TWO CLASSES OF LALESCU'S SEQUENCES

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This year is celebrating 116 years since the publication of Traian Lalescu's problem in Mathematical Gazette. Because a large number of Lalescu's type problems are recently published in international prestigious magazines (some of these can be found also in reference in [4] - http://www.mathproblems-ks.org/?wpfb_dl=20) we present two classes of Lalescu's sequences.

Traian Lalescu's sequence is $(L_n)_{n\geq 2}$ given by $L_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$, see [5]. Let be $(a_n)_{n\geq 1}$ is a sequence of real numbers strictly positive and $x \in R_+$. We define the sequence $M_n^{[x]}(a_1, a_2, \ldots, a_n)_{n\geq 1}$ of general term

$$(1)\quad M_n^{[x]}(a_1, a_2, \ldots, a_n) = \begin{cases} \left(\frac{a_1^x + a_2^x + \ldots + a_n^x}{n}\right)^{\frac{1}{x}}, x \in R_+^*, \\ \lim_{x\to 0} M_n^{[n]}(a_1, a_2, \ldots, a_n) = \sqrt[n]{a_1 a_2 \ldots a_n}, x = 0 \end{cases}$$

Next we consider a sequence $(r_n)_{n\geq 1}$ of real numbers strictly positive, convergent with $\lim_{n\to\infty} r_n = r \in R_+^*$ and also a sequence $(a_n)_{n\geq 1}$ of real numbers strictly positive having the property that there exists $k \in R_+$ such that

$$a_{n+1} = a_n + r_n \cdot n^k, \forall n \in N^* \tag{2}$$

Also, we will denote $a_1 \cdot a_2 \cdot \ldots \cdot a_n = a_n!$ and for any $x \in R_+$ we denote $s_n(x) = a_1^x + a_2^x + \ldots + a_n^x$.
Next we will calculate some limits useful in proving the theorems from this article.
1. From the definition mode of the sequence $(a_n)_{n\geq 1}$ it follows that $\frac{a_{n+1}-a_n}{n^k} = r_n, \forall n \in N^*$ wherefrom we deduce that

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{n^k} = r \tag{3}$$

2. Using Cesaro - Stolz's theorem we obtain
(4)

$$\lim_{n\to\infty} \frac{a_n}{n^{k+1}} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{(n+1)^{k+1} - n^{k+1}} = \lim_{n\to\infty} \left(\frac{a_{n+1} - a_n}{n^k} \cdot \frac{n^k}{(n+1)^{k+1} - n^{k+1}}\right) = \frac{r}{k+1}$$

3. Using Cauchy - D' Alembert's criterion it follows that

$$\lim_{n\to\infty} \frac{n^{k+1}}{\sqrt[n]{a_n!}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^{n(k+1)}}{a_n!}} = \lim_{n\to\infty} \left(\frac{(n+1)^{(n+1)(k+1)}}{a_{n+1}!} \cdot \frac{a_n!}{n^{n(k+1)}}\right) =$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)^{k+1}}{a_{n+1}} \cdot \left(1 + \frac{1}{n}\right)^{n(k+1)}\right) = \frac{k+1}{r} \cdot e^{k+1} \tag{5}$$

4. We notice that

$$\lim_{n\to\infty}\frac{a_n}{\sqrt[n]{a_n!}}=\lim_{n\to\infty}\left(\frac{a_n}{n^{k+1}}\cdot\frac{n^{k+1}}{\sqrt[n]{a_n!}}\right)=\frac{r}{k+1}\cdot\frac{k+1}{r}\cdot e^{k+1} \tag{6}$$

and that

$$\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}!}}{\sqrt[n]{a_n!}}=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}!}}{a_{n+1}}\cdot\frac{a_n}{\sqrt[n]{a_n!}}\cdot\frac{a_{n+1}}{a_n}\right)=e^{-(k+1)}\cdot e^{k+1}\cdot\lim_{n\to\infty}\frac{a_n+n^k\cdot r_n}{a_n}=$$

$$=1+r\cdot\lim_{n\to\infty}\frac{n^k}{a_n}=1+r\cdot\lim_{n\to\infty}\left(\frac{n^{k+1}}{a_n}\cdot\frac{1}{n}\right)=1+r\cdot\frac{k+1}{r}\cdot e^{k+1}\cdot 0=1 \tag{7}$$

