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SP.155. Let x,y,z be positive real numbers such that: $xyz=1$. Find the minimum value of:

$$P=\frac{x^3}{(2y^2-yz+2z^2)^2}+\frac{y^3}{(2z^2-zx+2x^2)^2}+\frac{z^3}{(2x^2-xy+2y^2)^2}+\frac{xy+yz+zx}{3}\quad (1)$$

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Solution by proposer

*** Lemma: Let a,b,c be positive real numbers we have inequality:**

$$a^4+b^4+c^4+abc(a+b+c)\geq ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\quad (2)$$

(2): $a^4+b^4+c^4+abc(a+b+c)\geq ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)$

$$\Leftrightarrow a^4+b^4+c^4+abc(a+b+c)-ab(a^2+b^2)-bc(b^2+c^2)-ca(c^2+a^2)\geq 0$$

$$\Leftrightarrow a^2(a^2-ab-ac+bc)+b^2(b^2-bc-ba+ca)+c^2(c^2-ca-cb+ab)\geq 0$$

$$\Leftrightarrow a^2(a-b)(a-c)+b^2(b-a)(b-c)+c^2(c-a)(c-b)\geq 0\quad (3)$$

- Supposed $a\geq b\geq c>0$.

+ We have: $\begin{cases}c\leq a\\ c\leq b\end{cases}\Leftrightarrow\begin{cases}c-a\leq 0\\ c-b\leq 0\end{cases}\Rightarrow (c-a)(c-b)\geq 0\Leftrightarrow c^2(c-a)(c-b)\geq 0\quad (4)$

+ Let: $a^2(a-b)(a-c)+b^2(b-a)(b-c)=(a-b)[a^2(a-c)-b^2(b-c)]$

$$\Leftrightarrow a^2(a-b)(a-c)+b^2(b-a)(b-c)=(a-b)[(a^3-b^3)-c(a^2-b^2)]$$

$$=(a-b)[(a-b)(a^2+ab+b^2)-c(a-b)(a+b)]$$

$$= (a-b)(a-b)(a^2+ab+b^2-ac-bc)$$

$$= (a-b)^2(a^2+ab+b^2-ac-bc) \quad \textbf{(5)}$$

– ***Because*** $a \geq b \geq c > 0$ ***then*** $a-c \geq 0; b-c \geq 0$

\+ ***Hence:*** $a^2+ab+b^2-ac-bc = a(a-c)+b(b-c)+ab \geq ab > 0; (a-b)^2 \geq 0; \forall a,b \in \mathbb{R}$

$\Rightarrow (a-b)^2(a^2+ab+b^2-ac-bc) \geq 0$. ***Let (5):*** $\Rightarrow a^2(a-b)(a-c)+b^2(b-a)(b-c) \geq 0$ ***(6)***

- Let (4), (6): $\Rightarrow a^2(a-b)(a-c)+b^2(b-a)(b-c)+c^2(c-a)(c-b) \geq 0$

$\Rightarrow$ ***Inequality (3) true*** $\Rightarrow$ ***(2) true and lemma get the result.***

**** Let*** $(a,b,c) = (x,y,z)$:

$$\Rightarrow x^4+y^4+z^4+xyz(x+y+z) \geq xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2) \quad \textbf{(7)}$$

- By AM-GM inequality we have:

$$xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2) \geq xy\cdot 2xy+yz\cdot 2yz+zx\cdot 2zx = 2(x^2y^2+y^2z^2+z^2x^2)$$

$$\Leftrightarrow xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2) \geq 2(x^2y^2+y^2z^2+z^2x^2) \quad \textbf{(8)}$$

- Let (7), (8): $\Rightarrow x^4+y^4+z^4+xyz(x+y+z) \geq 2(x^2y^2+y^2z^2+z^2x^2)$

$$\Leftrightarrow x^4+y^4+z^4+2(x^2y^2+y^2z^2+z^2x^2) \geq 4(x^2y^2+y^2z^2+z^2x^2)-xyz(x+y+z)$$

$$\Leftrightarrow (x^2+y^2+z^2)^2 \geq 4(x^2y^2+y^2z^2+z^2x^2)-xyz(x+y+z)$$

$$\Leftrightarrow \frac{(x^2+y^2+z^2)^2}{4(x^2y^2+y^2z^2+z^2x^2)-xyz(x+y+z)} \geq 1 \quad \textbf{(9)}$$

