SOME MATRICES IDENTITIES WITH FIBONACCI NUMBERS AND LUCAS NUMBERS

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Abstract. In this paper we present some inequalities with Fibbonaci numbers and Lucas numbers.

Fibonacci sequence: $(F_n)_{n\geq 0}, F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n, \forall n \in \mathbb{N}$.
Lucas sequence: $(L_n)_{n\geq 0}, L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n, \forall n \in \mathbb{N}$.

Theorem 1. Let k be a positive integer $E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $A(k) = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k+1} \end{pmatrix}$.
Then,

$$\prod_{k=1}^{n} A(k) = \left(\prod_{k=2}^{n} F_{k+2}\right) E$$

Proof. If $E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then for any matrix $B = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, $a, b \in \mathbb{R}_+^*$ we have

$$EB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} a+b & b+a \\ a+b & b+a \end{pmatrix} = (a+b)E$$

So, $A(1) = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = E$, and we deduce that

$$A(1)A(2) = E \begin{pmatrix} F_3 & F_2 \\ F_2 & F_3 \end{pmatrix} = (F_2 + F_3)E = F_4 E;$$

$$A(1)A(k) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k+1} \end{pmatrix} = (F_k + F_{k+1})E = F_{k+2}E.$$

Therefore, $A(1)A(2) = F_4E, A(1)A(2)A(3) = F_4EA(3) = F_4F_5E$ and by induction we obtain

$$\prod_{k=1}^{n} A(k) = F_4F_5 \ldots F_nF_{n+1}F_{n+2}E = \left(\prod_{k=2}^{n} F_{k+2}\right) E.$$

□

Theorem 2. Let k be a positive integer, $E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B(k) = \begin{pmatrix} F_k^2 & F_{k+1}^2 \\ F_{k+1}^2 & F_k^2 \end{pmatrix}$.
Then,

$$\prod_{k=1}^{n} B(k) = \left(\prod_{k=2}^{n} F_{2k+1}\right) E$$

Proof. If $E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then for any matrix $B = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, $a, b \in \mathbb{R}_+^*$ we have □

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$$EB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} a+b & b+a \\ a+b & b+a \end{pmatrix} = (a+b)E$$

So, $B(1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = E$, and we deduce that:

$$EB(k) = B(1)B(k) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_k^2 & F_{k+1}^2 \\ F_{k+1}^2 & F_k^2 \end{pmatrix} = (F_k^2 + F_{k+1}^2)E = F_{2k+1}E$$

Therefore, $B(1)B(2) = F_5E, B(1)B(2)B(3) = F_4EB(3) = F_5F_7E$ and by induction we obtain:

$$\prod_{k=1}^{n} B(k) = F_5F_7 \dots F_{2n+1}E = \left(\prod_{k=2}^{n} F_{2k+1} \right) E.$$

Theorem 3. Let k be a positive integer, $E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $C(k) = \begin{pmatrix} F_k & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix} \begin{pmatrix} L_k & L_{k+1} \\ L_{k+1} & L_k \end{pmatrix}$
Then,

$$\prod_{k=1}^{n} C(k) = \left(\prod_{k=1}^{n} L_{k+2} \right) \left(\prod_{k=2}^{n} F_{k+2} \right) E$$

Proof. If $E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, U = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, V = \begin{pmatrix} c & d \\ d & c \end{pmatrix}, a, b, c, d \in \mathbb{R}_+^*$. We have $EU = (a+b)E$ and $EUV = (a+b)(c+d)E$. Then,

$$C(1) = \begin{pmatrix} F_1 & F_2 \\ F_2 & F_1 \end{pmatrix} \begin{pmatrix} L_1 & L_2 \\ L_2 & L_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} L_1 & L_2 \\ L_2 & L_1 \end{pmatrix} = (L_1 + L_2)E = L_3E;$$

$$EC(k) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_k & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix} \begin{pmatrix} L_k & L_{k+1} \\ L_{k+1} & L_k \end{pmatrix} = (F_k + F_{k+1})(L_k + L_{k+1})E = F_{k+2}L_{k+2}E.$$

We obtain $C(1) = L_3E, C(1)C(2) = L_3EC(2) = L_3(EC(2)) = L_3F_4L_4E$;
$C(1)C(2)C(3) = L_3F_4L_4EC(3) = L_3F_4L_4F_5L_5E$, and by induction we deduce that:

$$\prod_{k=1}^{n} C(k) = \left(\prod_{k=1}^{n} L_{k+2} \right) \left(\prod_{k=2}^{n} F_{k+2} \right) E.$$

