Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

INEQUALITY IN TRIANGLE 931
ROMANIAN MATHEMATICAL MAGAZINE
2018

MARIN CHIRCIU

1) In ΔABC

$$\sum \frac{h_a}{h_b h_c} \le \frac{R}{2r^2}$$

Proposed by Bogdan Fustei - Romania

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC

$$\sum \frac{h_a}{h_b h_c} = \frac{s^4 + s^2(2r^2 - 8Rr) + r^2(4R+r)^2}{8s^2r^2R}$$

Proof.

Using $h_a = \dfrac{2S}{a}$, *we obtain:*

$$\sum \frac{h_a}{h_b h_c} = \sum \frac{\frac{2S}{a}}{\frac{2S}{b} \cdot \frac{2S}{c}} = \frac{1}{2S} \sum \frac{bc}{a} = \frac{s^4 + s^2(2r^2 - 8Rr) + r^2(4R+r)^2}{8s^2r^2R},$$

which follows from:

$$\sum \frac{bc}{a} = \frac{s^4 + s^2(2r^2 - 8Rr) + r^2(4R+r)^2}{4srR}$$

□

Back to the main problem:

*Using the **Lemma** the inequality can be written:*

$$\frac{s^4 + s^2(2r^2 - 8Rr) + r^2(4R+r^2)}{8s^2r^2R} \le \frac{R}{2r^2} \Leftrightarrow s^2(4R^2 + 8Rr - 2r^2 - s^2) \ge r^2(4R+r)^2$$

which follows from Gerretsen's inequality $16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2$.

It remains to prove that:

$$(16Rr - 5r^2)(4R^2 + 8Rr - 2r^2 - 4R^2 - 4Rr - 3r^2) \ge r^2(4R+r)^2 \Leftrightarrow$$

$$\Leftrightarrow 4R^2 - 9Rr + 2r^2 \ge 0 \Leftrightarrow (R - 2r)(R - r) \ge 0$$

obviously form Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasise an inequality having an opposite sense.

3) In ΔABC:

$$\sum \frac{h_a}{h_b h_c} \geq \frac{1}{r}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using the **Lemma** we can write:*

$$\frac{s^4 + s^2(2r^2 - 8Rr) + r^2(4Rr + r)^2}{8s^2r^2R} \geq \frac{1}{r} \Leftrightarrow s^2(s^2 + 2r^2 - 16Rr) + r^2(4R + r)^2 \geq 0$$

We distinguish the following cases:

Case 1). If $(s^2 + 2r^2 - 16Rr) \geq 0$, the inequality is obvious.

Case 2). If $(s^2 + 2r^2 - 16Rr) < 0$, the inequality can be rewritten:

$r^2(4R + r)^2 \geq s^2(16Rr - 2r^2 - s^2)$, which follows from Blundon-Gerretsen's

inequality: $16Rr - 5r^2 \leq s^2 \leq \dfrac{R(4R + r)^2}{2(2R - r)}$

It remains to prove that:

$$r^2(4R+r)^2 \geq \frac{R(4R + r)^2}{2(2R - r)}(16Rr - 2r^2 - 16Rr + 5r^2) \Leftrightarrow R \geq 2r \text{ (Euler's inequality).}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

The double inequality can be written:

4) In ΔABC:

$$\frac{1}{r} \leq \sum \frac{h_a}{h_b h_c} \leq \frac{R}{2r^2}$$

Proof.

*See inequalities **1)** and **3)**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

If we replace $\dfrac{h_a}{h_b h_c}$ with $\dfrac{r_a}{r_b r_c}$ we propose:

5) In ΔABC:

$$\frac{2}{r}\left(1 - \frac{r}{R}\right) \leq \sum \frac{r_a}{r_b r_c} \leq \frac{1}{r}\left(\frac{R}{r} - 1\right)$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.
6) In ΔABC:

$$\sum \frac{r_a}{r_b r_c} = \frac{1}{r}\left[\left(\frac{4R+r}{s}\right)^2 - 2\right]$$

Proof.

Using $r_a = \dfrac{S}{s-a}$, *using:*

$$\sum \frac{r_a}{r_b r_c} = \sum \frac{\frac{S}{s-a}}{\frac{S}{s-b}\cdot\frac{S}{s-c}} = \frac{1}{S}\sum \frac{(s-b)(s-c)}{s-a} = \frac{1}{rs}\cdot\frac{(4R+r)^2-2s^2}{s} = \frac{1}{r}\left[\left(\frac{4R+r}{s}\right)^2 - 2\right]$$

which follows from:

$$\sum \frac{(s-b)(s-c)}{s-a} = \frac{(4R+r)^2-2s^2}{s}$$

Back to the main problem.

