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If $2\sin^2 x + 2\sin^2 y = 1, x, y \in \left(0, \frac{\pi}{2}\right)$ ***then:***

$$2\tan x \tan y + 2\tan x + 2\tan y < 3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Vietnam, Solution 2 by Khanh Hung Vu-Vietnam

Solution 1 by Tran Hong-Vietnam

$$\sin^2 x + \sin^2 y = \frac{1}{2} \Leftrightarrow \frac{1}{2} = \frac{\tan^2 x}{1 + \tan^2 x} + \frac{\tan^2 y}{1 + \tan^2 y} \overset{(Schwarz)}{\geq}$$

$$\geq \frac{(\tan x + \tan y)^2}{2 + (\tan x + \tan y)^2 - 2\tan x \tan y} \Leftrightarrow 2(\tan x + \tan y)^2 \leq$$

$$\leq 2 + (\tan x + \tan y)^2 - 2\tan x \tan y \Leftrightarrow (\tan x + \tan y)^2 + 2\tan x \tan y \leq 2$$

$$\Rightarrow \tan x + \tan y + \tan x \tan y \leq \tan x + \tan y + \frac{2 - (\tan x + \tan y)^2}{2} = \frac{-u^2 + 2u + 2}{2} \leq \frac{3}{2}$$

$$\forall u \in (0; 2) \ \left(u = \tan x + \tan y, 0 < x, y < \frac{\pi}{4} \Rightarrow 0 < u < 2\right)$$

Solution 2 by Khanh Hung Vu-Vietnam

If $2\sin^2 x + 2\sin^2 y = 1, x, y \in \left(0; \frac{\pi}{2}\right)$ **then** $2\tan x \tan y + 2\tan x + 2\tan y < 3$ ***(1)***

We have $2\sin^2 x + 2\sin^2 y = 1 \Rightarrow \sin^2 x + \sin^2 y = \frac{1}{2} \Rightarrow 1 - \cos^2 x + 1 - \cos^2 y = \frac{1}{2}$

$$\Rightarrow \cos^2 x + \cos^2 y = \frac{3}{2} \Rightarrow \frac{1}{1+\tan^2 x} + \frac{1}{1+\tan^2 y} = \frac{3}{2} \quad (2)$$

Put $\tan x = a, \tan y = b \Rightarrow a, b \in (0; +\infty)$

We have the equation (2) equivalent to:

$$\frac{1}{1+a^2} + \frac{1}{1+b^2} = \frac{3}{2} \Rightarrow \frac{a^2+b^2+2}{a^2b^2+a^2+b^2+1} = \frac{3}{2} \Rightarrow$$

$$\Rightarrow 2(a^2+b^2+2) = 3(a^2b^2+a^2+b^2+1) \Rightarrow$$

$$\Rightarrow 3a^2b^2 + a^2 + b^2 = 1 \Rightarrow 3a^2b^2 + (a+b)^2 - 2ab = 1 \quad (3)$$

On the other hand, we have

$$(a+b)^2 \geq 4ab \Rightarrow -3a^2b^2 + 2ab + 1 \geq 4ab \Rightarrow -3a^2b^2 - 2ab + 1 \geq 0 \Rightarrow$$

$\Rightarrow 0 < ab \leq \frac{1}{3}$***. That means the equation (3) is equivalent to***

$a + b = \sqrt{-3a^2b^2 + 2ab + 1}$***. We have the inequality (1) equivalent to***

$$2ab + 2a + 2b < 3 \Rightarrow 2ab + 2\sqrt{-3a^2b^2 + 2ab + 1} < 3 \Rightarrow$$

$$\Rightarrow 2\sqrt{-3a^2b^2 + 2ab + 1} < 3 - 2ab \Rightarrow 4(-3a^2b^2 + 2ab + 1) < 4a^2b^2 - 12ab + 9 \Rightarrow$$

$$\Rightarrow 16a^2b^2 - 20ab + 5 > 0 \Rightarrow 16\left(ab - \frac{5+\sqrt{5}}{8}\right)\left(ab - \frac{5-\sqrt{5}}{8}\right) > 0$$

(True since $ab - \frac{5+\sqrt{5}}{8} < 0$ ***and*** $ab - \frac{5-\sqrt{5}}{8} < 0$ ***by*** $0 < ab \leq \frac{1}{3}$***)***

So, (1) is true $\Rightarrow 2\tan x \tan y + 2\tan x + 2\tan y < 3$