□

Theorem 4. Let k be a positive integer, $E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $D(k) = \begin{pmatrix} F_k & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix} \begin{pmatrix} F_k & F_k \\ F_{k+2} & F_{k+2} \end{pmatrix}$
Then,

$$\prod_{k=1}^{n} D(k) = L_2 \left(\prod_{k=2}^{n} F_{k+2}L_{k+1} \right) E.$$

Proof. We have
$D(1) = \begin{pmatrix} F_1 & F_2 \\ F_2 & F_1 \end{pmatrix} \begin{pmatrix} F_1 & F_1 \\ F_3 & F_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_1 & F_1 \\ F_3 & F_3 \end{pmatrix} = (F_1 + F_3)E = L_2E$ and we've used the well-known fact that $F_1 + F_3 = L_2$. Also, we have:

$$ED(k) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_k & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix} \begin{pmatrix} F_k & F_k \\ F_{k+2} & F_{k+2} \end{pmatrix} = (F_k + F_{k+1})(F_k + F_{k+2})E = F_{k+2}L_{k+1}E$$

where, we've used the well-known fact that: $F_k + F_{k+2} = L_{k+1}$. We obtain:

$$\prod_{k=1}^{n} D(k) = D(1)\prod_{k=2}^{n} D(k) = L_2 E \prod_{k=2}^{n} D(k) = L_2\left(\prod_{k=2}^{n} F_{k+2}L_{k+1}\right)E.$$

□

Theorem 5. Let k be a positive integer, $E = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ and $E(k) = \begin{pmatrix}F_k & F_{k+1}\\ F_{k+1} & F_k\end{pmatrix}\begin{pmatrix}L_{k+2} & L_k\\ L_k & L_{k+2}\end{pmatrix}$ Then,

$$\prod_{k=1}^{n} E(k) = 5^n \cdot \prod_{k=1}^{n} F_{k+1} \cdot \prod_{k+2}^{n} F_{k+2} \cdot E$$

Proof. We have
$E(1) = \begin{pmatrix}F_1 & F_2\\ F_2 & F_1\end{pmatrix}\begin{pmatrix}L_3 & L_1\\ L_1 & L_3\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}L_3 & L_1\\ L_1 & L_3\end{pmatrix} = (L_1+L_3)E = 5F_2\cdot E$, where we've used the fact that $L_1 + L_3 = 5F_2$. Also, we have:

$$E\cdot E(k) = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}F_k & F_{k+1}\\ F_{k+1} & F_k\end{pmatrix}\begin{pmatrix}L_{k+2} & L_k\\ L_k & L_{k+2}\end{pmatrix} = (F_k+F_{k+1})(L_k+L_{k+2})E = F_{k+2}\cdot 5F_{k+1}E$$

where we've used the well-known fact that $L_k + L_{k+2} = 5F_{k+1}$. Therefore,

$$\prod_{k=1}^{n} E(k) = E(1)\prod_{k=2}^{n} E(k) = \ldots = 5F_2(5F_3F_4)(5F_4F_5)\cdot\ldots\cdot(5F_{n+1}F_{n+2})E =$$

$$= 5^n \cdot \prod_{k=1}^{n} F_{k+1} \cdot \prod_{k=2}^{n} F_{k+2} \cdot E.$$

□

Theorem 6. Let k be a positive integer,
$E = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ and $F(k) = \begin{pmatrix}F_k^2 & F_{k+1}^2\\ F_{k+1}^2 & F_k^2\end{pmatrix}\begin{pmatrix}L_{k+1} & L_k\\ L_k & L_{k+1}\end{pmatrix}$. Then,

$$\prod_{k=1}^{n} F(k) = L_3 \cdot \prod_{k=3}^{n+2} L_k \cdot \prod_{k=2}^{n} F_{2k+1} \cdot E$$

Proof. We have $F(1) = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}L_2 & L_1\\ L_1 & L_2\end{pmatrix} = (L_1 + L_2)E = L_3 \cdot E$. Also, we have:

$$E\cdot F(k) = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}F_k^2 & F_{k+1}^2\\ F_{k+1}^2 & F_k^2\end{pmatrix}\begin{pmatrix}L_{k+1} & L_k\\ L_k & L_{k+1}\end{pmatrix} = (F_k^2+F_{k+1}^2)(L_k+L_{k+1})E = F_{2k+1}L_{k+2}E$$

where we've used the well-known fact that $F_k^2 + F_{k+1}^2 = F_{2k+1}$. Therefore,

$$\prod_{k=1}^{n} F(k) = F(1)\prod_{k=2}^{n} F(k) = L_3 \cdot (E \cdot F(2)) \cdot F(3) \cdot \ldots \cdot F(n) =$$