□

Using the ***Lemma*** *the inequality can be written:*

$$\frac{2}{r}\left(1-\frac{r}{R}\right) \le \frac{1}{r}\left[\left(\frac{4R+r}{s}\right)^2 - 2\right] \le \frac{1}{r}\left(\frac{R}{r}-1\right),$$ *which follows from*

Blundon-Gerretsen's inequality: $\dfrac{r(4R+r)^2}{R+r} \le s^2 \le \dfrac{R(4R+r)^2}{2(2R-r)}$

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum \dfrac{h_a}{h_b h_c}$ *and* $\sum \dfrac{r_a}{r_b r_c}$ *the following relationship holds:*

7) In ΔABC:

$$\sum \frac{h_a}{h_b h_c} \ge \left(\frac{2r}{R}\right)^2 \sum \frac{r_a}{r_b r_c}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the identities ***2)*** *and* ***6)*** *the inequality can be written:*

$$\frac{s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2}{8s^2r^2R} \ge \left(\frac{2r}{R}\right)^2 \cdot \frac{(4R+r)^2-2s^2}{rs^2} \Leftrightarrow$$

$$\Leftrightarrow s^2[Rs^2+2r(32r^2+Rr-4R^2)] \ge r^2(4R+r)^2(32r-R),$$

which follows from Gerretsen's inequality $s^2 \ge 16Rr-5r^2$.

It remains to prove that:

$$(16Rr-5r^2)[R(16Rr-5r^2)+2r(32r^2+Rr-4R^2)] \ge r^2(4R+r)^2(32r-R) \Leftrightarrow$$

$$\Leftrightarrow 9R^3-37R^2r+49Rr^2-22r^3 \ge 0 \Leftrightarrow (R-2r)(9R^2-19Rr+11r^2) \ge 0$$

obviously from Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

References

[1] Mihály Bencze, Daniel Sitaru, Marian Ursărescu, *Olympic Mathematical Energy.* Studis Publishing House, Iaşi, 2018.
[2] Daniel Sitaru, *Algebraic Phenomenon.* Paralela 45 Publishing House, Piteşti, 2017, ISBN 978-973-47-2523-6
[3] Daniel Sitaru, *Murray Klamkin's Duality Principle for Triangle Inequalities.* The Pentagon Journal-Volume 75 NO 2, Spring 2016.
[4] Daniel Sitaru, Claudia Nănuţi, *Generating Inequalities using Schweitzer's Theorem.* CRUX MATHEMATICORUM, Volume 42, NO. 1, January 2016.
[5] Daniel Sitaru, Claudia Nănuţi, *A "probabilistic" method for proving inequalities.* CRUX MATHEMATICORUM, Volume 43, NO. 7, September 2017.
[6] Daniel Sitaru, Mihály Bencze, *699 Olympic Mathematical Challenges.* Studis Publishing House, Iaşi, 2017.
[7] Daniel Sitaru, *Analytical Phenomenon.* Cartea Românească Publishing House, Piteşti, 2018.
[8] Daniel Sitaru, George Apostolopoulos, *The Olympic Mathematical Marathon.* Cartea Românească Publishing House, Piteşti, 2018.
[9] Daniel Sitaru, *Contest Problems.* Cartea Românească Publishing House, Piteşti, 2018.
[10] Mihály Bencze, Daniel Sitaru, *Quantum Mathematical Power.* Studis Publishing House, Iaşi, 2018.
[11] Daniel Sitaru, *A Class of Inequalities in triangles with Cevians.* The Pentagon Journal, Volume 77 NO. 2, Fall 2017
[12] Marin Chirciu, *Geometrical Inequalities.* Paralela 45 Publishing House, Piteşti, 2015
[13] Marin Chirciu, *Algebraic Inequalities.* Paralela 45 Publishing House, Piteşti, 2015
[14] Marin Chirciu, *Trigonometric Inequalities.* Paralela 45 Publishing House, Piteşti, 2015
[15] Daniel Sitaru, Marian Ursărescu, *Ice Math.* Studis Publishing House, Iaşi, 2019
[16] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, ROMANIA.

Email address: dansitaru63@yahoo.com