Hence,

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}!}}{\sqrt[n]{a_n!}}\right)=\lim_{n\to\infty}\left(\frac{a_{n+1}!}{a_n!}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}!}}\right)=\lim_{n\to\infty}\frac{a_{n+1}}{\sqrt[n+1]{a_{n+1}}}=e^{k+1} \tag{8}$$

5. For $x\in R_+^*$ we calculate

$$\lim_{n\to\infty}\frac{M_n^{[n]}(a_1,a_2,\dots,a_n)}{n^{k+1}}=\lim_{n\to\infty}\frac{\left(\frac{s_n(x)}{n}\right)^{\frac{1}{x}}}{n^{k+1}}=\lim_{n\to\infty}\left(\frac{s_n(x)}{n^{(k+1)\cdot x+1}}\right)^{\frac{1}{x}}=$$

$$=\left(\lim_{n\to\infty}\frac{s_{n+1}(x)-s_n(x)}{(n+1)^{(k+1)\cdot x+1}-n^{(k+1)\cdot x+1}}\right)^{\frac{1}{x}}=\left(\lim_{n\to\infty}\frac{n^{(k+1)\cdot x}}{(n+1)^{(k+1)\cdot x+1}-n^{(k+1)\cdot x+1}}\cdot\left(\frac{a_{n+1}}{n^{k+1}}\right)^{x}\right)^{\frac{1}{x}}=$$

$$=\left(\frac{1}{(k+1)\cdot x+1}\right)^{\frac{1}{x}}\cdot\lim_{n\to\infty}\frac{a_{n+1}}{n^{k+1}}=\Big(1+(k+1)\cdot x\Big)^{-\frac{1}{x}}\cdot\frac{r}{k+1} \tag{9}$$

and also

$$\lim_{n\to\infty}\frac{M_n^{[x]}(a_1,a_2,\dots,a_n)}{a_n}=\lim_{n\to\infty}\left(\frac{M_n^{[x]}(a_1,a_2,\dots,a_n)}{n^{k+1}}\cdot\frac{n^{k+1}}{a_n}\right)=$$

$$=\frac{r}{k+1}\cdot\Big(1+(k+1)\cdot x\Big)^{-\frac{1}{x}}\cdot\frac{k+1}{r}=\Big(1+(k+1)\cdot x\Big)^{-\frac{1}{x}} \tag{10}$$

respectively

(11)

$$\lim_{n\to\infty}\frac{M_n^{[n]}(a_1,a_2,\dots,a_n)}{M_n^{[y]}(a_1,a_2,\dots,a_n)}=\lim_{n\to\infty}\left(\frac{a_n}{M_n^{[y]}(a_1,a_2,\dots,a_n)}\cdot\frac{M_n^{[x]}(a_1,a_2,\dots,a_n)}{a_n}\right)=\frac{(1+y)^{\frac{1}{y}}}{(1+x)^{\frac{1}{x}}}$$

6. If $x\in R_+^*$ and $k\in R_+$, then

$$\lim_{n\to\infty}\frac{M_{n+1}^{[x]}(a_1,a_2,\dots,a_n,a_{n+1})}{M_n^{[x]}(a_1,a_2,\dots,a_n)}=1 \tag{12}$$

Indeed, for any $x\in R_+^*$ we have

$$\lim_{n\to\infty}\frac{M_{n+1}^{[x]}(a_1,a_2,\dots,a_n,a_{n+1})}{M_n^{[x]}(a_1,a_2,\dots,a_n)}=\lim_{n\to\infty}\left(\frac{M_{n+1}^{[x]}(a_1,a_2,\dots,a_n,a_{n+1})}{a_{n+1}}\cdot\frac{a_{n+1}}{a_n}\cdot\frac{a_n}{M_n^{[x]}(a_1,a_2,\dots,a_n)}\right)=$$

$$=\Big(1+(k+1)\cdot x\Big)^{-\frac{1}{x}}\cdot\Big(1+(k+1)\cdot x\Big)^{\frac{1}{x}}=1$$

7. If $x \in R_+^*$ and $k \in R_+$, then

$$\lim_{x\to\infty}\left(\frac{M_{n+1}^{[x]}(a_1,a_2,\dots,a_n,a_{n+1})}{M_n^{[x]}(a_1,a_2,\dots,a_n)}\right)^n = e^{k+1} \tag{13}$$

