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In $\Delta ABC, n_a, n_b, n_c$ are Nagel's cevians. Prove that:

$$n_a \geq \sqrt{s(s-a)}$$

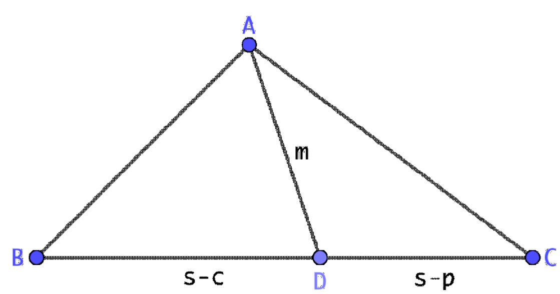
$$n_a^2 + n_b^2 + n_c^2 \geq s^2$$

When equality holds?

Proposed by Mustafa Tarek-Cairo-Egypt

Solution 1 by Marian Ursărescu-Romania, Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Marian Ursărescu-Romania



From Stewart's relation we have:

$$c^2(s-b) + b^2(s-c) = AD^2 \cdot a + (s-b)(s-c)a \quad \textbf{(1)}$$

$$s - a = x$$

Let $s - b = y \Rightarrow s = x + y + z$

$$s - c = z$$

and $a = y + z, b = x + z, c = x + y$ **(2)**

From (1)+(2)$\Rightarrow (x+y)^2y+(x+z)^2z=AD^2\cdot(y+z)+yz(y+z)\Rightarrow$

$$(x+y)^2y+(x+z)^2z=AD^2(y+z)+y^2z+yz^2\Rightarrow$$

$$AD^2(y+z)=y[(x+y)^2-z^2]+z[(x+z)^2-y^2]\Rightarrow$$

$$AD^2=\frac{y(x+y+z)(x+y-z)+z(x+y+z)(x+z-y)}{y+z}\Rightarrow$$

$$\left.\begin{array}{c}\dfrac{(x+y+z)(xy+y^2-yz+xz+z^2-yz)}{y+z}\\ \text{we must show } AD^2\geq(x+y+z)x\end{array}\right\}\Rightarrow$$

$$\frac{(x+y+z)(xy+xz+y^2+z^2-2yz)}{y+z}\geq(x+y+z)x\Leftrightarrow$$

$xy+xz+y^2+z^2-2yz\geq xy+xz\Leftrightarrow(y-z)^2\geq 0$, **equality when** $b=c$

$n_a^2+n_b^2+n_c^2\geq s(s-a+s-b+s-c)=s^2$, **equality when** $a=b=c$.

Solution 2 by Soumava Chakraborty-Kolkata-India

Let $n_a=x$ **intersect** BC **at** D, $BD=s-c$, $CD=s-b$. **By Stewart's theorem:**

$$b^2(s-c)+c^2(s-b)=ax^2+a(s-b)(s-c)$$

$$2b^2(a+b-c)+2c^2(c+a-b)=4ax^2+a(c+a-b)(a+b-c)$$

$$4ax^2=2b^2(a+b-c)+2c^2(c+a-b)-a(c+a-b)(a+b-c)\geq 4as(s-a)$$

$$2b^2(a+b-c)+2c^2(c+a-b)-a(c+a-b)(a+b-c)-$$

$$-a(a+b+c)(b+c-a)\geq 0$$

$$b^2+c^2-bc(b+c)+a(b-c)^2\geq 0$$

$$(a+b+c)(b-c)^2\geq 0$$

$$4ax^2\geq 4as(s-a)$$

$$n_a\geq\sqrt{s(s-a)}$$

$$n_a^2\geq s(s-a), n_b^2\geq s(s-b), n_c^2\geq s(s-c)$$

$$n_a^2+n_b^2+n_c^2\geq s(s-a+s-b+s-c)=s^2$$