NEW CLASSES OF SEQUENCES/FUNCTIONS AND FAMOUS LIMITS

D.M. BĂTINEŢU-GIURGIU, DANIEL SITARU, NECULAI STANCIU

Abstract. In this paper we present general methods to calculate the limits of some classes of functions and also for some classes of sequences.

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1. Main Results

In many cases, calculating limits of functions or sequences is quite difficult; therefore knowledge of additional properties of functions or sequences can lead us easier to determine those limits.

Definition 1.
$f : R_+^* \to R_+^*$ is B-function, if there exists $r \in R_+ = [0, \infty)$ such that

$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)\cdot x^{r+1}} = a \in R_+^* = (0,\infty) \text{ and there exists } \lim_{x\to\infty} \frac{(f(x))^{\frac{1}{x}}}{x}.$$

Definition 2.
The positive real sequence $(a_n)_{n\geq 1}$ is B-sequence, if there exists $r \in R_+$ such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot n^{r+1}} = a \in R_+^* \text{ and there exists } \lim_{n\to\infty} \frac{(a_n)^{\frac{1}{n}}}{n}$$

Definition 3.
$g : R_+^* \to R_+^*$ is G-function, if there exists $s \in R_+$ such that

$$\lim_{x\to\infty} \frac{g(x+1)-g(x)}{x^s} = b \in R_+^* \text{ and there exists } \lim_{x\to\infty} \frac{g(x)}{x^{s+1}}$$

Definition 4.
The positive real sequence $(b_n)_{n\geq 1}$ is G-sequence, if there exists $s \in R_+$ such that

$$\lim_{x\to\infty} \frac{b_{n+1}-b_n}{n^s} = b \in R_+^* \text{ and there exists } \lim_{x\to\infty} \frac{b_n}{n^{s+1}}.$$

Example 1.
The function $f : R_+^* \to R_+^*, f(x) = \Gamma(x+1)$, where $\Gamma : R_+^* \to R_+^*$ is the gamma function - is B-function. Indeed,

$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)\cdot x} = \lim_{x\to\infty} \frac{\Gamma(x+2)}{\Gamma(x+1)\cdot x} = \lim_{x\to\infty} \frac{\Gamma(x+1)\cdot(x+1)}{\Gamma(x+1)\cdot x} = \lim_{x\to\infty} \frac{x+1}{x} = 1, \text{ so } r=0 \text{ and } a=1.$$

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$$\lim_{x\to\infty}\frac{(f(x))^{\frac{1}{x}}}{x}=\lim_{x\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}=\lim_{\substack{x\to\infty\\ n\in N^*}}\frac{(\Gamma(n+1))^{\frac{1}{n}}}{n}=\lim_{x\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{x\to\infty}\sqrt[n]{\frac{n!}{n^n}}=\overbrace{=}^{\text{C-D'A}}$$

$$\overbrace{=}^{\text{C-D'A}}\lim_{x\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\lim_{x\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}$$

Example 2.
The function $h:R_+^*\to R_+^*, h(x)=x^{x+1}$ is B-function. Indeed,

$$\lim_{x\to\infty}\frac{h(x+1)}{h(x)\cdot x}=\lim_{x\to\infty}\frac{(x+1)^{x+2}}{x^{x+1}\cdot x}=\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{x+2}=e, \text{ so } r=0 \text{ and } a=e.$$

$$\lim_{x\to\infty}\frac{(h(x))^{\frac{1}{x}}}{x}=\lim_{x\to\infty}\frac{x^{\frac{x+1}{x}}}{x}=\lim_{x\to\infty}x^{\frac{1}{x}}=1.$$

Example 3.
The sequence $(a_n)_{n\geq 1}, a_n=(2n-1)!!$ is B-sequence. Indeed,

$$\lim_{x\to\infty}\frac{a_{n+1}}{a_n\cdot n}=\lim_{x\to\infty}\frac{(2n+1)!!}{(2n-1)!!\cdot n}=\lim_{x\to\infty}\frac{2n+1}{n}=2, \text{ so } r=0 \text{ and } a=2.$$

$$\lim_{x\to\infty}\frac{(a_n)^{\frac{1}{n}}}{n}=\lim_{x\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{x\to\infty}\sqrt[n]{\frac{a_n}{n^n}}\overbrace{=}^{C-D'A}=\lim_{x\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n}=$$

$$=\lim_{x\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=\lim_{x\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n=\frac{2}{e}/$$