Indeed, $\lim_{x\to\infty}\left(\frac{M_{n+1}^{[x]}(a_1,a_2,\dots,a_n,a_{n+1})}{M_n^{[x]}(a_1,a_2,\dots,a_n)}\right)^n = \lim_{x\to\infty}\left(\frac{s_{n+1}(x)}{n+1}\cdot\frac{n}{s_n(x)}\right)^{\frac{n}{x}} =$

$$= \lim_{n\to\infty}\left(\left(\frac{s_{n+1}(x)}{s_n(x)}\right)^n\cdot\left(\frac{n}{n+1}\right)^n\right)^{\frac{1}{x}} = e^{-\frac{1}{x}}\cdot\left(\lim_{n\to\infty}\left(\frac{s_{n+1}(x)}{s_n(x)}\right)^n\right)^{\frac{1}{x}}, \tag{14}$$

We notice that
(15)

$$\lim_{x\to\infty}\left(\frac{s_{n+1}(x)}{s_n(x)}\right)^n = \lim_{n\to\infty}\left(1+\frac{a_{n+1}^x}{s_n(x)}\right)^n = \lim_{x\to\infty}\left(\left(1+\frac{a_{n+1}^x}{s_n(x)}\right)^{s_n(x)\cdot a_{n+1}^{-x}}\right)^{\frac{n\cdot a_{n+1}^x}{s_n(x)}} = e^t$$

where

$$t = \lim_{x\to\infty}\frac{n\cdot a_{n+1}^x}{s_n(x)} = \lim_{x\to\infty}\left(\frac{a_{n+1}}{n^{k+1}}\right)^x\cdot\lim_{x\to\infty}\frac{n^{(k+1)\cdot x+1}}{s_n(x)} =$$

$$=\left(\frac{r}{k+1}\right)^x\cdot\lim_{x\to\infty}\frac{(n+1)^{(k+1)\cdot x+1}}{s_{n+1}(x)-s_n(x)} =\left(\frac{r}{k+1}\right)^x\cdot\lim_{x\to\infty}\frac{(n+1)^{(k+1)\cdot x+1}-n^{(k+1)\cdot x+1}}{n^{(k+1)\cdot x}}\cdot\lim_{x\to\infty}\frac{n^{(k+1)\cdot x}}{a_{n+1}^x} =$$

$$=\left(\frac{r}{k+1}\right)^x\cdot\left(\frac{k+1}{r}\right)^x\cdot\Big(1+(k+1)\cdot x\Big) = 1+(k+1)\cdot x \tag{16}$$

From (14), (15) and (16) we deduce that

$\lim_{n\to\infty}\left(\frac{M_{n+1}^{[x]}(a_1,a_2,\dots,a_n,a_{n+1})}{M_n^{[x]}(a_1,a_2,\dots,a_n)}\right)^n = e^{-\frac{1}{x}}\cdot e^{\frac{1}{x}+k+1} = e^{k+1}$ which follows from (13)

Next we will consider $k=0$, namely $a_{n+1}=a_n+r_n, \forall n\in N^*$ with $\lim_{n\to\infty}(a_{n+1}-a_n)=r>0$ and we will prove

Theorem 1. If the sequence of real numbers strictly positive $(a_n)_{n\geq1}$ verify the relationship $a_{n+1}=a_n+r_n, \forall n\in N^*$, then the sequence $(L_n(x))_{n\geq1}$ of general term

$L_n(x) = M_{n+1}^{[x]}(a_1,a_2,\dots,a_n,a_{n+1}) - M_n^{[x]}(a_1,a_2,\dots,a_n)$ is convergent and

$$L(x) = \lim_{n\to\infty}L_n(x) = \begin{cases} r\cdot(1+x)^{-\frac{1}{x}}, x\in R_+^*,\\ r\cdot e^{-1}, x=0\end{cases} \tag{17}$$

