SOME LIMITS OF TRAIAN LALESCU TYPE (II)

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Abstract. In this paper we present some limits of Lalescu type.

Theorem 1. Let $f, g : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be functions such that $\lim_{x\to\infty}(f(x+1) - f(x)) = a \in \mathbb{R}_+^*, \lim_{x\to\infty} \frac{g(x+1)}{xg(x)} = b \in \mathbb{R}_+^*$ and there exists $\lim_{x\to\infty} \frac{f(x)}{x}$ and $\lim_{x\to\infty} \frac{(g(x))^{\frac{1}{x}}}{x}$. For $t \in \mathbb{R}$, then:

$$\lim_{x\to\infty}(f(x))^{\cos^2 t}\left((g(x))^{\frac{\sin^2 t}{x+1}} - (g(x))^{\frac{\sin^2 t}{x}}\right) = \frac{a^{\cos^2 t}\cdot b^{\sin^2 t}}{e^{\sin^2 t}}\cdot \sin^2 t$$

Proof.

$$\lim_{x\to\infty}\frac{f(x)}{x} = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}^*}}\frac{f(n)}{n} \overset{\text{C-S}}{=} \lim_{n\to\infty}\frac{f(n+1)-f(n)}{(n+1)-n} = \lim_{n\to\infty}(f(n+1)-f(n)) = a.$$

By Cauchy-D'Alembert we have:

$$\lim_{x\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x} = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}^*}}\frac{(g(n))^{\frac{1}{n}}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{g(n)}{n^n}} \overset{\text{C-D'A}}{=} \lim_{n\to\infty}\left(\frac{g(n+1)}{(n+1)^{n+1}}\cdot\frac{n^n}{g(n)}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{g(n+1)}{ng(n)}\left(\frac{n}{n+1}\right)^{n+1}\right) = \frac{b}{e}. \text{ So, } \lim_{x\to\infty}(f(x))^{\cos^2 t}\left((g(x))^{\frac{\sin^2 t}{x+1}} - (g(x))^{\frac{\sin^2 t}{x}}\right) =$$

$$= \lim_{x\to\infty}\left(\frac{f(x)}{x}\right)^{\cos^2 t}\lim_{x\to\infty}\left(\frac{(g(x))^{\frac{1}{x}}}{x}\right)^{\sin^2 t}(u(x)-1)x^{\sin^2 t + \cos^2 t} =$$

$$= a^{\cos^2 t}\cdot\frac{b^{\sin^2 t}}{e^{\sin^2 t}}\cdot\lim_{x\to\infty}\left(\frac{u(x)-1}{\ln u(x)}\cdot\ln(u(x))^x\right) \text{ where, } u(x) = \left(\frac{(g(x+1))^{\frac{1}{x+1}}}{(g(x))^{\frac{1}{x}}}\right)^{\sin^2 t}$$

with $\lim_{x\to\infty} u(x) = 1$, so, $\lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)} = 1$. We also have:

$$\lim_{x\to\infty}(u(x))^x = \lim_{x\to\infty}\left(\frac{(g(x+1))^{\frac{x}{x+1}}}{g(x)}\right)^{\sin^2 t} = \lim_{x\to\infty}\left(\frac{g(x+1)}{g(x)}\cdot\frac{1}{(g(x+1))^{\frac{1}{x+1}}}\right)^{\sin^2 t} =$$

$$= \lim_{x\to\infty}\left(\frac{g(x+1)}{xg(x)}\cdot\frac{x+1}{(g(x+1))^{\frac{1}{x+1}}}\cdot\frac{x}{x+1}\right)^{\sin^2 t} = \left(b\cdot\frac{e}{b}\cdot 1\right)^{\sin^2 t} = e^{\sin^2 t}.$$

Therefore: $\lim_{x\to\infty}(f(x))^{\cos^2 t}\left((g(x))^{\frac{\sin^2 t}{x+1}} - (g(x))^{\frac{\sin^2 t}{x}}\right) = \frac{a^{\cos^2 t}\cdot b^{\sin^2 t}}{e^{\sin^2 t}}\cdot\ln e^{\sin^2 t} =$

$$= \frac{a^{\cos^2 t}\cdot b^{\sin^2 t}}{e^{\sin^2 t}}\cdot\sin^2 t.$$

□

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Theorem 2. Let $f, g : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that: $\lim_{x\to\infty}(f(x+1)-f(x)) = a \in \mathbb{R}_+^*$, $\lim_{x\to\infty} \frac{g(x+1)}{xg(x)} = b \in \mathbb{R}_+^*$ and there is $\lim_{x\to\infty} \frac{f(x)}{x}, \lim_{x\to\infty} \frac{(g(x))^{\frac{1}{x}}}{x}$. For $t \in \mathbb{R}$, then:

$$\lim_{x\to\infty}(f(x))^{\sin^2 t}\left((g(x))^{\frac{\cos^2 t}{x+1}} - (g(x))^{\frac{\cos^2 t}{x}}\right) = \frac{a^{\sin^2 t}\cdot b^{\cos^2 t}}{e^{\cos^2 t}}\cdot\cos^2 t$$

Proof. By Cesaro-Stolz theorem we have:

$$\lim_{x\to\infty}\frac{f(x)}{x} = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}^*}}\frac{f(n)}{n} \overset{\text{C-S}}{=} \lim_{n\to\infty}\frac{f(n+1)-f(n)}{(n+1)-n} = \lim_{n\to\infty}(f(n+1)-f(n)) = a$$

and by Cauchy-D'Alembert theorem we deduce that:

$$\lim_{x\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x} = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}^*}}\frac{(g(n))^{\frac{1}{n}}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{g(n)}{n^n}} \overset{\text{C-D'A}}{=} \lim_{n\to\infty}\left(\frac{g(n+1)}{(n+1)^{n+1}}\cdot\frac{n^n}{g(n)}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{g(n+1)}{ng(n)}\left(\frac{n}{n+1}\right)^{n+1}\right) = \frac{b}{e}.\text{ So, } \lim_{x\to\infty}(f(x))^{\sin^2 t}\left((g(x))^{\frac{\cos^2 t}{x+1}} - (g(x))^{\frac{\cos^2 t}{x}}\right) =$$

$$= \lim_{x\to\infty}\left(\frac{f(x)}{x}\right)^{\sin^2 t}\lim_{x\to\infty}\left(\frac{(g(x))^{\frac{1}{x}}}{x}\right)^{\cos^2 t}(u(x)-1)x^{\sin^2 t+\cos^2 t} =$$

$$= a^{\sin^2 t}\cdot\frac{b^{\cos^2 t}}{e^{\cos^2 t}}\cdot\lim_{x\to\infty}\left(\frac{u(x)-1}{\ln u(x)}\cdot\ln(u(x))^x\right),\text{ where } u(x) = \left(\frac{(g(x+1))^{\frac{1}{x+1}}}{(g(x))^{\frac{1}{x}}}\right)^{\cos^2 t}$$

with $\lim_{x\to\infty} u(x) = 1$, then $\lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)} = 1$. We have:

$$\lim_{x\to\infty}(u(x))^x = \lim_{x\to\infty}\left(\frac{(g(x+1))^{\frac{x}{x+1}}}{g(x)}\right)^{\cos^2 t} = \lim_{x\to\infty}\left(\frac{g(x+1)}{g(x)}\cdot\frac{1}{(g(x+1))^{\frac{1}{x+1}}}\right)^{\cos^2 t} =$$

$$= \lim_{x\to\infty}\left(\frac{g(x+1)}{xg(x)}\cdot\frac{x+1}{(g(x+1))^{\frac{1}{x+1}}}\cdot\frac{x}{x+1}\right)^{\cos^2 t} = \left(b\cdot\frac{e}{b}\cdot 1\right)^{\cos^2 t} = e^{\cos^2 t}.$$

Therefore: $\lim_{x\to\infty}(f(x))^{\sin^2 t}\left((g(x))^{\frac{\cos^2 t}{x+1}} - (g(x))^{\frac{\cos^2 t}{x}}\right) = \frac{a^{\sin^2 t}\cdot b^{\cos^2 t}}{e^{\cos^2 t}}\cdot\ln e^{\cos^2 t} =$

$$= \frac{a^{\sin^2 t}\cdot b^{\cos^2 t}}{e^{\cos^2 t}}\cdot\cos^2 t,\text{ and we are done.}$$

□

Theorem 3.

$$\lim_{x\to\infty}\left(x^{\cosh^2 t}\left(((\Gamma(x+1))^{-\frac{\sinh^2 t}{x}} - ((\Gamma(x+2))^{-\frac{\sinh^2 t}{x+1}}\right)\right) = e^{\sinh^2 t}\cdot\sinh^2 t$$

where $t \in \mathbb{R}$ and Γ is the Gamma function (Euler integral of second kind).