$$= L_3\cdot F_5\cdot L_3\cdot(E\cdot F(3))\cdot F(4)\cdot F(5)\cdot\ldots\cdot F(n) = \ldots = L_3F_5L_3F_7L_5F_9L_7\cdot\ldots\cdot(EF(n-1))F(n) =$$

$$= L_3F_5L_3F_7L_5F_9L_7\ldots L_{n+1}F_{2n-1}EF(n) = L_3F_5L_3F_7L_5F_9\ldots L_{n+1}F_{2n-1}F_{2n+1}L_{n+2} =$$

$$= L_3 \cdot \prod_{k=3}^{n+2} L_k \cdot \prod_{k=2}^{n} F_{2k+1} \cdot E.$$

□

Theorem 7. Let $(F_k)_{k\geq 0}, F_0 = 0, F_1 = 1, F_{k+2} = F_k + F_{k+1}, \forall k \in \mathbb{N}$;
$A = (a_{ij})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}, B = (b_{ik})_{\substack{1\leq j\leq n\\ 1\leq k\leq m}}, C = (c_{rs})_{1\leq r,s\leq m}, a_{ij} = F_j \cdot b_{jk} = F_j$,
$c_{rs} = F^2_{m-r+1}, \forall i, k = \overline{1, m}, \forall j = \overline{1, n}$ and $E = (e_{rs})_{1\leq r,s\leq m}, e_{rs} = 1, \forall r, s = \overline{1, m}$.
For p, q positive integers, then $(A \cdot B)^p \cdot C^q = m^{p-1} F^p_n F^q_m F^{p+1}_{n+1} F^{q+1}_{m+1}, \forall p, q \in \mathbb{N}^*$.

Proof. We have:

$$A = \begin{pmatrix} F_1 & F_2 & \dots & F_n \\ F_1 & F_2 & \dots & F_n \\ \dots & \dots & \dots & \dots \\ F_1 & F_2 & \dots & F_n \end{pmatrix}, B = \begin{pmatrix} F_1 & F_1 & \dots & F_1 \\ F_2 & F_2 & \dots & F_2 \\ \dots & \dots & \dots & \dots \\ F_n & F_n & \dots & F_n \end{pmatrix}, C = \begin{pmatrix} F^2_m & F^2_m & \dots & F^2_m \\ F^2_{m-1} & F^2_{m-1} & \dots & F^2_{m-1} \\ \dots & \dots & \dots & \dots \\ F^2_1 & F^2_1 & \dots & F^2_1 \end{pmatrix}$$

Then, $A \cdot B = \left(\sum_{k=1}^n F^2_k\right) E = F_n F_{n+1} E$, so,

$$(A \cdot B)^p = F^p_n F^p_{n+1} E^p = m^{p-1} F^p_n F^p_{n+1} E \tag{1}$$

Also, we have $E \cdot C = \left(\sum_{k=1}^m F^2_k\right) E = F_m F_{m+1} E$, so

(2)
$$EC^q = (EC)C^{q-1} = F_m F_{m+1} EC^{q-1} = F_m F_{m+1}(EC)C^{q-2} = F^2_m F^2_{m+1} EC^{q-2} = \dots = F^q_m F^q_{m+1} E$$

By (1) and (2) yields that $(A \cdot B)^p \cdot C^q = m^{p-1} F^p_n F^q_m F^{p+1}_{n+1} F^{q+1}_{m+1} E, \forall p, q \in \mathbb{N}^*$. □

Theorem 8. Let $(F_k)_{k\geq 0}, F_0 = 0, F_1 = 1, F_{k+2} = F_k + F_{k+1}, A = (a_{ij})_{1\leq i,j\leq n}$,
$B = (b_{ij})_{1\leq i,j\leq n}, C = (c_{ij})_{1\leq i,j\leq n}, a_{ij} = F_j, b_{ij} = F_i, c_{ij} = F^2_{\sigma(i)}, \forall i, j = \overline{1, n}$ and σ is a permutation of the set $1, 2, \dots, n$. For m, n positive integers and
$E = (e_{ij})_{1\leq i,j\leq n}, e_{ij} = 1, \forall i, j = \overline{1, n}$, then

$$(A \cdot B)^m \cdot C^p = n^{m-1} F^{m+p}_n F^{m+p}_{n+1} E, \forall m, p \in \mathbb{N}^*.$$

