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JP.183. In ΔABC the following relationship holds:

$$3s\left(\frac{2r}{R}\right)^2 \le \sum h_a^2\left(\frac{1}{b}+\frac{1}{c}\right) \le 3s$$

Proposed by Marin Chirciu – Romania

Solution 1 by Mustafa Tarek-Cairo-Egypt, Solution 2 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Mustafa Tarek-Cairo-Egypt

$$\sum h_a^2\left(\frac{1}{b}+\frac{1}{c}\right) = c\frac{4\Delta^2}{\Delta^2}\left(\frac{b+c}{bc}\right) = \frac{4\Delta^2}{abc}\sum\left(\frac{b+c}{a}\right) = \frac{sr}{R}\sum\left(\frac{b+c}{a}\right) \quad \textit{(a)}$$

$$\textit{(1)} \Leftrightarrow \sum\left(\frac{b+c}{a}\right) \ge 12\frac{r}{R}, \textit{ but } \frac{b}{a}+\frac{a}{b} \ge 2 \quad \textit{etc}$$

$$\therefore \sum\left(\frac{b+c}{a}\right) = \sum\left(\frac{b}{a}+\frac{a}{b}\right) \ge 6 \overset{??}{\ge} 12\frac{r}{R} \Rightarrow \textit{true } \frac{1}{2} \ge \frac{r}{R}$$

$$\textit{(1) (proved), we have } \frac{h_a}{w_a} = \frac{b+c}{a}\sin\frac{A}{2} \le 1 \quad \textit{etc}$$

$$\therefore \textit{(3)} = \frac{sr}{R}\sum\left(\frac{b+c}{a}\right) \le \frac{sr}{R}\sum\sin\frac{A}{2} = \frac{S}{R}\sum AI \overset{??}{\le} 3s$$

$$\textit{We must prove } \sum AI \le 3R, \textit{ to prove that we will prove that } \sum AI \overset{(1)}{\le} \sqrt{\sum ab} \overset{(5)}{\le} 2$$

$$\textit{(5)} \Leftrightarrow s^2 + 4Rr + r^2 \le 4R^2 + 4r^2 + 8Rr \leftrightarrow s^2 \le 4R^2 + 4Rr + 3r^2$$

$$\textit{True} \to \textit{(Gerretsen)} \to (5\textit{-proved})$$

$$\therefore \sum AI \le 2(R+r) \le 3R \leftrightarrow 2r \le R \to \textit{true (Euler)}$$

$$\therefore \sum AI \le 3R, \therefore 2 \quad \textit{(proved)}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\textit{Using } h_a = \frac{2S}{a} \textit{ we obtain:}$$

$$\sum h_a^2\left(\frac{1}{b}+\frac{1}{c}\right) = \sum\frac{b+c}{bc}\cdot\frac{4S^2}{a^2} = \frac{4S^2}{abc}\sum\frac{b+c}{a}$$

$$= \frac{4S^2}{4RS}\sum\frac{b+c}{a} = \frac{S}{R}\sum\frac{b+c}{a} = \Omega$$

$$\because \sum\frac{2s-a}{a} = 2s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - 3$$

$$= 2s\left(\frac{ab + bc + ca}{abc}\right) - 3 = 2s\left(\frac{s^2 + 4Rr + r^2}{4Rsr}\right) - 3$$

$$= \frac{s^2 + 4Rr + r^2}{2Rr} - 3 = \frac{s^2 - 2Rr + r^2}{2Rr}$$

$$\Rightarrow \Omega = \frac{sr}{R} \cdot \frac{s^2 - 2Rr + r^2}{2Rr} = \frac{s}{2R^2}(s^2 - 2Rr + r^2)$$

$$\Omega \geq 3s\left(\frac{2r}{R}\right)^2 \quad \textit{(1)}$$

$$\Leftrightarrow s^2 - 2Rr + r^2 \geq 24r^2 \Leftrightarrow s^2 - 2Rr \geq 23r^2$$

$$\therefore s^2 \geq 16Rr - 5r^2 \Rightarrow 14Rr - 5r^2 \geq 23r^2$$

$$\Leftrightarrow 14Rr \geq 28r^2 \Leftrightarrow R \geq 2r \textit{ (true)} \Rightarrow \textit{(1) true.}$$

$$\Omega \leq 3s \quad \textit{(2)}$$

$$\Leftrightarrow s^2 - 2Rr + r^2 \leq 6R^2$$

$$\therefore s^2 \leq 4R^2 + 4Rr + 3r^2 \Rightarrow 4R^2 + 2Rr + 4r^2 \leq 6R^2$$

$$\Leftrightarrow 2Rr + 4r^2 \leq 2R^2 \Leftrightarrow Rr + 2r^2 \leq R^2$$

$$\textit{(True} \because Rr + 2r^2 \overset{(Euler)}{\leq} R \cdot \frac{R}{2} + 2 \cdot \frac{R^2}{4} = R^2\textit{)} \Rightarrow \textit{(2) true.}$$