Example 4.
The function $g:R_+^*\to R_+^*, g(x)=x^2$ is G-function. Indeed,

$$\lim_{x\to\infty}\frac{g(x+1)-g(x)}{x}=\lim_{x\to\infty}\frac{(x+1)^2-x^2}{x}=\lim_{x\to\infty}\frac{2x+1}{x}=2, \text{ so } s=1 \text{ and } b=2.$$

$$\lim_{x\to\infty}\frac{g(x)}{x^{s+1}}=\lim_{x\to\infty}\frac{x^2}{x^2}=1.$$

Example 5.
The sequence $(b_n)_{n\geq 1}, b_n=n^{s+1}, s\in R_+^*, s\in R_+^*$ is G-sequence. Indeed,

$$\lim_{x\to\infty}\frac{b_{n+1}-b_n}{n^s}=\lim_{x\to\infty}\frac{(n+1)^{s+1}-n^{s+1}}{n^s}=s+1$$

$$\lim_{x\to\infty}\frac{b_n}{n^{s+1}}\overbrace{=}^{C-S}\lim_{x\to\infty}\frac{b_{n+1}-b_n}{(n+1)^{s+1}-n^{s+1}}=\lim_{x\to\infty}\frac{b_{n+1}-b_n}{n^s}\cdot\lim_{x\to\infty}\frac{n^s}{(n+1)^{s+1}-n^{s+1}}=(s+1)\cdot\frac{1}{s+1}=1.$$

In the following, we present some properties of classes of functions and sequences defined above.

(1) 1) If $f:R_+^*\to R_+^*$ is B-function, then $\displaystyle\lim_{x\to\infty}\frac{(f(x))^{\frac{1}{x}}}{x^{r+1}}=\frac{a}{e^{r+1}}$,

Indeed, $$\lim_{x\to\infty}\frac{(f(x))^{\frac{1}{x}}}{x^{r+1}}=\lim_{\substack{n\to\infty\\ n\in N^*}}\frac{(f(n))^{\frac{1}{n}}}{n^{r+1}}=\lim_{x\to\infty}\sqrt[n]{\frac{f(n)}{n^{n(r+1)}}}\overbrace{=}^{\text{C-D'A}}=\lim_{x\to\infty}\left(\frac{f(n+1)}{(n+1)^{(n+1)(r+1)}}\cdot\frac{n^{n(r+1)}}{f(n)}\right)=$$

$$=\lim_{x\to\infty}\frac{f(n+1)}{f(n)\cdot n^{r+1}}\cdot\left(\frac{n}{n+1}\right)^{(n+1)(r+1)}=\frac{a}{e^{r+1}}.$$

(2)

2) If $f : R_+^* \to R_+^*$ is B-function, then $\lim\limits_{x\to\infty} \dfrac{(f(x+1))^{\frac{1}{x+1}}}{(f((x))^{\frac{1}{x}}} = 1$ and $\lim\limits_{x\to\infty} \left(\dfrac{(f(x+1))^{\frac{1}{x+1}}}{(f(x))^{\frac{1}{x}}}\right)^x = e^{r+1}$

Indeed, $\lim\limits_{x\to\infty} \dfrac{(f(x+1))^{\frac{1}{x+1}}}{(f(x))^{\frac{1}{x}}} = \lim\limits_{x\to\infty} \dfrac{(f(x+1))^{\frac{1}{x+1}}}{(x+1)^{r+1}} \cdot \dfrac{x^{r+1}}{(f(x))^{\frac{1}{x}}} \cdot \left(\dfrac{x+1}{x}\right)^{r+1} = \dfrac{a}{e^{r+1}} \cdot \dfrac{e^{r+1}}{a} \cdot 1 = 1$ and

$$\lim_{x\to\infty} \left(\frac{(f(x+1))^{\frac{1}{x+1}}}{(f(x))^{\frac{1}{x}}}\right)^x = \lim_{x\to\infty} \frac{f(x+1)}{f(x)} \cdot \frac{1}{(f(x+1))^{\frac{1}{x+1}}} = \lim_{x\to\infty} \left(\frac{f(x+1)}{f(x)\cdot x^{r+1}} \cdot \frac{(x+1)^{r+1}}{(f(x+1))^{\frac{1}{x+1}}} \cdot \left(\frac{x}{x+1}\right)^{r+1}\right) =$$

$$= a \cdot \frac{e^{r+1}}{a} \cdot 1 = e^{r+1}.$$

(3)

