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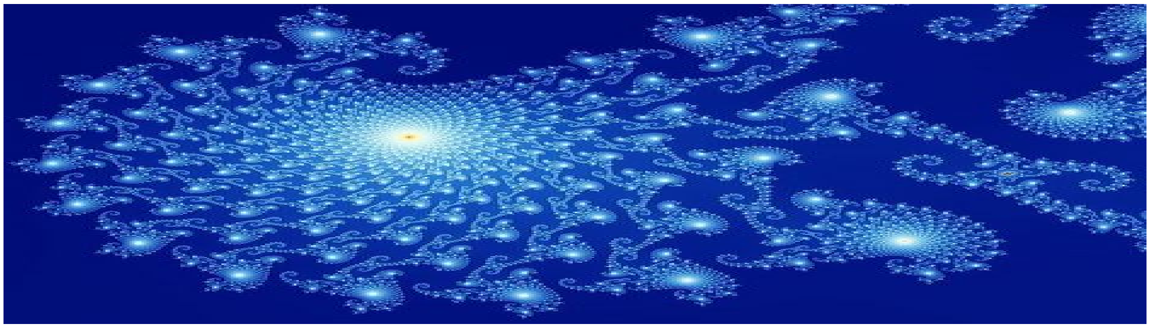

**If**

$$\Psi(x)=\sum_{n=0}^{\infty}\frac{(n^2+n+1)}{\binom{2n}{n}}x^n$$

**then show that**

$$\int_0^1 \Psi(1+x)\frac{dx}{1+x}=3+2\pi-\frac{7\pi}{9\sqrt{3}}+\ln(2)$$

***Proposed by Srinivasa Raghava-AIRMC-India***

***Solution by Obidah Alsharafy-Sana'a-Yemen***

$$If\ \psi[x]=\sum_{k=0}^{\infty}\frac{(1+k+k^2)}{\binom{2k}{k}}x^k \to \text{find: } \int_0^1\frac{\psi[1+x]}{1+x}dx$$

***Solve:***

$$I=\int_0^1\frac{\psi[1+x]}{1+x}dx,\ \text{let } (v=1+x)\to\int_1^2\frac{\psi[x]}{x}dx$$

$$I=\int_1^2\frac{\psi[x]}{x}dx\to\int_1^2\frac{dx}{x}+\sum_{k=1}^{\infty}\frac{(1+k+k^2)}{\binom{2k}{k}}\int_1^2 x^{k-1}\,dx\to$$

$$\to\log[2]+\sum_{k=1}^{\infty}\frac{(1+k+k^2)(k!)^2}{(2k)!\,k}(2^k-1)$$

$$I=\log[2]+\left\{\sum_{k=1}^{\infty}\frac{(k!)^2}{(2k)!\,k}(2^k-1)+\sum_{k=1}^{\infty}\frac{(k!)^2}{(2k)!}(2^k-1)+\sum_{k=1}^{\infty}\frac{(k!)^2k}{(2k)!}(2^k-1)\right\}$$

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$$\because\ f[x]=\frac{2\sqrt{x}\sin^{-1}\left[\frac{\sqrt{x}}{2}\right]}{\sqrt{4-x}}=\sum_{k=1}^{\infty}\frac{(k!)^2}{(2k)!\,k}x^k$$

$$\frac{df[x]}{dx}=\frac{1}{(4-x)}+\frac{4\sin^{-1}\left[\frac{\sqrt{x}}{2}\right]}{(-4+x)^{\frac{3}{2}}\sqrt{x}}=\sum_{k=1}^{\infty}\frac{(k!)^2}{(2k)!}x^{k-1}$$

$$\frac{d}{dx}\left(x\frac{df[x]}{dx}\right)=\frac{6}{(4-x)^2}+\frac{4\sqrt{x}\arcsin\left[\frac{\sqrt{x}}{2}\right]}{(4-x)^{\frac{5}{2}}}\left(1+\frac{2}{x}\right)=\sum_{k=1}^{\infty}\frac{(k!)^2k}{(2k)!}x^{k-1}$$

$$\Rightarrow I=\log[2]+\{f[2]-f[1]+2f'[2]-f'[1]+(4f''[2]+2f'[2])-(f''[1]+f'[1])\}$$

$$\to I=\log[2]+\left\{\frac{\pi}{2}-\frac{\pi}{3\sqrt{3}}+1+\frac{\pi}{2}-\frac{1}{3}-\frac{2\pi}{9\sqrt{3}}+3+\pi-\frac{2}{3}-\frac{2\pi}{9\sqrt{3}}\right\}$$

$$\to I=\log[2]+2\pi+3-\frac{7\pi}{9\sqrt{3}}$$