Proof.

We have: $\lim_{x\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}^*}}\frac{(\Gamma(n+1))^{\frac{1}{n}}}{n} = \lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} \overset{\text{C-D'A}}{=}$

$$= \lim_{n\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right) = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}.\text{ Let}$$

$$f(x)=x^{\cosh^2 t}\Big((\Gamma(x+1))^{-\frac{\sinh^2 t}{x}}-(\Gamma(x+2))^{-\frac{\sinh^2 t}{x+1}}=-x^{\cosh^2 t}(\Gamma(x+1))^{-\frac{\sinh^2 t}{x}}(u(x)-1)$$

where $u:\mathbb{R}_+^*\to\mathbb{R}, u(x)=\Big(\dfrac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\Big)^{-\sinh^2 t}$. We deduce that:

$$\lim_{x\to\infty}u(x)=\lim_{x\to\infty}\Big(\frac{\Gamma(x+2))^{\frac{1}{x+1}}}{x+1}\cdot\frac{x}{(\Gamma(x+1))^{\frac{1}{x}}}\cdot\frac{x+1}{x}\Big)^{-\sinh^2 t}=\Big(\frac{1}{e}\cdot e\cdot 1\Big)^{-\sinh^2 t}=1.$$

We have, $\displaystyle\lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)}=1$. Also, we have:

$$\lim_{x\to\infty}(u(x))^x=\lim_{x\to\infty}\Big(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\Big)^{-x\sinh^2 t}=\lim_{x\to\infty}\left(\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{-\sinh^2 t}=$$

$$=\lim_{x\to\infty}\left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{-\sinh^2 t}=e^{-\sinh^2 t}.\ \text{Therefore:}$$

$$\lim_{x\to\infty}f(x)=-\lim_{x\to\infty}\left(x^{\cosh^2 t}\Big(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}\cdot x\Big)^{-\sinh^2 t}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\right)=$$

$$=-\lim_{x\to\infty}\left(\Big(\frac{\Gamma(x+1)^{\frac{1}{x}}}{x}\Big)^{-\sinh^2 t}\cdot x^{\cosh^2 t-\sinh^2 t}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\right)=$$

$$=-\Big(\frac{1}{e}\Big)^{-\sinh^2 t}\lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)}\cdot\ln(\lim_{x\to\infty}(u(x))^x)=-e^{\sinh^2 t}\cdot 1\cdot\ln e^{-\sinh^2 t}=e^{\sinh^2 t}\cdot\sinh^2 t$$

□

Theorem 4.

$$\lim_{x\to\infty}\left(x^{\sin^2 t}\left(((\Gamma(x+2))^{\frac{\cos^2 t}{x+1}}-((\Gamma(x+1))^{\frac{\cos^2 t}{x}}\right)\right)=e^{\cos^2 t}\cdot\cos^2 t$$

where $t\in\mathbb{R}$ and Γ is the Gamma function (Euler integral of the second kind).

Proof.

We have: $\displaystyle\lim_{x\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}=\lim_{\substack{n\to\infty\\ n\in\mathbb{N}^*}}\frac{(\Gamma(n+1))^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{\text{C-D'A}}{=}$

$$=\lim_{n\to\infty}\Big(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\Big)=\lim_{n\to\infty}\Big(\frac{n}{n+1}\Big)^n=\frac{1}{e}.\ \text{We denote:}$$

$$f(x)=x^{\sin^2 t}\Big((\Gamma(x+2))^{\frac{\cos^2 t}{x+1}}-(\Gamma(x+1))^{\frac{\cos^2 t}{x}}\Big)=x^{\sin^2 t}(\Gamma(x+1))^{\frac{\cos^2 t}{x}}(u(x)-1)$$

where $u:\mathbb{R}_+^*\to\mathbb{R}, u(x)=\left(\dfrac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^{\cos^2 t}$. We deduce that:

$$\lim_{x\to\infty}u(x)=\lim_{x\to\infty}\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{x+1}\cdot\frac{x}{(\Gamma(x+1))^{\frac{1}{x}}}\cdot\frac{x+1}{x}\right)^{\cos^2 t}=\Big(\frac{1}{e}\cdot e\cdot 1\Big)^{\cos^2 t}=1$$

We have, $\displaystyle\lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)}=1$ and als we have:

$$\lim_{x\to\infty}(u(x))^x = \lim_{x\to\infty}\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^{x\cos^2 t} = \lim_{x\to\infty}\left(\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{\cos^2 t} =$$

$$= \lim_{x\to\infty}\left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{\cos^2 t} = e^{\cos^2 t}. \text{ Therefore:}$$

$$\lim_{x\to\infty} f(x) = -\lim_{x\to\infty}\left(x^{\sin^2 t}\left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}\cdot x\right)^{\cos^2 t}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\right) =$$

$$= -\lim_{x\to\infty}\left(\left(\frac{(\Gamma(x+1)^{\frac{1}{x}}}{x}\right)^{\cos^2 t}\cdot x^{\sin^2 t+\cos^2 t}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\right) =$$

$$= e^{\cos^2 t}\lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)}\cdot\ln(\lim_{x\to\infty}(u(x))^x) = e^{\cos^2 t}\cdot 1\cdot\ln e^{\cos^2 t} = e^{\cos^2 t}\cdot\cos^2 t.$$

□

Theorem 5. Let $\{g_n\}_{n\geq 0}$ be a sequence defined by $g_n = \frac{(n+2)^{n+1}}{(n+1)^n}$, x be a real number and the sequence $\{G_n(x)\}_{x\geq 1}$, defined by $G_n(x) = n^{\sin^2 x}(g_{n+1}^{\cos^2 x} - g_n^{\cos^2 x})$ then

$$\lim_{n\to\infty} G_n(x) = \cos^2 x\cdot e^{\cos^2 x}.$$

Proof. We have that:

$$G_n(x) = n^{1-\cos^2 x}(g_{n+1}^{\cos^2 x} - g_n^{\cos^2 x}) = n\cdot\left(\frac{g_n}{n}\right)^{\cos^2 x}(u_n - 1) =$$

$$= \left(\frac{g_n}{n}\right)^{\cos^2 x}\frac{u_n-1}{\ln u_n}\ln u_n^n = \frac{(n+2)^{(n+1)\cos^2 x}}{n^{\cos^2 x}(n+1)^{n\cos^2 x}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n =$$

$$= \left(\frac{n+2}{n+1}\right)^{(n+1)\cos^2 x}\left(\frac{n+1}{n}\right)^{\cos^2 x}\frac{u_n-1}{\ln u_n}\ln u_n^n, \forall n\in\mathbb{N}^* \tag{1}$$

where we denote $u_n = \left(\frac{g_{n+1}}{g_n}\right)^{\cos^2 x} = \left(\frac{(n+3)^{n+2}}{(n+2)^{n+1}}\cdot\frac{(n+1)^n}{(n+2)^{n+1}}\right)^{\cos^2 x}, \forall n\in\mathbb{N}.$

