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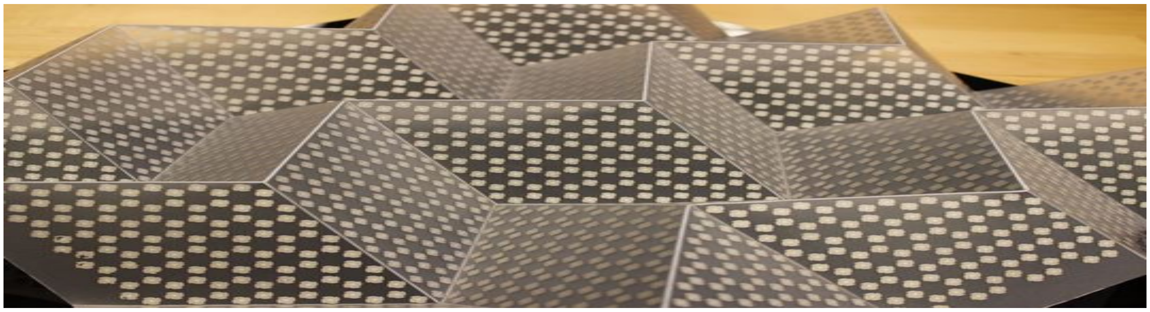

APPLICATIONS OF TAREK'S LEMMA

In $\Delta ABC, n_a, n_b, n_c$ – Nagel's cevians. Prove that:

$$\frac{n_a^2}{bc}+\frac{n_b^2}{ca}+\frac{n_c^2}{ab}\ge\frac{s^2+5r^2+2Rr}{8Rr}$$

$$\frac{n_a^2}{h_a}+\frac{n_b^2}{h_b}+\frac{n_c^2}{h_c}\ge\frac{s^2+5r^2+2Rr}{4r}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam, Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Tran Hong-Dong Thap-Vietnam

We have:

$n_a\ge m_a$ **(etc)** $\Rightarrow\sum\frac{n_a^2}{bc}\ge\sum\frac{m_a^2}{bc}$

Now, we prove: $\sum\frac{m_a^2}{bc}=\frac{s^2+2Rr+5r^2}{8Rr}$

In fact:

Since, $\sum am_a^2=\sum a\left(\frac{1}{4}b^2+\frac{1}{4}c^2+\frac{1}{2}bc\cos A\right)$

$$=\frac{1}{4}\sum(ab^2+a^2b)+\frac{1}{2}abc\sum\cos A$$

and: $\because(ab^2+a^2b)=(\sum a)(\sum bc)-3abc=2s(s^2-2Rr+r^2)$

$$\because abc=4Rrs;\sum\cos A=\frac{R+r}{R}$$

We get: $\sum am_a^2=\frac{s}{2}(s^2+2Rr+5r^2)$, *i.e.* $\sum\frac{m_a^2}{bc}=\frac{s^2+2Rr+5r^2}{8Rr}$

Proved.

We have:

$$\sum\frac{n_a^2}{h_a}=\frac{1}{2\Delta}\sum an_a^2\geq\frac{1}{2sr}\sum am_a^2=\frac{1}{2sr}\cdot\frac{s}{2}(s^2+2Rr+5r^2)$$

$$=\frac{s^2+2Rr+r^2}{4r}.\ \textit{Proved.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum\frac{n_a^2}{bc}\overset{(a)}{\geq}\frac{s^2+5r^2+2Rr}{8Rr}$$

$$\sum\frac{n_a^2}{h_a}\overset{(b)}{\geq}\frac{s^2+5r^2+2Rr}{4r}$$

Let n_a ***intersect*** BC ***at*** D

Then, $BD=s-c$ ***&*** $CD=s-b$

By Stewart's theorem,

$$b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow an_a^2\overset{(1)}{=}b^2(s-c)+c^2(s-b)-a(s-b)(s-c)$$

Similarly, $bn_b^2\overset{(2)}{=}c^2(s-a)+a^2(s-c)-b(s-c)(s-a)$ ***&***

$$cn_c^2\overset{(3)}{=}a^2(s-b)+b^2(s-a)-c(s-a)(s-b)$$

(1)+(2)+(3)$\Rightarrow\sum an_a^2=\sum b^2(s-c)+\sum c^2(s-b)-\sum a(s-b)(s-c)$

$$=s\sum b^2+s\sum c^2-\sum\left(b^2c+\sum bc^2\right)-\sum(as^2-s(b+c)+bc)$$

$$=4s(s^2-4Rr-r^2)-\sum bc(2s-a)-s^2(2s)+2s\sum ab-3abc$$

$$=4s(s^2-4Rr-r^2)-2s^3-2s\sum ab+2s\sum ab+3abc-3abc$$

$$=2s^3-16Rrs-4sr^2\Rightarrow\sum an_a^2\overset{(4)}{=}2s(s^2-8Rr-2r^2)$$

Now, (a)$\Leftrightarrow\frac{\sum an_a^2}{4Rrs}\geq\frac{s^2+2Rr+5r^2}{8Rr}\Leftrightarrow\frac{s^2-8Rr-2r^2}{2Rr}\geq\frac{s^2+2Rr+5r^2}{8Rr}$ ***(using (4))***

$$\Leftrightarrow 4s^2-32Rr-8r^2\geq s^2+2Rr+5r^2\Leftrightarrow 3s^2-34Rr-13r^2\geq 0$$

$$\Leftrightarrow(3s^2-48Rr+15r^2)+14r(R-2r)\geq 0$$

$\rightarrow$ ***true*** $\because 3s^2-48Rr+15r^2\overset{Gerretsen}{\geq}0$ ***&*** $R-2r\overset{Euler}{\geq}0\Rightarrow$ ***(a) is true***

Again, (b) $\Leftrightarrow \sum \frac{an_a^2}{ah_a} \geq \frac{s^2+2Rr+5r^2}{4r}$

$$\Leftrightarrow \sum \frac{an_a^2}{a\left(\frac{2rs}{a}\right)} \geq \frac{s^2+2Rr+5r^2}{4r} \Leftrightarrow 2\sum an_a^2 \geq s^3+2Rrs+5sr^2$$

$\Leftrightarrow 4s(s^2-8Rr-2r^2) \geq s(s^2+2Rr+5r^2)$ ***(using (4))***

$$\Leftrightarrow 3s^2-34Rr-13r^2 \geq 0 \Leftrightarrow (3s^2-48Rr+15r^2)+14r(R-2r) \geq 0$$

$\rightarrow$ ***true*** $\because 3s^2-48Rr+15r^2 \overset{Gerretsen}{\geq} 0$ ***&***

$R-2r \overset{Euler}{\geq} 0 \Rightarrow$ ***(b) is true.***

(Proved)