Romanian Mathematical Magazine
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SECLAMAN INEQUALITY
ROMANIAN MATHEMATICAL MAGAZINE
2018

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1) In ΔABC:

$$\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}\le 2R-r$$

Proposed by Dan Seclaman - Romania

Proof.

We prove the following lemma:

Lemma.
2) In ΔABC:

$$\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}=\frac{3(s^2-r^2-4Rr)}{2(4Rr+r)}$$

Proof.

$$\textit{Using } \sum m_a^2=\frac{3}{4}\sum a^2, \sum a^2=2(s^2-r^2-4Rr), \sum r_a=4R+r.$$

□

Let's get back to the main problem.

*Using the **Lemma** we write the inequality:*

$$\frac{3(s^2-r^2-4Rr)}{2(4R+r)}\le 2R-r \Leftrightarrow 3s^2\le(4R+r)^2$$

which follows from Gerretsen's inequality: $s^2\le 4R^2+4Rr+3r^2$.

It remains to prove that:

$$3(4R^2+4Rr+3r^2)\le(4R+r)^2 \Leftrightarrow R^2-Rr-2r^2\ge 0 \Leftrightarrow (R-2r)(2R+r)\ge 0,$$

obviously from Euler's inequality $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasises an inequality having an opposite sense.

3) In ΔABC:

$$\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}\ge 3r.$$

Proof.

Using the ***Lemma*** *the inequality can be written:*

$$\frac{3(s^2-r^2-4Rr)}{2(4R+r)} \geq 3r \Leftrightarrow s^2 \geq 12Rr+3r^2,$$

which follows from Gerretsen's inequality: $s^2 \geq 16Rr-5r^2$.

It remains to prove that:

$$16Rr-5r^2 \geq 12Rr+3r^2 \Leftrightarrow R \geq 2r \text{ (Euler's inequality)}.$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

4) In ΔABC:

$$\boldsymbol{3r \leq \frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c} \leq 2R-r.}$$

Proof.

See inequalities ***1)*** *and* ***3)****.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Replacing the sum $r_a+r_b+r_c$ *with* $h_a+h_b+h_c$ *we propose:*

5) In ΔABC:

$$\boldsymbol{\frac{3R}{2} \leq \frac{m_a^2+m_b^2+m_c^2}{h_a+h_b+h_c} \leq \frac{3R^2}{4r}}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma.
6) In ΔABC

$$\boldsymbol{\frac{m_a^2+m_b^2+m_c^2}{h_a+h_b+h_c} = 3R \cdot \frac{s^2-r^2-4Rr}{s^2+r^2+4Rr}}$$

Proof.

$$\text{Using } \sum m_a^2 = \frac{3}{4}\sum a^2, \sum a^2 = 2(s^2-r^2-4Rr), \sum h_a = \frac{s^2+r^2+4Rr}{2R}$$

□

Let's get back to the main problem:

The left side inequality:

*Using the **Lemma** the inequality from the left side can be written:*

$$3R\cdot\frac{s^2-r^2-4Rr}{s^2+r^2+4Rr}\geq\frac{3R}{2}\Leftrightarrow s^2\geq 12Rr+3r^2$$

which follows from Gerretsen's inequality: $s^2\geq 16Rr-5r^2$.

It remains to prove that:

$$16Rr-5r^2\geq 12Rr+3r^2\Leftrightarrow R\geq 2r \text{ (Euler's inequality)}$$

Equality holds if and only if the triangle is equilateral.

The inequality from the right:

$$3R\cdot\frac{s^2-r^2-4Rr}{s^2+r^2+4Rr}\leq\frac{3R^2}{4r}\Leftrightarrow s^2(R-4r)+r(4R^2+17Rr+4r^2)\geq 0$$

We distinguish the following cases:

Case 1) If $(R-4r)\geq 0$, *the inequality is obviously.*

Case 2) If $(R-4r)<0$, *the inequality can be rewritten:*

$$r(4R^2+17Rr+4r^2)\geq s^2(4r-R)$$

which follows from Gerretsen's inequality: $s^2\leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$r(4R^2+17Rr+4r^2)\geq(4R^2+4Rr+3r^2)(4r-R)\Leftrightarrow R^3-2R^2r+Rr^2-2r^3\geq 0$$

$\Leftrightarrow (R-2r)(R^2+r^2)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Between the sums $\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}$ *and* $\frac{m_a^2+m_b^2+m_c^2}{h_a+h_b+h_c}$ *we can write the relationship:*

7) In ΔABC:

$$\boldsymbol{\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}\leq\frac{m_a^2+m_b^2+m_c^2}{h_a+h_b+h_c}}$$

Proof.

The inequality is equivalent with:

$$\frac{1}{r_a+r_b+r_c}\leq\frac{1}{h_a+h_b+h_c}\Leftrightarrow h_a+h_b+h_c\leq r_a+r_b+r_c\Leftrightarrow\frac{s^2+r^2+4Rr}{2R}\leq 4R+r\Leftrightarrow$$

$s^2\leq 8R^2-2Rr-r^2$, *which follows from Gerretsen's inequality:* $s^2\leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$4R^2+4Rr+3r^2\leq 8R^2-2Rr-r^2\Leftrightarrow 2R^2-3Rr-2r^2\geq 0\leq(R-2r)(2R+r)\geq 0,$$

obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

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