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1) In ΔABC

$$\boldsymbol{\sum a^2 h_b h_c \leq 4(R+r)^4}$$

Proposed by Marian Ursărescu - Romania

Proof.

We prove the following lemma:

Lemma.
In ΔABC:

$$\boldsymbol{\sum a^2 h_b h_c = \frac{2r}{R} s^2 (s^2 - 3r^2 - 6Rr)}$$

Proof.

Using $h_a = \frac{2S}{a}$, *we obtain:*

$$\sum a^2 h_b h_c = \sum a^2 \cdot \frac{2S}{b} \cdot \frac{2S}{c} = 4S^2 \sum \frac{a^2}{bc} = \frac{2r}{R} s^2 (s^2 - 3r^2 - 6Rr)$$

which follows from: $\sum \frac{a^2}{bc} = \frac{s^2 - 3r^2 - 6Rr}{2Rr}$.

□

Let's get back to the main problem:

Using the ***Lemma*** *the inequality can be written:*

$$\frac{2r}{R} s^2 (s^2 - 3r^2 - 6Rr) \leq 4(R+r)^4 \Leftrightarrow s^2(s^2 - 3r^2 - 6Rr) \leq \frac{2R}{r}(R+r)^4$$

We have: $s^2(s^2 - 3r^2 - 6Rr) = s^4 - s^2(3r^2 + 6Rr)$ *and we use 1) and 2):*

1): $s^4 \leq s^2(4R^2 + 20Rr - 2r^2) - r(4R+r)^3$, *ture from:*

$$2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2 - 2Rr} \leq s^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr},$$

Bloundon-Rouche's inequality,

2): Blundon-Gerretsen: $s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$

We obtain:

$$s^2(s^2 - 3r^2 - 6Rr) = s^4 - s^2(3r^2 + 6Rr) \leq s^2(4R^2 + 20Rr - 2r^2) - r(4R+r)^3 - s^2(3r^2 + 6Rr) =$$

$$= s^2(4R^2 + 14Rr - 5r^2) - r(4R+r)^3 \leq \frac{R(4R+r)^2}{2(2R-r)}(4R^2 + 14Rr - 5r^2) - r(4R+r)^3 =$$

$$= (4R+r)^2 \frac{4R^3 - 2R^2r - Rr^2 + 2r^3}{2(2R-r)}.$$

It remains to prove that:

$$(4R+r)^2 \frac{4R^3 - 2R^2r - Rr^2 + 2r^3}{2(Rr-r)} \leq \frac{2R}{r}(R+r)^4 \Leftrightarrow$$

$$\Leftrightarrow 8R^6 - 36R^5r + 32R^4r^2 + 36R^3r^3 - 30R^2r^4 - 19Rr^5 - 2r^6 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(8R^5-20R^4r-8R^3r^2+20R^2r^3+10Rr^4+r^5) \geq 0$, *obviously from Euler* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasise an inequality having an opposite sense:

3) In ΔABC:

$$\sum a^2 h_b h_c \geq 324r^4$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma** we write the inequality:*

$\frac{2r}{R}s^2(s^2 - 3r^2 - 6Rr) \geq 324r^4$, *which follows from Gerretsen's inequality:*

$s^2 \geq 16Rr - 5r^2$. *It remains to prove that:*

$$\frac{2r}{R}(16Rr - 5r^2)(16Rr - 5r^2 - 3r^2 - 6Rr) \geq 324r^4 \Leftrightarrow 8R^2 - 17Rr + 2r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(8R-r) \geq 0$, *obviously from Euler's inequality:* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

4) In ΔABC:

$$324r^4 \leq \sum a^2 h_b h_c \leq 4(R+r)^4.$$

Proof.

See inequalities 1) and 3).

Equality holds if and only if the triangle is equilateral.

□

Remark.

If we replace $h_b h_c$ with $r_b r_c$ we propose:

5) In ΔABC:

$$12s^2r^2 \leq \sum a^2 r_b r_c \leq 6s^2 Rr$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.
6) In ΔABC:

$$\sum a^2 r_b r_c = 4s^2 r(R+r)$$

Proof.

Using $r_a = \frac{S}{s-a}$, *we obtain:*

$$\sum a^2 r_b r_c = \sum a^2 \cdot \frac{S}{s-b} \cdot \frac{S}{s-c} = S^2 \sum \frac{a^2}{(s-b)(s-c)} = 4s^2 r(R+r)$$

which follows from: $\sum \frac{a^2}{(s-b)(s-c)} = \frac{4(R+r)}{r}$.

□

Let's get back to the main problem:

Using the ***Lemma*** *the inequality holds:*

$12s^2r^2 \leq 4s^2r(R+r) \leq 6s^2Rr \Leftrightarrow 6r \leq 2(R+r) \leq 3R$, *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum a^2 h_b h_c$ *and* $\sum a^2 r_b r_c$ *the following relationship exists:*

7) In acute-angled ΔABC:

$$\sum a^2 r_b r_c \leq \sum a^2 h_b h_c$$

Proposed by Marin Chirciu - Romania

Proof.

Using the identities 2) and 6) we write the inequality:

$4s^2r(R+r) \leq \frac{2r}{R} s^2(s^2 - 3r^2 - 6Rr) \Leftrightarrow s^2 \geq 2R^2 + 8Rr + 3r^2$, *(Walker's inequality).*

true only for the acute-angled triangle.

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the sequence of inequalities:

1) In acute-angled ΔABC:

$$324r^4 \leq 12S^2 \leq \sum a^2 r_b r_c \leq \sum a^2 h_b h_c \leq 4(R+r)^4.$$

Proof.

See inequalities 1), 5), 7) and Mitrinovič's inequality $s^2 \geq 27r^2$.

Equality holds if and only if the triangle is equilateral.

□

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