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Find:

$$\Omega = \lim_{n\to\infty}\left(\log(2n+1) - \sum_{k=1}^{n}\left(\frac{1}{k[\sqrt{k}]}\cdot\left[\frac{\sqrt{k}+\sqrt{k+1}+\sqrt{k+2}}{3}\right]\right)\right),$$

$[*]$ **- GIF function**

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Solution by Pierre Mounir-Cairo-Egypt

By Jensen's inequality, we have:

$$\frac{\sqrt{k}+\sqrt{k+2}}{2} < \sqrt{\frac{k+(k+2)}{2}} \Rightarrow \sqrt{k}+\sqrt{k+2} < 2\sqrt{k+1} \Rightarrow$$

(We could've proved it using $\left(\sqrt{k+2}-\sqrt{k}\right)^2 > 0$***)***

$$\sqrt{k}+\sqrt{k+1}+\sqrt{k+2} < 3\sqrt{k+1} \Rightarrow$$

$$\sqrt{k} = \frac{\sqrt{k}+\sqrt{k}+\sqrt{k}}{3} < \frac{\sqrt{k}+\sqrt{k+1}+\sqrt{k+2}}{3} < \sqrt{k+1} < \sqrt{k}+1$$

$$\because x < y \Rightarrow [x] < [y]$$

$$\therefore [\sqrt{k}] < \left[\frac{\sqrt{k}+\sqrt{k+1}+\sqrt{k+2}}{3}\right] < [\sqrt{k}+1] = [\sqrt{k}]+1$$

$$\because \left[\frac{\sqrt{k}+\sqrt{k+1}+\sqrt{k+2}}{3}\right] \in \mathbb{Z} \Rightarrow \left[\frac{\sqrt{k}+\sqrt{k+1}+\sqrt{k+2}}{3}\right] = [\sqrt{k}]$$

$$\therefore \Omega = \lim_{n\to\infty}\left\{\ln(2n+1) - \sum_{k=1}^{n}\frac{1}{k[\sqrt{k}]}\left[\frac{\sqrt{k}+\sqrt{k+1}+\sqrt{k+2}}{3}\right]\right\}$$

$$= \lim_{n\to\infty}\left\{\ln(2n+1) - \ln n + \ln n - \sum_{k=1}^{n}\frac{1}{k}\right\}$$

$$= \lim_{n\to\infty}\ln\left(2+\frac{1}{n}\right) + \lim_{n\to\infty}\left\{\ln n - \sum_{k=1}^{n}\frac{1}{k}\right\}$$

$$= \ln 2 - \gamma$$