If $(a_n)_{n\geq 1}$ is B-sequence, then $\lim\limits_{x\to\infty} \dfrac{\sqrt[n]{a_n}}{n^{r+1}} = \dfrac{a}{e^{r+1}}$ and $\lim\limits_{x\to\infty} \left(\dfrac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^n = e^{r+1}$

Indeed, $\lim\limits_{x\to\infty} \dfrac{\sqrt[n]{a_n}}{n^{r+1}} = \lim\limits_{n\to\infty} \sqrt[n]{\dfrac{a_n}{n^{n(r+1)}}} \overbrace{=}^{\text{C-D'A}} = \lim\limits_{x\to\infty} \dfrac{a_{n+1}}{(n+1)^{(n+1)(r+1)}} \cdot \dfrac{n^{n(r+1)}}{a_n} =$

$$= \lim_{x\to\infty} \frac{a_{n+1}}{a_n \cdot n^{r+1}} \cdot \left(\frac{n}{n+1}\right)^{(n+1)(r+1)} = \frac{a}{e^{r+1}} \text{ and}$$

$$\lim_{x\to\infty} \left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^n = \lim_{x\to\infty} \frac{a_{n+1}}{a_n} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}}} = \lim_{x\to\infty} \frac{a_{n+1}}{a_n \cdot n^{r+1}} \cdot \frac{(n+1)^{r+1}}{\sqrt[n+1]{a_{n+1}}} \cdot \left(\frac{n}{n+1}\right)^{r+1} = a \cdot \frac{e^{r+1}}{a} \cdot 1 = e^{r+1}.$$

(4)

If $g : R_+^* \to R_+^*$ is G-function, then $\lim\limits_{x\to\infty} \dfrac{g(x)}{x^{s+1}} = \dfrac{b}{s+1}$; $\lim\limits_{x\to\infty} \dfrac{g(x+1)}{g(x)} = 1$ and $\lim\limits_{x\to\infty} \left(\dfrac{g(x+1)}{g(x)}\right)^x = e^{s+1}$

Indeed,

$$\lim_{x\to\infty} \frac{g(x)}{x^{s+1}} = \lim_{\substack{n\to\infty \\ n\in N^*}} \frac{g(n)}{n^{s+1}} \overbrace{=}^{C-S} \lim_{x\to\infty} \frac{g(n+1)-g(n)}{(n+1)^{s+1}-n^{s+1}} = \lim_{x\to\infty} \frac{g(n+1)-g(n)}{n^s} \cdot \frac{n^2}{(n+1)^{s+1}-n^{s+1}} = \frac{b}{s+1};$$

$$\lim_{x\to\infty} \frac{g(x+1)}{g(x)} = \lim_{x\to\infty} \frac{g(x+1)}{(x+1)^{s+1}} \cdot \frac{x^{s+1}}{g(x)} \cdot \left(\frac{x+1}{x}\right)^{s+1} = \frac{b}{s+1} \cdot \frac{s+1}{b} \cdot 1 = 1 \text{and then}$$

$$\lim_{x\to\infty} \left(\frac{g(x+1)}{g(x)}\right)^x = \lim_{x\to\infty} \left(\left(\left(1+\frac{g(x+1)-g(x)}{g(x)}\right)^{\frac{g(x)}{g(x+1)-g(x)}}\right)^{\frac{g(x+1)-g(x)}{x^s} \cdot \frac{x^{s+1}}{g(x)}}\right) = e^{b\cdot\frac{s+1}{b}} = e^{s+1}$$

(5)

5) If $(b_n)_{n\geq 1}$ is G-sequence, then $\lim\limits_{x\to\infty} \dfrac{b_n}{n^{s+1}} = \dfrac{b}{s+1}$, $\lim\limits_{x\to\infty} \dfrac{b_{n+1}}{b_n} = 1$ and $\lim\limits_{x\to\infty} \left(\dfrac{b_{n+1}}{b_n}\right)^n = e^{s+1}$

