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Solve for real numbers:

$$\log_{\cos^{-1} x}(\sin^{-1} x)\cdot\log(1+\cos^{-1} x)=\log_{\sin^{-1} x}(\cos^{-1} x)\cdot\log(1+\sin^{-1} x)$$

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$0<\sin^{-1} x,\cos^{-1} x\neq 1$, ***then we have:***

$$\log_{\cos^{-1} x}(\sin^{-1} x)\cdot\log(1+\cos^{-1} x)=\log_{\sin^{-1} x}(\cos^{-1} x)\cdot\log(1+\sin^{-1} x)$$

$$\Leftrightarrow \frac{\log(\sin^{-1} x)}{\log(\cos^{-1} x)}\log(1+\cos^{-1} x)=\frac{\log(\cos^{-1} x)}{\log(\sin^{-1} x)}\cdot\log(1+\sin^{-1} x)$$

$$\Leftrightarrow \frac{\log(1+\cos^{-1} x)}{\log^2(\cos^{-1} x)}=\frac{\log(1+\sin^{-1} x)}{\log^2(\sin^{-1} x)}\quad (*)$$

Let $f(t)=\frac{\log(1+t)}{\log^2 t};\ 0<t\neq 1\Rightarrow f'(t)=\frac{\frac{1}{1+t}\cdot\log t-2\frac{1}{t}\log(1+t)}{\log^3 t}$

$f'(t)>0\Leftrightarrow 0<t<1\Rightarrow f(t)\nearrow$ ***on*** $(0,1)$

$f'(t)<0\Leftrightarrow t>1\Rightarrow f(t)\searrow$ ***on*** $(1;+\infty)$

$$\overset{(*)}{\Rightarrow} f(\cos^{-1} x)=f(\sin^{-1} x)\Leftrightarrow \cos^{-1} x=\sin^{-1} x\Leftrightarrow \sin(\cos^{-1} x)=\sin(\sin^{-1} x)$$

$$\Leftrightarrow x=\sqrt{1-x^2}\Leftrightarrow x^2=\frac{1}{2}\overset{1\neq x>0}{\Leftrightarrow} x=\frac{\sqrt{2}}{2}$$