Therefore, $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\left(\frac{n+3}{n+2}\right)^{n+2}\left(\frac{n+1}{n+2}\right)^n\right)^{\cos^2 x} = \left(e\cdot\frac{1}{e}\right)^{\cos^2 x} = 1$, so

$$\lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1.$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{(n+3)^{n+2}}{(n+2)^{n+1}}\cdot\frac{(n+1)^n}{(n+2)^{n+1}}\right)^{n\cos^2 x} = \lim_{n\to\infty}\left(\left(\frac{(n+1)(n+3)}{(n+2)^2}\right)^{n+1}\cdot\frac{n+3}{n+1}\right)^{n\cos^2 x} =$$

$$= \lim_{n\to\infty}\left(\frac{n^2+4n+3}{n^2+4n+4}\right)^{(n^2+n)\cos^2 x}\cdot\lim_{n\to\infty}\left(\frac{n+3}{n+1}\right)^{n\cos^2 x} = e^{-\cos^2 x}\cdot e^{2\cos^2 x} = e^{\cos^2 x}$$

Taking limit in (1) with $n\to\infty$ we obtain that:

$$\lim_{n\to\infty} G_n(x) = e^{\cos^2 x}\cdot 1\cdot 1\cdot\ln(\lim_{n\to\infty} u_n^n) = e^{\cos^2 x}\cdot\ln e^{\cos^2 x} = \cos^2 x\cdot e^{\cos^2 x}$$

□

Observation. For $x=0$, yields that: $\lim_{n\to\infty} G_n(0)=\lim_{n\to\infty} G_n=e$, i.e. the limit of the sequences of Mihail Ghermănescu.

Theorem 6. For $\{a_n\}_{n\geq 0}, a_n=\frac{(n+2)^{n+1}}{(n+1)^n}, x\in(-\infty,\infty), \{b_n(x)\}_{n\geq 1}$, $b_n(x)=n^{\sin^2 x}(a_{n+1}^{\cos^2 x}-a_n^{\cos^2 x})$, then $\lim_{n\to\infty} b_n(x)=\cos^2 x\cdot e^{\cos^2 x}$.

Proof. We have that:

$$b_n(x)=n^{1-\cos^2 x}(a_{n+1}^{\cos^2 x}-a_n^{\cos^2 x})=n\cdot\Big(\frac{a_n}{n}\Big)^{\cos^2 x}(u_n-1)=$$

$$=\Big(\frac{a_n}{n}\Big)^{\cos^2 x}\frac{u_n-1}{\ln u_n}\ln u_n^n=\frac{(n+2)^{(n+1)\cos^2 x}}{n^{\cos^2 x}(n+1)^{n\cos^2 x}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n=$$

$$=\Big(\frac{n+2}{n+1}\Big)^{(n+1)\cos^2 x}\Big(\frac{n+1}{n}\Big)^{\cos^2 x}\frac{u_n-1}{\ln u_n}\ln u_n^n, \forall n\in\mathbb{N}^* \tag{1}$$

where we denote $u_n=\Big(\frac{a_{n+1}}{a_n}\Big)^{\cos^2 x}=\Big(\frac{(n+3)^{n+2}}{(n+2)^{n+1}}\cdot\frac{(n+1)^n}{(n+2)^{n+1}}\Big)^{\cos^2 x}, \forall n\in\mathbb{N}$.

Therefore, $\lim\limits_{n\to\infty} u_n=\lim\limits_{n\to\infty}\Big(\Big(\frac{n+3}{n+2}\Big)^{n+2}\Big(\frac{n+1}{n+2}\Big)^n\Big)^{\cos^2 x}=\Big(e\cdot\frac{1}{e}\Big)^{\cos^2 x}=1$, so,

$$\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1.$$

$$\lim_{n\to\infty} u_n^n=\lim_{n\to\infty}\Big(\frac{(n+3)^{n+2}}{(n+2)^{n+1}}\cdot\frac{(n+1)^n}{(n+2)^{n+1}}\Big)^{n\cos^2 x}=\lim_{n\to\infty}\Big(\Big(\frac{(n+1)(n+3)}{(n+2)^2}\Big)^{n+1}\cdot\frac{n+3}{n+1}\Big)^{n\cos^2 x}=$$

$$=\lim_{n\to\infty}\Big(\frac{n^2+4n+3}{n^2+4n+4}\Big)^{(n^2+n)\cos^2 x}\cdot\lim_{n\to\infty}\Big(\frac{n+3}{n+1}\Big)^{n\cos^2 x}=e^{-\cos^2 x}\cdot e^{2\cos^2 x}=e^{\cos^2 x}$$

Taking the limit in (1) with $n\to\infty$ we obtain that:

$$\lim_{n\to\infty} b_n(x)=e^{\cos^2 x}\cdot 1\cdot 1\cdot\ln(\lim_{n\to\infty} u_n^n)=e^{\cos^2 x}\cdot\ln e^{\cos^2 x}=\cos^2 x\cdot e^{\cos^2 x}$$

□

Theorem 7. Let the sequence $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}, a_n, b_n\in\mathbb{R}_+^*, \forall n\in\mathbb{N}^*$ such that

$$\lim_{n\to\infty}(a_{n+1}-a_n)=a\in\mathbb{R}_+^* \text{ and } \lim_{n\to\infty}\frac{b_{n+1}}{nb_n}=b\in\mathbb{R}_+^*. \text{ For } x\in\mathbb{R}, \text{ then}$$

$$\lim_{n\to\infty}\Big(a_n^{\cosh^2 x}\Big(\Big(\sqrt[n+1]{b_{n+1}}\Big)^{-\sinh^2 x}-\Big(\sqrt[n]{b_n}\Big)^{-\sinh^x}\Big)\Big)=-a^{\cosh^2 x}b^{-\sinh^2 x}e^{\sinh^2 x}\sinh^2 x$$

Proof. By Cesaro - Stolz lemma we have:

$\lim\limits_{n\to\infty}\frac{a_n}{n}=\lim\limits_{n\to\infty}\frac{a_{n+1}-a_n}{(n+1)-n}=\lim\limits_{n\to\infty}(a_{n+1}-a_n)=a$ and by Cauchy-D'Alembert

theorem we have: $\lim\limits_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}=\lim\limits_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}}=\lim\limits_{n\to\infty}\Big(\frac{b_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_n}\Big)=$

$$=\lim_{n\to\infty}\Big(\frac{b_{n+1}}{nb_n}\Big(\frac{n}{n+1}\Big)^{n+1}\Big)=\frac{b}{e}. \text{ So,}$$

$$\lim_{n\to\infty}\Big(a_n^{\cosh^2 x}\Big(\Big(\sqrt[n+1]{b_{n+1}}\Big)^{-\sinh^2 x}-\Big(\sqrt[n]{b_n}\Big)^{-\sinh^2 x}\Big)\Big)=$$

$$=\lim_{n\to\infty}\Bigg(\Big(\frac{a_n}{n}\Big)^{\cosh^2 x}\Big(\frac{\sqrt[n]{b_n}}{n}\Big)^{-\sinh^2 x}n^{\cosh^2 x-\sinh^2 x}(u_n-1)\Bigg)=$$

$= a^{\cosh^2 x} b^{-\sin^2 x} e^{\sinh^2 x} \lim_{n\to\infty} \Big(\frac{u_n - 1}{\ln u_n} \ln u_n^n \Big)$, where $u_n = \Big(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \Big)^{-\sinh^2 x}, \forall n \geq 2$

