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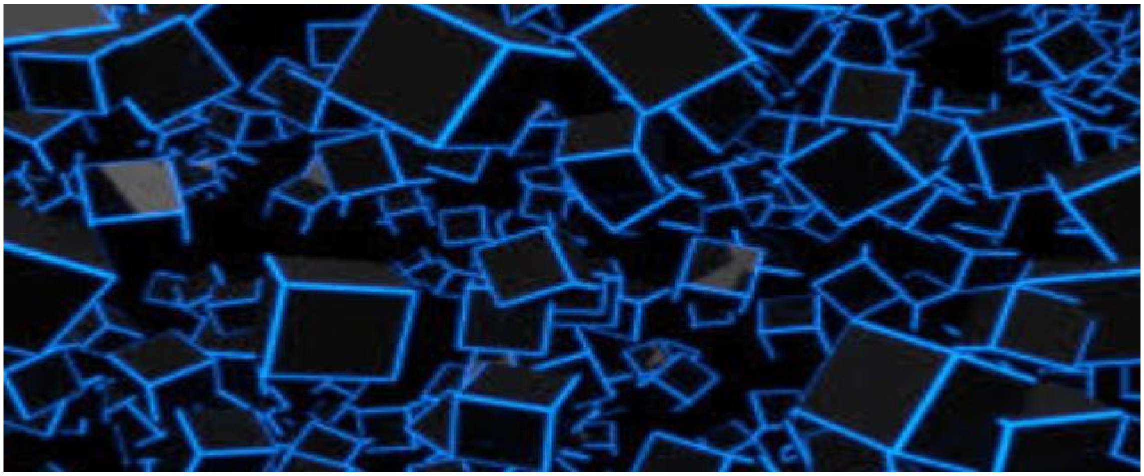

UP.151. *Given real numbers* $a_1, a_2, \dots, a_n \in [0,1]$.

Find the maximum and minimum possible value of

$$a_1 + a_2 + \dots + a_n + (1-a_1)(1-a_2)\dots(1-a_n).$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by proposer, Solution 2 by Michael Sterghiou-Greece

Solution 1 by proposer

We fix the variables $a_2, a_3, \dots, a_n$ *and rewrite the given expression as follows*

$$f(a_1) = [1-(1-a_2)\dots(1-a_n)]a_1 + (1-a_2)\dots(1-a_n) + (a_2 + \dots + a_n)$$

Now, we consider the following two cases:

Case 1. If $a_2, \dots, a_n$ *are all zero, then* $f(a_1) = 1$.

Case 2. If there is at least one of these numbers $a_2, \dots, a_n$ *is non zero. Then* $f(a_1)$ *is a first degree function (which is called linear function) with the variable* a_1 *and the leading coefficient is positive, hence* $f(0) \le f(a_1) \le f(1)$. *Similarly, we also have* $f(0) \le f(a_i) \le f(1)$ *for each* $i \in \{2, 3, \dots, n\}$. *Let* $F(a_1, a_2, \dots, a_n)$ *denotes the given expression, then we obtain* $1 = F(0, 0, \dots, 0) \le F(a_1, a_2, \dots, a_n) \le F(1, 1, \dots, 1) = n$.

Combining the above two cases we deduce that $\min F(a_1, a_2, \dots, a_n) = 1$ *when* $(a_1, a_2, \dots, a_n) = (x, 0, \dots, 0)$ *and its permutations;* $\max F(a_1, a_2, \dots, a_n) = n$ *when* $(a_1, a_2, \dots, a_n) = (1, 1, \dots, 1)$

Solution 2 by Michael Sterghiou-Greece

$$A=a_1+a_2+\cdots+a_n+(1-a_1)(1-a_2)\cdot\ldots\cdot(1-a_n)\quad (1)$$

From Weierstrass product inequality we have $\prod_{k=1}^{n}(1-a_k)\ge 1-\sum_{k=1}^{n}a_k$ *(*)* *which gives immediately* $\min A=1$ *when* $a_k=0\ \forall k\in\{1,2,\ldots,n\}$*. Now,*

$\prod_{k=1}^{n}(1-a_k)\le\left[\frac{\sum_{k=1}^{n}(1-a_k)}{n}\right]^n$ *from AM-GM given* $1-a_k\ge 0\ \ \forall k$ *or*

$\prod_{k=1}^{n}(1-a_k)\le\left(1-\frac{S_n}{n}\right)^n$ *where* $S_n=\sum_{k=1}^{n}a_k$ *(1) becomes* $A\le S_n+\left(1-\frac{S_n}{n}\right)^n$

Let $S_n=x, x\in[0,n]: f(x)=x+\left(1-\frac{x}{n}\right)^n$*.* $f'(x)=1-\left(1-\frac{x}{n}\right)^{n-1}>0$ *as* $\frac{x}{n}\le 1$,

$0\le\left(1-\frac{x}{n}\right)^{n-1}\le 1$ *so* $f(x)\uparrow$*. This means* $f(x)\le f(n)$ *or*

$S_n+\left(1-\frac{S_n}{n}\right)^n\le n+\left(1-\frac{n}{n}\right)^n=n$*. Therefore* $A\le n$ *and* $A_{\max}=n$ *when* $a_k=1$ $\forall k\in\{1,2,\ldots,n\}$

** proved easily by induction over* n*.*