Indeed, $\lim\limits_{x\to\infty} \dfrac{b_n}{n^{s+1}} \overbrace{=}^{C-S} \lim\limits_{x\to\infty} \dfrac{b_{n+1}-b_n}{(n+1)^{s+1}-n^s} = \lim\limits_{x\to\infty} \dfrac{b_{n+1}-b_n}{n^s} \cdot \dfrac{n^s}{(n+1)^{s+1}-n^{s+1}} = \dfrac{b}{s+1}$;

$$\lim_{x\to\infty} \frac{b_{n+1}}{b_n} = \lim_{x\to\infty} \frac{b_{n+1}}{(n+1)^{s+1}} \cdot \frac{n^{s+1}}{b_n} \cdot \left(\frac{n+1}{n}\right)^{s+1} = \frac{b}{s+1} \cdot \frac{s+1}{b} \cdot 1 = 1 \text{ and then}$$

$$\lim_{x\to\infty} \left(\frac{b_{n+1}}{b_n}\right)^n = \lim_{x\to\infty} \left(\left(\left(1+\frac{b_{n+1}-b_n}{b_n}\right)^{\frac{b_n}{b_{n+1}-b_n}}\right)^{\frac{b_{n+1}-b_n}{n^s} \cdot \frac{n^{s+1}}{b_n}}\right) = e^{b\cdot\frac{s+1}{b}} = e^{s+1}.$$

Theorem 1.
If $m,n,p,r,s \in R_+$ and $f,g : R_+^* \to R_+^*$ where f is B-function and g is G-function, then:

$$\lim_{x\to\infty}\left(\frac{(f(x+1))^{\frac{m}{x+1}}(g(x+1))^p}{(x+1)^{mr+ps+m+p-1}} - \frac{(f(x))^{\frac{m}{x}}(g(x))^p}{x^{mr+ps+m+p-1}}\right) = \frac{a^m b^p}{(s+1)^p e^{m(r+1)}}, \tag{6}$$

Proof.

We denote $B(x) = \frac{(f(x+1))^{\frac{m}{x+1}}(g(x+1))^p}{(x+1)^{mr+ps+m+p-1}} - \frac{(f(x))^{\frac{m}{x}}(g(x+1))^p}{(x+1)^{mr+ps+m+p-1}} - \frac{(f(x))^{\frac{m}{x}}(g(x))^p}{x^{mr+ps+m+p-1}} =$

$$= \frac{(f(x))^{\frac{m}{x}}(g(x))^p}{x^{mr+ps+m+p-1}}(u(x)-1) = \frac{(f(x))^{\frac{m}{x}}(g(x))^p}{x^{mr+ps+m+p-1}} \cdot \frac{u(x)-1}{\ln u(x)} \cdot \ln u(x) =$$

$$= \frac{(f(x))^{\frac{m}{x}}(g(x))^p}{x^{mr+ps+m+p-1}} \cdot \frac{u(x)-1}{\ln u(x)} \cdot \ln(u(x))^x =$$

$$= \left(\frac{(f(x))^{\frac{1}{x}}}{x^{r+1}}\right)^m \cdot \left(\frac{g(x)}{x^{x+1}}\right)^p \cdot \frac{u(x)-1}{\ln u(x)} \cdot \ln(u(x))^x \forall x \in R_+^* \tag{7}$$

Above we denote $u(x) = \left(\frac{(f(x+1))^{\frac{1}{x+1}}}{(f(x))^{\frac{1}{x}}}\right)^m \left(\frac{g(x+1)}{g(x)}\right)^p \left(\frac{x}{x+1}\right)^{mr+ps+m+p-1}, \forall x \in R_+^*$.

Then $\lim_{x\to\infty} u(x) = 1\cdot 1\cdot 1 = 1 \Rightarrow \lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)} = 1$ and

$$\lim_{x\to\infty}(u(x))^x = \lim_{x\to\infty}\left(\frac{(f(x+1))^{\frac{1}{x+1}}}{(f(x))^{\frac{1}{x}}}\right)^{mx} \cdot \lim_{x\to\infty}\left(\frac{g(x+1)}{g(x)}\right)^{px} \cdot \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{x(mr+ps+m+p-1)} =$$

$$= e^{m(r+1)} \cdot e^{p(s+1)} \cdot e^{mr+ps+m+p-1} = e$$

So, from (7) and above we obtain $\lim_{x\to\infty} B(x) = \frac{a^m b^p}{e^{m(r+1)}\cdot(s+1)^p}\cdot 1 \cdot \ln e = \frac{a^m b^p}{e^{m(r+1)}\cdot(s+1)^p}$, q.e.d. □

Theorem 2.
If $m,p,r,s \in R_+^*$ and $f,g : R_+^* \to R_+^*$, where f is B-function and g is G-function, then:

$$\lim_{x\to\infty}\left(\frac{(x+1)^{mr+ps+m+p+1}}{(f(x+1))^{\frac{m}{x+1}}(g(x+1))^p} - \frac{x^{mr+ps+m+p+1}}{(f(x))^{\frac{m}{x}}(g(x))^p}\right) = \frac{(s+1)^p e^{m(r+1)}}{a^m b^p} \tag{8}$$

Proof.

