Prove that:

$$\frac{\tan 36° + \tan 72°}{4 \cos 54°} = \varphi$$

Proposed by Alpaslan Ceran-Turkey

Solution 1 by Avishek Mitra-West Bengal-India, Solution 2 by Nelson Javier Villahererra Lopez-El Salvador

Solution 1 by Avishek Mitra-West Bengal-India

$$Let\ x = \frac{\pi}{10} \Rightarrow 5x = \frac{\pi}{2} \Rightarrow 2x = \frac{\pi}{2} - 3x \Rightarrow \sin 2x = \sin\left(\frac{\pi}{2} - 3x\right)$$

$$\Rightarrow \sin 2x = \cos 3x \Rightarrow 2 \sin x \cos x = 4 \cos^3 x - 3 \cos x$$

$$\Rightarrow 2 \sin x - 4 \cos^2 x + 3 = 0 \Rightarrow 4 \sin^2 x + 2 \sin x - 1 = 0$$

$$\Rightarrow \sin x = \frac{-2 \pm \sqrt{20}}{8} = \frac{-1 \pm \sqrt{5}}{4},\ \sin 18 = \frac{\sqrt{5}-1}{4} = \cos 72$$

$$\cos 18 = \sqrt{1 - \sin^2 18} = \frac{1}{4}\sqrt{20 + 2\sqrt{5}} = \sin 72,\ \tan 72 = \frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$$

$$Now, \cos 36 = 1 - 2 \sin^2 18 = \frac{\sqrt{5}+1}{4},\ \sin 36 = \sqrt{1 - \cos^2 36} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

$$\tan 36 = \frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1},\ \cos 54 = \sin 36 = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}$$

$$Hence = \frac{\tan 36 + \tan 72}{4 \cos 54} = \frac{\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1} + \frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}}{\sqrt{10-2\sqrt{5}}} = \frac{1}{\sqrt{5}+1} + \frac{1}{\sqrt{5}-1}\sqrt{\frac{10+2\sqrt{5}}{10-2\sqrt{5}}}$$

$$= \frac{1}{\sqrt{5}+1} + \frac{1}{\sqrt{5}-1} \cdot \frac{10+2\sqrt{5}}{4\sqrt{5}} = \frac{4\sqrt{5}(\sqrt{5}-1) + (10+2\sqrt{5})(\sqrt{5}+1)}{4\sqrt{5}\cdot 4}$$

$$= \frac{40+8\sqrt{5}}{16\sqrt{5}} = \frac{5+\sqrt{5}}{2\sqrt{5}} = \frac{5\sqrt{5}+5}{10} = \frac{\sqrt{5}+1}{2} = \phi \quad \text{(proved)}$$

Solution 2 by Nelson Javier Villahererra Lopez-El Salvador

$$\frac{\tan(36°) + \tan(72°)}{4\cos(54°)} = \frac{\sin(36°)\cos(72°) + \cos(36°)\sin(72°)}{4\sin(36°)\cos(36°)\cos(72°)} =$$

$$= \frac{\sin(108°)}{2\sin(72°)\cos(72°)} = \frac{\sin(108°)}{\sin(144°)} = \frac{\cos(18°)}{\cos(54°)} = \frac{\cos(18°)}{\sin(36°)} = \frac{1}{2\sin(18°)}$$

$$= \frac{1}{\sqrt{2-2\cos(36°)}} = \frac{1}{\sqrt{2 - \frac{(\sqrt{5}+1)}{2}}} = \sqrt{\frac{2}{3-\sqrt{5}}} = \sqrt{\frac{3+\sqrt{5}}{2}} = \sqrt{\frac{6+2\sqrt{5}}{4}} =$$

$$= \sqrt{\left(\frac{\sqrt{5}+1}{2}\right)^2} = \frac{\sqrt{5}+1}{2} = \varphi$$