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In $\Delta ABC, u=\cot A\cot B, v=\cot B\cot C, w=\cot C\cot A, x_n, y_n, z_n>0$

$$n\in\mathbb{N}, n\ge 1, \lim_{n\to\infty}\frac{x_{n+1}}{nx_n}=x, \lim_{n\to\infty}\frac{y_{n+1}}{ny_n}=y, \lim_{n\to\infty}\frac{z_{n+1}}{nz_n}=z, x,y,z>0$$

Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{x_{n+1}^u\cdot y_{n+1}^v\cdot z_{n+1}^w}-\sqrt[n]{x_n^u\cdot y_n^v\cdot z_n^w}\right)$$

Proposed by D.M.Batinetu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Marian Ursărescu-Romania, Solution 2 by Remus Florin Stanca-Romania

Solution 1 by Marian Ursărescu-Romania

In $\Delta ABC, \sum\cot A\cot B=1\Rightarrow u+v+w=1$

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[n]{x_n^u\cdot y_n^v\cdot z_n^w}}{n}\cdot n\left[\frac{\sqrt[n+1]{x_{n+1}^u\cdot y_{n+1}^v\cdot z_{n+1}^w}}{\sqrt[n]{x_n^u\cdot y_n^v\cdot z_n^w}}-1\right]\quad (1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{x_n^u\cdot y_n^v\cdot z_n^w}}{n}=\lim_{n\to\infty}\frac{\sqrt[n]{x_n^u}}{n^u}\cdot\frac{\sqrt[n]{y_n^v}}{n^v}\cdot\frac{\sqrt[n]{z_n^w}}{n^w}\quad (2)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{x_n^u}}{n^u}=\lim_{n\to\infty}\sqrt[n]{\frac{x_n^u}{n^{nu}}}=\left(\lim_{n\to\infty}\sqrt[n]{\frac{x_n}{n^n}}\right)^u\overset{C.D.}{=}$$

$$=\left(\lim_{n\to\infty}\frac{x_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{x_n}\right)^u=\left(\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\frac{n}{n+1}\cdot\frac{x_{n+1}}{n\times n}\right)^u=\left(\frac{x}{e}\right)^u\quad (3)$$

Similarly, $\lim_{n\to\infty}\frac{\sqrt[n]{y_n^v}}{n^v}=\left(\frac{y}{e}\right)^v$ *and* $\lim_{n\to\infty}\frac{\sqrt[n]{z_n^w}}{n^w}=\left(\frac{z}{e}\right)^w\quad (4)$

From (2)+(3)+(4)$\Rightarrow \lim_{n\to\infty}\frac{\sqrt[n]{x_n^u\cdot y_n^v\cdot z_n^w}}{n}=\frac{x^u\cdot y^v\cdot z^w}{e}$ *(5)*

Let $\alpha_n=\frac{\sqrt[n+1]{x_{n+1}^u\cdot y_{n+1}^v\cdot z_{n+1}^w}}{\sqrt[n]{x_n^u\cdot y_n^v\cdot z_n^w}}\Rightarrow$

$$\lim_{n\to\infty}n(\alpha_n-1)=\lim_{n\to\infty}n\left(e^{\ln\alpha_n}-1\right)=\lim_{n\to\infty}\frac{\left(e^{\ln\alpha_n}-1\right)}{\alpha_n}\cdot\ln\alpha_n$$

$$=\lim_{n\to\infty}n\ln\alpha_n=\lim_{n\to\infty}\ln\alpha_n^u=\ln\left(\lim_{n\to\infty}\alpha_n^u\right)=$$

$$=\ln\left(\lim_{n\to\infty}\frac{\left(\sqrt[n+1]{x_{n+1}^u\cdot y_{n+1}^v\cdot z_{n+1}^w}\right)^n}{x_n^u\cdot y_n^v\cdot z_n^w}\right)=$$

$$=\ln\left(\lim_{n\to\infty}\frac{x_{n+1}^u\cdot y_{n+1}^v\cdot z_{n+1}^w}{x_n^u\cdot y_n^v\cdot z_n^w}\cdot\frac{1}{\sqrt[n+1]{x_{n+1}^u\cdot y_{n+1}^v\cdot z_{n+1}^w}}\right)=$$

$$=\ln\left(\lim_{n\to\infty}\left(\frac{x_{n+1}}{nx_n}\right)^n\cdot\left(\frac{y_{n+1}}{ny_n}\right)^v\cdot\left(\frac{z_{n+1}}{nz_n}\right)^w\cdot\frac{n}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{x_{n+1}^n\cdot y_{n+1}^v\cdot z_{n+1}^w}}\right)$$

$$=\ln\left(x^u\cdot y^v\cdot z^w\cdot\frac{1}{\sqrt[n]{x_n^u\cdot y_n^v\cdot z_n^w}}\right)=\ln\left(x^u\cdot y^v\cdot z^w\cdot\frac{e}{x^u\cdot y^c\cdot z^w}\right)=$$

$$=\ln e=1\quad\textit{(6)}$$

From (1)+(5)+(6)$\Rightarrow\Omega=\frac{x^u\cdot y^v\cdot z^w}{e}$

Solution 2 by Remus Florin Stanca-Romania

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\cdot\tan B}=\tan(\pi-C)=-\frac{\tan\pi-\tan C}{1+\tan\pi\cdot\tan C}=-\tan C\Rightarrow$$