We denote $G(x) = \frac{(x+1)^{mr+ps+m+p+1}}{(f(x+1))^{\frac{m}{x+1}}(g(x+1))^p} - \frac{x^{mr+ps+m+p+1}}{(f(x))^{\frac{m}{x}}(g(x))^p} =$

$$= \frac{x^{mr+ps+m+p+1}}{(f(x))^{\frac{m}{x}}(g(x))^p}(v(x)-1) = \frac{x^{mr+ps+m+p+1}}{(f(x))^{\frac{m}{x}}(g(x))^p} \cdot \frac{v((x)-1}{\ln v(x)} \cdot \ln v(x) =$$

(9)

$$= \frac{x^{mr+ps+m+p}}{(f(x))^{\frac{m}{x}}(g(x))^p} \cdot \frac{v(x)-1}{\ln v(x)} \cdot \ln(v(x))^x = \left(\frac{x^{r+1}}{(f(x))^{\frac{1}{x}}}\right)^m \cdot \left(\frac{x^{s+1}}{g(x)}\right)^p \cdot \frac{v(x)-1}{\ln v(x)} \cdot \ln(v(x))^x, \forall x \in R_+^*$$

Above we denote $v(x) = \left(\frac{x+1}{x}\right)^{mr+ps+m+p+1} \cdot \left(\frac{(f(x))^{\frac{1}{x}}}{(f(x+1))^{\frac{1}{x+1}}}\right)^m \cdot \left(\frac{g(x)}{g(x+1)}\right)^p, \forall x \in R_+^*$.

Then $\lim_{x\to\infty} v(x) = 1\cdot 1\cdot 1 = 1 \Rightarrow \lim_{x\to\infty} \frac{v(x)-1}{\ln v(x)} = 1$ and

$$\lim_{x\to\infty}(v(x))^x = \lim_{x\to\infty}\left(\Big(\frac{x+1}{x}\Big)^{(mr+ps+m+p+1)x}\cdot\Big(\frac{f(x)}{f(x+1)}\Big)^m\cdot(f(x+1))^{\frac{m}{x+1}}\cdot\left(\Big(\frac{g(x)}{g(x+1)}\Big)^x\right)^p\right)=$$

$$= e^{mr+ps+m+p+1}\cdot\lim_{x\to\infty}\left(\Big(\frac{f(x)\cdot x^{r+1}}{f(x+1)}\Big)^m\cdot\left(\frac{f(x+1))^{\frac{1}{x+1}}}{(x+1)^{r+1}}\right)^m\cdot\Big(\frac{x+1}{x}\Big)^{m(r+1)}\cdot\left(\Big(\frac{g(x)}{g(x+1)}\Big)^x\right)^p\right)=$$

$$= e^{mr+ps+m+p+1}\cdot\frac{1}{a^m}\cdot\frac{a^m}{e^{m(r+1)}}\cdot 1\cdot\frac{1}{e^{p(s+1)}} = e.$$

Then from (9) we deduce that

$$\lim_{x\to\infty} G(x) = \frac{e^{m(r+1)}}{a^m}\cdot\frac{(s+1)^p}{b^p} = \frac{(s+1)^p\cdot e^{m(r+1)}}{a^m b^p},\ \text{q.e.d.}$$

□

Theorem 3.
If $m, p, r, s \in R_+$ and $f, g : R_+^* \to R_+^*$, where f is B-function and g is G-function, then:
(10)

$$\lim_{x\to\infty}\left(\frac{(f(x+1))^{\frac{m}{x+1}}}{(g(x+1))^p}\cdot(x+1)^{ps+p-mr-m+1}-\frac{(f(x))^{\frac{m}{x}}}{(g(x))^p}\cdot x^{ps+p-mr-m+1}\right)=\frac{a^m(s+1)^p}{b^p e^{m(r+1)}}$$

Proof.

