FEW SOLUTIONS AND TWO REFINEMENTS FOR BOGDAN FUSTEI'S INEQUALITY

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Abstract. In this paper we present 5 solutions and a refinement for an inequality proposed by Bogdan Fustei in Romanian Mathematical Magazine, March 2018:

In any triangle ABC the following relationship holds:

$$\sqrt{\frac{a}{s-a}}+\sqrt{\frac{b}{s-b}}+\sqrt{\frac{c}{s-c}}\geq 3\sqrt{2}$$

Solution 1 by Daniel Sitaru - Romania.

$$a=y+z, b=z+x, c=x+y$$

$$3\sqrt{2}=\frac{3}{\sqrt{xyz}}\cdot\sqrt{2xyz}=\frac{3}{\sqrt{xyz}}\cdot\sqrt[6]{8x^3y^3z^3}=\frac{3}{\sqrt{xyz}}\cdot\sqrt[6]{x^2y^2z^2\cdot 8xyz}\leq$$

$$\overbrace{\leq}^{CESARO}\frac{3}{\sqrt{xyz}}\cdot\sqrt[6]{x^2y^2z^2\cdot(y+z)(z+x)(x+y)}=$$

$$=\frac{1}{\sqrt{xyz}}\cdot 3\sqrt[3]{\sqrt{yz(y+z)}\cdot\sqrt{zx(z+x)}\cdot\sqrt{xy(x+y)}}\overbrace{\leq}^{GM\text{-}AM}\frac{1}{\sqrt{xyz}}\cdot\sum\sqrt{yz(y+z)}=$$

$$=\sum\sqrt{\frac{y+z}{x}}=\sum\sqrt{\frac{a}{s-a}}=\sqrt{\frac{a}{s-a}}+\sqrt{\frac{b}{s-b}}+\sqrt{\frac{c}{s-c}}$$

□

Solution 2 by Mehmet Sahin - Ankara - Turkey.
Using the function $f(x)=\sqrt{\frac{x}{s-x}}$, where $s=\frac{a+b+c}{2}$, $f(x)$ is a convex function in $(0,s)$. Using Jensen's Inequality:

$$f\Big(\frac{a+b+c}{3}\Big)\leq\frac{1}{3}[f(a)+f(b)+f(c)]$$

$$f(a)+f(b)+f(c)\geq 3f\Big(\frac{a+b+c}{3}\Big)\geq 3\sqrt{\frac{\frac{a+b+c}{3}}{\frac{a+b+c}{2}-\frac{a+b+c}{3}}}\geq 3\sqrt{2}\therefore$$

□

Solution 3 by Bogdan Fustei - Romania.
ABC any triangle. Starting from $\cos\frac{B-C}{2}\geq\sqrt{\frac{2r}{R}}$ (and analogs)

$$\cos\frac{B-C}{2}=\frac{h_a}{w_a}\geq\sqrt{\frac{2r}{R}}\Rightarrow\frac{1}{w_a}\geq\frac{1}{h_a}\sqrt{\frac{2r}{R}}\text{ (and analogs)}$$

$$\Rightarrow \frac{1}{w_a}+\frac{1}{w_c}+\frac{1}{w_c}\geq\sqrt{\frac{2r}{R}}\Big(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\Big);\text{ But }\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{R}$$

$$\Rightarrow \frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\geq\frac{1}{r}\sqrt{\frac{2r}{R}}=\sqrt{\frac{2r}{Rr^2}}=\sqrt{\frac{2}{Rr}}$$

We have the following inequality:

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\geq\sqrt{\frac{2}{Rr}}$$

$$\left.\begin{array}{l}\cos\frac{B-C}{2}=\frac{b+c}{a}\sin\frac{A}{2}\\ \sin\frac{A}{2}=\sqrt{\frac{rr_a}{2Rh_a}}\end{array}\right\}\Rightarrow\cos\frac{B-C}{2}=\frac{b+c}{a}\sqrt{\frac{r}{2R}}\cdot\sqrt{\frac{r_a}{h_a}}\geq\sqrt{\frac{2r}{R}}$$

$$\frac{b+c}{a}\sqrt{\frac{r_a}{h_a}}\geq\sqrt{2}\cdot\sqrt{2}=2$$

We have the inequality: $\sqrt{\frac{r_a}{h_a}}\geq 2\cdot\frac{a}{b+c}$ (and the analogs).

$$\sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{r_b}{h_b}}+\sqrt{\frac{r_c}{h_c}}\geq 2\Big(\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\Big)$$

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}\text{ - Nesbit's Inequality }\Rightarrow\sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{r_b}{h_b}}+\sqrt{\frac{r_c}{h_c}}\geq 3$$

We know the inequality:

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{w_a}{r_b+r_c}+\frac{w_b}{r_a+r_c}+\frac{w_c}{r_a+r_c}\Rightarrow$$

$$\Rightarrow\sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{r_b}{h_b}}+\sqrt{\frac{r_c}{h_c}}\geq 2\Big(\frac{w_a}{r_b+r_c}+\frac{w_b}{r_a+r_c}+\frac{w_c}{r_a+r_c}\Big)$$

$$\frac{r_a}{h_a}=\frac{S}{s-a}\cdot\frac{1}{h_a}=\frac{h_a\cdot a}{2(s-a)}\cdot\frac{1}{h_a}=\frac{a}{2(s-a)}\Rightarrow\frac{r_a}{h_a}=\frac{a}{2(s-a)}\text{ (and the analogs)}$$

$$\Rightarrow\sqrt{\frac{r_a}{h_a}}=\frac{1}{\sqrt{2}}\sqrt{\frac{a}{s-a}}\Rightarrow\sqrt{\frac{a}{s-a}}+\sqrt{\frac{b}{s-b}}+\sqrt{\frac{c}{s-c}}\geq 3\sqrt{2}$$

$$\Rightarrow\sqrt{\frac{a}{s-a}}+\sqrt{\frac{b}{s-b}}+\sqrt{\frac{c}{s-c}}\geq 2\sqrt{2}\Big(\frac{w_a}{r_b+r_c}+\frac{w_b}{r_a+r_c}+\frac{w_c}{r_a+r_b}\Big)\geq 2\sqrt{2}\cdot\frac{3}{2}=3\sqrt{2}$$

□

Solution 4 by Neculai Stanciu, Titu Zvonaru - Romania.

After squaring, the inequality we have to prove can be written:

$$\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}+2\left(\sqrt{\frac{ab}{(s-a)(s-b)}}+\sqrt{\frac{bc}{(s-b)(s-c)}}+\sqrt{\frac{ca}{(s-c)(s-a)}}\right)\geq 18.$$

$\Leftrightarrow \frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}+2\Big(\csc\frac{A}{2}+\csc\frac{B}{2}+\csc\frac{C}{2}\Big)\geq 18$, which follows from the known

$$\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\geq 6\text{ and }\csc\frac{A}{2}+\csc\frac{B}{2}+\csc\frac{C}{2}\geq 6$$

(see point 2.51 from ”Geometric Inequalities” of O. Bottema, Groningen, 1969).

For completeness, the inequality $\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\geq 6$ it follows from the identity

$$\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}-6=\frac{(a-b)^2}{(s-a)(s-b)}+\frac{(s-c)^2}{(s-b)(s-c)}+\frac{(c-a)^2}{(s-c)(s-a)}$$

□

Solution 5 by Neculai Stanciu, Titu Zvonaru - Romania.

We will use the notation $\sum$ for $\sum\limits_{cyc}$. We have:

$$\sum\sqrt{\frac{a}{s-a}}-3\sqrt{2}=\sum\left(\sqrt{\frac{a}{s-a}}-\sqrt{2}\right)=\sum\frac{\frac{a}{s-a}-2}{\sqrt{\frac{a}{s-a}}+\sqrt{2}}=$$

$$=\sum\frac{2a-b-c}{\sqrt{s(s-a)}+(s-a)\sqrt{2}}=\sum\frac{a-b}{\sqrt{a(s-a)}+(s-a)\sqrt{2}}+\sum\frac{a-c}{\sqrt{a(s-a)}+(s-a)\sqrt{2}}=$$

$$=\sum\frac{a-b}{\sqrt{a(s-a)}+(s-a)\sqrt{2}}+\sum\frac{s-a}{\sqrt{b(s-b)}+(s-b)\sqrt{2}}=$$

$$=\sum\frac{(a-b)[\sqrt{b(s-b)}-\sqrt{a(s-a)}+(s-b)\sqrt{2}-(s-a)\sqrt{2}]}{(\sqrt{a(s-a)}+(s-a)\sqrt{2})(\sqrt{b(s-b)}+(s-b)\sqrt{2})}$$

Because $\sqrt{s(s-b)}-\sqrt{a(s-a)}=\dfrac{s(s-b)-a(s-a)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}=\dfrac{a^2-b^2-s(a-b)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}=$

$$=\frac{(a-b)(a+b-s)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}=\frac{(a-b)(s-c)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}\text{ and}$$

$(s-b)\sqrt{2}-(s-a)\sqrt{2}=(a-b)\sqrt{2}$, it follows that:

$$\sum\sqrt{\frac{a}{s-a}}-3\sqrt{2}=\sum\frac{(a-b)^2\left(\frac{s-c}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}+\sqrt{2}\right)}{(\sqrt{a(s-a)}+(s-a)\sqrt{2})(\sqrt{b(s-b)}+(s-b)\sqrt{2})}\geq 0\text{ and}$$

the desired inequality is proved. We have equality if and only if $a=b=c$.