Proof. We have:

$$A = \begin{pmatrix} F_1 & F_2 & \dots & F_n \\ F_1 & F_2 & \dots & F_n \\ \dots & \dots & \dots & \dots \\ F_1 & F_2 & \dots & F_n \end{pmatrix}, B = \begin{pmatrix} F_1 & F_1 & \dots & F_1 \\ F_2 & F_2 & \dots & F_2 \\ \dots & \dots & \dots & \dots \\ F_n & F_n & \dots & F_n \end{pmatrix}$$

$$C = \begin{pmatrix} F^2_{\sigma(1)} & F^2_{\sigma(1)} & \dots & F^2_{\sigma(1)} \\ F^2_{\sigma(2)} & F^2_{\sigma(2)} & \dots & F^2_{\sigma(2)} \\ \dots & \dots & \dots & \dots \\ F^2_{\sigma(n)} & F^2_{\sigma(n)} & \dots & F^2_{\sigma(n)} \end{pmatrix}$$

Then $A \cdot B = (\sum_{k=1}^n F^2_k) E = F_n F_{n+1} E$, so,

$$(A \cdot B)^m = F^m_n F^m_{n+1} E^m = n^{m-1} F^m_n F^m_{n+1} E \tag{1}$$

Also, we have $E \cdot C = (\sum_{k=1}^n F^2_k) E = F_n F_{n+1} E$, so,

(2)
$$EC^p = (EC)C^{p-1} = F_n F_{n+1} EC^{p-1} = F_n F_{n+1}(EC)C^{p-2} = F^2_n F^2_{n+1} EC^{p-2} = \dots = F^p_n F^p_{n+1} E$$

By (1) and (2) yields that $(A \cdot B)^m \cdot C^p = n^{m-1} F^{m+p}_n F^{m+p}_{n+1} E, \forall m, p \in \mathbb{N}^*$ □

Theorem 10. Let $(F_k)_{k\geq 0}, F_0 = 0, F_1 = 1, F_{k+2} = F_k + F_{k+1}, \forall k \in \mathbb{N}$ and $A = (a_{ij})_{1\leq i,j\leq n}, B = (b_{ij})_{1\leq i,j\leq n}, C = (c_{ij})_{1\leq i,j\leq n}, a_{ij} = F_j, b_{ij} = F_i,$ $c_{ij} = F_{n-i+1}^2, \forall i,j = \overline{1,n}$. For m, n positive integers and $E = (e_{ij})_{1\leq i,j\leq n}$ then $(A\cdot B)^m \cdot C^p = n^{m-1}F_n^{m+p}F_{n+1}^{m+p}E, \forall m,n \in p \in \mathbb{N}^*$.

Proof. We have:

$$A = \begin{pmatrix} F_1 & F_2 & \dots & F_n \\ F_1 & F_2 & \dots & F_n \\ \dots & \dots & \dots & \dots \\ F_1 & F_2 & \dots & F_n \end{pmatrix}, B = \begin{pmatrix} F_1 & F_1 & \dots & F_1 \\ F_2 & F_2 & \dots & F_2 \\ \dots & \dots & \dots & \dots \\ F_n & F_n & \dots & F_n \end{pmatrix}$$

$$C = \begin{pmatrix} F_n^2 & F_n^2 & \dots & F_n^2 \\ F_{n-1}^2 & F_{n-1}^2 & \dots & F_{n-1}^2 \\ \dots & \dots & \dots & \dots \\ F_{n-1}^2 & F_{n-1}^2 & \dots & F_{n-1}^2 \end{pmatrix}$$

Then, $A\cdot B = \left(\sum_{k=1}^n F_k^2\right)E = F_nF_{n+1}E$, so,

$$(A\cdot B)^m = F_n^m F_{n+1}^m E^m = n^{m-1}F_n^m F_{n+1}^m E \tag{1}$$

Also, we have $E\cdot C = \left(\sum_{k=1}^n F_k^2\right)E = F_nF_{n+1}E$, so,

(2)
$$EC^p = (EC)C^{p-1} = F_nF_{n+1}EC^{p-1} = F_nF_{n+1}(EC)C^{p-2} = F_n^2F_{n+1}^2EC^{p-2} = \dots = F_n^pF_{n+1}^pE$$

By (1) and (2) yields that $(A\cdot B)^m \cdot C^p = n^{m-1}F_n^{m+p}F_{n+1}^{m+p}E, \forall m,p \in \mathbb{N}^*$. □

Theorem 11. Let $(F_k)_{k\geq 0}, F_0 = 0, F_1 = 1, F_{k+2} = F_k + F_{k+1}, \forall k \in \mathbb{N}$ and $A = (a_{ij})_{1\leq i,j\leq n}, B = (b_{ij})_{1\leq i,j\leq n}, C = (c_{ij})_{1\leq i,j\leq n}, a_{ij} = F_j, b_{ij} = F_i,$ $c_{ij} = F_{n-i+1}^2, \forall i,j = \overline{1,n}$. For m positive integers and $E = (e_{ij})_{1\leq i,j\leq n}$, $e_{ij} = 1, \forall i,j = \overline{1,n}$, then $(A\cdot B)^m \cdot C = n^{m-1}F_n^{m+1}F_{n+1}^{m+1}E, \forall m,p \in \mathbb{N}^*$.