We denote $D(x) = \frac{(f(x+1))^{\frac{m}{x+1}}}{g(x+1))^p}\cdot(x+1)^{ps+p-mr-m+1}-\frac{(f(x))^{\frac{m}{x}}}{(g(x))^p}\cdot x^{ps+p-mr-m+1} =$

$$= \frac{(f(x))^{\frac{m}{x}}}{(g(x))^p}\cdot x^{ps+p-mr-m+1}\cdot(t(x)-1) = \frac{(f(x))^{\frac{m}{x}}}{(g(x))^p}\cdot x^{ps+p-mr-m+1}\cdot\frac{t(x)-1}{\ln t(x)}\cdot\ln t(x) =$$

$$= \left(\frac{(f(x))^{\frac{1}{x}}}{x^{r+1}}\right)^m\cdot\Big(\frac{x^{s+1}}{g(x)}\Big)^p\frac{t(x)-1}{\ln t(x)}\cdot\ln(t(x))^x, \forall x\in R_+^* \tag{11}$$

Above we denote $t(x) = \Big(\frac{x+1}{x}\Big)^{ps+p-mr-m+1}\cdot\left(\frac{(f(x+1))^{\frac{1}{x+1}}}{(f(x))^{\frac{1}{x}}}\right)^m\cdot\Big(\frac{g(x)}{g(x+1)}\Big)^p, \forall x\in R_+*$

Then $\lim_{x\to\infty} t(x) = 1\cdot 1\cdot 1 \Rightarrow \lim_{x\to\infty}\frac{t(x)-1}{\ln t(x)} = 1$ and

$$\lim_{x\to\infty}(t(x))^x = \lim_{x\to\infty}\left(\Big(\frac{x+1}{x}\Big)^{(mr+ps+m+p+1)x}\cdot\left(\frac{(f(x+1))^{\frac{1}{x+1}}}{(f(x))^{\frac{1}{x}}}\right)^{mx}\cdot\left(\Big(\frac{g(x)}{g(x+1)}\Big)^x\right)^p\right)=$$

$$= e^{ps+p-mr-m+1}\cdot e^{m(r+1)}\cdot\frac{1}{e^{p(s+1)}} = e.$$

Therefore from above and (11) yields that

$$\lim_{x\to\infty} D(x) = \frac{a^m(s+1)^p}{b^p e^{m(r+1)}},\ \text{q.e.d.}$$

□

Theorem 4.
If $m,p,r,s\in R_+$ and $f,g:R_+^*\to R_+^*$, where f is B-function and g is G-function, then
(12)
$$\lim_{x\to\infty}\left(\frac{(g(x+1)^p}{(f(x+1))^{\frac{m}{x+1}}}\cdot(x+1)^{mr+m-ps-p+1}-\frac{(g(x))^p}{(f(x))^{\frac{m}{x}}}\cdot x^{mr+m-ps-p+1}\right)=\frac{b^p\cdot e^{m(r+1)}}{a^m\cdot(s+1)^p}$$

Proof.

We denote $E(x)=\dfrac{(g(x+1)^p}{(f(x+1))^{\frac{m}{x+1}}}\cdot(x+1)^{mr+m-ps-p+1}-\dfrac{(g(x))^p}{(f(x))^{\frac{m}{x}}}\cdot x^{mr+m-ps-p+1}=$

$$=\frac{(g(x))^p}{(f(x))^{\frac{m}{x}}}\cdot x^{mr+m-ps-p+1}\cdot(w(x)-1)=\frac{(g(x))^p}{(f(x))^{\frac{m}{x}}}\cdot x^{mr+m-ps-p+1}\cdot\frac{w(x)-1}{\ln w(x)}\cdot\ln w(x)=$$

$$=\left(\frac{g(x)}{x^{s+1}}\right)^p\cdot\left(\frac{x^{r+1}}{(f(x))^{\frac{1}{x}}}\right)^m\cdot\frac{w(x)-1}{\ln w(x)}\cdot\ln(w(x))^x,\forall x\in R_+^* \tag{13}$$

Abobe we denote $w(x)=\left(\dfrac{x+1}{x}\right)^{mr+m-ps-p+1}\cdot\left(\dfrac{g(x+1)}{g(x)}\right)^p\cdot\left(\dfrac{(f(x))^{\frac{1}{x}}}{(f(x+1))^{\frac{1}{x+1}}}\right)^m,\forall x\in R_+^*$.