$\Rightarrow\tan A+\tan B=-\tan C+\tan A\cdot\tan B\cdot\tan C\Rightarrow\tan A\cdot\tan B\cdot\tan C=\tan A+\tan B+\tan C$ *(1)*

$$u=\cot A\cdot\cot B=\frac{1}{\tan A\cdot\tan B}\overset{(1)}{=}\frac{\tan C}{\tan B+\tan B+\tan C}\Rightarrow$$

$\Rightarrow v=\frac{\tan A}{\tan B+\tan C+\tan A}$ *and* $w=\frac{\tan B}{\tan A+\tan B+\tan C}\overset{adding}{\Rightarrow}u+v+w=\frac{\tan A+\tan B+\tan C}{\tan A+\tan B+\tan C}=1$

$$\Rightarrow u+v+w=1\quad\textit{(2)}$$

$$\Omega=\lim_{n\to\infty}\sqrt[n]{x_n^uy_n^vz_n^w}\left(\frac{\sqrt[n+1]{x_{n+1}^u\cdot y_{n+1}^v\cdot z_{n+1}^w}}{\sqrt[n]{x_n^uy_n^vz_n^w}}-1\right)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{x_n}{n^n}}=\lim_{n\to\infty}e^{\frac{\ln\frac{x_n}{n^n}}{n}}\overset{Stolz\ Cesaro}{=}\lim_{n\to\infty}\frac{x_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{x_n}=\frac{1}{e}\cdot\lim_{n\to\infty}\frac{x_{n+1}}{nx_n}=\frac{x}{e}$$

$\Rightarrow \lim_{n\to\infty} \frac{\sqrt[n]{y_n}}{n} = \frac{y}{e}$ *and* $\lim_{n\to\infty} \frac{\sqrt[n]{z_n}}{n} = \frac{z}{e} \Rightarrow \lim_{n\to\infty} \frac{\sqrt[n]{x_n^u}}{n^u} = \left(\frac{x}{e}\right)^u$; $\lim_{n\to\infty} \frac{\sqrt[n]{y_n^v}}{n^v} = \left(\frac{y}{e}\right)^v$ *and*

$$\lim_{n\to\infty} \frac{\sqrt[n]{z_n^w}}{n^w} = \left(\frac{z}{e}\right)^w$$

$$\Omega = \lim_{n\to\infty} \frac{\sqrt[n]{x_n^u}}{n^u} \cdot \frac{\sqrt[n]{y_n^v}}{n^v} \cdot \frac{\sqrt[n]{z_n^w}}{n^w} \cdot n^{u+v+w} \cdot \left(\frac{\sqrt[n+1]{x_{n+1}^u \cdot y_{n+1}^v \cdot z_{n+1}^w}}{\sqrt[n]{x_n^u \cdot y_n^v \cdot z_n^w}} - 1\right) \stackrel{(2)}{=} \frac{x^u \cdot y^v \cdot z^w}{e} \cdot$$

$$\cdot \lim_{n\to\infty} n \cdot \left(\frac{\sqrt[n+1]{x_{n+1}^u \cdot y_{n+1}^v \cdot z_{n+1}^w}}{\sqrt[n]{x_n^u \cdot y_n^v \cdot z_n^w}} - 1\right) = \frac{x^u \cdot y^v \cdot z^w}{e} \cdot \lim_{n\to\infty} n \cdot \ln\left(\frac{\sqrt[n+1]{x_{n+1}^u \cdot y_{n+1}^v \cdot z_{n+1}^w}}{\sqrt[n]{x_n^u \cdot y_n^v \cdot z_n^w}}\right)$$

$$=$$

$$= \frac{x^u \cdot y^v \cdot z^w}{e} \cdot \lim_{n\to\infty} \ln\left(\left(\frac{x_{n+1}}{x_n}\right)^u \cdot \left(\frac{y_{n+1}}{y_n}\right)^v \cdot \left(\frac{z_{n+1}}{z_n}\right)^w \cdot \frac{1}{\sqrt[n+1]{x_{n+1}^u \cdot y_{n+1}^v \cdot z_{n+1}^w}}\right) =$$

$$= \frac{x^u \cdot y^v \cdot z^w}{e} \cdot \lim_{n\to\infty} \ln\left(\left(\frac{x_{n+1}}{nx_n}\right)^u \cdot \left(\frac{y_{n+1}}{ny_n}\right)^v \cdot \left(\frac{z_{n+1}}{nz_n}\right)^w \cdot \left(\frac{n}{\sqrt[n]{x_n}}\right)^u \cdot \left(\frac{n}{\sqrt[n]{y_n}}\right)^v \cdot \left(\frac{n}{\sqrt[n]{z_n}}\right)^w\right) =$$

$$= \frac{x^u \cdot y^v \cdot z^w}{e} \cdot \ln\left(x^u \cdot y^v \cdot z^w \cdot \frac{e}{x^u \cdot y^v \cdot z^w}\right) = \frac{x^u \cdot y^v \cdot z^w}{e} \Rightarrow \Omega = \frac{x^u \cdot y^v \cdot z^w}{e}$$