We have: $\lim_{n\to\infty} u_n = 1$, $\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$ and $u_n^n = \lim_{n\to\infty} \Big(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \Big)^{-n \sinh^2 x} =$

$$= \Big(\lim_{n\to\infty} \Big(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \Big)^n \Big)^{-n \sinh^2 x} =$$

$$= \Big(\lim_{n\to\infty} \frac{b_{n+1}}{n b_n} \cdot \frac{n+1}{\sqrt[n+1]{b_{n+1}}} \cdot \frac{n}{n+1} \Big)^{-\sinh^2 x} = \Big(b \cdot \frac{e}{b} \cdot 1 \Big)^{-\sinh^2 x} = e^{-\sinh^2 x}. \text{ So,}$$

$$= \lim_{n\to\infty} \Big(a_n^{\cosh^2 x} \Big(\Big(\sqrt[n+1]{b_{n+1}} \Big)^{-\sinh^2 x} \Big) \Big) = a^{\cosh^2 x} b^{-\sinh^2 x} e^{\sinh^2 x} \cdot 1 \cdot \ln(\lim_{n\to\infty} u_n^n) =$$

$$= a^{\cosh^2 x} b^{-\sin^2 x} e^{\sinh^2 x} \cdot 1 \cdot \ln(e^{-\sinh^2 x}) = -a^{\cosh^2 x} b^{-\sinh^2 x} e^{\sinh^2 x} \sinh^2 x$$

and we are done. □

Theorem 8. Let $x \in \mathbb{R}$, and $(L_n(x))_{n\geq 2}$, given by

$L_n(x) = n^{\cos^2 x} \Big(\Big(\sqrt[n+1]{(n+1)!} \Big)^{\sin^2 x} - \Big(\sqrt[n]{n!} \Big)^{\sin^2 x} \Big)$, then $\lim_{n\to\infty} L_n(x) = \frac{\sin^2 x}{e^{\sin^2 x}}$.

Proof. We have:
(1)

$$L_n(x) = n^{\cos^2 x} \Big(\Big(\sqrt[n+1]{(n+1)!}^{\sin^2 x} - \Big(\sqrt[n]{n!} \Big)^{\sin^2 x} \Big) = n^{\cos^2 x} \Big(\sqrt[n]{n!} \Big)^{\sin^2 x} (u_n - 1) =$$

$$= \Big(\frac{\sqrt[n]{n!}}{n} \Big)^{\sin^2 x} n^{\sin^2 x + \cos^2 x} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n = \Big(\frac{\sqrt[n]{n!}}{n} \Big)^{\sin^2 x} n \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n,$$

where

$u_n = \Big(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n}} \Big)^{\sin^2 x}$, and $\lim_{n\to\infty} \Big(\frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \frac{n}{\sqrt[n]{n!}} \cdot \frac{n+1}{n} \Big)^{\sin^2 x} = \Big(\frac{1}{e} \cdot e \cdot 1 \Big)^{\sin^2 x} = 1.$

So, $\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$. Also, we have:

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \Big(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \Big)^{n \sin^2 x} = \lim_{n\to\infty} \Big(\frac{(n+1)!}{n!} \cdot \frac{1}{\sqrt[n+1]{(n+1)!}} \Big)^{\sin^2 x} =$$

$$= \lim_{n\to\infty} \Big(\frac{n+1}{\sqrt[n+1]{(n+1)!}} \Big)^{\sin^2 x} = e^{\sin^2 x}, \text{ then by (1) we obtain:}$$

$$\lim_{n\to\infty} L_n(x) = \Big(\frac{1}{x} \Big)^{\sin^2 x} \cdot 1 \cdot \ln e^{\sin^2 x} = \frac{\sin^2 x}{e^{\sin^2 x}}.$$

□

Observation. If $x = \frac{\pi}{2}$, then $\sin x = 1, \cos x = 0$ and
$L_n\Big(\frac{\pi}{2}\Big) = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$, i.e. we obtain the sequence of Traian Lalescu.

$$\lim_{n\to\infty} L_n\Big(\frac{\pi}{2}\Big) = \lim_{n\to\infty} L_n = \frac{1}{e}.$$

Theorem 9. Let $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}, x_n \in \mathbb{R}_+^*, y \in \mathbb{R}$ such that
$\lim_{n\to\infty}(x_{n+1} - x_n) = x \in \mathbb{R}_+^*, \lim_{n\to\infty} y_n = y \in \mathbb{R}$ and
$\lim_{n\to\infty}((y_n - y)n) = z \in \mathbb{R}^*$. Then $\lim_{n\to\infty}(x_{n+1} y - x_n y_n) = x(y - z)$

Proof. Let
(1)
$(u_n)_{n\geq 1}, u_n = x_{n+1}y - x_n y_n = x_{n+1}y - x_n y - x_n(y_n - y) = y(x_{n+1} - x_n) - x_n(y_n - y), \forall n \in \mathbb{N}^*$

We have:

$$\lim_{n\to\infty}\frac{x_n}{n} = \lim_{n\to\infty}\frac{x_{n+1}-x_n}{(n+1)-n} = \lim_{n\to\infty}(x_{n+1}-x_n) = x \tag{2}$$

then:

$$\lim_{n\to\infty} u_n = y\lim_{n\to\infty}(x_{n+1}-x_n) - \lim_{n\to\infty}\frac{x_n}{n}\cdot\lim_{n\to\infty}(n(y_n-y)) = yx - xz = x(y-z).$$

□

Theorem 10. Let $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}, x_n, y_n \in \mathbb{R}_+^*$, such that $\lim_{n\to\infty}\frac{x_n}{n} = x \in \mathbb{R}_+^*$ and $\lim_{n\to\infty}(y_{n+1}-y_n) = y \in \mathbb{R}_+^*$. Then $\lim_{n\to\infty}\left(\frac{y_{n+1}}{y_n}\right)^{x_n} = e^x$.

Proof. We have that:

$$\lim_{n\to\infty}\frac{y_n}{n} = \lim_{n\to\infty}\frac{y_{n+1}-y_n}{(n+1)-n} = \lim_{n\to\infty}(y_{n+1}-y_n) = y.$$

$$\lim_{n\to\infty}\left(\frac{y_{n+1}}{y_n}\right)^{x_n} = \lim_{n\to\infty}\left(\left(1+\frac{y_{n+1}-y_n}{y_n}\right)^{\frac{y_n}{y_{n+1}-y_n}}\right)^{\frac{x_n(y_{n+1}-y_n)}{y_n}} =$$

$$= e^{\lim_{n\to\infty}\frac{x_n}{n}\cdot\frac{n}{y_n}(y_{n+1}-y_n)} = e^{x\cdot\frac{1}{y}\cdot y} = e^x.$$

□

Theorem 11. For m a positive integer

$$\lim_{n\to\infty}\left(n\left(me - \sum_{k=1}^{m}\left(1+\frac{1}{n}\right)^{n+k}\right)\right) = -\frac{m^2 e}{2}$$

Proof. Denote $e_n(k) = \left(1+\frac{1}{n}\right)^{n+k}$ and then we have to calculate:

$$\lim_{n\to\infty}\left(n\left(me - \sum_{k=1}^{m}\left(1+\frac{1}{n}\right)^{n+k}\right)\right) = \sum_{k=1}^{m}\lim_{n\to\infty}(n(e-e_n(k))) \tag{1}$$