Then $\lim_{x\to\infty}w(x)=1\cdot1\cdot1=1\Rightarrow\lim_{x\to\infty}\frac{w(x)-1}{\ln w(x)}=1$ and

$$\lim_{x\to\infty}(w(x))^x=\lim_{x\to\infty}\left(\left(\frac{x+1}{x}\right)^{(mr+m-ps-p+1)x}\cdot\left(\frac{(f(x))^{\frac{1}{x}}}{(f(x+1))^{\frac{1}{x+1}}}\right)^{mx}\cdot\left(\left(\frac{g(x)}{g(x+1)}\right)^x\right)^p\right)=$$
$$=e^{mr+m-ps-p+1}\cdot\frac{1}{e^{m(r+1)}}\cdot e^{p(s+1)}=e.$$

Hence, $\lim_{x\to\infty}E(x)=\frac{b^p}{(s+1)^p}\cdot\frac{e^{m(r+1)}}{a^m}\cdot1\cdot\ln e=\frac{b^pe^{m(r+1)}}{a^m(s+1)^p}$, q.e.d. □

2. Applications

The following applications are particular cases of limits calculated above and which also can be used to compute the limits of functions and sequences of this type. For other applications of general methods from above see [1], [2], [3], [4] and [5].

A1. If $f:R_+^*\to R_+^*,f(x)=\Gamma(x+1)$, then by example 1 we deduce that f is B-function with $a=1,r=0$. If $g:R_+^*\to R_+^*$ is G-function, then by theorem 1

$$\lim_{x\to\infty}\left(\frac{(f(x+1))^{\frac{m}{x+1}}(g(x+1))^p}{(x+1)^{mr+ps+m+p-1}}-\frac{(f(x))^{\frac{m}{x}}(g(x))^p}{x^{mr+ps+m+p-1}}\right)=$$

$$=\lim_{x\to\infty}\left(\frac{(\Gamma(x+1))^{\frac{m}{x+1}}(g(x+1)^p}{(x+1)^{ps+m+p-1}}-\frac{(\Gamma(x+1))^{\frac{m}{x}}(g(x))^p}{x^{ps+m+p-1}}\right)=\frac{b^p}{(s+1)^pe^m} \tag{14}$$

If we take $p=0$, then by (14) we deduce

$$\lim_{x\to\infty}\left(\frac{(\Gamma(x+1))^{\frac{m}{x+1}}}{(x+1)^{m-1}}-\frac{(\Gamma(x+1))^{\frac{m}{x}}}{x^{m-1}}\right)=\frac{1}{e^m} \tag{15}$$

If we take $m=1$, then by (15) we obtain

$$\lim_{x\to\infty}\left((\Gamma(x+1))^{\frac{1}{x+1}}-(\Gamma(x+1))^{\frac{1}{x}}\right)=\frac{1}{e} \tag{16}$$

If we take $x = n \in N^* - \{1\}$, then $\Gamma(x+1) = \Gamma(x+1) = \Gamma(n+1) = n!$, so

$$(17) \qquad \lim_{x\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}\cdot(g(x+1))^p}{(n+1)^{ps+m+p-1}} - \frac{\sqrt[n]{n!}\cdot(g(n))^p}{n^{ps+m+p-1}}\right) = \frac{b^p}{(s+1)^p e^m}$$

If we take $p = 0$ and $m = 1$, then by (17) yields that

$$(18) \qquad \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) = \frac{1}{e}$$

i.e. the limit of Traian Lalescu's sequence (Problem 579 from Romanian Mathematical Gazette, Vol. VI (1900-1901)).

A2. If $f : R_+^* \to R_+^*, f(x) = x^{x+1}$, then by example 2 we deduce that f is B-function with $a = e, r = 0$. If $g : R_+^* \to R_+^*$ is G-function, then by theorem 1

(19)

$$\lim_{x\to\infty}\left(\frac{(x+1)^{\frac{m(x+2)}{x+1}}(g(x+1)^p}{(x+1)^{ps+m+p-1}} - \frac{x^{\frac{m(x+1)}{x}}(g(x))^p}{x^{ps+m+p-1}}\right) = \frac{a^m b^p}{(s+1)^p e^m} = \frac{e^m b^p}{(s+1)^p e^m} = \frac{b^p}{(s+1)^p}$$

If we take $p = 0$, then by (19) we deduce

$$(20) \qquad \lim_{x\to\infty}\left(\frac{(x+1)^{\frac{m(x+2)}{x+1}}}{(x+1)^{m-1}} - \frac{x^{\frac{m(x+1)}{x}}}{x^{m-1}}\right) = \lim_{x\to\infty}\left((x+1)^{1+\frac{m}{x+1}} - x^{1+\frac{m}{x}}\right) = 1$$

