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Find:

$$\Omega=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(\tan^{-1}\left(\frac{3}{k^2-k-1}\right)\tan^{-1}\left(\frac{2}{8(n-k+1)^2-4n+4k-5}\right)\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Naren Bhandari-Bajura-Nepal

$$\Omega=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(\tan^{-1}\left(\frac{3}{k^2-k-1}\right)\tan^{-1}\left(\frac{2}{8(n-k+1)^2-4n+4k-5}\right)\right)\right)$$

We can observe that for $k=n-k+1$ *the sum becomes*

$$\Omega=\sum_{k=1}^{\infty}\left(\sum_{k=n-k+1}^{n}\left(\tan^{-1}\left(\frac{3}{k^2-k-1}\right)\tan^{-1}\left(\frac{2}{8k^2-4k-1}\right)\right)\right)$$

which further can be decomposed into two infinite arctangent sum le.

$$\Omega=\left(\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{1}{k^2-k-1}\right)\right)\left(\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{2}{8k^2-4k-1}\right)\right)$$

Now, note that:

$$\lim_{M\to\infty}\sum_{k=1}^{M}\tan^{-1}\left(\frac{3}{k^2-k-1}\right)=\lim_{M\to\infty}\sum_{k=1}^{M}\tan^{-1}\left(\frac{(k+1)+(2-k)}{1+(2-k)(k+1)}\right)$$

$$= \lim_{M\to\infty}\sum_{k=1}^{M}\left(\tan^{-1}(k+1)+\tan^{-1}(2-k)\right)$$

Since sum is telescoping sum and gives us the partial sum as

$$\Omega_1 = \lim_{M\to\infty}\left(\tan^{-1}(M+4)+\tan^{-1}(M+5)+\tan^{-1}(M+6)\right)-\tan^{-1}(0) = \frac{3\pi}{2}$$

as $k+1+2-k=3$

Since $8k^2-4k-1$ ***cannot be factored into two linear factors so that sum becomes telescoping to make it multiple and divide by any number now***

$$\Omega_2 = \sum_{k=1}^{\infty}\tan^{-1}\left(\frac{2}{8k^2-4k-1}\right)$$

$$= \lim_{N\to\infty}\sum_{k=1}^{N}\tan^{-1}\left(\frac{10}{40k^2-20k-5}\right)$$

As

$$40k^2-20k-5 = 4(k+1)+(6k+1)(6k-5)$$

$$10 = \frac{6k+1}{2(k+1)}-\frac{6k-5}{2k}$$

thus

$$\Omega_2 = \sum_{k=1}^{N}\tan^{-1}\left(\frac{\frac{6k+1}{2(k+1)}-\frac{6k-5}{2k}}{1+\frac{(6k+1)(6k-5)}{4k(k+1)}}\right)$$

$$= \sum_{k=1}^{N}\left(\tan^{-1}\left(\frac{6k+1}{2(k+1)}\right)-\tan^{-1}\left(\frac{6k-5}{2k}\right)\right)$$

Observe that Ω_2 ***is telescoping sum giving us the partial sum as***

$$\Omega_2 = \lim_{N\to\infty}\left(\left(\tan^{-1}\left(\frac{6N+1}{2(N+1)}\right)\right)-\tan^{-1}\left(\frac{1}{2}\right)\right)$$

$$= \tan^{-1}(3)-\tan^{-1}\left(\frac{1}{2}\right)$$

Therefore,

$$\Omega = \frac{\pi}{2}\left(\tan^{-1}(3)-\tan^{-1}\left(\frac{1}{2}\right)\right) = \frac{\pi}{2}\tan^{-1}(1) = \frac{3\pi^2}{8}$$

Note: The principle branch

$-\frac{\pi}{2} \le \tan^{-1} x \le \frac{\pi}{2}$ ***thus***

$-\pi \le \tan^{-1} x + \tan^{-1} y \le \pi$ ***. As***

$\Omega_1 = \frac{3\pi}{2} > \pi$***. Therefore,***

$\Omega_1 = \frac{3\pi}{2} - \pi = \frac{\pi}{2}$***. So, the answer is*** $\frac{\pi^2}{8}$