We have: $\lim\limits_{n\to\infty}(n(e-e_n(k)) = \lim\limits_{n\to\infty}(n(e-e_n u_n))$, where $e_n = \left(1+\frac{1}{n}\right)^n, u_n = \left(1+\frac{1}{n}\right)^k$ where $\lim\limits_{n\to\infty} e_n = e$, $\lim\limits_{n\to\infty} u_n = 1$, $\lim\limits_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1$, $\lim\limits_{n\to\infty} u_n^n = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{kn} = e^k$, $\lim\limits_{n\to\infty}(n(e-e_n)) = \frac{e}{2}$.
Therefore:

$$\lim_{n\to\infty}(n(e-e_n(k))) = \lim_{n\to\infty}(n(e-e_n u_n)) = \lim_{n\to\infty}(n((e-e_n)+e_n-e_n u_n)) =$$

$$= \lim_{n\to\infty}(n(e-e_n)) - \lim_{n\to\infty} e_n(n(u_n-1)) = \frac{e}{2} - e\lim_{n\to\infty}(n(u_n-1)) =$$

$$= \frac{e}{2} - e\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}\lim_{n\to\infty}\ln u_n^n = \frac{e}{2} - e\cdot 1\cdot\ln\lim_{n\to\infty}u_n^n = \frac{e}{2} - e\cdot\ln e^k = \frac{e}{2} - ke \tag{2}$$

By (1) and (2) it follows that:

$$\lim_{n\to\infty}\left(n\left(me-\sum_{k=1}^{m}\left(1+\frac{1}{n}\right)^{n+k}\right)\right)=\sum_{k=1}^{m}\lim_{n\to\infty}(n(e-e_n(k)))=\sum_{k=1}^{m}\left(\frac{e}{2}-ke\right)=$$

$$=\frac{e}{2}\sum_{k=1}^{m}(1-2k)=\frac{e}{2}\left(m-2\cdot\frac{m(m+1)}{2}\right)=-\frac{m^2e}{2},$$

and we are done. □

Theorem 12. Let $(x_n)_{n\geq1}$ be a sequence which satisfy $-\ln(mn+x_n)+\sum_{k=1}^{mn}\frac{1}{k}=\gamma$, where m is positive integer and γ is Euler-Mascheroni's constant, then $\lim_{n\to\infty}x_n=\frac{1}{2}$.

Proof. It is well-known that $\lim_{n\to\infty}\gamma_n=\gamma$ and $\gamma_n=-\ln n+\sum_{k=1}^{n}\frac{1}{k}$. Hence,

$$-\ln(mn+x_n)+\sum_{k=1}^{mn}\frac{1}{k}=\gamma\Leftrightarrow-\ln(mn+x_n)+\ln(mn)-\ln(mn)+\sum_{k=1}^{mn}\frac{1}{k}=\gamma\Leftrightarrow$$

$$\Leftrightarrow-\ln\frac{mn-x_n}{mn}+\gamma_{mn}=\gamma\Leftrightarrow\gamma_{mn}-\gamma=\ln\left(1+\frac{x_n}{mn}\right)\Leftrightarrow e^{\gamma_{mn}-\gamma}-1=\frac{x_n}{mn}\Leftrightarrow$$

$$\Leftrightarrow x_n=mn(e^{\gamma_{mn}-\gamma}-1)=mn\cdot\frac{e^{\gamma_{mn}-\gamma}-1}{\gamma_{mn}-\gamma}\cdot(\gamma_{mn}-\gamma),$$

So, we obtain that:

$$\lim_{n\to\infty}x_n=m\cdot\lim_{n\to\infty}\frac{e^{\gamma_{mn}-\gamma}-1}{\gamma_{mn}-\gamma}\cdot\lim_{n\to\infty}(n(\gamma_{mn}-\gamma))=\lim_{n\to\infty}(mn(\gamma_{mn}-\gamma))=\frac{1}{2}.$$

We prove that: $\lim_{n\to\infty}(mn(\gamma_{mn}-\gamma))=\frac{1}{2}$. Indeed, by Young's inequalities we have that:

$$\frac{1}{2(mn+1)}<\gamma_{mn}-\gamma<\frac{1}{2mn}\Leftrightarrow\frac{mn}{2(mn+1)}<mn(\gamma_{mn}-\gamma)<\frac{1}{2},\forall n\in\mathbb{N}^*,$$

where by taking to limit with $n\to\infty$ we deduce that $\lim_{n\to\infty}(mn(\gamma_{mn}-\gamma))=\frac{1}{2}$. □

Observation. The limit of the sequence $(x_n)_{n\geq1}$ is independent of $m\in\mathbb{N}^*$

Theorem 13. Let $\{x_n\}_{n\geq1}$ defined by $-\ln(m(n+x_n))+\sum_{k=1}^{mn}\frac{1}{k}=\gamma,\forall n\in\mathbb{N}^*$ where $m\in\mathbb{N}^*$ and γ is Euler-Mascheroni's constant, then $\lim_{n\to\infty}x_n=\frac{1}{2m}$.

Proof. It is well-known that $\lim_{n\to\infty}\gamma_n=\gamma$ and $\gamma_n=-\ln n+\sum_{k=1}^{n}\frac{1}{k}$. Therefore we have that:

$$-\ln(m(n+x_n))+\sum_{k=1}^{mn}\frac{1}{k}=\gamma\Leftrightarrow-\ln(m(n+x_n))+\ln(mn)-\ln(mn)+\sum\sum_{k=1}^{mn}\frac{1}{k}=\gamma\Leftrightarrow$$

$$\Leftrightarrow-\ln\frac{m(n+x_n)}{mn}+\gamma_{mn}=\gamma\Leftrightarrow\gamma_{mn}-\gamma=\ln\left(1+\frac{x_n}{n}\right)\Leftrightarrow e^{\gamma_{mn}-\gamma}-1=\frac{x_n}{n}\Leftrightarrow$$

$$\Leftrightarrow x_n=n(e^{\gamma_{mn}-1}-1)=\frac{e^{\gamma_{mn}-\gamma}-1}{\gamma_{mn}-\gamma}\cdot n(\gamma_{mn}-\gamma).$$

So, we obtain that:

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}\frac{\gamma_{mn}-1}{\gamma_{mn}-\gamma}\cdot\lim_{n\to\infty}(n(\gamma_{mn}-\gamma))=\lim_{n\to\infty}(n(\gamma_{mn}-1))\tag{1}$$

By Young's inequalities we have that:

$$\frac{1}{2(mn+1)} < \gamma_{mn} - \gamma < \frac{1}{2mn} \Leftrightarrow \frac{mn}{2(mn+1)} < mn(\gamma_{mn} - \gamma) < \frac{1}{2}, \forall n \in \mathbb{N}^*,$$

from where taking limit with $n \to \infty$ we deduce that:

$$\lim_{n\to\infty} (mn(\gamma_{mn} - \gamma)) = \frac{1}{2} \tag{2}$$

From (1) and (2) it follows that:

$$\lim_{n\to\infty} x_n = \frac{1}{2m}, m \in \mathbb{N}^*.$$

□

Theorem 14. Let $(a_n)_{n\geq 1}$ be an arithmetic progression and $m \in \mathbb{N}^*$, then:

$$\lim_{n\to\infty}\left(\left(\sum_{k=1}^{m}\left(1+\frac{1}{n}\right)^{n+a_k} - me\right)\cdot n\right) = \frac{me}{2}(a_1 + a_m - 1)$$

Proof. We denote $e_n(a_k) = \left(1+\frac{1}{n}\right)^{n+a_k} = e_n \cdot \left(1+\frac{1}{n}\right)^{a_k}$ where $e_n = \left(1+\frac{1}{n}\right)^n$ $\lim_{n\to\infty} e_n = e$. We have to calculate:

$$\lim_{n\to\infty}\left(\left(\sum_{k=1}^{m} e_n(a_k) - me\right)n\right) = \sum_{k=1}^{m} \lim_{n\to\infty}((e_n(a_k) - e)n) \tag{1}$$