Proof. For any $x\in R_+^*$ avem

$$L(x) = \lim_{x\to\infty}\Big(M_n^{[x]}(a_1,a_2,\dots,a_n)\cdot(u_n(x)-1)\Big) \tag{18}$$

where $u_n(x) = \frac{M_{n+1}^{[x]}(a_1,a_2,\dots,a_n,a_{n+1})}{M_n^{[x]}(a_1,a_2,\dots,a_n)}$ and according to (12) we have
$\lim_{n\to\infty}u_n(x)=1$ and hence $\lim_{n\to\infty}\frac{u_n(x)-1}{\ln u_n(x)}=1$.
So, $L(x) = \lim_{n\to\infty}\Big(M_n^x(a_1,a_2,\dots,a_n)\cdot\frac{u_n(x)-1}{\ln u_n(x)}\cdot\ln u_n(x)\Big) =$

$\lim_{n\to\infty}\left(\frac{M_n^{[x]}(a_1,a_2,\ldots,a_n)}{n}\cdot\frac{u_n(x)-1}{\ln u_n(x)}\cdot\frac{u_n(x)-1}{\ln u_n(x)}\cdot\ln(u_n(x))^n\right)$, wherefrom if we take into account relationships (9), and (13) we deduce that

$$L(x)=r\cdot(1+x)^{-\frac{1}{x}}\cdot 1\cdot\ln e=r\cdot(1+x)^{-\frac{1}{x}}=\frac{r}{(1+)^{\frac{1}{x}}}\text{ for } x\in R_+^*.$$

If $x=0$, then $L(0)=\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}!}-\sqrt[n]{a_n!}\right)=\lim_{n\to\infty}\left(\sqrt[n]{a_n!}\cdot(u_n(0)-1)\right)=$

$$=\lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n!}}{n}\cdot\frac{u_n(0)-1}{\ln u_n(0)}\cdot\ln(u_n(0))^n\right) \tag{19}$$

If we take into account relationships (5) and (8) from relationship (19) we obtain

$$L(0)=\frac{r}{e}\cdot 1\cdot\ln e=\frac{r}{e} \tag{20}$$

With these theorem it is proved. □

Observation 1. If $r_n=1,\forall n\in N^*$ and $a_1=1$, then $a_n=n,\forall n\in N^*$ and relationship (20) becomes

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)=e^{-1} \tag{21}$$

we have obtained the limit of Traina Lalescu's sequence.

Let be $(r_n)_{n\geq 1}$ a sequence of real strictly positive numbers such that $\lim_{n\to\infty} r_n=r\in R_+$ and $(a_n)_{n\geq 1}: a_1\in R_+^*, a_{n+1}=a_n\cdot r_n\cdot n^k,\forall n\in N^*$, $k\in R$, and

$$M_n^{[x]}(a_1,a_2,\ldots,a_n)=\left(\frac{1}{n}\sum_{i=1}^{n}a_i^x\right)^{\frac{1}{x}},\forall n\in N^*, \tag{22}$$

We have the next

Theorem 2.

$$\lim_{n\to\infty}\left(n^{-k}\left(M_{n+1}^{[x]}-M_n^{[x]}\right)\right)=r\cdot\left(1+(k+1)\cdot x\right)^{-\frac{1}{x}} \tag{23}$$

Proof. We have $B_n(x)=n^{-k}\left(M_{n+1}^{[x]}-M_n^{[x]}\right)=n^{-k}\cdot M_n^{[x]}\cdot(u_n(x)-1)=$

$$=\frac{M_n^{[x]}}{n^k}\cdot\frac{u_n(x)-1}{\ln u_n(x)}\cdot\ln(u_n(x))=\frac{M_n^{[x]}}{n^{k+1}}\cdot\frac{u_n(x)-1}{\ln u_n(x)}\cdot\ln(u_n(x))^n,\forall n\geq 2,\text{ where } u(x)=\frac{M_{n+1}^{[x]}}{M_n^{[x]}},\forall n\geq 2.$$

According with (9) we have $\lim_{n\to\infty}u_n(x)=\lim_{n\to\infty}\left(\frac{M_{n+1}^{[x]}}{(n+1)^{k+1}}\cdot\frac{n^{k+1}}{M_n^{[x]}}\cdot\left(\frac{n+1}{n}\right)^{k+1}\right)=$

$$=\frac{r}{k+1}\cdot\left(1+(k+1)\cdot x\right)^{-\frac{1}{x}}\cdot\frac{k+1}{r}\cdot\frac{1}{(1+(k+1)\cdot x)^{-\frac{1}{x}}}\cdot 1=1.$$

