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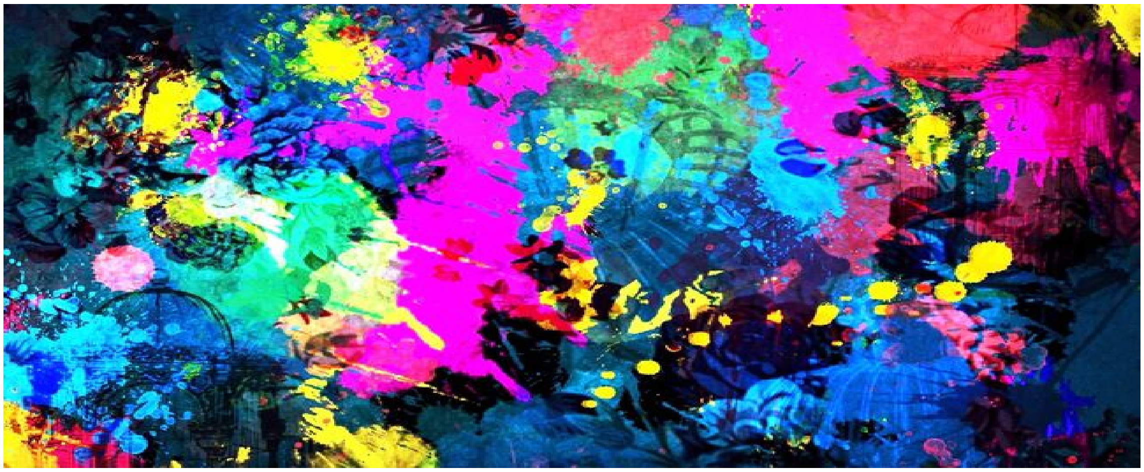

Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{1\le i<j<k\le n}\frac{i\cdot j\cdot k}{(1+i^2)(1+j^2)(1+k^2)}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Remus Florin Stanca-Romania, Solution 2 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Remus Florin Stanca-Romania

Let $a_k=\frac{k}{k^2+1}\Rightarrow\Omega=\lim_{n\to\infty}\left(\frac{1}{n^3}\cdot\sum_{1\le i<j<k\le n}\frac{1}{n^3}a_ia_ja_k\right)=$

$$=\lim_{n\to\infty}\frac{a_1(a_2(a_3+\cdots+a_n)+\cdots+a_{n-1}a_n)+\cdots+a_{n-2}(a_{n-2}\cdot a_n)}{n^3}\overset{Stolz\ Cesaro}{=}$$

$$=\lim_{n\to\infty}\frac{a_1a_{n+1}(a_2+\cdots+a_n)+\cdots+a_{n-1}a_{n+1}a_n}{(n+1)^3-n^3}=$$

$$=\lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{a_{n+1}(a_1(a_2+\cdots+a_n)+\cdots+a_{n-1}a_n)}{\left(\frac{n+1}{n}-1\right)\left(\left(\frac{n+1}{n}\right)^0+\left(\frac{n+1}{n}\right)^1+\left(\frac{n+1}{n}\right)^2\right)}=$$

$$=\frac{1}{3}\cdot\lim_{n\to\infty}\frac{a_{n+1}(a_1(a_2+\cdots+a_n)+\cdots+a_{n-1}a_n)}{n^2}=$$

$$=\frac{1}{3}\lim_{n\to\infty}\frac{\frac{n+1}{(n+1)^2+1}(a_1(a_2+\cdots+a_n)+\cdots+a_{n-1}a_n)}{n^2}=$$

$$= \frac{1}{3}\lim_{n\to\infty} \frac{a_1(a_2+\cdots+a_n)+\cdots+a_{n-1}a_n}{n(n^2+2n+2)} \overset{Stolz\ Cesaro}{=}$$

$$= \frac{1}{3}\lim_{n\to\infty} \frac{a_{n+1}(a_1+\cdots+a_n)}{2+2(2n+1)+n^2+2n+1+n^2+n^2+n} =$$

$$= \frac{1}{3}\lim_{n\to\infty} \frac{\frac{n+1}{(n+1)^2+1}(a_1+\cdots+a_n)}{n\left(3n+7+\frac{5}{n}\right)} = \frac{1}{3}\cdot\lim_{n\to\infty} \frac{a_1+\cdots+a_n}{(n^2+2n+2)\left(3n+7+\frac{5}{n}\right)} =$$

$$= \frac{1}{3}\lim_{n\to\infty} \frac{a_1+\cdots+a_n}{3n^3+7n^2+5n+6n^2+14n+10+6n+14+\frac{10}{n}} \overset{Stolz\ Cesaro}{=}$$

$$= \frac{1}{3}\lim_{n\to\infty} \frac{\frac{n+1}{(n+1)^2+1}}{3\left((n+1)^2+n^2+n(n+1)\right)+7(2n+1)+5+6(2n+1)+14+6-\frac{10}{n(n+1)}} =$$

$$= \frac{0}{\infty} = 0 \Rightarrow \Omega = 0$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

For $i,j,k \geq 1$ we have:

$$(1+i^2)(1+j^2)(1+k^2) \overset{AM-GM}{\geq} 2i\cdot 2j\cdot 2k = 8ijk$$

$$\Rightarrow \frac{ijk}{(1+i^2)(1+j^2)(1+k^2)} \leq \frac{1}{8}$$

$$\Rightarrow 0 < \Omega_n = \frac{1}{n^3}\sum_{1\leq i<j<k\leq n} \frac{ijk}{(1+i^2)(1+j^2)(1+k^2)} \leq \frac{1}{n^3}\sum_{1\leq i<j<k\leq n}\frac{1}{8} \leq \frac{1}{n^3}\cdot\frac{1}{8}\cdot n = \frac{1}{8n^2} \to 0\ (n\to\infty)$$

$$\Rightarrow \Omega = \lim_{n\to\infty}\Omega_n = 0$$