If we take $m = 1$, then by (20) we infer that

$$(21) \qquad \lim_{x\to\infty}\left((x+1)^{1+\frac{1}{x+1}} - x^{1+\frac{1}{x}}\right) = \lim_{x\to\infty}\left((x+1)\cdot(x+1)^{\frac{1}{x+1}} - x\cdot x^{\frac{1}{x}}\right) = 1$$

If we take $x = n \in N^* - \{1\}$, then by (21) we obtain

$$(22) \qquad \lim_{n\to\infty}\left((n+1)\cdot\sqrt[n+1]{n+1} - n\cdot\sqrt[n]{n}\right) = 1$$

i.e. the limit of Romeo T. Ianculescu' s sequence (Problem 2042 from Romanian Mathematical Gazette, Vol. XIX, (1913-1914)).

A3. If $f : R_+^* \to R_+^*, f(x) = \Gamma(x+1)$, then by example 1 we deduce that f is B-function with $a = 1, r = 0$. If $g : R_+^* \to R_+^*$ is G-function, then by theorem 2

$$(23) \qquad \lim_{x\to\infty}\left(\frac{(x+1)^{ps+m+p+1}}{(\Gamma(x+2))^{\frac{m}{x+1}}(g(x+1))^p} - \frac{x^{ps+m+p+1}}{(\Gamma(x+1))^{\frac{m}{x}}(g(x))^p}\right) = \frac{(s+1)^p e^m}{b^p}$$

If we take $p = 0$, then by (23) we deduce that

$$(24) \qquad \lim_{x\to\infty}\left(\frac{(x+1)^{m+1}}{(\Gamma(x+2))^{\frac{m}{x+1}}} - \frac{x^{m+1}}{(\Gamma(x+1))^{\frac{m}{x}}}\right) = e^m$$

If we take $m = 1$, then by (23) yields that

$$(25) \qquad \lim_{x\to\infty}\left(\frac{(x+1)^2}{(\Gamma(x+2))^{\frac{1}{x+1}}} - \frac{x^2}{(\Gamma(x+1))^{\frac{1}{x}}}\right) = e$$

If we take $x = n \in N^* - \{1\}$ and taking account that $\Gamma(x+1) = \Gamma(n+1) = n!$ then by 25 we obtain

$$(26) \qquad \lim_{x\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}\right) = e,$$

i.e. the limit of Bătineţu-Giurgiu's sequence (Problem C:890 from Romanian Mathematical Gazette, Vol. XCIV, 1989).

Remark. For certain values of the numbers $m, p, r, s \in R_+$ and for particular functions f, g using the theorems 1-4, the reader can get the limits from many problems publishead in math journals - see for example the list of problems from references of [6]

Finally, we let readers to apply the general methods from this paper to solve the following:

Problem 11771, The American Mathematical Monthly, April, 2014;
Problem 11875, The American Mathematical Monthly, December, 2015;
Problem 11889, The American Mathematical Monthly, February, 2016;
Problem 11935, The American Mathematical Monthly, October, 2016;
Problem 1300, Pi Mu Epsilon journal, Spring, 2015;
Problem 5285, School Science and Mathematics journal, January, 2014;
Problem 5398, School Science and Mathematics journal, April, 2016;
Problem 5405, School Science and Mathematics journal, May, 2016;
Problem 5411, School Science and Mathematics journal, October, 2016;
Problem 75, Mathproblems journal, Vol.3, Issue 3, 2013;
Problem 135, Mathproblems journal, Vol.5, Issue 3, 2015;
Problem V5-2, Asymmetry, March, 2014;
Problem W4, József Wildt International Mathematical Competition, 2016;
Problem W5, József Wildt International Mathematical Competition, 2016;
Problem 3867, Crux Mathematicorum, September, 2013;
Problem 4127, Crux Mathematicorum, March, 2016;
Problem B-1151, The Fibonacci Quarterly, August, 2014;
Problem H-758, The Fibonacci Quarterly, August, 2014;
Problem B-1160, The Fibonacci Quarterly, November, 2014;
Problem B-1170, The Fibonacci Quarterly, May, 2015.
Problem H-793, The Fibonacci Quarterly, August, 2016.

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Mathematics Department "Matei Basarab" National College, Bucharest, Romania
E-mail address: dmb_g@yahoo.com

Mathematics Department National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Mehedinti, Romania
E-mail address: dansitaru63@yahoo.com

Mathematics Department "George Emil Palade" School, Buzau, Romania
E-mail address: stanciuneculai@yahoo.com