So $\lim_{n\to\infty}\frac{u_n(x)-1}{\ln u_n(x)}=1$ and according with (13)

$$\lim_{n\to\infty}(u_n(x))^n=\lim_{n\to\infty}\left(\frac{M_{n+1}^{[x]}}{M_n[x]}\right)^n=e^{k+1} \tag{24}$$

Then, $\lim_{n\to\infty}B_n(x)=\frac{r}{k+1}\cdot\left(1+(k+1)\cdot x\right)^{-\frac{1}{x}}\cdot 1\cdot\ln e^{k+1}=r\left(1+(k+1)\cdot x\right)^{-\frac{1}{x}}$,
q.e.d. □

Theorem 3.

$$\lim_{n\to\infty}\left(\frac{M_{n+1}^{[x]}}{(n+1)^k}-\frac{M_n^{[x]}}{n^k}\right)=\frac{r}{k+1}\cdot\Big(1+(k+1)\cdot x\Big)^{-\frac{1}{x}} \tag{25}$$

Proof. We have

$$L_n(x)=\frac{M_{n+1}^{[x]}}{(n+1)^k}-\frac{M_n^{[x]}}{n^k}=\frac{M_n^{[x]}}{n^k}\cdot(v_n(x)-1)=\frac{M_n^{[x]}}{n^k}\cdot\frac{v_n^x-1}{\ln v_n(x)}\cdot\ln v_n(x)=$$

$$=\frac{M_n^{[x]}}{n^{k+1}}\cdot\frac{v_n^x-1}{\ln v_n(x)}\cdot\ln(v_n(x))^n,\forall n\geq 2,\text{ where } v_n(x)=\frac{M_{n+1}^{[x]}}{M_n^{[x]}}\left(\frac{n}{n+1}\right)^k,\forall n\geq 2.$$

So, $\lim_{n\to\infty} v_n(x)=\lim_{x\to\infty}\left(\frac{M_{n+1}^{[x]}}{(n+1)^{k+1}}\cdot\frac{n^{k+1}}{M_n^{[x]}}\cdot\frac{n+1}{n}\right)$ and with relationship (9) it follows $\lim_{n\to\infty} v_n(x)=1$ wherefrom

$$\lim_{n\to\infty}\frac{v_n(x)-1}{\ln v_n(x)}=1 \text{ and } \lim_{n\to\infty}(v_n(x))^n=\lim_{x\to\infty}\left(\frac{M_{n+1}^{[x]}}{M_n^{[x]}}\right)^n\cdot\lim_{x\to\infty}\Big(\frac{n}{n+1}\Big)^{n\cdot k}=e^{k+1}\cdot e^{-k}=e.$$

Then, $\lim_{x\to\infty} L_n(x)=\frac{r}{k+1}\cdot\Big(1+(k+1)\cdot x\Big)^{-\frac{1}{x}}\cdot 1\cdot\ln e=\frac{r}{k+1}\cdot\Big(1+(k+1)\cdot x\Big)^{-\frac{1}{x}}$, q.e.d. □

Observation 2. From theorem 3 it follows that

$$\lim_{x\to\infty}\Big(\lim_{x\to\infty}L_n(x)\Big)=\frac{r}{k+1}\cdot\lim_{x\to\infty}\Big(1+(k+1)\cdot x\Big)^{-\frac{1}{x}}=$$

$$=\frac{r}{k+1}\cdot\lim_{x\to 0}\left(\Big(1+(k+1)x\Big)^{-\frac{1}{(k+1)x}}\right)^{-(k+1)}=\frac{r}{k+1}\cdot e^{-(k+1)},$$

wherefrom from $k=0$ and $r=1$ we obtain again the limit of Traian Lalescu's sequence, namely

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)=\frac{1}{e}$$

Comment. Other methods for calculating the limits of some Traian Lalescu's sequences can be found in [2]
http://elib.mi.sanu.ac.rs/files/journals/tm/31/tm1624.pdf
Other classes of Lalescu's sequences can be consulted in [3]
http://www.kappamuepsilon.org/pages/a/Pentagon/Vol_73_Num_2_Spring_2014.pdf
This year is celebrating 116 years since the publication of Traian Lalescu's problem in Mathematical Gazette. A large number of Lalescu's type problems, published in prestigious international magazines, can be found also in [4]
http://www.mathproblems-ks.org/?wpfb_dl=20

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