Proof. We have:

$$A = \begin{pmatrix} F_1 & F_2 & \dots & F_n \\ F_1 & F_2 & \dots & F_n \\ \dots & \dots & \dots & \dots \\ F_1 & F_2 & \dots & F_n \end{pmatrix}, B = \begin{pmatrix} F_1 & F_1 & \dots & F_1 \\ F_2 & F_2 & \dots & F_2 \\ \dots & \dots & \dots & \dots \\ F_n & F_n & \dots & F_n \end{pmatrix}$$

$$C = \begin{pmatrix} F_n^2 & F_n^2 & \dots & F_n^2 \\ F_{n-1}^2 & F_{n-1}^2 & \dots & F_{n-1}^2 \\ \dots & \dots & \dots & \dots \\ F_{n-1}^2 & F_{n-1}^2 & \dots & F_{n-1}^2 \end{pmatrix}$$

Then, $A\cdot B = \left(\sum_{k=1}^n F_k^2\right)E = F_nF_{n+1}E$, where we've used the well-known $\sum_{k=1}^n F_k^2 = F_nF_{n+1}$, so

$$(A\cdot B)^m = F_n^m F_{n+1}^m E^m = n^{m-1}F_n^m F_{n+1}^m E \tag{1}$$

(2) Also, we have $E\cdot C = \left(\sum_{k=1}^n F_k^2\right)E = F_nF_{n+1}E$

By (1) and (2) yields that $(A \cdot B)^m \cdot C = n^{m-1} F_n^{m+1} F_{n+1}^{m+1} E, \forall m, p \in \mathbb{N}^*$. □

Theorem 12. Let $(F_k)_{k \geq 0}, F_0 = 0, F_1 = 1, F_{k+2} = F_k + F_{k+1}, \forall k \in \mathbb{N}$ and $A = (a_{ij})_{1 \leq i,j \leq n}, B = (b_{ij})_{1 \leq i,j \leq n}, C = (c_{ij})_{1 \leq i,j \leq n}, a_{ij} = F_j, b_{ij} = F_i$, $c_{ij} = F_{2i-1}, \forall i, j = \overline{1,n}$. For m, p positive integers and $E = (e_{ij})_{1 \leq i,j \leq n}, e_{ij} = 1, \forall i, j = \overline{1,n}$, then $(A \cdot B)^M \cdot C^p = n^{m-1} F_n^M F_{n+1}^m F_{2n}^p E, \forall m, p \in \mathbb{N}^*$.

Proof. We have:

$$A = \begin{pmatrix} F_1 & F_2 & \dots & F_n \\ F_1 & F_2 & \dots & F_n \\ \dots & \dots & \dots & \dots \\ F_1 & F_1 & \dots & F_n \end{pmatrix}, B = \begin{pmatrix} F_1 & F_1 & \dots & F_1 \\ F_2 & F_2 & \dots & F_2 \\ \dots & \dots & \dots & \dots \\ F_n & F_n & \dots & F_n \end{pmatrix}$$

$$C = \begin{pmatrix} F_1 & F_1 & \dots & F_1 \\ F_3 & F_3 & \dots & F_3 \\ \dots & \dots & \dots & \dots \\ F_{2n-1} & F_{2n-1} & \dots & F_{2n-1} \end{pmatrix}$$

Then, $A \cdot B = \left(\sum_{k=1}^n F_k^2 \right) E = F_n F_{n+1} E$, so,

$$(A \cdot B)^m = F_n^m F_{n+1}^m E^m = n^{m-1} F_n^m F_{n+1}^m E \tag{1}$$

Also, we have $E \cdot C = \left(\sum_{k=1}^n F_{2k-1} \right) E = F_{2n} E$, so,

$$EC^p = (EC)C^{p-1} = F_{2n} EC^{p-1} = \dots F_{2n}^p E \tag{2}$$

By (1) and (2) yields that $(A \cdot B)^m \cdot C^p = n^{m-1} F_n^m F_{n+1}^m F_{2n}^p E, \forall m, p \in \mathbb{N}^*$. □

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