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Find all $n \in \mathbb{N}$ such that $\Omega(n) \in \mathbb{N}$:

$$\Omega(n) = \sqrt[n]{\left(\log_n\left(\frac{n!}{(n-2)!}\right)^2\right)^2} + \log_n\left(\sqrt{\binom{2n}{3}}\right)$$

Proposed by Ajao Yinka-Nigeria

Solution by Michael Sterghiou-Greece

$$\Omega(n) = \sqrt[n]{\left(\log_n\left(\frac{n!}{(n-2)!}\right)^2\right)^2} + \log_n\left(\sqrt{\binom{2n}{3}}\right) \quad (1)$$

$n \geq 2$ *else (1) is not defined.*

(1) is written as: $(2\log_n[n(n-1)])^{\frac{2}{n}} + \frac{1}{2}\log_n\left[\frac{2}{3}n(n-1)(2n-1)\right]$

or $\underbrace{[2(1+\log_n(n-1))]^{\frac{2}{n}}}_{\Omega_1(n)} + \underbrace{\frac{1}{2}\left(\log_n\frac{2}{3} + 1 + \log_n(n-1) + \log_n(2n-1)\right)}_{\Omega_2(n)} \quad (2)$

$\Omega_1(n) < [2\cdot(1+1)]^{\frac{2}{n}} = 16^{\frac{1}{n}}$ *(as $\log_n(n-1) < 1$). Also,*

$\Omega_1(n) > 1 \ \left(2^{\frac{2}{n}} > 1 \ and \ (1+\log_n(n-1)) > 1\right)$ *so,* $1 < \Omega_1(n) < 16^{\frac{1}{n}}$

For $n > 9 \rightarrow 1 < \Omega_1(n) < \frac{4}{3}$

$$\Omega_2(n) > 1 \textit{ and } \left.\begin{matrix} \log_n \frac{2}{3} < 0 \quad n \ge 2 \\ \log_n(n-1) < 1 \quad n \ge 2 \\ \log_n(2n-1) < \frac{4}{3} \quad n \ge 6 \\ 1 \le 1 \quad n \ge 2 \end{matrix}\right\} \Omega_2(n) < \frac{1}{2}\left(0 + 2 + \frac{4}{3}\right) = \frac{5}{3}$$

Therefore for $n > 9: 2 < \Omega_1(n) + \Omega_2(n) = \Omega(n) < \frac{4}{3} + \frac{5}{3} = 3$

and $\Omega(n)$ ***cannot be natural. By trial and error for all*** $n: 2 \le n \le 9$ ***we conclude that only*** $\Omega(2) = 3 \in \mathbb{N}$

[Answer: $n = 2$***]***