Greece

JP.130. Solve the equation in real numbers: $3\sqrt[3]{x^2-x+1}+\sqrt[4]{\frac{x^8+1}{2}}=2(x^4-3x+4)$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.131. Solve the system of equations in positive real numbers:

$$\begin{cases}3\left(\sqrt[3]{x^2}+\sqrt[3]{y^2}+\sqrt[3]{z^2}\right)+21=10(xy+yz+zx)\\ x+y+z=3\end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.132. Let $x, y\in\left(0,\frac{\pi}{2}\right)$. Denote $k=2-\min\{\sin^2 x,\cos^2 x\}$. Prove that:

$$\left(\frac{\sin^2 x}{1-\cos y}+\frac{\cos^2 x}{1-\sin y}\right)\left(\frac{\sin^2 x}{1+\cos y}+\frac{\cos^2 x}{1+\sin y}\right)\leq\left(\frac{\pi^2}{4}+\frac{\cos^2 x}{\cos^2 y}\right)^k.$$

Proposed by Mihalcea Andrei Ştefan – Romania

JP.133. Let a, b, c be positive real numbers. Prove that:

$$\sqrt[8]{\frac{a^8+b^8}{2}}+\sqrt[8]{\frac{b^8+c^8}{2}}+\sqrt[8]{\frac{c^8+a^8}{2}}\leq(a+b+c)^{10}\left(\frac{1}{9a}+\frac{1}{9b}+\frac{1}{9c}\right)^9$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.134. Let $a, b, c>0$ such that: $a^2+b^2+c^2=3abc$. Find the maximum of the expression:

$$P = \frac{ab}{2a^6 - a^5 + b^4 + a^2 + 1} + \frac{bc}{2b^6 - b^5 + c^4 + b^2 + 1} + \frac{ca}{2c^6 - c^5 + a^4 + c^2 + 1}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.135. Let x, y, z be positive real numbers such that: $x + y + z = 3$. Prove that:

$$\frac{x^3y^3}{x^4 + y^3 - x + 2} + \frac{y^3z^3}{y^4 + z^3 - y + 2} + \frac{z^3x^3}{z^4 + x^3 - z + 2} \leq \frac{x^4 + y^4 + z^4 + 3xyz}{6}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEMS FOR SENIORS

SP.121. Let x, y, z be positive real numbers such that: $x + y + z + 3 = 3$. Prove that:

$$\frac{x^4}{5 - 3\sqrt[3]{y}} + \frac{y^4}{5 - 3\sqrt[3]{z}} + \frac{z^4}{5 - 3\sqrt[3]{x}} + \frac{\sqrt{x} + \sqrt{y} + \sqrt{z}}{2} \geq 3$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.122. If $z_1, z_2, z_3 \in \mathbb{C}$ are different in pairs and $|z_1| = |z_2| = |z_3| = 1$ then:

$$|z_1 - z_3| + |z_2 - z_3| \leq 3 + |z_1 + z_2|$$

Proposed by Marian Ursărescu – Romania

SP.123. If $A \in M_n(\mathbb{R}), \det A \neq 0, A^T = A$ then: $\det(A^2 + A^{-2} + 2I_n) \geq 4^n$

Proposed by Marian Ursărescu – Romania

SP.124. Let a, b, c be the side lengths of a triangle ABC with inradius r and circumradius R. Prove that:

$$\frac{3}{2} \leq \frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} \leq \frac{2R - r}{2r}$$

Proposed by George Apostolopoulos – Messolonghi - Greece

SP.125. Let triangle ABC have exradii r_a, r_b, r_c altitudes h_a, h_b, h_c and a, b, c be the lengths of the sides. Prove that

$$\left(\frac{h_a}{r_a}\right)^2 + \left(\frac{h_b}{r_b}\right)^2 + \left(\frac{h_c}{r_c}\right)^2 \leq \frac{1}{2}\left(\frac{a^4 + b^4}{c^4} + \frac{b^4 + c^4}{a^4} + \frac{c^4 + a^4}{b^4}\right)$$

Proposed by George Apostolopoulos – Messolonghi - Greece

SP.126. Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC with circumradius R. Prove that:

$$\frac{1}{m_a m_b}+\frac{1}{m_b m_c}+\frac{1}{m_c m_a}+\frac{3}{m_a m_b+m_b m_c+m_c m_a}\geq 4\frac{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}}{m_a+m_b+m_c}$$