**** By Cauchy Schwarz inequality we have:***

$$\frac{x^3}{(2y^2-yz+2z^2)^2}+\frac{y^3}{(2z^2-zx+2x^2)^2}+\frac{z^3}{(2x^2-xy+2y^2)^2}$$

$$= \frac{\left(\frac{x^2}{2y^2-yz+2z^2}\right)^2}{x}+\frac{\left(\frac{y^2}{2z^2-zx+2x^2}\right)^2}{y}+\frac{\left(\frac{z^2}{2x^2-xy+2y^2}\right)^2}{z} \geq$$

$$\geq \frac{\left(\frac{x^2}{2y^2-yz+2z^2}+\frac{y^2}{2z^2-zx+2x^2}+\frac{z^2}{2x^2-xy+2y^2}\right)^2}{x+y+z} \quad \textbf{(10)}$$

- Other: $\frac{x^2}{2y^2-yz+2z^2}+\frac{y^2}{2z^2-zx+2x^2}+\frac{z^2}{2x^2-xy+2y^2}$

$$= \frac{x^4}{2x^2y^2-x^2yz+2x^2z^2}+\frac{y^4}{2y^2z^2-y^2zx+2y^2x^2}+\frac{z^4}{2z^2x^2-z^2xy+2z^2y^2} \geq$$

$$\ge \frac{(x^2+y^2+z^2)^2}{(2x^2y^2-x^2yz+2x^2z^2)+(2y^2z^2-y^2zx+2y^2x^2)+(2z^2+x^2-z^2xy+2z^2y^2)}$$

$$\Leftrightarrow \frac{x^2}{2y^2-yz+2z^2}+\frac{y^2}{2z^2-zx+2x^2}+\frac{z^2}{2x^2-xy+2y^2} \ge \frac{(x^2+y^2+z^2)^2}{4(x^2y^2+y^2z^2+z^2x^2)-xyz(x+y+z)} \quad (11)$$

- Let (9), (11): $\Rightarrow \frac{x^2}{2y^2-yz+2z^2}+\frac{y^2}{2z^2-zx+2x^2}+\frac{z^2}{2x^2-xy+2y^2} \ge 1 \quad (12)$

- Let (10), (12):

$$\Rightarrow \frac{x^3}{(2y^2-yz+2z^2)^2}+\frac{y^3}{(2z^2-zx+2x^2)^2}+\frac{z^3}{(2x^2-xy+2y^2)^2} \ge \frac{1}{x+y+z}$$

$$\Rightarrow P = \frac{x^3}{(2y^2-yz+2z^2)^2}+\frac{y^3}{(2z^2-zx+2x^2)^2}+\frac{z^3}{(2x^2-xy+2y^2)^2}+\frac{xy+yz+zx}{3} \ge$$

$$\ge \frac{1}{x+y+z}+\frac{xy+yz+zx}{3} \quad (13)$$

- By inequality: $(mn+np+pm)^2 \ge 3mnp(m+n+p)$ **and AM-GM inequality and:** $xyz=1$**. We have:**

$$\frac{1}{x+y+z}+\frac{xy+yz+zx}{3} = \left(\frac{1}{x+y+z}+\frac{xy+yz+zx}{9}+\frac{xy+yz+zx}{9}\right)+\frac{xy+yz+zx}{9} \ge$$

$$\ge 3\cdot\sqrt[3]{\frac{1}{x+y+z}\cdot\frac{xy+yz+zx}{9}\cdot\frac{xy+yz+zx}{9}}+\frac{3\cdot\sqrt[3]{xy\cdot yz\cdot zx}}{9}$$

$$\Rightarrow \frac{1}{x+y+z}+\frac{xy+yz+zx}{3} \ge 3\cdot\sqrt[3]{\frac{(xy+yz+zx)^2}{81(x+y+z)}}+\frac{3\sqrt[3]{(xyz)^2}}{9} \ge 3\sqrt[3]{\frac{3xyz(x+y+z)}{81(x+y+z)}}+\frac{3\cdot 1}{9}$$

$$\Rightarrow \frac{1}{x+y+z}+\frac{xy+yz+zx}{3} \ge 3\sqrt[3]{\frac{3\cdot 1}{81}}+\frac{3}{9} = 1+\frac{1}{3} = \frac{4}{3} \Leftrightarrow \frac{1}{x+y+z}+\frac{xy+yz+zx}{3} \ge \frac{4}{3} \quad (14)$$

- Let (13), (14): $\Rightarrow P \ge \frac{4}{3} \Rightarrow P_{\min} = \frac{4}{3}$**. Equality occurs if:**

$$\Leftrightarrow \begin{cases} x=y=z>0 \\ xyz=1 \\ \frac{1}{2y^2-yz+2z^2}=\frac{1}{2z^2-zx+2x^2}=\frac{1}{2x^2-xy+2y^2} \\ \frac{1}{x+y+z}=\frac{xy+yz+zx}{9} \end{cases} \Leftrightarrow x=y=z=1.$$