We denote $u_n(a_k) = \left(1+\frac{1}{n}\right)^{a_k}$, then $\lim_{n\to\infty} u_n(a_k) = 1$ and $\lim_{n\to\infty} \frac{u_n(a_k) - 1}{\ln u_n(a_k)} = 1$

$$\lim_{n\to\infty} u_n^n(a_k) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{na_k} = e^{a_k}, \forall k = \overline{1,m}.$$

$$\lim_{n\to\infty}((e_n(a_k)-e)n) = \lim_{n\to\infty}((e_n \cdot u_n(a_k)-e)n) = \lim_{n\to\infty}((e_n(u_n(a_k)-1)+e_n-e)n) =$$

$$e\lim_{n\to\infty}((u_n(a_k)-1)n)+\lim_{n\to\infty}((e_n-e)n) = e\lim_{n\to\infty}\left(\frac{u_n(a_k)-1}{\ln u_n(a_k)}\cdot \ln u_n^n(a_k)\right)+\lim_{n\to\infty}((e_n-e)n) =$$

$$= e\cdot 1\cdot \ln e^{a_k} + \lim_{n\to\infty}((e_n - e)n) = a_k e + \lim_{n\to\infty}((e_n - e)n) \tag{2}$$

We know that: $\frac{e}{2n+2} < e - e_n < \frac{e}{2n+1}, \forall n \in \mathbb{N}^*$ (Polya - Szegö inequalities). So,

$$\frac{ne}{2n+2} < n(e - e_n) < \frac{ne}{2n=1}, \forall n \in \mathbb{N}^* \text{ therefore}$$

$$\lim_{n\to\infty}(n(e_n - e)) = -\frac{e}{2} \tag{3}$$

By (2) and (3) we deduce that:

$$\lim_{n\to\infty}((e_n(a_k) - e)n) = a_k e - \frac{e}{2} = \frac{e}{2}(2a_k - 1), \forall k = \overline{1,m} \tag{4}$$

By (1) and (4) we obtain that:

$$\lim_{n\to\infty}\left(\left(\sum_{k=1}^{m} e_n(a_k) - me\right)n\right) = \frac{e}{2}\sum_{k=1}^{m}(2a_k - 1) = e\sum_{k=1}^{m} a_k - \frac{me}{2} = e\cdot\frac{(a_1+a_2)m}{2} - \frac{me}{2} =$$

$$= \frac{me}{2}(a_1 + a_m - 1).$$

□

Theorem 15. Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$, be real sequences with $a_n \neq a_{n+1}$, $b_n \neq b_{n+1}$, such that $\lim_{n\to\infty} a_n = a \in \mathbb{R}, \lim_{n\to\infty} b_n = b \in \mathbb{R}$, $\lim_{n\to\infty}(n(a_{n+1} - a_n)) = c \in \mathbb{R}, \lim_{n\to\infty}(n(b_{n+1} - b_n)) = d \in \mathbb{R}$. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions with continuous derivatives on $\mathbb{R}$, then:

$$\lim_{n\to\infty}(n \cdot (f(a_{n+1}g(b_{n+1}) - f(a_n)g(a_n))) = c \cdot f'(a) \cdot g(b) + d \cdot f(a) \cdot g'(b).$$

Proof. Applying Lagrange's theorem, for the function f on each interval $[a_n, a_{n+1}]$ yields that there exists x_n between a_n and a_{n+1} such that:

$$f(a_{n+1}) - f(a_n) = (a_{n+1} - a_n)f'(x_n), \forall n \in \mathbb{N}^* \tag{1}$$

Since x_n is between a_n and $a_{n+1}, \forall n \in \mathbb{N}^*$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = a$ we have that $\lim_{n\to\infty} x_n = a$. By (1) we have:
$n(f(a_{n+1}) - f(a_n)) = n(a_{n+1} - a_n)f'(x_n), \forall n \in \mathbb{N}^*$ so we deduce that:

$$\lim_{n\to\infty}(n(f(a_{n+1} - f(a_n)) = c \lim_{n\to\infty} f'(x_n) = cf'(\lim_{n\to\infty} x_n) = c \cdot f'(a) \tag{2}$$

Analogously we deduce that:

$$\lim_{n\to\infty}(n(g(b_{n+1}) - g(b_n))) = d \cdot g'(b) \tag{3}$$

Also, we have that:

$n(f(a_{n+1})g(b_{n+1}) - f(a_n)g(b_n)) = n((f(a_{n+1} - f(a_n))g(b_{n+1}) + f(a_n)(g(b_{n+1} - g(b_n))),$

hence taking to limit with $n \to \infty$ and taking into account by (2), (3) we obtain:

$$\lim_{n\to\infty}(n \cdot (f(a_{n+1})g(b_{n+1}) - f(a_n)g(a_n)) = c \cdot f'(a) \cdot g(b) + d \cdot f(a) \cdot g'(b),$$

which should be calculated, and we are done. □

Theorem 16. Let $e_n = \left(1 + \frac{1}{n}\right)^n, n \in \mathbb{N}^*$ and $\{x_n\}_{n\in\mathbb{N}^*}$ defined by $x_n = (n!) \cdot (e_1 \cdot e_2 \cdot \ldots \cdot e_n), n \in \mathbb{N}^*$, then $\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}}\right) = 1$.

Proof. We have:

$$B_n = \frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}} = \frac{n^2}{\sqrt[n]{x_n}}(u_n - 1) = \frac{n}{\sqrt[n]{x_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \forall n \in \mathbb{N}^* - \{1\},$$

where $u_n = \left(\frac{n+1}{n}\right)^2 \cdot \frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}, \forall n \geq 2$. We have:

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{x_n}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{x_n}} = \lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{x_{n+1}} \cdot \frac{x_n}{n^n}\right) = \lim_{n\to\infty}\left(e_n \cdot \frac{(n+1)x_n}{x_{n+1}}\right) =$$

$$= e \cdot \lim_{n\to\infty} \frac{n!(n+1)e_1e_2\ldots e_n}{(n+1)!e_1e_2\ldots e_ne_{n+1}} = e \cdot \lim_{n\to\infty} \frac{1}{e_{n+1}} = e \cdot \frac{1}{e} = 1.$$

Also, we have that:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 \cdot \lim_{n\to\infty}\left(\frac{\sqrt[n]{x_n}}{n} \cdot \frac{n+1}{\sqrt[n+1]{x_{n+1}}} \cdot \frac{n}{n+1}\right) = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

and then $\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$, so,

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} e_n^2 \cdot \lim_{n\to\infty} \frac{x_n}{x_{n+1}} \cdot \sqrt[n+1]{x_{n+1}} = e^2 \cdot \lim_{n\to\infty} \frac{\sqrt[n+1]{x_{n+1}}}{(n+1)e_{n+1}} = e^2 \cdot \frac{1}{e} = e.$$

Hence: $\lim_{n\to\infty} B_n = \lim_{n\to\infty} \frac{n}{\sqrt[n]{x_n}} \cdot \lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} \cdot \ln(\lim_{n\to\infty} u_n^n) = 1 \cdot 1 \cdot \ln e = 1$, and we are done.

□

Theorem 17. Let $a, b \in \mathbb{R}_+, \gamma_n(a,b) = -\ln(n+a) + \sum_{k=1}^{n} \frac{1}{k+b}$ with $\lim_{n\to\infty} \gamma_n(a,b) = \gamma(a,b) \in \mathbb{R}$, then $\lim_{n\to\infty} \Big(\ln \frac{e}{n+a} + \sum_{k=1}^{n} \frac{1}{k+b} - \gamma(a,b) \Big)^n = e^{b-a+\frac{1}{2}} = e^{b-a} \cdot \sqrt{e}$.

