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83 IDENTITY IN TRIANGLE
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1) In ΔABC:

$$\sum \frac{h_a + h_b}{r_a + r_b} = 2\left(1 + \frac{r}{R}\right)$$

Proposed by Bogdan Fustei - Romania

Proof.

Using $h_a = \frac{2S}{a}$ *and* $r_a = \frac{S}{s-a}$ *we obtain:*

$$\sum \frac{h_a + h_b}{r_a + r_b} = \sum \frac{\frac{2S}{a} + \frac{2S}{b}}{\frac{S}{s-a} + \frac{S}{s-b}} = \frac{2}{abc} \sum (a+b)(s-a)(s-b) = 2\left(1 + \frac{r}{R}\right)$$

which follows from: $abc = 4srR$ *and* $\sum (a+b)(s-a)(s-b) = 4sr(R+r)$

□

Remark.

Let's emphasises a double inequality with the above sum:

2) In ΔABC

$$\frac{6r}{R} \le \sum \frac{h_a + h_b}{r_a + r_b} \le 3$$

Proof.

Using identity ***1)*** *the inequality can be written:* $\frac{6r}{R} \le 2\left(1 + \frac{r}{R}\right) \le 3,$

which follows from Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Reversing the fraction from the above sum we propose:

3) In ΔABC:

$$3 \le \sum \frac{r_b + r_c}{h_b + h_c} \le \frac{3R}{2r}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.
4) In ΔABC:

$$\sum \frac{r_b + r_c}{h_b + h_c} = \frac{R}{r} \cdot \frac{s^2 + 5r^2 + 8Rr}{s^2 + r^2 + 2Rr}$$

Proof.

Using $h_a = \frac{2S}{a}$ *and* $r_a = \frac{S}{s-a}$ *we obtain:*

$$\sum \frac{r_b + r_c}{h_b + h_c} = \sum \frac{\frac{S}{s-b} + \frac{S}{s-c}}{\frac{2S}{b} + \frac{2S}{c}} = \frac{abc}{2} \sum \frac{1}{(b+c)(s-b)(s-c)} = \frac{R}{r} \cdot \frac{s^2 + 5r^2 + 8Rr}{s^2 + r^2 + 2Rr}$$

which follows from: $abc = 4srR$ *and* $\sum \frac{1}{(b+c)(s-b)(s-c)} = \frac{s^2 + 5r^2 + 8Rr}{2r^2 s(s^2 + r^2 + 2Rr)}$

□

Let's get back to the main problem.

The left side inequality:

Using the ***Lemma****, the left side inequality can be written:*

$$\frac{R}{r} \cdot \frac{s^2 + 5r^2 + 8Rr}{s^2 + r^2 + 2Rr} \geq 3 \Leftrightarrow s^2(R - 3r) + r(8R^2 - Rr - 3r^2) \geq 0$$

We distinguish the following cases:

Case 1). If $(R - 3r) \geq 0$*, the inequality is obvious.*

Case 2). If $(R-3r) < 0$*, we rewrite the inequality:* $r(8R^2 - Rr - 3r^2) \geq s^2(3r - R)$*,*

which follows from Gerretsen's inequality: $s^2 \leq 4R^2 + 4Rr + 3r^2$.

It remains to prove that:

$$r(8R^2 - Rr - 3r^2) \geq (4R^2 + 4Rr + 3r^2)(3r - R) \Leftrightarrow 2R^3 - 5Rr^2 - 6r^3 \geq 0$$

$\Leftrightarrow (R - 2r)(2R^2 + 4Rr + 3r^2) \geq 0$*, obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

The right hand inequality:

Using ***Lemma*** *the right hand inequality can be written:*

$$\frac{R}{r} \cdot \frac{s^2 + 5r^2 + 8Rr}{s^2 + r^2 + 2Rr} \leq \frac{3R}{2r} \Leftrightarrow s^2 \leq 12Rr + 3r^2,$$

which follows from Gerretsen's inequality: $s^2 \geq 16Rr - 5r$.

It remains to prove that:

$$16Rr - 5r^2 \geq 12Rr + 3r^2 \Leftrightarrow R \geq 2r \text{ (Euler's inequality).}$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\sum \frac{h_b + h_c}{r_b + r_c}$ *and* $\sum \frac{r_b + r_c}{h_b + h_c}$ *the relationship can be written:*

5) In ΔABC:

$$\sum \frac{h_b + h_c}{r_b + r_c} \leq \sum \frac{r_b + r_c}{h_b + h_c}$$

Proof.

Using the sums $\sum \frac{h_b + h_c}{r_b + r_c} = 2\left(1 + \frac{r}{R}\right)$ *and* $\sum \frac{r_b + r_c}{h_b + h_c} = \frac{R}{r} \cdot \frac{s^2 + 5r^2 + 8Rr}{s^2 + r^2 + 2Rr}$

we write the inequality: $2\left(1 + \frac{r}{R}\right) \leq \frac{R}{r} \cdot \frac{s^2 + 5r^2 + 8Rr}{s^2 + r^2 + 2Rr} \Leftrightarrow$

$$\Leftrightarrow s^2(R^2 - 2Rr - 2r^2) + r(8R^3 + R^2r - 6Rr^2 - 2r^3) \geq 0$$

We distinguish the following cases:

Case 1) If $(R^2 - 2Rr - 2r^2) \geq 0$*, the inequality is obviously.*

Case 2). If $(R^2 - 2Rr - 2r^2) < 0$*, inequality can be written:*

$$r(8R^3 + R^2r - 6Rr^2 - 2r^3) \geq s^2(2r^2 + 2Rr - r^2),$$

which follows from Gerretsen's inequality: $s^2 \leq 4R^2 + 4Rr + 3r^2$.

It remains to prove that:

$$r(8R^3 + R^2r - 6Rr^2 - 2r^2) \geq (4R^2 + 4Rr + 3r^2)(2r^2 + 2Rr - R^2) \Leftrightarrow$$

$$\Leftrightarrow R^4 + R^3r - 3R^2r^2 - 5Rr^3 - 2r^4 \geq 0 \Leftrightarrow (R - 2r)(R + r)^3 \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

The sequence of inequalities can be written:

6) In ΔABC:

$$\frac{6r}{R} \leq \sum \frac{h_a + h_b}{r_a + r_b} \leq 3 \leq \sum \frac{r_b + r_c}{h_b + h_c} \leq \frac{3R}{2r}$$

Proposed by Marin Chirciu - Romania

Proof.

*See inequalities **2)** and **3)**.*

Equality holds if and only if the triangle is equilateral.

□

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