Proposed by George Apostolopoulos – Messolonghi - Greece

SP.127. If $A,B\in M_n(\mathbb{R}), n\in\mathbb{N}, n\geq 2, x\in\mathbb{Z}^*, \det(AB-BA)\neq 0, A^2+B^2=2\sin\frac{\pi}{x}AB$ then nx is an even number. ***Proposed by Marian Ursărescu – Romania***

SP.128. Let x,y,z be positive real numbers such that: $x+y+z=3$. Find the minimum of the expression:

$$P=\frac{x^3}{y\left(\sqrt{2(y^4+z^4)}+yz\right)^2}+\frac{y^3}{z\left(\sqrt{2(z^4+x^4)}+zx\right)^2}+\frac{z^3}{x\left(\sqrt{2(x^4+y^4)}+xy\right)^2}+\frac{\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}}{27}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.129. Let a,b,c be positive real numbers such that: $a+b+c=3$. Prove that:

$$\frac{a^3}{b^2\left(\sqrt[4]{\frac{b^8+c^8}{2}}+5bc\right)}+\frac{b^3}{c^2\left(\sqrt[4]{\frac{c^8+a^8}{2}}+5ca\right)}+\frac{c^3}{a^2\left(\sqrt[4]{\frac{a^8+b^8}{2}}+5ab\right)}+\frac{ab+bc+ca}{2(a^2+b^2+c^2)}\geq 1$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.130. Let a,b,c be positive real numbers such that: $a+b+c=3$. Prove that:

$$\frac{a^3}{b^2(b^2+bc+c^2)}+\frac{b^3}{c^2(c^2+ca+a^2)}+\frac{c^3}{a^2(a^2+ab+b^2)}+\frac{ab+bc+ca}{a^2+b^2+c^2}\geq 2$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.131. Let $a,b,c>0$ with sum 1. Prove that $\sum ab^2\geq(\sum ab)^2-10\sum ab+12abc$

Proposed by Mihalcea Andrei Ştefan – Romania

SP.132. Let a,b be two positive numbers. Prove that:

$$\frac{(1+ab)(ab-a-b-1)}{(a^2+1)(b^2+1)}+\frac{4(a^2+b^2+a+b)}{(2+a+b)(a^2+b^2+a+b)-2(a+b)^2}\leq 1$$

Proposed by Mihalcea Andrei Ştefan – Romania

SP.133. Let $a,b,c>0$ with sum 1. Show that $2\sqrt{abc}\sum\frac{a}{1+a^2}\leq\frac{1+\sum a^2}{3+\sum a^2}$

Proposed by Mihalcea Andrei Ştefan – Romania

SP.134. Let $ABCD$ be a cyclic quadrilateral with perimeter 2. Denote $AB=a, BC=b, CD=c, DA=d$. Show that $4\leq\sum\tan\frac{A}{2}<\frac{2(a+c)(b+d)}{\sqrt{\prod(1-a)}}$.

Proposed by Mihalcea Andrei Ştefan – Romania

SP.135. If $a, b, c > 1$ then:

$$\frac{\log_a b}{a+b+c} + \frac{\log_b c}{b+c+d} + \frac{\log_c d}{c+d+a} + \frac{\log_d a}{d+a+b} \geq \frac{16}{3(a+b+c+d)}$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

UNDERGRADUATE PROBLEMS

UP.121. Show that:

$$\int_0^{\infty} x^{p-a}\left(\frac{2}{1+\sqrt{1+4x}}\right)^n dx = \left(\frac{n}{n-p}\right)B(n-2p, p)$$

Proposed by Shivam Sharma – New Delhi – India

UP.122.

$$a_n = \sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{k^2+k+1}\right)$$

Find:

$$\Omega = \lim_{n\to\infty} n\left(a_n^{\frac{\pi}{4}} - \left(\frac{n}{4}\right)^n\right)$$

Proposed by Marian Ursărescu – Romania

UP.123. Find:

$$\Omega = \lim_{n\to\infty} \prod_{k=1}^{n}\left(\frac{\sqrt{n^4+k^2}+k}{\sqrt{n^4+k^2}-k}\right)$$

Proposed by Marian Ursărescu – Romania

UP.124. Find all continuous functions such that:

$$\int_{e^x}^{e^{2x}} f(t)\,dt = \int_1^{e^x} f(t)\,dt$$

Proposed by Marian Ursărescu – Romania

UP.125. Prove that:

$$\frac{4i\pi}{\sqrt{5}} \int_{-1}^{1} \frac{1+x+x^2}{1+x+x^2+x^3+x^4}\,dx = \frac{1+\sqrt{5}}{2} \int_{-\infty}^{\infty} \frac{(1+x+x^2)\ln(x)}{1+x+x^2+x^3+x^4}\,dx$$