Proof.

We have: $\lim_{n\to\infty} \Big(\ln \frac{e}{n+a} + \sum_{k=1}^{n} \frac{1}{k+b} - \gamma(a,b) \Big)^n = \lim_{n\to\infty} (1 + \gamma_n(a,b) - \gamma(a,b))^n =$

(1)

$= \lim_{n\to\infty} (1 + \gamma_n(a,b) - \gamma(a,b))^{\frac{1}{\gamma_n(a,b)-\gamma(a,b)} \cdot n(\gamma_n(a,b)-\gamma(a,b))} = e^{\lim_{n\to\infty}(n(\gamma_n(a,b)-\gamma(a,b))}$

We calculate (with Cesaro-Stolz's theorem and then the theorem of L'Hospital):

$$\lim_{n\to\infty} (n(\gamma_n(a,b) - \gamma(a,b))) = \lim_{n\to\infty} \frac{\gamma_n(a,b) - \gamma(a,b)}{\frac{1}{n}} \overset{\text{C-S}}{\underset{\frac{0}{0}}{=}}$$

$$= \lim_{n\to\infty} \frac{\gamma_{n+1}(a,b) - \gamma_n(a,b)}{\frac{1}{n+1} - \frac{1}{n}} = \lim_{n\to\infty} (n^2+n)(\gamma_n(a,b) - \gamma_{n+1}(a,b)) =$$

$$= \lim_{n\to\infty} (n^2+n)(\gamma_n(a,b) - \gamma_{n+1}(a,b) = \lim_{n\to\infty} (n^2+n)\Big(-\ln(n+a) + \ln(n+1+a) - \frac{1}{n+1+b} \Big) =$$

$$= \lim_{n\to\infty} (n^2+n)\Big(\ln \frac{n+1+a}{n+a} - \frac{1}{n+1+b} \overset{\text{L'H}(\infty\cdot 0)}{\underset{x=\frac{1}{n}, x\to 0, x>0}{=}} b-a+\frac{1}{2}$$ and by (1) we obtain that:

$$\lim_{n\to\infty} \Big(\ln \frac{e}{n+a} + \sum_{k=1}^{n} \frac{1}{k+b} - \gamma(a,b) \Big)^n = e^{b-a+\frac{1}{2}} = e^{b-a} \cdot \sqrt{e}.$$

□

Theorem 18. Let $a \in (0,\infty)$ and $e(x) = \Big(1 + \frac{1}{x}\Big)^x$, for any $x \in (0,\infty)$, then:

$$\lim_{n\to\infty} \Bigg(\lim_{x\to\infty} \Bigg(\Big(\frac{x}{n}\Big)^2 \Big(\sum_{k=1}^{n} e(x+ka) - ne(x) \Big) \Bigg) \Bigg) = \frac{ae}{4}.$$

Proof. First we prove that:

(1) $$\lim_{x\to\infty} (x^2(e(x+a) - e(x+b))) = \frac{(a-b)e}{2}, \forall a, b \in \mathbb{R}_+, a > b$$

For this we consider the function $f : [x+b, x+a] \to \mathbb{R}, f(x) = e(x)$ and for this we apply the theorem of Lagrange on $[x+b, x+a]$. Yields exists $c(x) \in (x+b, x+a)$ such that:

(2) $$f(x+a) - f(x+b) = (a-b)e'(c(x)) = (a-b)e(c(x))\Big(\ln\Big(1 + \frac{1}{c(x)}\Big) - \frac{1}{1+c(x)} \Big)$$

Since $c(x) \in (x+b, x+a)$ we have: $x+b < c(x) < x+a \Leftrightarrow 1 + \frac{b}{x} < \frac{c(x)}{x} < 1 + \frac{a}{x}$, so, $\lim_{x\to\infty} \frac{c(x)}{x} = 1$. Denoting $t = \frac{1}{c(x)}$, we have $t \to 0$ and by (2) follows that:

$$\lim_{x\to\infty} (x^2(e(x+a) - e(x+b))) = \lim_{x\to\infty} \Big(\frac{x}{c(x)} \cdot \frac{x}{c(x)} (c(x))^2 (a-b) e(c(x)) \Big(\ln\Big(1 + \frac{1}{c(x)}\Big) - \frac{1}{c(x)+1} \Big) \Big) =$$

$$= (a-b)\cdot 1\cdot e\cdot \lim_{t\to 0}\frac{\ln(1+t)-\frac{1}{1+t}}{t^2} \overset{\text{L'H}}{=} (a-b)\cdot e\cdot \lim_{t\to 0}\frac{\frac{1}{t+1}-\frac{1}{(t+1)^2}}{2t} =$$

$$= \frac{(a-b)}{2}\cdot e\cdot \lim_{t\to 0}\frac{t+1-1}{t(t+1)^2} = \frac{(a-b)e}{2}\cdot \lim_{t\to 0}\frac{1}{(t+1)^2} = \frac{(a-b)e}{2} \tag{3}$$

If $b=0$ then we have:

$$\lim_{x\to\infty}(x^2(e(x+a)-e(x))) = \frac{ae}{2} \tag{4}$$

We deduce that:

(5)

$$\lim_{x\to\infty}\left(\left(\sum_{k=1}^{n} e(x+ka)-ne(x)\right)x^2\right) = \sum_{k=1}^{n}\lim_{x\to\infty}(x^2(e(x+ka)-e(x))) = \sum_{k=1}^{n}\frac{kae}{2} = \frac{aen(n+1)}{4}$$

Hence,

$$\lim_{n\to\infty}\Big(\lim_{x\to\infty}\Big(\Big(\frac{x}{n}\Big)^2\Big(\sum_{k=1}^{n} e(x+ka)-ne(x)\Big)\Big)\Big) = \lim_{n\to\infty}\frac{1}{n^2}\cdot\sum_{k=1}^{n}\big(\lim_{x\to\infty}(x^2(e(x+ka)-e(x)))\big) =$$

$$= \lim_{n\to\infty}\frac{aen(n+1)}{4n^2} = \frac{ae}{4}$$

□

Theorem 19. Let $(x_n)_{n\geq 1}$, be a positive real sequence, such that $\lim_{n\to\infty}(x_{n+1}-x_n) = x > 0$, then $\lim_{n\to\infty}\Big(\frac{(n+1)x_{n+1}}{\sqrt[n+1]{(2n+1)!!}} - \frac{nx_n}{\sqrt[n]{(2n-1)!!}}\Big) = \frac{xe}{2}$.

Proof. Let $(y_n)_{n\geq 1}$,

$$y_n = \frac{(n+1)x_{n+1}}{\sqrt[n+1]{(2n+1)!!}} - \frac{nx_n}{\sqrt[n]{(2n-1)!!}} = \frac{nx_n}{\sqrt[n]{(2n-1)!!}}(u_n-1) =$$

$$= \frac{x_n}{\sqrt[n]{(2n-1)!!}}\cdot\frac{u_n-1}{\ln u_n}\cdot \ln u_n^n, \forall n\in\mathbb{N}^*-\{1\}, \text{ where}$$

$$u_n = \frac{n+1}{n}\cdot\frac{x_{n+1}}{x_n}\cdot\frac{\sqrt[n]{(2n-1)!!}}{\sqrt[n+1]{(2n+1)!!}} = \frac{x_{n+1}}{n+1}\cdot\frac{n}{x_n}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{\sqrt[n]{(2n-1)!!}}{\sqrt[n+1]{(2n+1)!!}}.$$

We have $\lim_{n\to\infty}\frac{x_n}{n} = \lim_{n\to\infty}\frac{x_{n+1}-x_n}{(n+1)-n} = x$ and

