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If $0 < x, y, z < \frac{\pi}{6}$ then:

$$(\sin^2 x)^{\sin\left(\frac{y+z}{2}\right)\cos\left(\frac{y-z}{2}\right)} + (\sin^2 y)^{\sin\left(\frac{z+x}{2}\right)\cos\left(\frac{z-x}{2}\right)} + (\sin^2 z)^{\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} > 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Avishek Mitra-West Bengal-India, Solution 2 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Avishek Mitra-West Bengal-India

$$\Rightarrow (\sin^2 x)^{\sin\left(\frac{y+z}{2}\right)\cdot\cos\left(\frac{y-z}{2}\right)} = (\sin x)^{(\sin y + \sin z)}$$

Need to prove $\Rightarrow \sum_{cyc} (\sin x)^{(\sin y + \sin z)} > 1$

$$\Leftrightarrow (1 + \sin x - 1)^{(\sin y + \sin z)} \overset{Bernoulli}{>} 1 + (\sin x - 1)(\sin y + \sin z)$$

$$= 1 + \sin x \cdot \sin y + \sin x \cdot \sin y - \sin z$$

$$\Rightarrow \sum_{cyc} (\sin x)^{(\sin y + \sin z)} > 3 + 2\sum_{cyc} \sin x \sin y - 2(\sin x + \sin y + \sin z)$$

$$> 3 + 2\sum_{cyc} \sin x (\sin y - 1) > 3 - 2\sum_{cyc} \sin x \left(\cos\frac{y}{2} - \sin\frac{y}{2}\right)^2$$

$$\because x < \frac{\pi}{6} \Rightarrow \sin x < \frac{1}{2} \Rightarrow \left(\cos\frac{y}{2} - \sin\frac{y}{2}\right)^2 < \frac{1}{2}$$

$$\Leftrightarrow \sum_{cyc} \sin x \left(\cos \frac{y}{2} - \sin \frac{y}{2}\right)^2 < \frac{3}{4} \Rightarrow 2\sum_{cyc} \sin x \left(\cos \frac{y}{2} - \sin \frac{y}{2}\right)^2 < \frac{3}{2}$$

Surely $\Rightarrow \sum_{cyc} (\sin x)^{\sin y + \sin z} > 1$ ***(proved)***

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\sin\left(\frac{y+z}{2}\right)\cos\left(\frac{y-z}{2}\right) = \frac{1}{2}[\sin y + \sin z]$$

$$\sin\left(\frac{x+z}{2}\right)\cos\left(\frac{z-x}{2}\right) = \frac{1}{2}[\sin z + \sin x]$$

$$\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{1}{2}[\sin x + \sin y]$$

$$\Rightarrow LHS \geq (\sin x)^{(\sin y + \sin z)} + (\sin y)^{[\sin z + \sin x]} + (\sin z)^{[\sin x + \sin y]}$$

$$= (\sin x)^{\sin y}(\sin x)^{\sin z} + (\sin y)^{\sin z} \cdot (\sin y)^{\sin x} + (\sin z)^{\sin x} \cdot (\sin z)^{\sin y}$$

Let $X = \sin x\,; Y = \sin y\,; Z = \sin z$

$$\left(X, Y, Z \in \left(0; \frac{1}{2}\right)\right)$$

We prove that: $X^Y \cdot X^Z + Y^Z \cdot Y^X + Z^X \cdot Z^Y > 1$

Using AM-GM:

$$X^Y \cdot X^Z + Y^Z \cdot Y^X + Z^X \cdot Z^Y \geq 3\sqrt[3]{(X^Y \cdot Y^Z \cdot Z^X)(Y^X \cdot X^Z \cdot Z^X)}$$

But $X^Y \cdot Y^Z \cdot Z^X > \frac{1}{3\sqrt{3}}; Y^X \cdot X^Z \cdot Z^X > \frac{1}{3\sqrt{3}}$

Now, we want to prove: $X^Y \cdot Y^Z \cdot Z^X > \frac{1}{3\sqrt{3}}$ ***(Similarly:*** $Y^X \cdot X^Z \cdot Z^X > \frac{1}{3\sqrt{3}}$***)***

$$\Leftrightarrow Y \ln X + Z \ln Y + X \ln Z > -\ln(3\sqrt{3})$$

Using Jensen's inequality with $f(t) = \ln(\sin t)\,, t \in \left(0, \frac{\pi}{2}\right)$

$$Y \ln X + Z \ln Y + X \ln Z \geq (X + Y + Z) \ln\left(\frac{XY + YZ + ZX}{X + Y + Z}\right)$$

$$> \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) \ln\left(\frac{\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} + \frac{1}{2} + \frac{1}{2}}\right) = \frac{3}{2} \ln \frac{1}{2} = -\frac{3}{2} \ln 2 > -\ln(3\sqrt{3})$$

(Because $g(n) = n \ln \frac{\alpha}{n} \searrow \left(0; \frac{1}{2}\right)$***)***

Hence, $X^Y \cdot X^Z + Y^Z \cdot Y^X + Z^X + Z^X \cdot Z^Y > 1$ ***(Proved)***