Proposed by K. Srinivasa Raghava – AIRMC – India

UP.126. Prove that:

$$\int_0^1 \int_0^1 \frac{\sqrt{x+y}+\sqrt{xy}}{\left(x\sqrt{y}+y\sqrt{x}\right)}\,dy\,dy = \ln\left(17-12\sqrt{2}\right)+\sqrt{2}\ln\left(17+12\sqrt{2}\right)-\frac{4}{3}\left(1-3\sqrt{2}+2\ln(2)\right)$$

Proposed by K. Srinivasa Raghava – AIRMC – India

UP.127. Prove that:

$$\sum_{k=1}^{\infty}\int_0^1 \frac{1}{(\cos(\pi x)^2+ik\sin(\pi x)^2)^3}\,dx = \frac{e^{\frac{3\pi i}{4}}}{8}\left(3\zeta\left(\frac{5}{2}\right)+2i\zeta\left(\frac{3}{2}\right)-3\zeta\left(\frac{1}{2}\right)\right)$$

Proposed by K. Srinivasa Raghava – AIRMC – India

UP.128. Prove that:

$$\int_0^{\pi} \operatorname{arccot}(\cot(x)^5)\cot(x)\,dx = \frac{\pi}{5}\left(4\ln(2)-3\ln(5)+\ln\left(25+11\sqrt{5}\right)\right)$$

Proposed by K. Srinivasa Raghava – AIRMC – India

UP.129. Let R_+^* be the set of real positive numbers, let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be two sequences of real positive numbers with $\lim_{n\to\infty}(a_{n+1}-a_n) = a \in R_+^*$, $\lim_{n\to\infty}(b_{n+1}-b_n) = b \in R_+^*$, let $P_n = \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}$ and we denote $P_n! = P_1P_2 \ldots P_n$, for any positive integer n. If $u, v \in R$ with $u+v=1$. Evaluate:

$$\lim_{n\to\infty}\left(b_{n+1}^u\sqrt[n+1]{(P_{n+1}!)^v}-b_n^u\sqrt[n]{(P_n!)^v}\right)$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

UP.130. If $a, b, c, d \in [1,\infty)$; $x \in \mathbb{R}$ then:

$$(ac)^{\sin x}\cdot(bd)^{\cos x} \leq 2^{\sqrt{2\left(\log_2^2 a+\log_2^2 b+\log_2^2 c+\log_2^2 d\right)}}$$

Proposed by Daniel Sitaru – Romania

UP.131. Prove that in any triangle ABC the following relationship holds:

$$2^{\sin A}+2^{\sin B}+2^{\sin C}+2^{\cos A}+2^{\cos B}+2^{\cos C} > \frac{6}{\left(\sqrt{2}\right)^{\sqrt{2}}}$$

Proposed by Daniel Sitaru – Romania

UP.132. If $a, b > 0$; $m, n \geq 1$ then:

$$\int_0^b \left(\int_0^a (\sin x \sin y)^{2n} (\cos x \cos y)^{2m} dx \right) dy \leq \frac{ab}{4^{m+n}} \left(\frac{m}{n}\right)^{m-n}$$

Proposed by Daniel Sitaru – Romania

UP.133. Prove that if: $0 \leq b \leq a \leq \frac{\pi}{4}$ then:

$$\int_b^a \left(\int_b^a \left(\frac{\sin^2(x+y) + \sin^2(x-y) - 1}{1 + 2\sin x \cos y} \right) dx \right) dy \geq (a-b)(\sin^2 a - \sin^2 b + b - a)$$

Proposed by Daniel Sitaru – Romania

UP.134. In $\Delta\, ABC$ the following relationship holds:

$$\frac{a^2(m_a + m_b)}{h_c} + \frac{b^2(m_b + m_c)}{h_a} + \frac{c^2(m_c + m_a)}{h_b} \geq 8\sqrt{3}S$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

UP.135. In $a, b, c > 0$ then:

$$\frac{a + c^2}{xb + yc} + \frac{b + a^2}{zc + ya} + \frac{c + b^2}{xa + yb} \geq \frac{3}{x + y} + \frac{a + b + c}{x + y}$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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NOTĂ: Pentru a publica probleme propuse, articole și note matematice în RMM puteți trimite materialele pe mailul: dansitaru63@yahoo.com
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