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If $x, y > 0, xy \geq \frac{1}{8}$ then:

$$\frac{x^2}{\sin\frac{3\pi}{11}} + \frac{y^2}{\sin\frac{4\pi}{11}} > \frac{1}{\left(\cos\frac{2\pi}{11} + \sin\frac{5\pi}{11}\right)^2}$$

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By BCS inequality and AM-GM inequality, we have:

$$\frac{x^2}{\sin\frac{3\pi}{11}} + \frac{y^2}{\sin\frac{4\pi}{11}} \geq \frac{(x+y)^2}{\sin\frac{3\pi}{11}+\sin\frac{4\pi}{11}} \geq \frac{4xy}{\sin\frac{3\pi}{11}+\sin\frac{4\pi}{11}} \geq \frac{1}{2\left(\sin\frac{3\pi}{11}+\sin\frac{4\pi}{11}\right)} \quad (\text{Since } xy \geq \tfrac{1}{8}) \ (1)$$

We need to prove that $\frac{1}{2\left(\sin\frac{3\pi}{11}+\sin\frac{4\pi}{11}\right)} > \frac{1}{\left(\cos\frac{2\pi}{11}+\cos\frac{5\pi}{11}\right)^2}$ *(2)*

Put $t = \frac{\pi}{11} \Rightarrow 11t = \pi \Rightarrow 4t = \pi - 7t \Rightarrow \sin 4t = \sin(\pi - 7t) = \sin 7t$

We have inequality (2) equivalent to $\frac{1}{2(\sin 3t+\sin 4t)} > \frac{1}{(\cos 2t+\sin 5t)^2}$ *(3)*

We have $(\cos 2t + \sin 5t)^2 = \left(\sin\left(\frac{\pi}{2} - 2t\right) + \sin 5t\right)^2 = \left(2\sin\left(\frac{\pi}{4} + \frac{3t}{2}\right)\cos\left(\frac{\pi}{4} - \frac{7t}{2}\right)\right)^2$

$$\Rightarrow (\cos 2t + \sin 5t)^2 = 4\sin^2\left(\frac{\pi}{4} + \frac{3t}{2}\right)\cos^2\left(\frac{\pi}{5} - \frac{7t}{2}\right) = \left[1 - \cos\left(\frac{\pi}{2} + 3t\right)\right]\left[1 + \cos\left(\frac{\pi}{2} - 7t\right)\right]$$

$$\Rightarrow (\cos 2t + \sin 5t)^2 = (1 + \sin 3t)(1 + \sin 7t) \ (4)$$

We have $(\sin\sin 3t - 1)(\sin 7t - 1) > 0 \Rightarrow \sin 3t \cdot \sin 7t - \sin 3t - \sin 7t + 1 > 0$

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$$\Rightarrow \sin 3t \cdot \sin 7t + \sin 3t + \sin 7t + 1 > 2(\sin 3t + \sin 7t)$$

$$\Rightarrow (1 + \sin 3t)(1 + \sin 7t) > 2(\sin 3t + \sin 7t) \Rightarrow (1 + \sin 3t)(1 + \sin 7t) > 2(\sin 3t + \sin 4t)$$

$$\overset{(4)}{\Rightarrow} (\cos 2t + \sin 5t)^2 > 2(\sin 3t + \sin 4t) \Rightarrow \textit{(3) true} \Rightarrow \textit{(2) true}$$

$$\textit{From (1) and (2)} \Rightarrow \frac{x^2}{\sin\frac{3\pi}{11}} + \frac{y^2}{\sin\frac{4\pi}{11}} > \frac{1}{\left(\cos\frac{2\pi}{11}+\sin\frac{5\pi}{11}\right)^2} \quad \textit{(Q.E.D.)}$$