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!}{n^n}} = \lim_{n\to\infty}\Big(\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}\Big) =$$

$$= \lim_{n\to\infty}\frac{2n+1}{n+1}\Big(\frac{n}{n+1}\Big)^n = \frac{2}{e}$$

Therefore $\lim_{n\to\infty}u_n = x\cdot\frac{1}{x}\cdot 1\cdot\frac{2}{e}\cdot\frac{e}{2} = 1$, so, $\lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1$. We have:

$$\lim_{n\to\infty}u_n^n = \lim_{n\to\infty}\Big(\frac{n+1}{n}\Big)^n\cdot\lim_{n\to\infty}\Big(\frac{x_{n+1}}{x_n}\Big)^n\cdot\lim_{n\to\infty}\frac{(2n-1)!!}{(2n+1)!!}\sqrt[n+1]{(2n+1)!!} =$$

$$= e\cdot\lim_{n\to\infty}\left(\Big(1+\frac{x_{n+1}-x_n}{x_n}\Big)^{\frac{x_n}{x_{n+1}-x_n}}\right)^{\frac{n(x_{n+1}-x_n)}{x_n}}\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{(2n+1)!!}}{2n+1} = e\cdot e\cdot\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{(2n-1)^n}} =$$

$$= e^2\cdot\lim_{n\to\infty}\Big(\frac{(2n+1)!!}{(2n+1)^{n+1}}\cdot\frac{(2n-1)^n}{(2n-1)!!}\Big) = e^2\cdot\lim_{n\to\infty}\Big(\frac{2n-1}{2n+1}\Big)^n = e^2\cdot e^{-1} = e.$$

Therefore,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{x_n}{\sqrt[n]{(2n-1)!!}} \cdot \lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} \cdot \ln(\lim_{n\to\infty} u_n^n) =$$

$$= \lim_{n\to\infty} \frac{x_n}{n} \cdot \lim_{n\to\infty} \frac{n}{\sqrt[n]{(2n-1)!!}} \cdot \ln e = x \cdot \frac{e}{2} \cdot 1 = \frac{xe}{2}, \text{ and we are done.}$$

□

Theorem 20. Let $f : (0, \infty) \to (0, \infty)$ be a function with $\lim_{x\to\infty} \frac{f(x)}{x} = a \in (0, \infty)$ and $t \in \mathbb{R}$ then

$$\lim_{n\to\infty} \left((n+1)^{\sin^2 t} \cdot \sqrt[n+1]{(f(1)f(2)\dots f(n)f(n+1))^{\cos^2 t}} - n^{\sin^2 t} \cdot \sqrt[n]{f(1)f(2)\dots f(n))^{\cos^2 t}} \right) = \left(\frac{a}{e}\right)^{\cos^2 t}$$

Proof.

$$\text{We denote: } f(n)! = \prod_{k=1}^{n} f(k), \text{ and}$$

$$B_n = (n+1)^{\sin^2 t} \sqrt[n+1]{(f(n+1)!)^{\cos^2 t}} - n^{\sin^2 t} \sqrt[n]{(f(n)!)^{\cos^2 t}} = n^{\cos^2 t} \sqrt[n]{(f(n)!)^{\cos^2 t}} (v_n - 1) =$$

$$= \left(\frac{f(n)!}{n}\right)^{\cos^2 t} \cdot \frac{v_n - 1}{\ln v_n} \cdot \ln v_n^n, \forall n \in \mathbb{N}^* - \{1\}, \text{ where } v_n = \left(\frac{n+1}{n}\right)^{\sin^2 t} \left(\frac{\sqrt[n+1]{f(n+1)!}}{\sqrt[n]{f(n)!}}\right)^{\cos^2 t},$$

$\forall n \in \mathbb{N}^* - \{1\}$. We have: $\lim_{n\to\infty} \frac{f(n)}{n} = a$, and by Cauchy-D'Alembert criteria we deduce that

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{f(n)!}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{f(n)!}} \overset{\text{C-D'A}}{=} \lim_{n\to\infty} \left(\frac{(n+1)^{n+1}}{f(n+1)!} \cdot \frac{f(n)!}{n^n} \right) = \lim_{n\to\infty} e_n \cdot \lim_{n\leftrightarrow\infty} \frac{n+1}{f(n+1)} = \frac{e}{a}$$

where $e_n = \left(1 + \frac{1}{n}\right)^n, \forall n \in \mathbb{N}^*$ with $\lim_{n\to\infty} e_n = e$. Also, we have that:

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{\sin^2 t} \cdot \lim_{n\to\infty} \left(\frac{\sqrt[n+1]{f(n+1)!}}{\sqrt[n]{f(n)!}} \right)^{\cos^2 t} =$$

$$= 1 \cdot \lim_{n\to\infty} \left(\frac{\sqrt[n+1]{f(n+1)!}}{n+1} \cdot \frac{n}{\sqrt[n]{f(n)!}} \cdot \frac{n+1}{n} \right)^{\cos^2 t} = \left(\frac{e}{a} \cdot \frac{a}{e} \cdot 1\right)^{\cos^2 t} = 1,$$

So, $\lim_{n\to\infty} \frac{v_n - 1}{\ln v_n} = 1$. Yields that: $\lim_{n\to\infty} v_n^n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{n\sin^2 t} \cdot \lim_{n\to\infty} \left(\frac{\sqrt[n+1]{f(n+1)!}}{\sqrt[n]{f(n)!}} \right)^{n\cos^2 t} =$

$$= e^{\sin^2 t} \cdot \lim_{n\to\infty} \left(\frac{f(n+1)!}{f(n)!} \cdot \frac{1}{\sqrt[n+1]{f(n+1)!}} \right)^{\cos^2 t} = e^{\sin^2 t} \cdot \lim_{n\to\infty} \left(\frac{f(n+1)}{\sqrt[n+1]{f(n+1)!}} \right)^{\cos^2 t} =$$

$$= e^{\sin^2 t} \cdot \lim_{n\to\infty} \left(\frac{f(n+1)}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{f(n+1)!}} \right)^{\cos^2 t} = e^{\sin^2 t} \cdot \left(a \cdot \frac{e}{a}\right)^{\cos^2 t} = e^{\sin^2 t} \cdot e^{\cos^2 t} = 1.$$

$$\text{Therefore: } \lim_{n\to\infty} B_n = \left(\frac{a}{e}\right)^{\cos^2 t} \cdot 1 \cdot \ln e = \left(\frac{a}{e}\right)^{\cos^2 t}$$

□

References

[1] D.M. Bătineţu-Giurgiu, Neculai Stanciu, *The numbers of Fibonacci and Lucas - Identities - Proofs with few words (I).* MateInfo.ro Magazine, November 2017, 12-19.
[2] D.M. Bătineţu-Giurgiu, Neculai Stanciu, *The numbers of Fibonacci and Lucas - Identities - Proofs with few words (II).* MateInfo.ro Magazine, December 2017, 4-15.
[3] D.M. Bătineţu-Giurgiu, Daniel Sitaru, Neculai Stanciu, *About Ionescu - Nesbitt's Inequality.* Romanian Mathematical Magazine, No. 19, September 2017, 18-19.
[4] Mihály Bencze, Daniel Sitaru, *699 Olympic Mathematical Challenges.* Studis Publishing House, Iaşi 2017.
[5] Mihály Bencze, Daniel Sitaru, *Quantum Mathematical Power.* Studis Publishing House, Iaşi 2018.
[6] Mihály Bencze, Daniel Sitaru, Marian Ursărescu, *Olympic Mathematical Energy.* Studis Publishing House, Iaşi 2018.
[7] Octogon Mathematical Magazine, 2003-2018
[8] Romanian Mathematical Magazine, www.ssmrmh.ro

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