Remark. Thus more elaborate, the second solution allows obtaining a refinement of the inequality from enunciation. Because:

$$(\sqrt{a(s-a)}+(s-a)\sqrt{2})(\sqrt{b(s-b)}+(s-b)\sqrt{2})=$$

$$=\sqrt{(s-a)(s-b)}(\sqrt{ab}+\sqrt{2a(s-b)}+\sqrt{2b(s-a)}+\sqrt{2(s-a)\cdot 2(s-b)})\leq$$

$$\leq\sqrt{(s-a)(s-b)}\cdot\frac{a+b+2a+s-b+2b+s-a+2s-2a+2s-2b}{2}=$$

$=3s\sqrt{(s-a)(s-b)}$, we deduce the inequality:

$$\sum\sqrt{\frac{a}{s-a}}-3\sqrt{2}\geq\frac{\sqrt{2}}{3s}\sum\frac{(s-b)^2}{\sqrt{(s-a)(s-b)}}$$

□

Refinement by Neculai Stanciu, Titu Zvonaru - Romania

Prove that in any triangle the following inequality holds:

$$\sum\sqrt{\frac{a}{s-a}}-3\sqrt{2}\geq\frac{\sqrt{2}}{3s}\sum\frac{(a-b)^2}{\sqrt{(s-a)(s-b)}}$$

Proof.

$$\sum\sqrt{\frac{a}{s-a}}-3\sqrt{2}=\sum\Big(\sqrt{\frac{a}{s-a}}-\sqrt{2}\Big)=\sum\frac{\frac{a}{s-a}-2}{\sqrt{\frac{a}{s-a}}+\sqrt{2}}=$$

$$=\sum\frac{2a-b-c}{\sqrt{a(s-a)}+(s-a)\sqrt{2}}=\sum\frac{a-b}{\sqrt{a(s-a)}+\sqrt{(s-a)\sqrt{2}}}+\sum\frac{a-c}{\sqrt{a(s-a)}+(s-a)\sqrt{2}}=$$

$$=\sum\frac{a-b}{\sqrt{a(s-a)}+(s-a)\sqrt{2}}+\sum\frac{b-a}{\sqrt{b(s-b)}+(s-b)\sqrt{2}}=$$

$$=\sum\frac{(a-b)[\sqrt{b(s-b)}-\sqrt{a(s-a)}+(s-b)\sqrt{2}-(s-a)\sqrt{2}]}{(\sqrt{a(s-a)}+(s-a)\sqrt{2})(\sqrt{b(s-b)}+(s-b)\sqrt{2})}$$

Because $\sqrt{b(s-b)}-\sqrt{a(s-a)}=\dfrac{b(s-b)-a(s-a)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}=\dfrac{a^2-b^2-s(s-b)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}=$

$$=\frac{(a-b)(a+b-p)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}=\frac{(a-b)(s-c)}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}\text{ and}$$

$(s-b)\sqrt{2}-(s-a)\sqrt{2}=(a-b)\sqrt{2}$, it follows that:

$$\sum\sqrt{\frac{a}{s-a}}-3\sqrt{2}=\sum\frac{(a-b)^2\Big(\frac{s-c}{\sqrt{a(s-a)}+\sqrt{b(s-b)}}+\sqrt{2}\Big)}{(\sqrt{a(s-a)}+(s-a)\sqrt{2})(\sqrt{b(s-b)}+(s-b)\sqrt{2})}\geq 0$$

Because:

$$(\sqrt{a(s-a)}+(s-a)\sqrt{2})(\sqrt{b(s-b)}+(s-b)\sqrt{2})=$$

$$=\sqrt{(s-a)(s-b)}(\sqrt{ab}+\sqrt{2a(s-b)}+\sqrt{2b(s-a)}+\sqrt{2(s-a)\cdot 2(s-b)})\leq$$

$$\leq\sqrt{(s-a)(s-b)}\cdot\frac{a+b+2a+s-b+2b+s-a+2s-2a+2s-2b}{2}=$$

$=3s\sqrt{(s-a)(s-b)}$, we deduce the inequality:

$$\sum\sqrt{\frac{a}{s-a}}-3\sqrt{2}\geq\frac{\sqrt{2}}{3s}\sum\frac{(a-b)^2}{\sqrt{(s-a)(s-b)}},\text{ q.e.d.}$$

We have equality if and only if $a=b=c$. □

Remark. The inequality proposed above refines the inequality:

$$\sqrt{\frac{a}{s-a}}+\sqrt{\frac{b}{s-b}}+\sqrt{\frac{c}{s-c}}\geq 3\sqrt{2}.$$

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